Data Mining 2021
Logistic Regression
Text Classification

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Two types of approaches to classification

In (probabilistic) classification we are interested in the conditional distribution

$$P(Y \mid x),$$

so that, for example, when we observe $X = x$ we can predict the class y with the highest probability for that value of X.

There are two basic approaches to modeling $P(Y \mid x)$:

- Generative Models (use Bayes' rule):

$$P(Y = y \mid x) = \frac{P(x \mid Y = y)P(Y = y)}{P(x)} = \frac{P(x \mid Y = y)P(Y = y)}{\sum_{y'} P(x \mid Y = y')P(Y = y')}$$

- Discriminative Models: model $P(Y \mid x)$ directly.

Generative Models

Examples of generative classification methods:

- Naive Bayes classifier (discussed in the previous lecture)
- Linear/Quadratic Discriminant Analysis (not discussed)
- ...

Discriminative Models

Discriminative methods only model the *conditional* distribution of Y given X. The probability distribution of X itself is not modeled.

For the binary classification problem:

$$P(Y = 1 \mid X) = f(X, \beta)$$

where $f(X, \beta)$ is some function of features X and parameters β.

Discriminative Models

Examples of discriminative classification methods:

- Linear probability model
- Logistic regression
- Feed-forward neural networks
- ...

Discriminative Models: linear probability model

Consider the linear regression model

$$\mathbb{E}[Y \mid x] = \beta^\top x \qquad\qquad Y \in \{0, 1\},$$

where

$$\beta^\top x = \sum_{j=0}^{m} \beta_j x_j, \qquad\qquad \text{with } x_0 \equiv 1.$$

But

$$\begin{aligned} \mathbb{E}[Y \mid x] &= 1 \cdot P(Y = 1 \mid x) + 0 \cdot P(Y = 0 \mid x) \\ &= P(Y = 1 \mid x) \end{aligned}$$

So the model assumes that

$$P(Y = 1 \mid x) = \beta^\top x$$

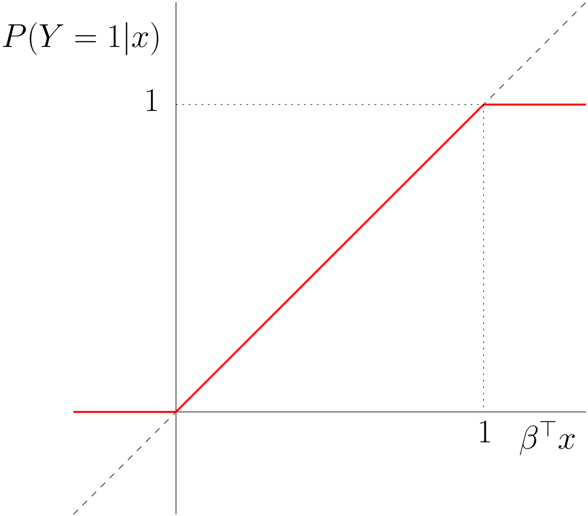
Notation

$$\beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} \qquad x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{pmatrix}$$

with $x_0 \equiv 1$, so

$$\beta^\top x = \sum_{j=0}^{m} \beta_j x_j = \beta_0 + \beta_1 x_1 + \ldots + \beta_m x_m$$

Linear response function



Logistic regression

The linear probability model allows negative "probabilities" and "probabilities" bigger than one.

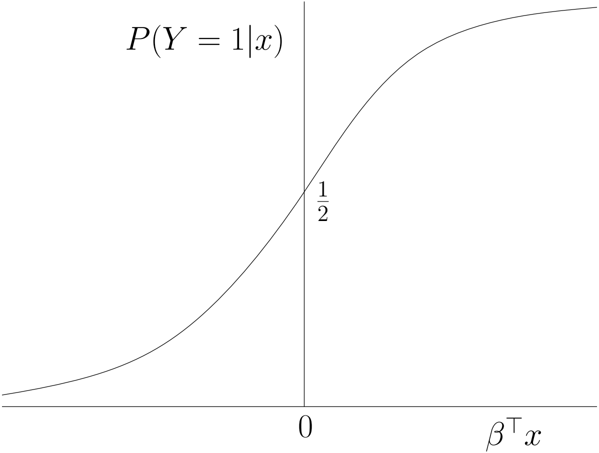
Logistic response function

$$\mathbb{E}[Y \mid x] = P(Y = 1 \mid x) = \frac{e^{\beta^\top x}}{1 + e^{\beta^\top x}}$$

