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Constructor theory of information

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We propose a theory of information expressed solely in terms of which transformations of physical systems are possible and which are impossible—i.e. in constructor-theoretic terms. It includes conjectured, exact laws of physics expressing the regularities that allow information to be physically instantiated. Although these laws are directly about information, independently of the details of particular physical instantiations, information is not regarded as an *a priori* mathematical or logical concept, but as something whose nature and properties are determined by the laws of physics alone. This theory solves a problem at the foundations of existing information theory, namely that information and distinguishability are each defined in terms of the other. It also explains the relationship between classical and quantum information, and reveals the single, constructor-theoretic property underlying the most distinctive phenomena associated with the latter, including the lack of in-principle distinguishability of some states, the impossibility of cloning, the existence of pairs of variables that cannot simultaneously have sharp values, the fact that measurement processes can be both deterministic and unpredictable, the irreducible perturbation caused by measurement, and locally inaccessible information (as in entangled systems).

1. Information

In some respects, information is a qualitatively different sort of entity from all others in terms of which the physical sciences describe the world. It is not, for instance, a function only of tensor fields on space–time (as general relativity requires all physical quantities to be), nor is it a quantum-mechanical observable.

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But in other respects, information does resemble some entities that appear in laws of physics: the theory of computation and statistical mechanics seem to refer directly to it without regard to the specific media in which it is instantiated, just as conservation laws do for the electromagnetic four-current or the energy–momentum tensor. We call that the *substrate-independence* of information. Information can also be moved from one type of medium to another while retaining all its properties qua information. We call this its *interoperability* property; it is what makes human capabilities such as language and science possible, as well as the possibility of biological adaptations that use symbolic codes, such as the genetic code.

In addition, information has a counter-factual character: an object in a particular physical state cannot be said to carry information unless it *could have been* in a different state. As Weaver [1] put it

this word ‘information’ in communication theory relates not so much to what you *do* say, as to what you *could* say. . . .

Information is also of the essence in *preparation* and *measurement*, both of which are necessary for testing scientific theories. The output of a measurement is information; the input of a preparation includes information, specifying an attribute with which a physical system is to be prepared.

All these applications of information involve *abstraction* in that one entity is represented symbolically by another. But, information is not abstract in the same sense as, say, the set of all prime numbers, for—as we understand it in this paper—it only exists when it is physically instantiated. So, the laws governing information, like those governing computation—but unlike those governing prime numbers—are laws of physics. In this paper, we conjecture what those laws are.

Previous attempts to incorporate information into physics at a fundamental level [2], or at least into quantum theory [3–5], have regarded information itself as an *a priori* mathematical or logical concept. Our approach is the opposite, namely that the nature and properties of information follow entirely from the laws of physics. We begin by expressing in exact terms the regularities that are required in the laws of physics for information—as informally conceived, with its characteristic properties, such as interoperability—to be instantiated in physical objects. We also explain the properties of what is informally referred to as ‘quantum information’ in terms of those regularities. Those regularities, thus expressed, have the status of new, *conjectured laws (principles) of physics*.

So we are not trying to derive quantum theory, or any other existing law, from anything. Our proposal is that the status of information in physics is analogous to that of (say) energy: given the laws of motion of physical objects, neither the concept of energy nor the conservation law for the energy–momentum tensor is necessary for making any prediction from initial data, yet our understanding of the physical world would be radically incomplete without them. The conservation law *explains* some aspects of motion as consequences of a universal regularity in nature—which is why we expect as-yet-undiscovered laws of motion to conserve the energy–momentum tensor too. Different physical systems that conserve energy can have very different laws of motion, so we do not expect to *derive* new laws of motion from the universal conservation law. It is a *principle*—a law of physics that constrains other laws rather than the behaviour of physical objects directly. Likewise, our theory of information consists of proposed principles of physics that explain the regularities in physical systems that are associated with information, including quantum information. This theory also constitutes a framework, not dependent on quantum theory, in which information under a broad range of theories (including ‘post-quantum’ theories, or theories supporting hypercomputation) could be understood, provided that they obey the principles. None of the existing theories of information is adequate for any of these purposes.

The classical theory [6] was developed to analyse the physics of communication, where the objective is for a *receiver* to receive a *message* from a *transmitter* through a *medium*. The receiver, transmitter and medium are physical systems, but the message is not. It is information, initially instantiated in the transmitter, then in the medium, then in the receiver. The overall

process constitutes a measurement of one of the transmitter’s physical variables, representing the message. Essential to Shannon’s notion of communication is that a message is one of at least two *possible* messages, which are *accurately distinguishable* by measurement, and that the receiver is able to retransmit the information to a further receiver while retaining an instance of it. Much of Shannon’s theory is about unreliable transmission and measurement, and inefficient representations, and how to compose them into more reliable and efficient ones. But, here we are concerned with the fundamental issues that remain even in the limiting case when all error rates have been reduced to their physically possible minima, and there is no redundancy in the message being transmitted. In that limit, receiving the message means *distinguishing* it with perfect reliability from all the other possible messages.

Here, we shall need to address what ‘distinguishing’ means physically. Shannon’s theory does not. But it implicitly assumes, among other things, that the communication media are governed by classical physics. (Hence, the type of information it describes is now called *classical* information.) Quantum information processing, in general, involves states that are different, but cannot be reliably distinguished (e.g. non-orthogonal pure states of a medium). So to implement our agenda of expressing precisely the regularities in the laws of physics that are informally understood under the heading of information, we must provide a theory, not specifically rooted in classical or quantum physics, of what ‘distinguishing’ and ‘distinguishability’ require of the physical system whose states are to be distinguished—what interactions the laws of physics must allow it to undergo.

A process that distinguishes perfectly whether a message is x or y , and retransmits the message while retaining a copy, is a *non-perturbing measurement*, whose effect we may summarize operationally as

message	receiver		message	receiver
x	b	\rightarrow	x	x
y	b	\rightarrow	y	y

(1.1)

Here, x and y in the leftmost column are two possible states of some physical system, representing one of the two possible messages x and y . The process transforms an initially blank state b of some medium of the receiver to another state x or y respectively, thus making a copy of the message while leaving the original message unchanged.

But, this does not, in fact, distinguish message x from message y unless the resulting *receiver* states x and y are themselves distinguishable. Thus, (1.1), considered as a definition of the ‘distinguishing’ done by a receiver of a communication, is *recursive*. We need a base for that recursion that is itself an operational property of the receiver in the state x or y . For example, the criterion that ‘the receiver states x and y must be orthogonal’ would not do, because we should then also have to specify operationally *what it is* about orthogonal states that would make them usable for communication—and so on. None of the existing theories of information provides that. The theory that we shall present here does (§4).

Quantum information theory, as it stands, never gets round to specifying what exactly it is referring to as ‘quantum information’, nor its relation to classical information. It is not, despite the name, a theory of a new type of information, but only a collection of quantum phenomena that violate the laws of classical information. For example, certain prohibitions that quantum theory imposes, such as the impossibility of cloning, violate the kind of interoperability that is assumed in Shannon’s theory. A new theory of information is needed for that reason too, within physics but at a deeper level than both quantum theory and Shannon’s theory. We shall provide that, via *constructor theory* [7].

After setting out as much of it as we shall need (§2), we shall begin our search for a deeper theory of information by expressing, in exact, constructor-theoretic terms, the concepts of computation, measurement and classical information that are already assumed, informally, to be instantiated in the physical world (§§3–5). Then, in §6, we express the regularities that are informally associated with *classical* information as exact, purely constructor-theoretic principles of physics—which turn out to be elegant and natural. In particular, we express prediction and

testing in constructor-theoretic terms. In §7, we introduce *superinformation* media as information media on which certain tasks, with a natural constructor-theoretic definition, are *impossible*. In §8, we show that the most distinctive qualitative features of quantum information follow from the impossibility of those tasks. In other words, quantum information is an instance of superinformation.

