Linear Temporal Logic

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Reading for this lecture: Wojtek Jamroga, *Logical Methods for the Specification and Verification of Multiagent Systems*, Chapter 3.1.1 and 3.1.2.

Linear Time: LTL

- **LTL**: **Linear Temporal Logic**
- reasoning about a single computation, or run, of a system
- time is linear: just one possible future path is considered
- Model: a path
- important distinction: computational vs. behavioral structure

Linear Time: LTL

Definition (Models of LTL)

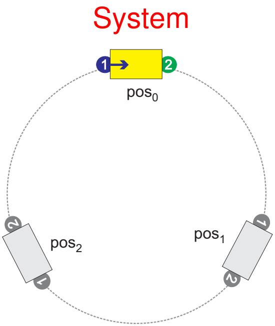
A **model of LTL** is a sequence of time moments (states). We call such models **paths**, and denote them by λ.

Evaluation of atomic propositions at particular time moments is also needed.

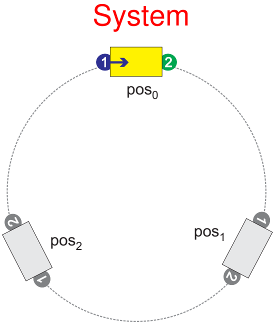
Notation:

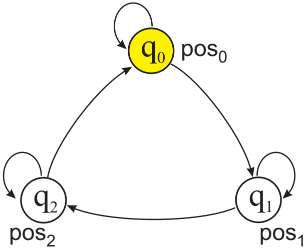
- $\lambda[i]$: ith time moment (starting from 0)
- $\lambda[i \ldots j]$: all time moments between i and j
- $\lambda[i \ldots \infty]$: all timepoints from i on

Computational vs. Behavioral Structures



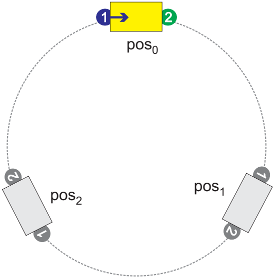
Computational vs. Behavioral Structures



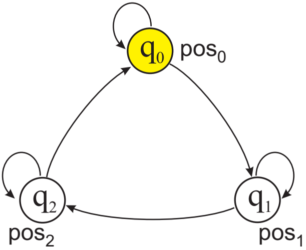


Computational vs. Behavioral Structures

System



Computational str.



Computational vs. Behavioral Structures

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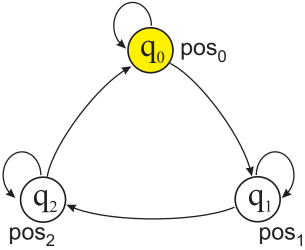
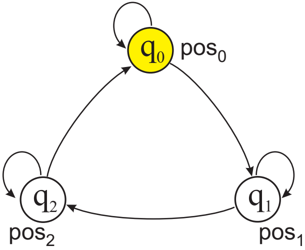
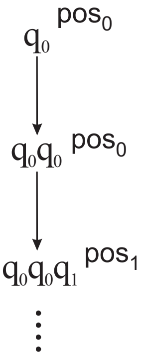


Computational vs. Behavioral Structures

Computational str.



Behavioral str.



Computational vs. Behavioral Structures: Beware

In LTL, models are defined as behavioral structures!

...But input to the verification problem is defined by the computational structure.

Linear Time: LTL

Definition (Semantics of LTL)

$\lambda \models p$ iff p is true at moment $\lambda[0]$ (that is, $\lambda[0] \in \mathcal{V}(p)$);

Linear Time: LTL

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$\lambda \models X\varphi$	iff $\lambda[1] \models \varphi$;

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$\lambda \models X\varphi$	iff $\lambda[1] \models \varphi$; No!

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$\lambda \models G\varphi$	iff $\lambda[i..\infty] \models \varphi$ for all $i \geq 0$;

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$\lambda \models G\varphi$	iff $\lambda[i..\infty] \models \varphi$ for all $i \geq 0$;
$\lambda \models \varphi U \psi$	iff $\lambda[i..\infty] \models \psi$ for some $i \geq 0$, and $\lambda[j..\infty] \models \varphi$ for all $0 \leq j < i$.