or (divide numerator and denominator by $e^{\beta^\top x}$)

$$P(Y = 1 \mid x) = \frac{1}{1 + e^{-\beta^\top x}} = (1 + e^{-\beta^\top x})^{-1}$$

Logistic Response Function



Linearization: the logit transformation

Since $P(Y = 1 \mid x)$ and $P(Y = 0 \mid x)$ have to add up to 1, we have:

$$P(Y = 1 \mid x) = \frac{e^{\beta^\top x}}{1 + e^{\beta^\top x}} \quad \Rightarrow \quad P(Y = 0 \mid x) = \frac{1}{1 + e^{\beta^\top x}}$$

Hence,

$$\frac{P(Y = 1 \mid x)}{P(Y = 0 \mid x)} = e^{\beta^\top x}$$

Therefore:

$$\ln \left\{ \frac{P(Y = 1 \mid x)}{P(Y = 0 \mid x)} \right\} = \beta^\top x$$

The ratio

$$\frac{P(Y = 1 \mid x)}{P(Y = 0 \mid x)}$$

is called the *odds*.

Linear Separation

Assign to class 1 if $P(Y = 1 \mid x) > P(Y = 0 \mid x)$, i.e. if

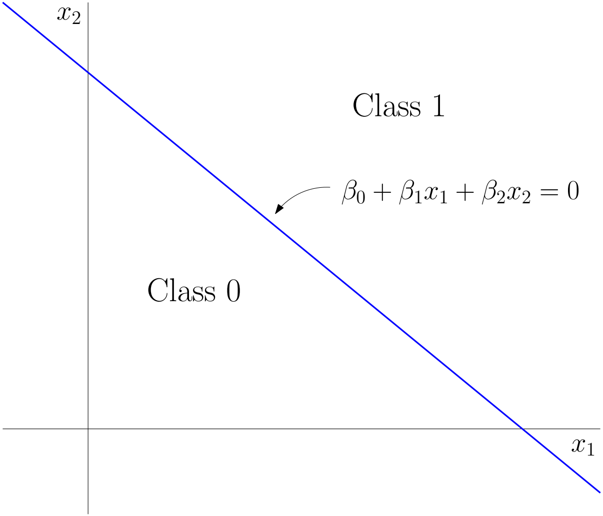
$$\frac{P(Y = 1 \mid x)}{P(Y = 0 \mid x)} > 1$$

This is true if

$$\ln \left\{ \frac{P(Y = 1 \mid x)}{P(Y = 0 \mid x)} \right\} > 0$$

So assign to class 1 if $\beta^\top x > 0$, and to class 0 otherwise.

Linear Decision Boundary



Maximum Likelihood Estimation

Coin tossing example:

$Y = 1$ if heads, $Y = 0$ if tails. $p = P(Y = 1)$.
One coin flip

$$P(y) = p^y (1 - p)^{1-y}$$

Note that $P(1) = p$, $P(0) = 1 - p$ as required.
Sequence of n independent coin flips

$$P(y_1, y_2, ..., y_n) = \prod_{i=1}^{n} p^{y_i} (1 - p)^{1-y_i}$$

which defines the likelihood function when viewed as a function of p.

Maximum Likelihood Estimation

In a sequence of 10 coin flips we observe $y = (1, 0, 1, 1, 0, 1, 1, 1, 1, 0)$.