2. Constructor theory

The laws of constructor theory are all principles, which we conjecture are obeyed by all other laws of physics, so we shall call theories about the latter, *subsidiary* theories. Principles, being laws about other laws, do not make direct assertions about the outcomes of measurements. They are nevertheless experimentally testable: a principle P is refuted if some law violating P survives experimental tests, whereas all rival laws conforming to P are refuted.¹

The *basic principle of constructor theory* is that

- I. All other laws of physics are expressible entirely in terms of statements about which physical transformations are possible and which are impossible, and why.

This is in contrast with the *prevailing conception* of fundamental physics, which seeks to explain the world in terms of initial conditions and laws of motion, and whose basic dichotomy is therefore between what happens and what does not.

Constructor theory describes the world in terms of transformations involving two kinds of physical systems, playing different roles. One is the agent causing the transformation, which we refer to as the *constructor*, whose defining characteristic is that it remains unchanged in its ability to cause the transformation again. The other consists of the subsystems—which we refer to as the *substrates*—which are transformed from having some physical attribute to having another. Schematically

$$\text{Input attribute of substrates} \xrightarrow{\text{constructor}} \text{Output attribute of substrates}, \quad (2.1)$$

where the constructor and the substrates jointly are isolated.

By *attribute*, we mean any property that the substrate could possibly have (whether measurable or not), according to the subsidiary theory describing it—such as being red, or being spherical. We represent an attribute formally as the set of states in which the substrate has it; for instance, the attribute ‘being red’ is the set of all states of the substrate in which it is red.

Unlike Shannon’s theory, constructor theory of information makes no mention of probability. Indeed, principle I rules out any reference to probability in fundamental laws of physics (see below).

An *intrinsic* attribute of a substrate is one that does not refer to any other specific substrate. For example, the distance x of an object S_1 from another object S_2 is an intrinsic attribute of the combined system $S_1 \oplus S_2$ but not of either S_1 or S_2 . In quantum theory, ‘entangled with each other’ is a possible intrinsic attribute of a pair of qubits; having a particular density operator is an intrinsic attribute of a system, whereas the rest of its quantum state (see below) describes entanglement relationships with other systems.

Any set of disjoint attributes, we shall call a physical *variable*. Whenever a substrate is in a state with attribute $x \in X$, where X is a variable, X is *sharp*, with the *value* x .² This is a generalization of the usual notion of physical variable. In the usual notion, the attributes (say, ‘red’, ‘blue’, ...))

¹For example, early observations of beta decay satisfied the principle of the conservation of energy under the assumption that neutrinos were emitted, but violated it under the assumption that no undetected particle was emitted. The conservation law would have been refuted if the neutrino theory had failed experimental tests, whereas some testable explanation predicting the destruction of energy survived. The supposed principle of parity invariance was refuted in just that way.

²Not all variables of a given substrate need be sharp in a given state. For instance, in quantum theory, a useful variable of a spin-1/2 particle consists of the two attributes: z-component of the spin being 1/2 and −1/2. Each of those attributes contains one pure state. In most states—superpositions and mixtures of those two states—that variable is not sharp.

in a variable ('colour') are physically related in a way that makes it natural to group them together. But, under constructor theory, it is for the subsidiary theories to say what is natural. They determine, for instance, whether two attributes overlap—e.g. whether red spheres can exist in nature or not. Constructor theory does not prejudice that, and therefore in principle allows variables such as {'red', 'spherical'}.

That individual physical systems (and not just the entire physical world) *have* states and attributes in the sense we have described is guaranteed by Einstein's *principle of locality* [8], which has a precise expression in constructor-theoretic form:

- II. There exists a mode of description such that the state of the combined system $S_1 \oplus S_2$ of any two substrates S_1 and S_2 is the pair (x, y) of the states x of S_1 and y of S_2 , and any construction undergone by S_1 and not S_2 can change only x and not y .

In quantum theory, the Heisenberg picture is such a mode of description (see [9]).³ Principle II rules out, for example, nonlinear modifications of the Schrödinger equation.

The basic entities of constructor theory are specifications of only the input–output pairs in (2.1), with the constructor abstracted away

Input attributes of substrates \rightarrow Output attributes of substrates.

We call these *construction tasks*, or *tasks* for short. In general, a task \mathfrak{A} on a substrate S is a set

$$\mathfrak{A} = \{x_1 \rightarrow y_1, x_2 \rightarrow y_2, \dots\}$$

where the x_1, x_2, \dots and the y_1, y_2, \dots are attributes of S . We call the $\{x_i\} = \text{In}(\mathfrak{A})$ the *legitimate input attributes* of \mathfrak{A} and the $\{y_i\} = \text{Out}(\mathfrak{A})$ its *legitimate output attributes*. The *transpose* of a task $\mathfrak{A} = \{x_1 \rightarrow y_1, x_2 \rightarrow y_2, \dots\}$ is $\mathfrak{A}^\sim = \{y_1 \rightarrow x_1, y_2 \rightarrow x_2, \dots\}$.

In constructor theory, the notion of time does not appear. There is a notion of input and output, but not of before and after. That is for the subsidiary theories to provide, and is not fundamental to constructor theory. However, in this paper, where we are discussing information-processing tasks, it is harmless to interpret the inputs as preceding the outputs (and harmless not to do so).

A constructor is *capable of performing* a task \mathfrak{A} if, whenever presented with substrates having an attribute in $\text{In}(\mathfrak{A})$, it delivers them with one of the corresponding attributes from $\text{Out}(\mathfrak{A})$ (regardless of what it does if the substrate is in any other state).

No perfect constructors exist in nature. Approximations to them, such as catalysts or robots, have non-zero error rates and also deteriorate with repeated use. But, we call a task \mathfrak{A} *possible* (which we write as \mathfrak{A}^\vee) if the laws of nature impose no limit, short of perfection, on how accurately \mathfrak{A} could be performed, nor on how well things that are capable of approximately performing it could retain their ability to do so again. Otherwise, \mathfrak{A} is *impossible* (which we write as \mathfrak{A}^\times).

Accordingly, we must also understand the principle I as requiring subsidiary theories to provide a measure of the accuracy with which any approximate constructor they describe performs tasks (including its own maintenance); and provide a meaning to whether an infinite sequence of tasks $\mathfrak{A}_1, \mathfrak{A}_2, \dots$ on a system $S \oplus E$ (where E might be the environment of S) converges to a limiting task \mathfrak{A} on S alone.

As we have said, principle I also rules out theories that are stochastic at a fundamental level. For instance, it implies that 'generalized probabilistic theories' [12], can at most be descriptions at an emergent level, because, translated into constructor-theory-like terms, they would include statements about something being possible or impossible *with a given probability*. Hence, mere statements about what transformations are possible or impossible could not express the full content of such a theory. For a discussion of the emergence of probability in deterministic quantum theory, see Deutsch [13] and Wallace [14].

³Note that the local states (in our sense) of a system S in the Heisenberg picture are the observables of S ; the global 'state vector' is unchangeable. So the controversy about whether the locality of quantum physics conceals residual 'non-locality' [10,11] is not relevant here because principle II only requires *changeable* quantities to be local.