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$\lambda \models \varphi\, U\, \psi$	iff $\lambda[i..\infty] \models \psi$ for some $i \geq 0$, and $\lambda[j..\infty] \models \varphi$ for all $0 \leq j < i$.

$\lambda \models \neg\varphi$	iff not $\lambda \models \varphi$;
$\lambda \models \varphi \wedge \psi$	iff $\lambda \models \varphi$ and $\lambda \models \psi$.

Linear Time: LTL

Definition (Semantics of LTL)

$\lambda \models p$	iff p is true at moment $\lambda[0]$ (that is, $\lambda[0] \in \mathcal{V}(p)$);
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$\lambda \models \neg\varphi$	iff not $\lambda \models \varphi$;
$\lambda \models \varphi \wedge \psi$	iff $\lambda \models \varphi$ and $\lambda \models \psi$.

Note that:

$$G\varphi \equiv \neg F \neg \varphi$$

$$F\varphi \equiv \neg G \neg \varphi$$

$$F\varphi \equiv \top\, U\, \varphi$$

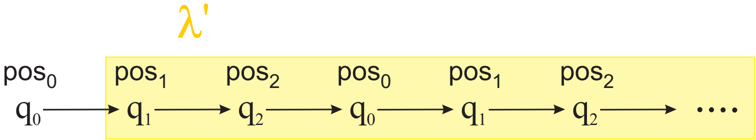
Semantics of LTL: $X\,\mathrm{pos}_1$

$$\lambda$$

$$\begin{array}{ccccccccccccc}
\mathrm{pos}_0 & & \mathrm{pos}_1 & & \mathrm{pos}_2 & & \mathrm{pos}_0 & & \mathrm{pos}_1 & & \mathrm{pos}_2 & & \\
q_0 & \longrightarrow & q_1 & \longrightarrow & q_2 & \longrightarrow & q_0 & \longrightarrow & q_1 & \longrightarrow & q_2 & \longrightarrow & \cdots
\end{array}$$

$$\lambda \models X\,\mathrm{pos}_1$$

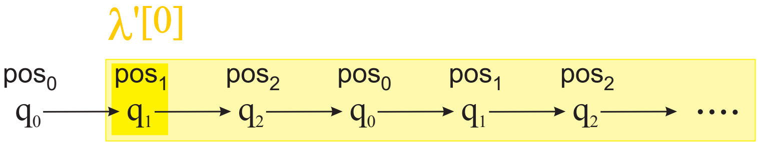
Semantics of LTL: $X\,\mathsf{pos}_1$



$$\lambda \models X\,\mathsf{pos}_1$$

$$\lambda' = \lambda[1..\infty] \models \mathsf{pos}_1$$

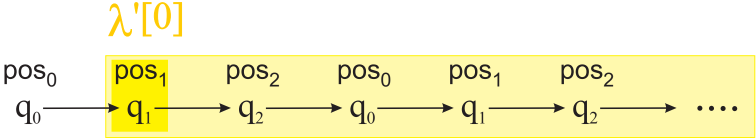
Semantics of LTL: $X\mathsf{pos}_1$



$$\lambda \models X\mathsf{pos}_1$$

$$\lambda' = \lambda[1..\infty] \models \mathsf{pos}_1$$
$$\lambda'[0] \in \mathcal{V}(\mathsf{pos}_1)$$

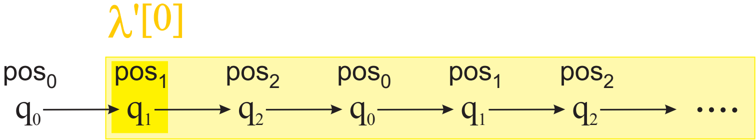
Semantics of LTL: $pos_0 \; U \; pos_1$



$$\lambda \models pos_0 \; U \; pos_1$$

$$\lambda' = \lambda[1..\infty] \models pos_1$$
$$\forall i < 1, \lambda[i..\infty] \models pos_0$$
$$\lambda'[0] \in \mathcal{V}(pos_1)$$
$$\lambda[0] \in \mathcal{V}(pos_0)$$