The corresponding likelihood function is

$$\begin{aligned} L(y \mid p) &= p \cdot (1-p) \cdot p \cdot p \cdot (1-p) \cdot p \cdot p \cdot p \cdot p \cdot (1-p) \\ &= p^7 (1-p)^3 \end{aligned}$$

The corresponding log-likelihood function is

$$\ell(y \mid p) = \ln L(y \mid p) = \ln(p^7(1-p)^3) = 7 \ln p + 3 \ln(1-p)$$

Find the value of p that maximizes this function.

Computing the maximum

To determine that value, we take the derivative, equate it to zero and solve for p.

Recall that

$$\frac{d \ln x}{dx} = \frac{1}{x}$$

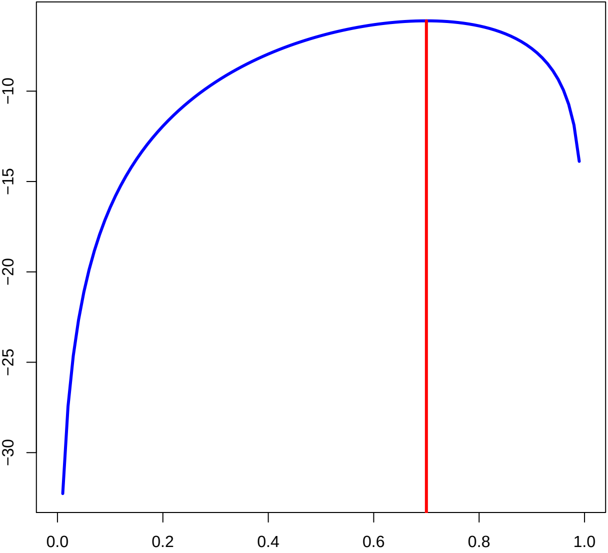
So the derivative of the log-likelihood function with respect to p is:

$$\frac{d\ell(y \mid p)}{dp} = \frac{7}{p} - \frac{3}{1-p}$$

Equating to zero, and solving for p yields maximum likelihood estimate $\hat{p} = 0.7$.

This is just the relative frequency of heads in the sample!

Log-likelihood function for $y = (1, 0, 1, 1, 0, 1, 1, 1, 1, 0)$



ML estimation for logistic regression

Logistic regression is similar to the coin tossing example, except that now the probability of success p_i depends on x_i and β:

$$\begin{aligned} p_i &= P(Y=1 \mid x_i) = (1+e^{-\beta^\top x_i})^{-1} \\ 1-p_i &= P(Y=0 \mid x_i) = (1+e^{\beta^\top x_i})^{-1} \end{aligned}$$

we can represent its probability distribution as follows

$$P(y_i) = p_i^{y_i}(1-p_i)^{1-y_i} \qquad\qquad y_i \in \{0,1\};\quad i=1,\ldots,n$$

ML estimation for logistic regression

Example

i	x_i	y_i	$P(y_i)$
1	8	0	$(1+e^{\beta_0+8\beta_1})^{-1}$
2	12	0	$(1+e^{\beta_0+12\beta_1})^{-1}$
3	15	1	$(1+e^{-\beta_0-15\beta_1})^{-1}$
4	10	1	$(1+e^{-\beta_0-10\beta_1})^{-1}$

The likelihood function is:

$$(1+e^{\beta_0+8\beta_1})^{-1} \times (1+e^{\beta_0+12\beta_1})^{-1} \times (1+e^{-\beta_0-15\beta_1})^{-1} \times (1+e^{-\beta_0-10\beta_1})^{-1}$$

ML Estimation: find values of β_0 and β_1 that maximize this probability.