Tasks may be composed into networks to form other tasks, as follows. The *parallel composition* $\mathfrak{A} \otimes \mathfrak{B}$ of two tasks \mathfrak{A} and \mathfrak{B} is the task whose net effect on a composite system $\mathbf{M} \oplus \mathbf{N}$ is that of performing \mathfrak{A} on \mathbf{M} and \mathfrak{B} on \mathbf{N} . When $\text{Out}(\mathfrak{A}) = \text{In}(\mathfrak{B})$, the *serial composition* $\mathfrak{B}\mathfrak{A}$ is the task whose net effect is that of performing \mathfrak{A} and then \mathfrak{B} on the same substrate. A *regular network* of tasks is a network without loops whose nodes are tasks and whose lines are their substrates, where the legitimate input states at the end of each line are the legitimate output states at its beginning. Loops are excluded, because a substrate on a loop is a constructor.

It may be that a task $\{(x, z) \rightarrow (y, w)\}$ cannot be decomposed into $\{x \rightarrow y\} \otimes \{z \rightarrow w\}$, because the individual attributes are not intrinsic, and therefore the operands of that parallel composition are not valid tasks. However, if $\{(x, z) \rightarrow (y, w)\}$ and $\{x \rightarrow y\}$ are valid tasks, then $\{z \rightarrow w\}$ must be too.

A *constructor-theoretic statement* is one that refers only to substrates and which tasks on them are possible or impossible—not to constructors. Constructor theory is the theory that the (other) laws of physics can be expressed *without* referring explicitly to constructors.

A task refers to an isolated system of constructor and substrates. But, we are also often interested in what is possible or impossible, regardless of the resources required. So if $(\mathfrak{A} \otimes \mathfrak{T})^\checkmark$ for some task \mathfrak{T} on some generic, naturally occurring substrate (as defined in §6), we shall say that \mathfrak{A} is *possible with side-effects*, which we write as \mathfrak{A}^\checkmark . The ‘side-effect’ is the effect of the task \mathfrak{T} .

3. Computation

Our theory of information rests on first understanding *computation* in constructor-theoretic terms. This will allow us to express information in terms of computation, not vice versa as is usually done. This is the key to providing the base for the recursive definition of ‘distinguishing’ that we described in §1.

A *reversible computation* $\mathfrak{C}_\Pi(S)$ is the task of performing, with or without side-effects, a permutation Π over some set S of at least two possible attributes of some substrate

$$\mathfrak{C}_\Pi(S) = \bigcup_{x \in S} \{x \rightarrow \Pi(x)\}.$$

For example, swapping two pure quantum states (say, with density operators $|\psi\rangle\langle\psi|$ and $|\varphi\rangle\langle\varphi|$) constitutes a reversible computation, and may be a possible task even if they are not orthogonal.⁴ (Thus, by a ‘reversible computation’ \mathfrak{C}_Π we mean a logically reversible, i.e. one-to-one, task. Because side-effects are allowed, this definition does not require the physical processes instantiating the computations \mathfrak{C}_Π and $\tilde{\mathfrak{C}}_\Pi$ to be reversible.)

It is then natural to define a *computation variable* as a set S of two or more possible attributes for which $\mathfrak{C}_\Pi^\checkmark$ for all permutations Π over S , and a *computation medium* as a substrate with at least one computation variable.

Note again that, in this paper, we are not taking computation to be an *a priori* concept and seeking necessary and sufficient conditions for a physical process to instantiate it (cf. [15]). We are conjecturing objective regularities in nature: *principles of physics*. These happen to be conveniently expressed in terms of the tasks we have called ‘computations’ and the property that we shall call ‘information’. We think that these correspond reasonably closely to the intuitive concepts with those names, but our claims in no way depend on that being so.

4. Information

As we mentioned in §1, the intuitive concept of information is associated with that of copying. We shall now express this association exactly and without relying on the notion of distinguishability, in terms of computations as defined in §3.

⁴The unitary that swaps them is $|\langle\varphi|\psi\rangle|/(1 - |\langle\varphi|\psi\rangle|^2) (|\psi\rangle\langle\psi| + |\varphi\rangle\langle\varphi| - (|\varphi\rangle\langle\psi|/\langle\psi|\varphi\rangle) - |\psi\rangle\langle\varphi|/\langle\varphi|\psi\rangle)$.

We first consider computations involving two instances of the same substrate \mathbf{S} . The *cloning task* for a set S of possible attributes of \mathbf{S} is the task

$$\mathfrak{R}_S(x_0) = \bigcup_{x \in S} \{(x, x_0) \rightarrow (x, x)\} \quad (4.1)$$

on $\mathbf{S} \oplus \mathbf{S}$, where x_0 is some fixed (independent of x) attribute with which it is possible to prepare \mathbf{S} from generic, naturally occurring resources (§6).

A set S is *clonable* if $\mathfrak{R}_S(x_0) \checkmark$ for some such x_0 . The usual notion of cloning, as in the no-cloning theorem [16], is (4.1) with S as the set of *all* attributes of \mathbf{S} .

An *information variable* is a clonable computation variable. It is then natural to define an *information attribute* as one that is a member of an information variable, and an *information medium* as a substrate that has at least one information variable.

In addition, a substrate \mathbf{S} *instantiates classical information* if some information variable S of \mathbf{S} is sharp, and if giving it any of the other attributes in S was possible. Note that the constructor-theoretic mode of explanation has allowed this to be expressed as an exact, intrinsic property of the substrate. The classical information capacity of \mathbf{S} is the logarithm of the cardinality of its largest information variable. The principle of locality II implies the convenient property that the combined classical information capacity of disjoint substrates is the sum of their capacities.

Thus, we have provided the purely constructor-theoretic notion of classical information that we promised. But we have emancipated it from its dependence on classical physics.

5. Measurement

We can now do the same for distinguishability and measurement. A variable X of a substrate \mathbf{S} is *distinguishable* if

$$\left(\bigcup_{x \in X} \{x \rightarrow \psi_x\} \right) \checkmark, \quad (5.1)$$

where the $\{\psi_x\}$ constitute an information variable—which, by (4.1), implies the possibility of a subsequent non-perturbing measurement, as required for communication. That provides the base for the recursive definition (1.1), because we have defined ‘information variable’ above without reference to distinguishability, and only to the possibility or impossibility of tasks (specifically, permutations and cloning).

If a pair of attributes $\{x, y\}$ is distinguishable we shall write $x \perp y$ (and $x \not\perp y$ if not). If $x \perp y$, we say either ‘ x is distinguishable from y ’ or ‘ $\{x, y\}$ is a distinguishable variable’.

If the original substrate continues to exist and the process (5.1) stores its result in a second, *output substrate* (which must therefore be an information medium), (5.1) is the condition for the *input variable* X to be *measurable*

$$\left(\bigcup_{x \in X} \{(x, x_0) \rightarrow (y_x, 'x')\} \right) \checkmark, \quad (5.2)$$

where the output substrate is initially prepared with a ‘receptive’ attribute x_0 . When X is sharp, the output substrate ends up with an information attribute x of an *output variable*, which represents the abstract outcome ‘it was x ’. (We give quoted labels such as ‘ x ’ to attributes in the output variable corresponding to those in X .) Thus, measurement is like cloning a variable (4.1) except that the output substrate is an information medium rather than a second instance of the cloned substrate.

A constructor is a *measurer* of X if there is some choice of its output variable, labelling and receptive state, under which it is capable of performing (5.2). Consequently, a measurer of X is automatically a measurer of a range of other variables, because one can interpret it as such by re-labelling its outputs. (Such re-labellings must be possible tasks because they are classical computations on a finite set—see §6.) For example, a measurer of X measures any subset of X , or any *coarsening* of X (a variable whose members are unions of attributes in X).

