Infinite Games: Example Sheet 2

Otto Pyper

(14) (i) Suppose $f : \omega^\omega \to \omega^\omega$ is continuous. We construct a coherent $c : \omega^{<\omega} \to \omega^{<\omega}$ defined inductively as follows:

- $c(\emptyset) = \emptyset$
- Let $s \in \omega^\omega$ and $m \in \omega$, and suppose $c(s)$ is defined. If there exists some m' such that $x \in [sm] \implies f(x) \in [c(s)m']$ then let $c(sm) = c(s)m'$. Otherwise, let $c(sm) = c(s)$.

This construction can be done for any f and always gives c increasing; we will show that continuity of f implies boundedness of c:

Suppose c is not unbounded. Then there exists some x for which $c(x \restriction n) = c(x \restriction N)$, of length M, for all $n \geq N$ for some $N \in \omega$. Thus for each $n \geq N$ there exists some $y_n \in \omega^\omega$ such that $x \restriction n = y_n \restriction n$ but $f(x) \restriction M+1 \neq f(y_n) \restriction M+1$. But then $y_n \to x$, and so $f(y_n) \to f(x)$, contradiction.

Lastly, we have that $f = f_c$. Indeed, for any $x \in \omega^\omega$, $c(x \restriction n) \subseteq f(x)$ for all n, and hence $\bigcup c(x \restriction n) \subseteq f(x)$, but the union lies in ω^ω and so they are equal. Hence $f = f_c$.

(ii) We have a coherent c such that $f = f_c$; let $x_n \to x \in \omega^\omega$. Suppose for contradiction that there is some M for which $f(x_n) \restriction M \neq f(x) \restriction M$ for infinitely many n.

Since c is unbounded, there is some $M' \in \omega$ such that $\bigcup_{m \in M'} c(x \restriction m)$ has length M. But now since c is increasing, if $x_n \restriction M' = x \restriction M'$ then $f(x) \restriction M = f(x') \restriction M$; this must happen eventually for all x_n, since $x_n \to x$. Contradiction.

(15) (i) $\implies$ (ii): Suppose player I has played string x_n and player II has played string y_n, which has pass-concatenation y_n^*, and assume that $x \in [x_n] \implies f(x) \in [y_n^*]$. Now suppose player I plays $m \in \omega^\omega$. Either we have some m' such that $x \in [x_n m] \implies f(x) \in [y_n^* m]$, in which case II plays m', or no such m' exists, in which case player II passes. Such an m' must eventually exist by continuity of f (same as above).

Perhaps more precisely: $f = f_c$ for some c. Define τ for player II as follows. Suppose player I has played a sequence $x = x_0 x_1 \dots x_n$ and player II has played $y = y_0 y_1 \dots y_{n-1}$. If there exists $m \in \omega^\omega$ such that $y^* m \subseteq c(x)$, then $\tau(x_0 y_0 \dots x_n) = m$. Otherwise $\tau(x_0 y_0 \dots x_n) = \text{pass}$.

Then at each stage we have $y^* \subseteq c(x)$, so taking the union we see that the run y'^* of player II lies in the union of all the $c(x)$s, which equals $f(x')$ for x' the run of player I.

(ii) $\implies$ (i): Suppose player II has a winning strategy τ. Define c coherent by letting $c(s)$ be the pass-concatenation of $\tau(s_i) : i \leq \ell h(s)$. Suppose player I produces x, player II produces y. Then for each n, we have $c(x \restriction n) = (y \restriction n)^*$. Then $f_c(x) = \bigcup c(x \restriction n) = \bigcup (y \restriction n)^* = y^* = f(x)$. So $f = f_c$.

If we do not require the possibility of pass moves, we still have that (ii) $\implies$ (i) by the argument above, but the other direction is more complicated.

Suppose player I starts by playing m. Then player II must play the start of a sequence inside $f([m])$. Suppose the earlier condition stated does not hold. Then $\forall m' \exists x (x \in [m] \wedge f(x) \notin [m'])$. So if player II plays m', then player I picks such a run x such that $x \restriction 1 = m$ but $f(x) \notin [m']$. Then player I wins.