Logistic Regression: likelihood function

Since the y_i observations are assumed to be independent (e.g. random sampling):

$$P(y) = \prod_{i=1}^{n} P(y_i) = \prod_{i=1}^{n} p_i^{y_i}(1 - p_i)^{1-y_i}$$

Or, taking the natural log:

$$\begin{aligned} \ln P(y) &= \ln \prod_{i=1}^{n} p_i^{y_i}(1 - p_i)^{1-y_i} \\ &= \sum_{i=1}^{n} \{y_i \ln p_i + (1 - y_i) \ln(1 - p_i)\} \end{aligned}$$

Logistic Regression: log-likelihood function

For the logistic regression model we have

$$
\begin{aligned}
p_i &= (1 + e^{-\beta^\top x_i})^{-1} \\
1 - p_i &= (1 + e^{\beta^\top x_i})^{-1}
\end{aligned}
$$

so filling in gives

$$
\ell(y \mid \beta) = \sum_{i=1}^{n} \left\{ y_i \ln \left(\frac{1}{1 + e^{-\beta^\top x_i}} \right) + (1 - y_i) \ln \left(\frac{1}{1 + e^{\beta^\top x_i}} \right) \right\}
$$

- Non-linear function of the parameters.
- No closed form solution (no nice formulas for the parameter estimates).
- Likelihood function globally concave so relatively easy optimization problem (no local maxima).

Fitted Response Function

Substitute maximum likelihood estimates into the response function to obtain the *fitted response function*

$$\hat{P}(Y = 1 \mid x) = \frac{e^{\hat{\beta}^\top x}}{1 + e^{\hat{\beta}^\top x}}$$

Example: Programming Assignment

Model the probability of successfully completing a programming assignment.

Explanatory variable: "programming experience".
We find $\hat{\beta}_0 = -3.0597$ and $\hat{\beta}_1 = 0.1615$, so

$$\hat{P}(Y = 1 \mid x) = \frac{e^{-3.0597+0.1615x}}{1 + e^{-3.0597+0.1615x}}$$

14 months of programming experience:

$$\hat{P}(Y = 1 \mid x = 14) = \frac{e^{-3.0597+0.1615(14)}}{1 + e^{-3.0597+0.1615(14)}} \approx 0.31$$

Interpretation

We have

$$\ln\left\{\frac{\hat{P}(Y=1\mid x)}{\hat{P}(Y=0\mid x)}\right\} = -3.0597 + 0.1615x,$$

so with every additional month of programming experience, the log odds increase with 0.1615.

The odds are multiplied by $e^{0.1615} \approx 1.175$ so with every additional month of programming experience, the odds increase with 17.5%.

When x increases with one unit, the odds are multiplied by e^{β_1} because:

$$e^{\beta_0+\beta_1(x+1)} = e^{\beta_0+\beta_1 x+\beta_1} = e^{\beta_0+\beta_1 x} \times e^{\beta_1},$$

since $e^{a+b} = e^a \times e^b$.

Interpretation

Note that the effect of an increase in x on the probability of success depends on the value of x:

- An increase from 14 to 24 months of programming experience leads to an increase of the probability of success from 0.31 to 0.69.
- An increase from 34 to 44 months of programming experience leads to an increase of the probability of success from 0.92 to 0.98.

Example: Programming Assignment

	month.exp	success	fitted
1	14	0	0.310262
2	29	0	0.835263
3	6	0	0.109996
4	25	1	0.726602
5	18	1	0.461837
6	4	0	0.082130
7	18	0	0.461837
8	12	0	0.245666
9	22	1	0.620812
10	6	0	0.109996
11	30	1	0.856299
12	11	0	0.216980
13	30	1	0.856299
14	5	0	0.095154
15	20	1	0.542404

	month.exp	success	fitted
16	13	0	0.276802
17	9	0	0.167100
18	32	1	0.891664
19	24	0	0.693379
20	13	1	0.276802
21	19	0	0.502134
22	4	0	0.082130
23	28	1	0.811825
24	22	1	0.620812
25	8	1	0.145815

Allocation Rule

Probability of the classes is equal when

$$-3.0597 + 0.1615x = 0$$

Solving for x we get $x \approx 18.95$.

Allocation Rule:

$x \geq 19$: *predict* $y = 1$

$x < 19$: *predict* $y = 0$

If a person has 19 months or more programming experience, predict success, otherwise predict failure.