If $y_x \subseteq x$ in (5.2), the measurement is *non-perturbing*, which is the type of measurement typically needed in computation and communication. It follows from the definition of information variables that the task of measuring them non-perturbatively is always possible.

6. Conjectured principles of physics bearing on information

Crucially, the most important properties of information do not follow from the definitions we have given. Here, we seek the constructor-theoretic principles of physics that determine those properties. Of these, perhaps the most fundamental one cannot even be stated in the prevailing conception of fundamental physics, but it has an elegant expression in constructor theory. It is the *interoperability principle*

- III. The combination of two substrates with information variables S_1 and S_2 is a substrate with information variable $S_1 \times S_2$,

where multiplication symbol denotes the Cartesian product of sets. Note that this principle requires certain interactions to exist in nature. For instance, it would rule out theories of dark matter in which dark information media exist, but interactions between the dark and normal sectors do not allow information to be copied arbitrarily accurately between them.

Exploring the properties of distinguishability in more detail leads us to conjecture further such principles. Suppose that all attributes in a variable X are pairwise distinguishable—i.e. $(\forall x \in X, \forall y \in X, x \neq y) x \perp y$. It does not follow logically from the definitions that X is a distinguishable variable. For example, if each x represents some number of photons in a cavity and any number of photons can be distinguished from any other number, it does not follow that an instrument that measures the total number is possible. But, in quantum physics, it (plausibly) is. We shall conjecture that this is a special case of a deeper principle: we expect that whenever there is a regularity among observable phenomena in a substrate (such as a set of its attributes being pairwise distinguishable), that is always because the phenomena are related by a unifying explanation—i.e. that they can all be distinguished by measuring some variable, in this case, the total number of photons. This principle, too, has an elegant, purely constructor-theoretic expression:

- IV. If every pair of attributes in a variable X is distinguishable, then so is X .

And similarly,

- V. If every state with attribute y is distinguishable from an attribute x , then so is y .

Although we conjecture that IV and V hold universally in nature, none of our conclusions in §8 depends on their being universal. It would suffice if they held only for a special class of substrates. The same also holds for the remaining conjectured principles (VI–VIII).

Two of these could be considered simplifying assumptions rather than grand conjectures about the nature of reality. First, because we are concerned with the nature and properties of information, not its long-term future, we assume that unlimited resources are available for conversion into information storage devices. In constructor-theoretic terms, we express this as:

- VI. Any number of instances of any information medium, with any one of its information-instantiating attributes, is preparable from naturally occurring substrates.

We call such substrates, which must therefore exist in unlimited numbers, ‘generic resources’. In describing a task, we use the symbol g to represent a suitable generic resource or resources for the task. Thus, for every task \mathfrak{A} , we have $\mathfrak{A} \Rightarrow \{g \rightarrow C_{\mathfrak{A}}\}^{\checkmark}$, where ‘ \Rightarrow ’ denotes implication and $C_{\mathfrak{A}}$

is arbitrarily close to being a constructor for \mathcal{A} . The assumption VI implies that there are generic (approximations to) constructors in nature too. Thus, we also have $\mathcal{A} \preceq \Leftrightarrow (\exists h) (\mathcal{A} \times \{g \rightarrow h\})^\vee$.

Finally, we assume that unlimited resources are available for information *processing* too. We express this in constructor-theoretic terms as the conjectured *composition principle* [7]:

VII. Every regular network of possible tasks is a possible task,

though here we need only assume that it holds for information-processing tasks.

Cosmological models in which the resources available for computers and computations are finite violate VI and VII; but that need not concern us here, because it would not affect our definitions of computation, information or distinguishability, nor our results in §§6–8. That is because these limitations would only cause certain possible tasks to become impossible, whereas none of our results take the form of statements that a task must be possible.

Provided that information exists at all, principles III, VI and VII imply that for every function f from a finite set of integers to itself, the task of computing f is possible.

(a) Measurement of non-sharp variables

We have defined measurement of a variable X by the measurer's effect (5.2) when X is sharp. When we refer to a 'sharp' output we shall mean that the output variable is sharp. But, the term 'measurement' is also used to describe cases where X is not sharp. In such cases, the output variable need not be sharp either. Exploring these will motivate another constructor-theoretic principle about information.

We first define a convenient tool, the *bar* operation: let x be any attribute; \bar{x} (' x -bar') is the union of all attributes (i.e. the set of all states) that are distinguishable from x . If X is a variable, principle IV allows us to assign the natural meaning $\bar{X} \equiv \bigcup_{x \in X} \bar{x}$. (Thus, any expression topped by a bar denotes an attribute.) When \bar{X} is empty, we call X a *maximal* variable.

The bar operation has the property $\bar{\bar{x}} \equiv \bar{x}$. For we have $x \subseteq \bar{x}$ (because distinguishability is symmetric), i.e. $\bar{x} = x \cup y$ for some y . Similarly, $\bar{\bar{x}} = \bar{x} \cup z$ for some z that does not overlap with \bar{x} . Because, by definition, z contains only states distinguishable from \bar{x} ; and because $x \subseteq \bar{x}$, each such state is also distinguishable from x . Hence, z is empty and $\bar{\bar{x}} \equiv \bar{x}$.

Any variable of the form $\{x, \bar{x}\}$ we shall call a *Boolean variable*. Principle IV and the definition of bar trivially imply that every Boolean variable is distinguishable.⁵ In addition, every Boolean variable is maximal, because no attribute y can be distinguishable from both x and \bar{x} , for if $y \perp x$, we would have $y \subseteq \bar{x}$ and hence $y \not\perp \bar{x}$.

Now, consider an attribute $\{a\}$ in which X is non-sharp. A trivial case is when $\{a\} \subseteq \bar{X}$; in that case, by principle IV, there is a distinguishable variable that includes X and is sharp. But suppose instead that $\{a\} \not\subseteq \bar{X}$, so that $\{a\} \not\subseteq \bar{X}$ (which includes all cases when X maximal, because then \bar{X} includes all states). Whether it is then still possible for X not to be sharp in the state a , and what it means if that is so, is up to the subsidiary theories. (For instance, in quantum theory, it would mean that a was a superposition or mixture of the states with attributes in X . The generalization of this to constructor theory is discussed in §8e.) But, it must at least imply that the output variable of every measurer of X is either non-sharp, or sharp with some value x , where $x \in X$. That means that the measurer could mistake the attribute $\{a\}$ for one having an attribute in X , for if some measurer of X could not make such a mistake, it could distinguish $\{a\}$ from all attributes in X , contradicting $\{a\} \not\subseteq \bar{X}$.

Consider the case when a particular measurer of X is presented with $\{a\} \subseteq \bar{X}$ and the output variable is sharp with, say, the value ' x '. That means that that measurer can mistake $\{a\}$ for only

⁵For the sake of uniformity of notation, we include cases where \bar{x} is empty, even though $\{x, \bar{x}\}$ is not literally *variable* in those cases. But, because, the empty set is an attribute that no substrate can possibly have, every state is distinguishable from it; hence, in such cases, \bar{x} is the set of all states. A trivial constructor that produces a fixed result, which can then be labelled ' x ' in (5.2) then qualifies as a distinguisher of $\{x, \bar{x}\}$.