Indeed, if there is *any* position s for which this is the case, that is, if player I has played s and player II t and there is some m such that for all m', there exists $x \in [sm]$ such that $f(x) \notin [tm']$, then player I wins.

So in order for II to be able to win, we have this more strict condition on f, which is that $f = f_c$ for some coherent c satisfying $\ell h(c(s)) \geq \ell h(s)$. Equivalently, there is some coherent c with $\ell h(c(s)) = \ell h(s)$ for all s.

(16) $\mathbf{\Sigma}^2_0(\mathbb{Q})$ is the collection of sets which are countable unions of closed sets, *i.e.* every $A \subseteq \mathbb{Q}$ since singletons are closed. So $\mathbf{\Pi}^2_0(\mathbb{Q}) = \mathbb{PQ}$ also, hence $\mathbf{\Delta}^0_2(\mathbb{Q}) = \mathbb{PQ}$.

Let $B \subseteq \mathbb{Q}$ be dense with dense complement. Now suppose that there is some $A \in \mathbf{\Delta}^0_2$ such that $A \cap \mathbb{Q} = B$. We must have $A = \bigcap_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n$ for open sets A_n and closed sets B_n.

We have $B \subseteq A_n$ for each n. Consider A_n^c. This is nowhere dense: if it is dense in some interval (a, b), then there is some $q \in B$ that lies within the interval, and hence some ε such that $(q - \varepsilon, q + \varepsilon) \subseteq A_n \cap (a, b)$. So A_n^c not dense in $(q - \varepsilon, q + \varepsilon)$. So A^c is meagre.

We have $A^c = \bigcap_{n \in \mathbb{N}} B_n^c$, B_n^c open also. But B^c dense, so applying exactly the same argument as above gives that A is meagre. Hence $A \cup A^c$ is meagre, but $\mathcal{A} \cup A^c = \mathbb{R}$, which is not meagre by the Baire Category Theorem.

(17) We can write $F = \bigcup_{n \in \mathbb{N}} F_n$, where $F_n = [-n, n] \cap F$ is closed and bounded, hence has a sequentially closed image, which is thus closed in a metric space. Hence $f[F] = \bigcup f[F_n]$ is a countable union of closed sets, hence is $\mathbf{\Sigma}^0_2$.

(18) Base case: need to show $\mathbf{\Sigma}^1_1$ closed under countable union (intersection is easy).

Let A_n be a family of analytic sets, such that $A_n = pB_n$ for $B_n \in \mathbf{\Pi}^0_1(\omega^\omega \times \omega^\omega)$. Define $C_n = \{(nx, y) : (x, y) \in B_n\}$. Then C_n is closed, since convergent sequences in C_n clearly inherit from convergent sequences in B_n. But now $\bigcup C_n$ is closed because a sequence can only converge if the first letter is eventually constant, meaning the sequence must get stuck in exactly one of the disjoint C_ns. Then $\bigcup A_n = p \bigcup C_n$ is analytic.

Now suppose $\mathbf{\Pi}^1_n$ is closed under countable union. Then if A_n is a family of $\mathbf{\Sigma}^1_{n+1}$ sets, $A_n = pB_n$, $\bigcup A_n = \bigcup pB_n = p \bigcup B_n \in \mathbf{\Sigma}^1_{n+1}$. Exact same argument for countable intersection. The $\mathbf{\Sigma} \to \mathbf{\Pi}$ case is clear, and similarly for $\mathbf{\Delta}$s.

We have $\mathbf{\Pi}^0_1 = \mathbf{\Pi}^1_0 \subseteq \mathbf{\Delta}^1_1$, hence $\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Delta}^1_1$ also. It is then easy to see inductively that every Borel class is a subset of $\mathbf{\Delta}^1_1$.

In (17), we found that $\mathbf{\Sigma}^1_1(\mathbb{R}) \subseteq \mathbf{\Sigma}^0_2(\mathbb{R})$. Hence $\mathbf{\Delta}^1_1 \subseteq\neq \mathbf{\Sigma}^1_1 \subseteq \mathbf{\Sigma}^0_2$, so not every Borel set is $\mathbf{\Delta}^1_1$ in $\mathbb{R}$. The problem here, morally, is that in $\mathbb{R}$ bounded and closed $\iff$ compact; continuous images of compact sets are compact, which are closed in metric spaces. But ω^ω does not have the property that closed and bounded $\implies$ compact, so this issue does not occur (indeed, every set is bounded, the entire space itself is closed and not compact).

