# Epistemic Logic

Natasha Alechina Brian Logan

Utrecht University

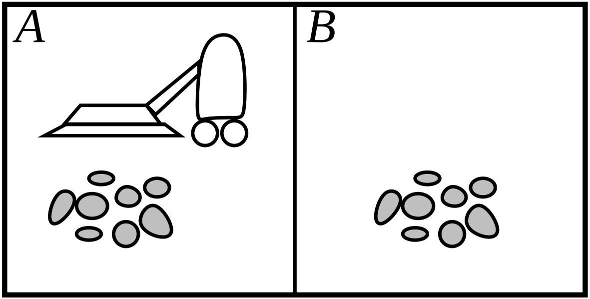
n.a.alechina@uu.nl b.s.logan@uu.nl

# Why imperfect information

- agents can only select actions based on the information they have
- different agents have different information
- representing this is important for communication, coordinating actions, or keeping opponents in the dark
- just as temporal logic describes the system dynamics ...
- epistemic logic describes the agents' knowledge

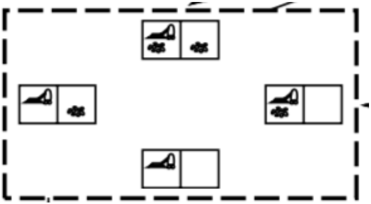
# Motivating example: representing imperfect information

Recall the vacuum cleaner the from Russell and Norvig's textbook:



# Motivating example: representing imperfect information

- suppose the agent has a location sensor but no dirt sensor
- then it will ‘know’ that it is in room A, but will consider several states of the world possible regarding the dirt:



# Possible worlds

- this is a key idea behind representing incomplete information: the agent considers several states of the world possible
- or (equivalently) it cannot distinguish between several possible worlds
- this indistinguishability relation is the main idea behind both epistemic logic and in representing agent systems where agents may have imperfect information

# Epistemic logic

- Epistemic Logic (EL) is a logic for reasoning about knowledge
- its models are Kripke models
- they look very much like state transition systems, but instead of transition relation they have an indistinguishability relation (for every agent in the system)

# Kripke Models

## Definition (Kripke model)

Let $\mathcal{PV}$ be a set of atomic propositions ($p, q, r, \ldots$) and *Agt* a finite set of agents.

A **Kripke model** $\mathcal{M} = \langle St, \{\sim_i (i \in Agt)\}, \mathcal{V} \rangle$ consists of

- a non-empty set of states *St*
- an indistinguishability relation $\sim_i$ for each agent *i* (which is reflexive, transitive and symmetric)
- a valuation of propositions $\mathcal{V} : \mathcal{PV} \to 2^{St}$

# Language of epistemic logic

## Definition (Syntax of EL)

agent I
knows that φ

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid K_i\varphi$$

- where $p \in \mathcal{PV}$ and $i \in Agt$

- sometimes useful to define $\bar{K}_i\varphi := \neg K_i\neg\varphi$ (for $i$ considers $\varphi$ possible)

# Possible World Semantics

## Definition (Semantic Clauses)

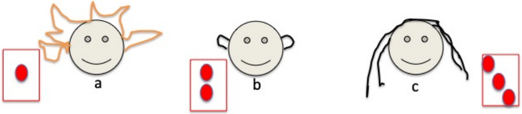
- $\mathcal{M}, q \models p$ iff $q \in \mathcal{V}(p)$;
- $\mathcal{M}, q \models \neg\varphi$ iff not $\mathcal{M}, q \models \varphi$;
- $\mathcal{M}, q \models \varphi \land \psi$ iff $\mathcal{M}, q \models \varphi$ and $\mathcal{M}, q \models \psi$;
- $\mathcal{M}, q \models K_i\varphi$ iff, for every $q' \in St$ such that $q \sim_i q'$, we have $\mathcal{M}, q' \models \varphi$.

# Epistemic logic

- $K_i\varphi$ means that $i$ knows that $\varphi$ is true
- $\mathcal{M}, q \models K_i\varphi$ iff $\varphi$ holds in all worlds that for the agent $i$ are indistinguishable from $q$

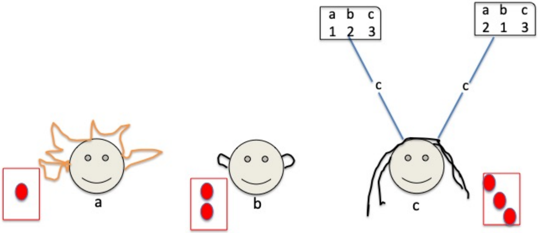
# Example: agents playing cards

Propositions: $a1$ for ‘agent a has a card with 1 dot’, $b1$,$c1$, $a2$, etc.