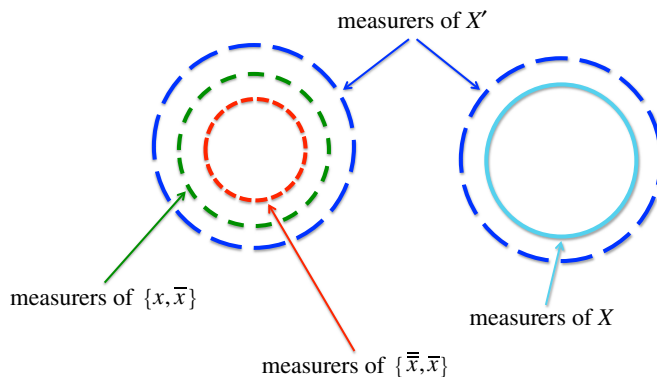


Figure 1. Hierarchy of measurers of variables of which x is a member. (Online version in colour.)

one attribute in X , namely x . Thus, it is also a measurer of another variable, namely the variant of X with the attribute x replaced by $x \cup \{a\}$. Similarly, we can construct a variable Z by augmenting each attribute $x \in X$ with *all* the states $a \in \bar{\bar{X}}$ that can be mistaken only for x by that measurer. And thus we see that the measurer was really a measurer of Z all along, with the additional property that whenever it produces any sharp output ' z ' on measuring its substrate, the input substrate really had the attribute z . It is logically possible that applying this procedure for some other measurer of X would yield a different Z . But, we propose the following *principle of the consistency of measurement*:

VIII. Whenever a measurer of a variable X would produce a sharp output when presented with the attribute $\{a\} \subseteq \bar{\bar{X}}$, all other measurers of X would too.⁶

Consequently, those other measurers of X would have to produce the *same* sharp output ' x ' for the given input $\{a\} \subseteq \bar{\bar{X}}$. For suppose that one of them produced an output ' x ' for that input. That would make $\{a\}$ distinguishable from all attributes in X other than x . Likewise, if another produced a different sharp output ' x '. So then, $\{a\}$ would be distinguishable from all $x \in X$, and would therefore (from IV) be included in $\bar{\bar{X}}$ and hence not in $\bar{\bar{X}}$, which is a contradiction.

Thus, principle VIII implies that for any measurable variable X there is a unique variable Z such that all measurers of X are measurers of Z . Z therefore has the property that whenever a measurer of Z produces a sharp output ' z ' the input substrate really has the attribute z . We shall call such variables *observables*, and *information observables* if they are information variables. (Quantum-mechanical observables, in the Heisenberg picture, are indeed observables by this definition.) Properties of observables are crucial to our results about superinformation (§8).

(b) Properties of observables

We now obtain a necessary and sufficient condition for a variable to be an observable. Consider *any* attribute x , and *any* measurable variable X of which it is a member. Now, let χ be the union of all attributes in X , and consider the variable $X' = \{x, \chi - x\}$, a coarsening of X . It must be measurable, because any measurer of $\{x, \bar{x}\}$ measures it. In this section and in §8e, we shall repeatedly rely on the relations between measurers of X , X' , and the two Boolean variables $\{x, \bar{x}\}$ and $\{\bar{\bar{x}}, \bar{x}\}$, which are represented in figure 1.

⁶Despite mentioning constructors (measurers), VIII is a purely constructor-theoretic statement, for it could be rephrased as 'if the task of measuring X while producing an output ' x ' ($x \in X$) for an input $\{a\} \subseteq \bar{\bar{X}}$ is possible, the task of measuring X while not producing that output for the input $\{a\}$ is impossible'.

First, we prove that all such measurers must produce a sharp 'x' when presented with a substrate with the attribute $\{a\} \subseteq \bar{x}$. By VIII, every measurer of $\{x, \bar{x}\}$ must deliver a sharp output 'x' for any input $\{a\} \subseteq \bar{x}$, because all measurers of $\{\bar{x}, \bar{x}\}$ are measurers of $\{x, \bar{x}\}$ and would deliver a sharp ' \bar{x} ', which could be reinterpreted as 'x'. Any measurer of $\{x, \bar{x}\}$ measures X' , and because the former produces a sharp output x when presented with $\{a\} \subseteq \bar{x}$, by principle VIII, so do all measurers of X' . In addition, any measurer of X is a measurer of X' . Therefore, in particular, any measurer of X must give a sharp output 'x', when presented with any input $\{a\} \subseteq \bar{x}$.

The converse is also true: if any measurer of a variable X (containing x) produces a sharp 'x' when presented with a state $\{a\} \subseteq \bar{X}$, then $\{a\} \subseteq \bar{x}$. For suppose that given $\{a\} \in \bar{X}$, a measurer of X produces a sharp outcome 'x'. Then, all of them produce a sharp 'x', by VIII. Again, any measurer of X is a measurer of X' ; therefore, by VIII, all measurers of X' must give the same sharp output 'x' when presented with $\{a\} \subseteq \bar{X}$. By definition of \bar{x} , any measurer of $\{x, \bar{x}\}$ is a measurer of X' . Therefore, again by VIII, all measurers of $\{x, \bar{x}\}$ must give the sharp output 'x' when presented with $\{a\} \in \bar{X}$. This implies that $\{a\} \perp x$, i.e. $\{a\} \subseteq \bar{x}$.

This elucidates the physical meaning of \bar{x} : it is the set of all states a such that, for any variable X containing the attribute x , a measurer of X produces a sharp output 'x' when presented with a substrate with the attribute $\{a\} \subseteq \bar{X}$. And it also provides our necessary and sufficient condition for a variable to be an observable, namely all its attributes x satisfy $x = \bar{x}$. This is also why sharpness of observables will play a central role in our discussion below. For example, in Everettian quantum theory, an observable being sharp means that it has the same value in all universes in which that instance of the substrate exists.

(c) Prediction, testing and ensembles

Under constructor theory, a *testable prediction* is a statement that the output variable of a certain possible measurement will be sharp, with a certain value. That requirement of sharpness does not come from constructor theory. It is entailed by the logic of testability, and is why, for instance, probabilistic predictions have to be tested by repeated measurements. For example, consider the testing of a fundamentally stochastic (and therefore incompatible with constructor theory) law of motion, which predicts the *probability* of an outcome of a measurement, but not the outcome. Because probabilities of individual outcomes are not measurable on individual systems, such a prediction can be tested only if it is interpreted as a manner of speaking about non-probabilistic predictions about measurements on ensembles, which can then be approximated by average results of finite sets of measurements on individual systems.

Likewise, the constructor theory of measurement and testing must concern itself with how to obtain sharp outputs in situations where the individual outputs of measurements are not sharp, either, because a subsidiary theory predicts that, or at a practical level, because perfectly accurate measurers do not exist. And again, most methods of doing that involve using multiple instances of a given substrate.

We denote by $S^{(n)}$ a physical system $\overbrace{S \oplus S \oplus \dots S}^{n \text{ instances}}$ consisting of n instances of S , and we denote by $x^{(n)}$ the attribute $\overbrace{(x, x, \dots x)}^{n \text{ terms}}$ of $S^{(n)}$. Evidently, the set $\{x^{(n)} | x \in S\}$ is an information variable of $S^{(n)}$ whenever S is an information variable of S . And each $x^{(n)}$ is a redundant instantiation of the information x .