(19) Hmm...

(20) A_n, B_m are Borel separable by C_{nm}. Then A_n and $\bigcup B_n$ are Borel separable by $D_n \coloneqq \bigcap_m C_{nm}$, since Borel sets are closed under countable intersection (they all lie in some $\mathbf{\Pi}^0_\alpha$, say, with α countable since ω_1 regular).

Then $\bigcup A_n, \bigcup B_n$ are separable by $\bigcup_n D_n$, which is again Borel.

(21) The first part is done in stages, by building up a sequence of inseparable trees to find a sequence that must lie in their intersection.

Given this, we already have Borel $\subseteq \mathbf{\Delta}^1_1$ and need to show that $A \subseteq \mathbf{\Delta}^1_1 \implies A$ is Borel. But A, A^c are both analytic, hence Borel separable - but the only separating set is A. Hence A is Borel.

(22) Suppose that R is Lebesgue-measurable, and let $x \in A$. Let $A^{<x} = \{y : yRx\}$. $A^{<x}$ has order-type less than κ, and since κ is initial we must then have that $A^{<x}$ has strictly smaller cardinality than κ and is hence null.

Thus $A^{>x}$ is not null for any x, since otherwise $A^{\leq x} = A \backslash A^{>x}$ is a smaller non-null set. So $\{x : A^{>x} \text{ not null}\} = A$, which is not null.

So then $N := \{x : A^{<x} \text{ not null}\}$ is also not null. But $A^{<x}$ is null for all x, so this set is empty and hence null. Contradiction.

(23) (i) $\underline{\Longleftarrow}$: If such a p exists, then player I plays p, and $[p] \backslash A = \bigcup_{n \in \omega} A_n$, for A_n all nowhere dense. In particular then, for all $q \in \omega^< < \omega$, there exists $x_1 \in [pq]$ such that x_1 cannot be approached from within A_1. Thus there is some m for which $[x_1 \restriction m] \cap A_1 = \emptyset$. Player I then plays to any position $\ni x_1 \restriction m$, and then $[x_1 \restriction m] \backslash A \subseteq \bigcup_{n>1} A_n$.

Player I then repeats this process, eliminating the losing runs in set A_n on move $n+1$. So if x is the run and player I has lost, then $x \in A_m$ for some m - but this cannot be the case by player I's strategy.

Note that specifying the strategy entirely here has used choice.

$\underline{\Longrightarrow}$ Suppose player I has a winning strategy σ, and let $p = \sigma(\emptyset)$. Consider the game $G^{**}(A^c_p)$. The second player of this game has a winning strategy by following σ in the first game, hence by (ii) A^c_p is meagre.

$A^c_p = \{x : px \notin A\} = \bigcup A_n$, A_n nowhere dense. Then $pA_n := \{px : x \in A_n\}$ is also nowhere dense, hence $\bigcup pA_n = pA^c_p = \{px : px \notin A\} = [p] \backslash A$ is meagre.

(ii) $\underline{\Longleftarrow}$: If A is meagre, then $A = \bigcup A_n$, A_n nowhere dense. Suppose player I plays p. Then by the same process as in (i), player II can ensure the run does not lie in A_n for any n; so player II has a winning strategy.

$\underline{\Longrightarrow}$: Suppose player II has a winning strategy τ, and that A is not meagre. Then if $A = \bigcup_{n \in \omega} A_n$, there must be some A_n that is somewhere dense. That is to say, A_n is dense in some $[p]$, $p \in \omega^{<\omega}$.

Then $A = \bigcup_n [n] \cap A$, so wlog say $[n] \cap A$ is somewhere dense in $[n]$. Then there is some $p \ni n$ such that A is dense in $[p]$. Then A is dense in $[pq]$ for any q.

(24) not too bad, I think the answer is $\mathrm{AC}_{X^{<\omega}}(X)$

(25) just a bit fiddly