Semantics of LTL: $F\,\mathrm{pos}_1$



$$\lambda \models F\,\mathrm{pos}_1$$

$$\lambda' = \lambda[1..\infty] \models \mathrm{pos}_1$$
$$\lambda'[0] \in \mathcal{V}(\mathrm{pos}_1)$$

Semantics of LTL: $G\,F\,\mathrm{pos}_1$

λ

$$\begin{array}{ccccccccccccc}
\mathrm{pos}_0 & & \mathrm{pos}_1 & & \mathrm{pos}_2 & & \mathrm{pos}_0 & & \mathrm{pos}_1 & & \mathrm{pos}_2 & & \\
q_0 & \longrightarrow & q_1 & \longrightarrow & q_2 & \longrightarrow & q_0 & \longrightarrow & q_1 & \longrightarrow & q_2 & \longrightarrow & \cdots
\end{array}$$

$$\lambda \models G\,F\,\mathrm{pos}_1$$

Semantics of LTL: $G\,F\,\mathrm{pos}_1$

$\lambda[0..\infty]$

pos_0	pos_1	pos_2	pos_0	pos_1	pos_2	
$q_0 \longrightarrow$	$q_1 \longrightarrow$	$q_2 \longrightarrow$	$q_0 \longrightarrow$	$q_1 \longrightarrow$	$q_2 \longrightarrow$	$\cdots\cdot$

$$\lambda \models G\,F\,\mathrm{pos}_1$$

$$\lambda[0..\infty] \models F\,\mathrm{pos}_1$$

Semantics of LTL: $G\,F\,\mathrm{pos}_1$

$\lambda[0..\infty]$

pos_0	pos_1	pos_2	pos_0	pos_1	pos_2	
$q_0 \longrightarrow$	$q_1 \longrightarrow$	$q_2 \longrightarrow$	$q_0 \longrightarrow$	$q_1 \longrightarrow$	$q_2 \longrightarrow$	$\cdots$

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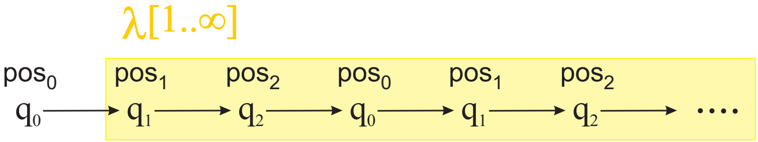

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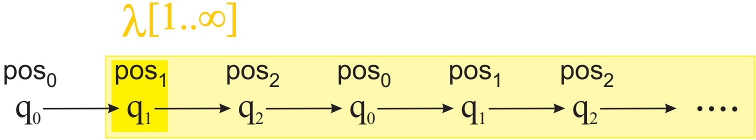

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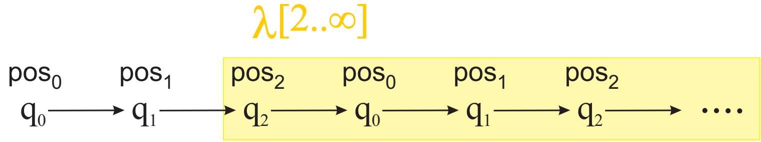

$$\lambda \models G\,F\,\mathrm{pos}_1$$

$$\lambda[0..\infty] \models F\,\mathrm{pos}_1$$
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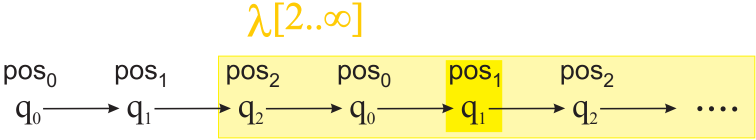

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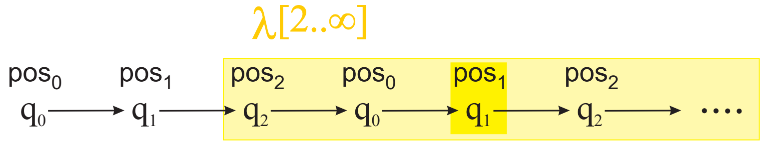

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$$\ldots$$

Example: specifying reward functions

- generate a reward function for an RL agent from a temporal logic specification
- e.g., ‘bring gems to the shed, always avoid zombies’

from Camacho et al. LTL and Beyond: Formal Languages for Reward Function Specification in Reinforcement Learning. IJCAI 2019: 6065-6073

Minecraft: specification of behaviour

- E1 Collect wood and iron in any order, and use the factory afterwards
- E2 If it is night time, stay in the shed until daylight
- E3 Always avoid zombies
- E4 While there are gems on the ground, put them in your bag. When your bag is full, deliver the gems to the shed, and get an empty bag.