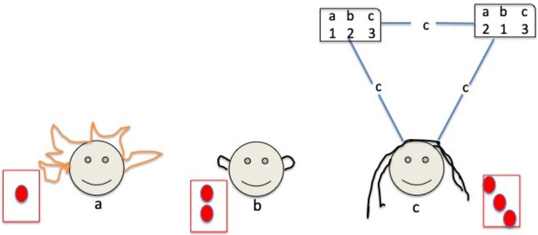
# Example: agents playing cards

*c* considers two states possible



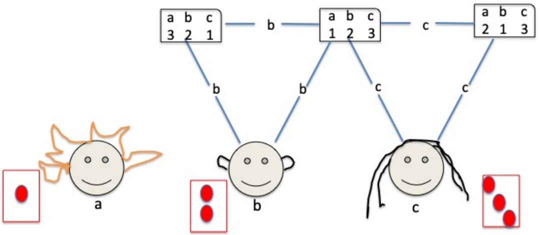
# Example: agents playing cards

$K_c c3$, $\neg K_c b2$



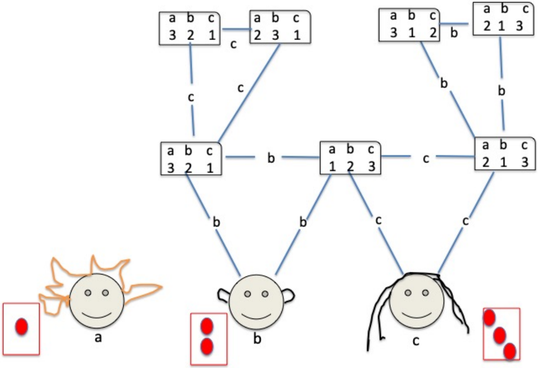
# Example: agents playing cards

*b* is similar



# Example: agents playing cards

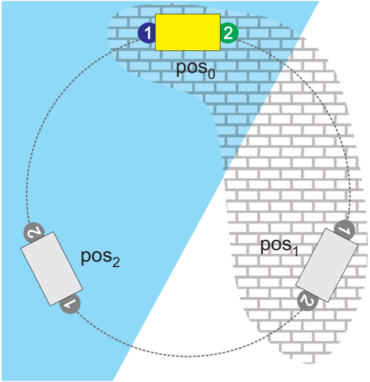
Reasoning about each other's knowledge: $K_c(K_b b1 \vee K_b b2)$, $K_c \neg K_b(c3)$



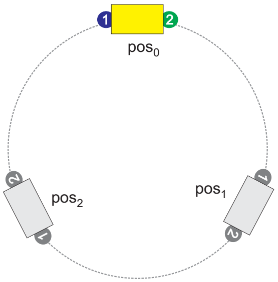

The Kripke model is not complete!

# Example: Robots and Carriage

Suppose robot 1 perceives only the colour of the surface, robot 2 only the texture



# How do we model this as a Kripke model?



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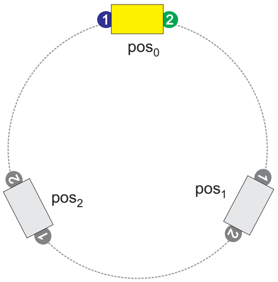


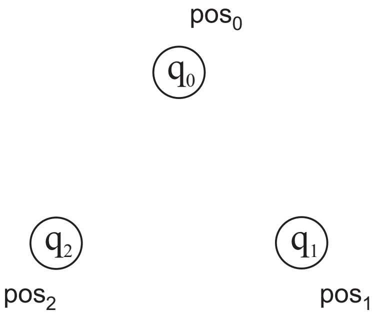

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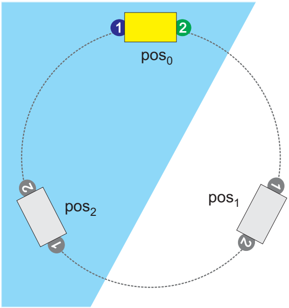