Programming Assignment: Confusion Matrix

Cross table of observed and predicted class label:

	0	1
0	11	3
1	3	8

Row: observed, Column: predicted

Error rate: 6/25=0.24

Default (predict majority class): 11/25=0.44

How to in R

```
> prog.logreg <- glm(success ~ month.exp, data=prog.dat, family=binomial)
> summary(prog.logreg)


Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -3.05970    1.25935  -2.430   0.0151 *
month.exp    0.16149    0.06498   2.485   0.0129 *
---

Number of Fisher Scoring iterations: 4


> table(prog.dat$success, as.numeric(prog.logreg$fitted > 0.5))


     0  1
  0 11  3
  1  3  8
```

Regularization

- If we have a large number of predictors, even a linear model estimated with maximum likelihood can be prone to overfitting.
- This can be controlled by punishing large (positive or negative) weights. The coefficient estimates are *shrunken* towards zero.
- Add a penalty term for the size of the coefficients to the objective function.
- With LASSO penalty:

$$E(\beta) = -\ell(\beta) + \lambda \sum_{j=1}^{m} |\beta_j|,$$

  where $E(\beta)$ is the new error function that we want to minimize, and $-\ell(\beta)$ is the negative log-likelihood function.
- The value of λ is usually selected by cross-validation.

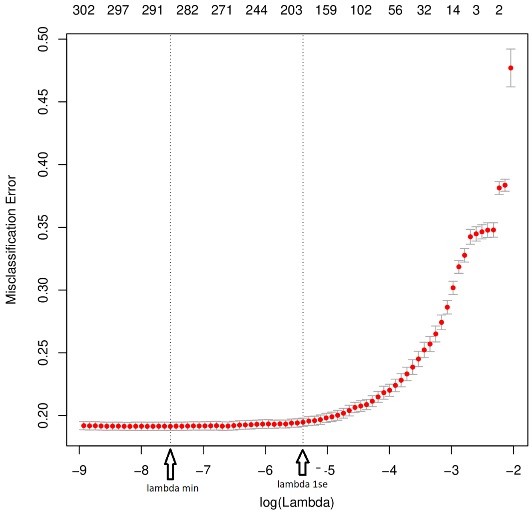
Application of Logistic Regression to Movie Reviews

```
# logistic regression with lasso penalty

> reviews.glmnet <- cv.glmnet(as.matrix(train.dtm),labels[index.train],
                      family="binomial",type.measure="class")
> plot(reviews.glmnet)

> coef(reviews.glmnet,s="lambda.1se")
309 x 1 sparse Matrix of class "dgCMatrix"
                         1
bad           -0.613843496
beautiful      0.378249156
best           0.400765691
better        -0.193594713
boring        -0.904918921
excellent      0.874061528
fun            0.390055537
funny          .
minutes       -0.381871597
perfect        0.757174138
poor          -0.726663951
script        -0.461754268
stupid        -0.555516834
supposed      -0.611473721
terrible      -0.830472064
wonderful      0.697696588
worst         -1.431738320
```

Cross-Validation on lambda



Application of Logistic Regression to Movie Reviews

```
# make predictions on the test set
> reviews.logreg.pred <- predict(reviews.glmnet,
   newx=as.matrix(test.dtm),s="lambda.1se",type="class")
# show confusion matrix
> table(reviews.logreg.pred,labels[-index.train])

reviews.logreg.pred    0    1
                  0 3468  704
                  1 1032 3796
# compute accuracy: about 81% correct
> (3468+3796)/9000
[1] 0.8071111
```

Including Bigrams

The bigrams in

```
the spy who loved me
```

are:

```
the spy
spy who
who loved
loved me
```

but not for example

```
spy loved
```

The two words need to be next to each other.