Minecraft: specification of behaviour

E1 Collect wood and iron in any order, and use the factory afterwards

$$F(\text{got\_wood} \wedge F\,\text{used\_factory}) \wedge F(\text{got\_iron} \wedge F\,\text{used\_factory})$$

E2 If it is night time, stay in the shed until daylight

$$G(\text{is\_night} \rightarrow \text{at\_shed})$$

E3 Always avoid zombies

$$G\,\neg\text{near\_zombie}$$

Minecraft: specification of behaviour 2

E4 While there are gems on the ground, put them in your bag. When your bag is full, deliver the gems to the shed, and get an empty bag.

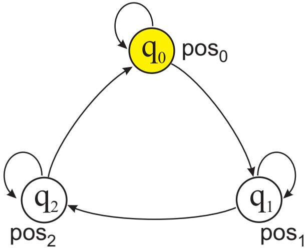
$$G(\neg \text{is\_night} \wedge \neg \text{near\_zombie} \rightarrow (\text{gems} \wedge \neg \text{bag\_full} \rightarrow X\,\text{got\_gem}) \wedge (\text{bag\_full} \rightarrow F(\text{at\_shed} \wedge \neg \text{bag\_full})))$$

Semantics of LTL in Computational Structures

Definition (Semantics of LTL in Transition Systems)

$M, q \models \varphi \quad$ iff $\lambda \models \varphi$ for every path λ in M starting from q.

Example: Robot and carriage



$M, q_0 \not\models G\,\mathrm{pos}_0$

(but also: $M, q_0 \not\models \neg G\,\mathrm{pos}_0$!)

$M, q_0 \not\models F\,\mathrm{pos}_1$

(but also: $M, q_0 \not\models \neg F\,\mathrm{pos}_1$!)

$M, q_0 \models (\neg G\,\mathrm{pos}_0) \rightarrow F\,\mathrm{pos}_1$

Verification using LTL

Definition (Model checking problem for LTL)

Given a finite state transition system M, a state q in M, and an LTL formula φ, check whether $M, q \models \varphi$.

Remember that we need to check whether $\lambda \models \varphi$ for every path λ in M starting from q.

Complexity of the LTL model checking problem

- LTL model checking is usually done using Büchi automata (automata over infinite strings)
- Given M, q and φ, two automata are constructed:
  - $\mathcal{A}_{\neg\varphi}$ that accepts all paths satisfying $\neg\varphi$
  - $\mathcal{A}_{M,q}$ that accepts all paths in M starting from q
- then a check is performed for whether the sets of paths accepted by $\mathcal{A}_{\neg\varphi}$ has a non-empty intersection with the set of paths accepted by $\mathcal{A}_{M,q}$

Complexity of the LTL model checking problem cont.

- the non-emptiness check is done by constructing a product automaton of $\mathcal{A}_{\neg\varphi}$ and $\mathcal{A}_{M,q}$ and checking non-emptyness of its language (the paths it accepts)
- the non-emptyness check can be done in linear time in the size of the product automaton
- $\mathcal{A}_{M,q}$ is of size linear in $|M|$, but unfortunately $\mathcal{A}_{\neg\varphi}$ is exponential in $|\varphi|$
- so the whole procedure is polynomial in $|M|$ but exponential in $|\varphi|$
- the problem itself is PSPACE-complete, so it is very unlikely one can do much better

What cannot be expressed in LTL?

- LTL can express very useful properties of agent behaviour
- any kind of regular expression pattern can be expressed in LTL
- however it cannot express that it is possible for an agent to have a choice between two or more actions
- sometimes we want to express that something may happen
- next week: branching time temporal logic

Reading for week 2

- Wojtek Jamroga, *Logical Methods for the Specification and Verification of Multiagent Systems*, Chapter 3.1.3, 3.1.4 (only the beginning; modal μ-calculus optional), 3.2.1.