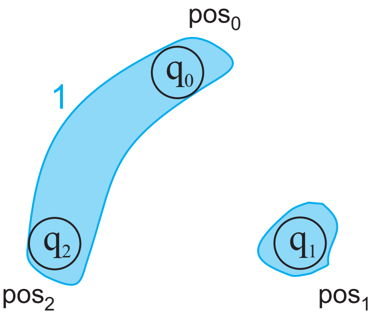

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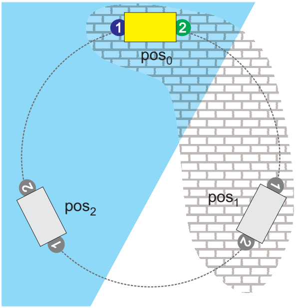


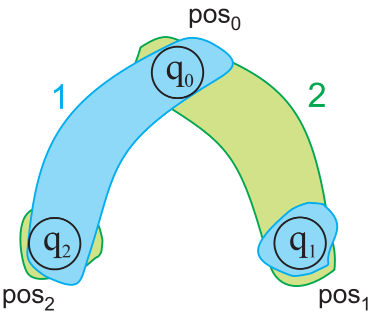

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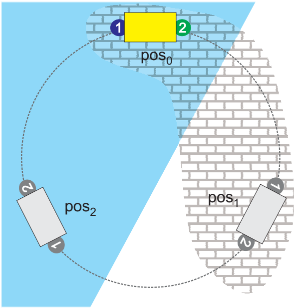
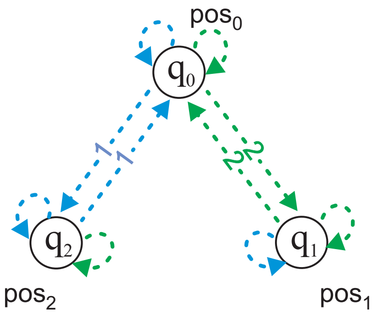
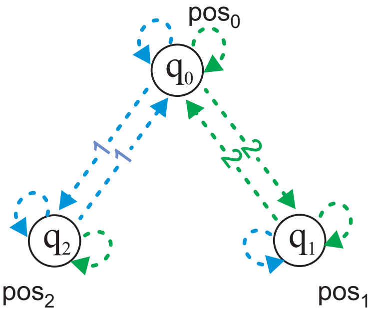


# Some formulas




$\text{pos}_2 \rightarrow \neg K_1 \text{pos}_2$
$\text{pos}_2 \rightarrow K_1 \neg \text{pos}_1$
$\text{pos}_2 \rightarrow K_2 K_1 \neg \text{pos}_1$

# Some properties of $K_i$

- knowledge is veridical: for every $\varphi$ and $i$, $K_i\varphi \rightarrow \varphi$ is valid, that is, true in all states in all models
- this is because $\sim_i$ is reflexive: since $q \sim_i q$, and $\varphi$ is required to be true in all states related to $q$ by $\sim_i$, it is true in $q$
- knowledge is consistent $\neg K_i\bot$ is valid (for the same reason)
- knowledge is introspective $K_i\varphi \rightarrow K_iK_i\varphi$ is valid (this is because $\sim_i$ is transitive) and $\neg K_i\varphi \rightarrow K_i\neg K_i\varphi$ (because $\sim_i$ is symmetrical)
- aside: correspondence theory between properties of modalities (like knowledge) and first order properties of Kripke models is a fascinating subject (Johan van Benthem)

# Logical Omniscience

- a less appealing property of $K_i$ is logical omniscience: each agent knows all tautologies and all logical consequences of its knowledge
- $K_i\top$ is valid (because $\top$ is true in all states reachable by $\sim_i$)
- $K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$ is valid (if $\varphi \to \psi$ and $\varphi$ are true in all states reachable by $\sim_i$, then $\psi$ has to be true as well)
- if an agent is a vacuum cleaner (no logical reasoning about tautologies) this is not a problem
- but it means epistemic logic is not very suitable for modelling human-like resource-bounded reasoners

  see, Alechina and Logan Ascribing beliefs to resource bounded agents. AAMAS 2002: 881–888

# Applications of epistemic logic

- analysing distributed systems (in the next lecture)
- verifying protocols where knowledge of participants is important (for example cryptographic protocols)
- epistemic planning: how to plan a sequence of actions to achieve some epistemic state

  see, Bolander and Andersen Epistemic planning for single- and multi-agent systems. Journal of Applied Non-Classical Logics, 21(1), 9–34