If, in conventional, probabilistic, information theory, S is a discrete set and each instance of S has a probability $p < 1/2$ of having been changed from the correct value x by uncorrelated perturbations, the probability that a plurality of the values stored in n instances will not instantiate x falls exponentially with n . Although probabilities are not allowed at the fundamental level in constructor theory, redundancy can still play effectively the same role, in following way. Denote by $S^{(\infty)}$ an unlimited supply of instances of S . Suppose that they all have the same intrinsic, preparable attribute, either x or y , and let us call a sequence of experiments on $S^{(n)}$, as n increases

without limit, an ‘experiment on an ensemble $S^{(\infty)}$ ’. Recall from §2 that a possible task is one for which the laws of nature impose no limit, short of perfection, on how well the task can be performed. Because our assumption VI implies that there is no limit on n , it follows that for any two intrinsic attributes x and y of a substrate S , $x^{(\infty)}$ and $y^{(\infty)}$ are either interchangeable in all experiments on $S^{(\infty)}$ (using generic resources), or distinguishable. For given such a supply, performing *all* possible experiments on (instances of) S , infinitely often, is a possible task. So if that cannot tell the difference between $x^{(\infty)}$ and $y^{(\infty)}$, nothing (that uses only generic resources) can. If something can, we call x and y *ensemble distinguishable*. Thus, we propose the principle:

IX. Any two disjoint, intrinsic attributes are ensemble distinguishable.

7. Superinformation

So far, in this paper, we have proposed a set of purely constructor-theoretic principles that capture in an exact, subsidiary-theory-independent way the behaviour of ‘classical’ information. From here on, we propose no further principles. We investigate what happens if the subsidiary theories impose a single further *prohibition* on what tasks are possible. This turns out to allow substrates to instantiate what we call *superinformation*, and we shall show that quantum information is an instance of it.

It follows from our theory so far that every subset with at least two members of an information observable S , is also an information observable. But, the converse does not hold: the union of two information observables, even if their attributes are mutually disjoint, is not necessarily an information observable. It is through that loophole that all the non-classical content of constructor information theory flows.

A *superinformation medium* \mathbf{M} is an information medium with at least two information observables that contain only mutually disjoint attributes and whose union S is not an information observable. For example, in quantum physics, any set of two orthogonal states of a qubit constitutes an information observable, but no union of two or more such sets does: its members are not all distinguishable. \mathbf{M} instantiates *superinformation* if it has one of the attributes in S but could have had any of the others. A *supercomputation* is a task that maps S to itself.

8. Properties of superinformation

In what follows, let \mathbf{M} be a superinformation medium and let X and Y be two of its information observables whose union $X \cup Y$ consists of mutually disjoint attributes but is not an information observable. (Hence, $X \cup Y$ is not even an information variable.)

(a) Not all information attributes of a superinformation medium are distinguishable

There must exist information attributes $x \in X$ and $y \in Y$ such that $x \not\perp y$. For suppose that $x \perp y$ for all $x \in X$ and $y \in Y$. This would imply that all attributes in $X \cup Y$ are pairwise distinguishable and hence, from principle IV, that $X \cup Y$ is a measurable variable. Each attribute in $X \cup Y$ is also preparable, because by VI that is true of X and Y separately. It would follow that all permutations on $X \cup Y$ are possible tasks. For, to perform Π , given \mathbf{M} with any attribute $z \in X \cup Y$, one would first measure which attribute that is, thereby preparing some information medium with the information attribute ‘ z ’. Then, one would compute $\Pi('z')$ on that medium (which must be possible because it is a computation medium). Then, one would use that result to prepare, from generic substrates, another instance of \mathbf{M} with the attribute $\Pi(z)$, which again must be possible by principle VI.

The remaining condition for $X \cup Y$ to be an information variable would be met too: the cloning task is possible (with side-effects). For if $X \cup Y$ is a distinguishable set of \mathbf{M} , $(X \cup Y) \times (X \cup Y)$ is a distinguishable set of $\mathbf{M} \oplus \mathbf{M}$: one can distinguish its members by performing a distinguishing operation on each instance of S in parallel and then combining the sharp outputs with a logical

or operation. Moreover, $X \times X$, $X \times Y$, $Y \times X$ and $Y \times Y$ are all information variables of $\mathbf{M} \oplus \mathbf{M}$, by the interoperability principle III. Therefore, each attribute in their union $(X \cup Y) \times (X \cup Y)$ is preparable. These two facts imply, by the argument above, that all permutations of $(X \cup Y) \times (X \cup Y)$ are possible with side-effects. Because those permutations include the cloning tasks on $X \cup Y$, it follows that $X \cup Y$ is an information variable, which contradicts the condition for \mathbf{M} to be a superinformation medium.

So, there must exist a pair of attributes $x \in X$ and $y \in Y$ of \mathbf{M} that are not distinguishable.

(b) Undetectability of sharpness

It is impossible to measure whether the observable X or Y is sharp, even given that one of them is. For it were possible, that would also distinguish between the above-mentioned x and y . (Of course sharpness is ensemble measurable, by IX, because none of the attributes in X intersects any attribute in Y .)

(c) Superinformation cannot be cloned

Suppose that the cloning task (4.1) was possible for $S = X \cup Y$. Then, in particular, any variable $\{x, y\}$ with $x \in X$ and $y \in Y$ could be cloned. Then, given our generic resources assumption VII, if $z \in \{x, y\}$, it is possible to apply the cloning operation to the substrate any number of times, and the output would be a composite medium $\mathbf{M} \oplus \mathbf{M} \oplus \mathbf{M} \dots$ with the attribute $(z, z, z \dots)$. Thus, preparing $z^{(\infty)}$ would be a possible task. The attributes x and y are intrinsic information attributes, so by assumption VI they are preparable, and therefore by principle IX the two attributes $z^{(\infty)}$ are ensemble distinguishable: $x^{(\infty)} \perp y^{(\infty)}$. Thus, by preparing $z^{(\infty)}$ from $\{x, y\}$, one could distinguish x from y .

If all such pairs were clonable, it would follow that all attributes in $X \cup Y$ were pairwise distinguishable, contrary to our result in §8*a*, so the assumption that the superinformation variable $X \cup Y$ is clonable is false.

(d) Pairs of observables not simultaneously preparable or measurable

The sets $x \cap y$ are all empty by the defining property of a superinformation medium. So when the substrate has any of the attributes $y \in Y$, X cannot be sharp, and vice versa; hence, it is impossible to prepare \mathbf{M} with its observables X and Y both sharp. For the same reason, simultaneously measuring X and Y is impossible.

(e) Unpredictability of deterministic processes

Superinformation media exhibit the counterintuitive property of evolving deterministically yet unpredictably.

Unpredictability arises when a measurer of X acts when X is non-sharp. Suppose that X is maximal, whereby $y \subseteq \overline{\overline{X}}$. If a substrate with attribute y is presented to a measurer of X , the output variable cannot be sharp. For if it were, with value ' x ', the properties of observables (§6), would imply that $y \subseteq x$; yet x and y are non-overlapping by hypothesis. Thus, no prediction of the form 'the outcome will be x ', where $x \in X$, can be true, because that would imply that the measurer, when presented with y , produces a sharp output. That already means that the outcome of a measurement of X , when the substrate has the attribute y , is unpredictable. Remarkably, constructor information theory also provides a definite physical meaning for this unpredictability, even though the relevant subsidiary theories must give no meaning to probabilities.

As a guide to the general constructor-theoretic case, consider first an example from quantum theory. For the observable X , let us take the number of photons in a cavity, $\hat{N} = |1\rangle\langle 1| + 2|2\rangle\langle 2| + 3|3\rangle\langle 3| + \dots$. The only way that \hat{N} can fail to be sharp when the photon field in the cavity has

some attribute y , is that y contains superpositions or mixtures of two or more eigenstates of \hat{N} —let us say, those with eigenvalue less than 3. In such a case, when \hat{N} is measured, no prediction of the form ‘the outcome will be n ’ (for some eigenvalue n of \hat{N}) would be true. But, there are other predictions that must be true: any pure state $|\psi\rangle$ in y has the property that the expectation value of the projector $|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|$ is 1 in $|\psi\rangle$. Therefore, the prediction that a measurer of $|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|$ would yield the outcome ‘1’, when presented with $|\psi\rangle$ would be true. Thus, because $|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|$ is a Boolean observable whose meaning is ‘whether there are fewer than three photons in the cavity’ (with 1 denoting yes and 0 no), the fact that the outcome of a measurement of \hat{N} would be less than 3 is predictable, even though predicting the outcome itself is an impossible task. This is the physical meaning of unpredictability in the quantum case.