Including Bigrams

```
# extract both unigrams and bigrams
> train.dtm2 <- DocumentTermMatrix(reviews.all[index.train],
    control = list(tokenize = UniBiTokenizer))
# more than one million uni+bigrams!
> dim(train.dtm2)
[1]   16000 1346555
# remove terms that occur in less than 1% of documents
> train.dtm2 <- removeSparseTerms(train.dtm2,0.99)
# after removing sparse terms only 1,753 left
> dim(train.dtm2)
[1] 16000  1753
```

Code for UniBiTokenizer:

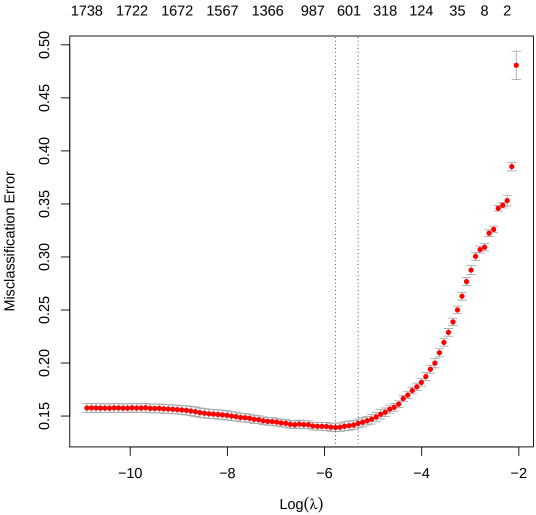
```
UniBiTokenizer <-
function (x) {
unlist(lapply(ngrams(words(x), 1:2), paste, collapse = " "),
       use.names = FALSE)
}
```

Including Bigrams

```
# fit regularized logistic regression model
# use cross-validation to evaluate different lambda values
> reviews.glmnet2 <- cv.glmnet(as.matrix(train.dtm2),labels[index.train],
    family="binomial",type.measure="class")

# show coefficient estimates for lambda-1se
# (only a selection of the bigram coefficients is shown here)
> coef(reviews.glmnet2,s="lambda.1se")
bad movie             -9.580669e-02
cant believe          -1.280761e-01
character development  .
great film             .
great movie            2.145233e-01
highly recommend       5.419558e-01
main character        -1.065261e-01
make sense            -1.737418e-01
one worst             -4.623925e-01
special effects        .
supporting cast        .
waste time            -6.520532e-02
well done              2.860367e-01
whole movie           -1.981443e-01
year old              -5.041887e-02
```

Cross-Validation on lambda



Including Bigrams

```
# create document term matrix for the test data,
# using the training dictionary
> test.dtm2 <- DocumentTermMatrix(reviews.all[-index.train],
    control = list(tokenize=UniBiTokenizer,dictionary=Terms(train.dtm2)))

# make predictions using lambda.1se
> reviews.glmnet.pred <- predict(reviews.glmnet2,newx=as.matrix(test.dtm2),
                                 s="lambda.1se",type="class")
# accuracy improved due to including more unigrams and including bigrams!
> table(reviews.glmnet.pred,labels[-index.train])
  reviews.glmnet.pred    0    1
                    0 3751  534
                    1  749 3966
> (3751+3966)/9000
[1] 0.8574444
```

The Second Assignment: Text Classification

Text Classification for the Detection of Opinion Spam.

- We analyze fake and genuine hotel reviews.
- The genuine reviews have been collected from several popular online review communities.
- The fake reviews have been obtained from Mechanical Turk.
- There are 400 reviews in each of the categories: positive truthful, positive deceptive, negative truthful, negative deceptive.
- We will focus on the negative reviews and try to discriminate between truthful and deceptive reviews.
- Hence, the total number of reviews in our data set is 800.

The Second Assignment: Text Classification

Analyse the data with:

1. Multinomial naive Bayes (generative linear classifier),
2. Regularized logistic regression (discriminative linear classifier),
3. Classification trees, (flexible classifier) and
4. Random forests (ensemble of classification trees).

The Second Assignment: Text Classification

- This is a data analysis assignment, not a programming assignment.
- You will need to program a little to perform the experiments.
- You only need to hand in a report of your analysis, no code!
- You are free to use whatever tools you want.
- We can provide support for R and Python.
- The report should describe the analysis you performed in such a way that the reader would be able reproduce it.
- Carefully read the assignment before you start!