The same logic applies equally in constructor theory, as we shall now explain, without the apparatus of projectors, expectation values and probabilities.

Let χ again be the union of all attributes in X , and χ_y be the union of all attributes in X that are not distinguishable from y . (By the defining properties of superinformation, there must exist at least one such attribute.) Our goal is now to explain that the Boolean observable $\{\overline{\chi_y}, \overline{\chi_y}\}$ plays the same role as the above projector. We start by showing that it must be sharp with value $\overline{\chi_y}$ when the substrate has the attribute y .

Consider any state $a \in y$. Any measurer of the Boolean variable $\{\chi - \chi_y, \overline{\chi - \chi_y}\}$, when presented with a substrate with the attribute $\{a\}$, will produce a sharp output ‘ $\chi - \chi_y$ ’, because $y \perp \chi - \chi_y$ (by definition of χ_y). Any measurer of $\{\chi - \chi_y, \overline{\chi - \chi_y}\}$ is also a measurer of the (maximal) variable $\{\chi - \chi_y, \chi_y\}$, (because $\chi_y \subseteq \chi - \chi_y$). Hence by principle VIII, all measurers of $\{\chi_y, \chi - \chi_y\}$ must give a sharp output ‘ χ_y ’ when presented with $\{a\} \subseteq y$. By the properties of observables (§6), we conclude that $\{a\} \subseteq \overline{\chi_y}$. Because this is true for any state in y , it must be true of y too: $y \subseteq \overline{\chi_y}$. Therefore, again by the result of §6, when the attribute is y , $\{\overline{\chi_y}, \overline{\chi_y}\}$ must be sharp with value $\overline{\chi_y}$.

Consider now the observable $X_y = \{x \in X : x \not\subseteq y\}$ (noting that $\overline{\chi_y} = \overline{X_y}$). First, it must contain at least two attributes. For suppose all the attributes in X except x were distinguishable from y . Then $\chi_y = x$, so that y would be included in \bar{x} . But this is a contradiction, because this would imply that $y \subseteq x$ (because X is an observable).

We can now see why $\{\overline{\chi_y}, \overline{\chi_y}\}$ generalizes the projector in the quantum example. One way of measuring $\{\chi_y, \overline{\chi_y}\}$ is to measure X first and then to perform a computation on the output ‘ x ’ that would, if X were sharp in the input, determine whether $x \in X_y$ or not. All such measurers must, by the principle of consistency of measurement VIII, produce a sharp ‘ χ_y ’ when presented with any attribute in $\overline{\chi_y}$; hence, they are all measurers of $\{\overline{\chi_y}, \overline{\chi_y}\}$ too. Thus, $\{\overline{\chi_y}, \overline{\chi_y}\}$ is a Boolean observable whose meaning is ‘whether the outcome is one of the x with $x \not\subseteq y$ ’. Because $y \subseteq \overline{\chi_y}$, that process, by principle VIII, must yield the same sharp output ‘ $\overline{\chi_y}$ ’, meaning ‘yes’, as any other measurer of $\{\overline{\chi_y}, \overline{\chi_y}\}$ does when presented with y . As in the quantum case, this provides the physical meaning of unpredictability: any constructor (including any observer) that measures X on a substrate with attribute y and then computes (or recollects) whether the outcome was one of the ‘ x ’ with $x \not\subseteq y$, by the above procedure, will reach the conclusion ‘yes’ (corresponding to ‘ $\overline{\chi_y}$ ’); and will thereby have the same attribute as it would have if X had been sharp with some value $x \in X_y$. Yet no prediction ‘the outcome will be x ’ with $x \in X$ will be true.

Again, it is up to the subsidiary theories to *explain* this deterministic unpredictability. In Everettian quantum theory, the explanation is that the measurer differentiates, during the measurement, into multiple instances, sharply agreeing that the output was one of the $x \in X$, $x \not\subseteq y$, but not all perceiving the same one.

(f) Irreducible perturbation of one observable caused by measuring another

The observable X_y contains two or more attributes, none of which is distinguishable from y nor overlaps with y . We shall now show that any measurer of X_y must cause an irreducible

perturbation of the substrate for some input attributes. In particular

$$\left(\bigcup_{x \in X_y} \{(x, x_0) \rightarrow (x, 'x')\} \bigcup \{(y, x_0) \rightarrow (y, k)\} \right)^x, \quad (8.1)$$

for all x_0 and k . In words, no device can both measure X_y non-perturbatively if X_y is sharp and leave the substrate unperturbed if the input has attribute y . Moreover, the perturbation is irreducible: nothing can subsequently undo it while leaving the outcome of the measurement in any information variable, for if it could, the overall process would be a counter-example to (8.1).

To prove (8.1), suppose that the task in (8.1) were possible. Because y is an information attribute (hence, it is intrinsic) and x_0 is preparable from generic resources, k would be an intrinsic attribute because, by hypothesis, it could be produced from generic resources in the combination (y, k) with the intrinsic attribute y .

Moreover, k would be disjoint from all the ' x '. For suppose, there were a state in $k \cap 'x'$, for some ' x '. That would imply that there exists a state a in y with the property that $\{a\}$, when presented to any measurer of X_y , produced an output with value ' x '. Thus, by the results in §6, it would follow that $\{a\} \subseteq x$ (because X_y is an observable), but this is a contradiction, as y does not intersect any x .

Therefore, k would be ensemble distinguishable from x . Thereby it would be possible, by performing the above task an unlimited number of times on the same instance of the substrate together with successive instances of the target with attribute x_0 , to produce an ensemble with attribute $k^{(\infty)}$ or ' $x'^{(\infty)}$ ', and so k would be distinguishable from ' x ', and so y would be distinguishable from x , contradicting the supposition.

(g) Consistency of consecutive measurements of a non-sharp observable

In quantum theory, consecutive non-perturbing measurements of an observable, even if it is not sharp, yield the same outcomes, in the sense that a measurement of whether they are the same always yields the sharp outcome 'true'. We now show that this also holds for arbitrary subsidiary theories that conform to constructor theory.

A somewhat roundabout way of measuring the variable $\{\chi_y, \overline{\chi_y}\}$ of \mathbf{M} is the following. First, apply two non-perturbing measurers of X in succession to the substrate, recording each output in one of two receptive information media \mathbf{M}' and \mathbf{M}'' . Then, present \mathbf{M}' to a computer that performs

$$\bigcup_{x \in X'_y} \{x \rightarrow x\} \bigcup \bigcup_{x \in (X' - X'_y)} \{x \rightarrow \Pi(x)\}, \quad (8.2)$$

where Π is any permutation with no fixed point. (Here, we have suppressed the quotation marks to avoid clutter, replacing ' x ' by x , writing X for the output variable of \mathbf{M}' ; similarly, we shall write X'' for that of \mathbf{M}'' , and so on.) This leaves the attributes in X'_y unchanged and changes those in $X' - X'_y$. Finally, present \mathbf{M}' , \mathbf{M}'' and a third, receptive, information medium \mathbf{R} to a computer that performs

$$\bigcup_{v \in X', w' \in X''} \{(v, w', x_0) \rightarrow (v, w', v = w')\}. \quad (8.3)$$

The question mark means that when X' of \mathbf{M}' and X'' of \mathbf{M}'' are sharp, the output observable $\{ 'true', 'false' \}$ of \mathbf{R} indicates whether their values v and w' are equal or not.

Call the combined measurer that performs this overall process \mathbf{N} . It is also a measurer of $\{\chi - \chi_y, \chi_y\}$, because when \mathbf{M} has attribute $x \in X_y$, the output observable of \mathbf{R} is sharp with the value 'true', which can also be interpreted as ' χ_y '; and similarly when \mathbf{M} has attribute $x \in X - X_y$, it is sharp with the value 'false', meaning ' $\chi - \chi_y$ '. By principle VIII, \mathbf{N} is also a measurer of $\{\chi_y, \overline{\chi_y}\}$.

Now, suppose that \mathbf{N} is presented with a substrate with the attribute y . Because $y \subseteq \overline{\chi_y}$ (§8e), by principle VIII the measurer \mathbf{N} must deliver a sharp outcome ' χ_y ' (i.e. 'true'), which is the same as it would deliver if the outputs of the two consecutive measurements of X were both sharp with

the same value—even though, as we have seen, neither of them can be sharp when y is the input, just as in quantum theory.

(h) Quantization

The states of a quantum system form a continuum, and their dynamical evolution is continuous in both space and time. With hindsight, we can now see that the ‘quantization’ after which quantum theory is named really refers to a property of quantum *information*. The discrete and the continuous are linked, in quantum theory, in a manner that was not previously guessed at, but is easily understood in terms of constructor information theory: each information observable of a quantum physical system has only a discrete set of attributes, but there is a continuous infinity of such observables, no union of which is an information observable. (E.g. the union of the sets of eigenstates of two complementary observables is not the set of eigenstates of any observable.) So in quantum physics, classical information is discrete, and superinformation (quantum information) is continuous.

(i) Coherence and locally inaccessible information

Another feature of quantum theory that is due to its permitting superinformation is the distinction between coherent and incoherent processes. Let w be a set of states of a superinformation medium \mathbf{M} . A computation or supercomputation \mathcal{C} on a proper subset v of w is *coherent* with respect to w if it is possible to perform it reversibly on w . (And coherence with respect to a set of attributes is defined analogously.) That is to say, there exists a task \mathfrak{A} , with \mathfrak{A}^\vee and $\mathfrak{A}^{\sim\vee}$, whose legitimate input set contains w and whose restriction to the subtask with legitimate input set v is \mathcal{C} . \mathbf{M} is a *coherent medium* with respect to information variables $\{S_i\}$ of \mathbf{M} if each permutation task on each S_i can be performed coherently with respect to the union of all the attributes in the $\{S_i\}$.

For example, the qubits of a universal quantum computer constitute a coherent superinformation medium, because they are a superinformation medium and all reversible classical computations in some computation basis can be performed reversibly on the set of all their pure states.

Quantum entanglement is an example of a phenomenon that depends on coherence. It is usually characterized in terms of probabilistic quantities such as the correlations referred to in Bell’s theorem. But underlying those quantitative measures is a qualitative property: the presence of *locally inaccessible information*⁷ [9]. In our terminology, that means that some combined system $\mathbf{M}_1 \oplus \mathbf{M}_2$ has information variables that cannot be measured by measuring Cartesian products (nor subsets thereof) of variables of \mathbf{M}_1 and \mathbf{M}_2 .

So, let \mathbf{M}_1 be a superinformation medium, as in §7. Again let X be maximal. Consider any twofold observable $B_1 = \{0', 1'\} \subseteq Y$. We proved in §8e that for any attribute $y \in Y$ there are at least two attributes in X which are not distinguishable from y . Let $A_1 = \{0, 1\} \subseteq X$ be the observable including two attributes that are not distinguishable from $0'$, and define A_2 and B_2 for \mathbf{M}_2 analogously. Now, suppose that $\mathbf{M}_1 \oplus \mathbf{M}_2$ is coherent with respect to the set $(A_1 \times A_2) \cup (B_1 \times A_2)$. Because $A_1 \times A_2$ and $B_1 \times A_2$ are information observables, we have, in particular, $\mathfrak{T} \not\prec$ where

$$\mathfrak{T} = \{(0, 0) \rightarrow (0, 0), \quad (1, 0) \rightarrow (1, 1), \quad (0, 1) \rightarrow (0, 1), \quad (1, 1) \rightarrow (1, 0)\}. \quad (8.4)$$

\mathfrak{T} is the controlled-not computation. The coherence condition implies that in addition, $(\mathfrak{T} \cup \mathfrak{T}') \not\prec$ and $(\mathfrak{T} \cup \mathfrak{T}')^{\sim\vee} \not\prec$, where

$$\mathfrak{T}' = \{(0', 0) \rightarrow \psi_{(1)}, \quad (0', 1) \rightarrow \psi_{(2)}, \quad (1', 0) \rightarrow \psi_{(3)}, \quad (1', 1) \rightarrow \psi_{(4)}\}, \quad (8.5)$$

and $\psi_{(1)} \dots \psi_{(4)}$ are four distinct states of $\mathbf{M}_1 \oplus \mathbf{M}_2$. Because of the principle of locality **II**, these must have the ordered-pair form $\psi_{(1)} = (a_{(1)}, b_{(1)})$, etc., where $a_{(1)} \dots a_{(4)}$ are attributes (not necessarily intrinsic) of \mathbf{M}_1 and $b_{(1)} \dots b_{(4)}$ of \mathbf{M}_2 .

⁷Entanglement is not the only property of quantum information for which locally inaccessible information is responsible. There is also the misleadingly named ‘non-locality without entanglement’ of [17].

Now, $C = \{(0, 1), (1, 1), (0', 0), (1', 0)\}$ is an information variable. (Evidently, one can measure it by first measuring A_2 and then either A_1 or B_1 according to the output of that measurement, which is necessarily sharp.)

The effect of performing the task $\mathfrak{T} \cup \mathfrak{T}'$ on the substrate when C is sharp must be to make the variable $D\{(0, 1), (1, 0), \psi_{(1)}, \psi_{(3)}\}$ sharp with the corresponding values under (8.4) and (8.5). And D must also be an information variable because, by the coherence property, $(\mathfrak{T} \cup \mathfrak{T}')^\sim$ is also possible.

Now, consider the subset $\{(0, 1), \psi_{(1)}\}$ of D . This is an information variable that cannot be measured by measuring products of variables of \mathbf{M}_1 and \mathbf{M}_2 . For if it could, then $a_1 \perp 0$ or $b_1 \perp 1$. But if $a_1 \perp 0$ then $\psi_{(1)} \perp (0, 0)$, which in turn would imply that $(0', 0) \perp (0, 0)$ and hence that $0' \perp 0$, contrary to construction. Similarly, $b_1 \perp 1 \Rightarrow \psi_{(1)} \perp (1, 1) \Rightarrow (0', 0) \perp (1, 0) \Rightarrow 0' \perp 1$, again contrary to construction.

So the information variable $\{(0, 1), \psi_{(1)}\}$ holds locally inaccessible information, as promised. Note that the failure of local distinguishability is possible, because superinformation media obey the principle of locality II, not disobey it as has been supposed.

9. Concluding remarks

The constructor theory of information relies only on the fundamental constructor-theoretic dichotomy between possible and impossible tasks. All its definitions and conjectured principles are constructor-theoretic. It reconciles apparently contradictory features of information: that of being an abstraction, yet governed by laws of physics; of being physical, yet counter-factual. And it robustly unifies the theories of quantum and classical information.

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