Topics in Combinatorics Revision Questions

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1. **Prove that if X is a random variable, then $\mathbb{P}[X \geq EX] > 0$ and $\mathbb{P}[X \leq EX] > 0$**
2. State and prove a proposition about average degrees in bipartite graphs.
3. What is this method called, roughly speaking?
4. State and prove another proposition about average degrees, concerning an inequality instead of an equality.
5. Show that for a planar graph, $E \geq 3F/2$.
6. Define $\partial_s \mathcal{A}$, in context. What is this called?
7. State and prove a lower bound on $|\partial_S \mathcal{A}|$, representing it in two different ways.
8. State Sperner's Theorem.
9. **Prove Sperner's Theorem.**
10. State the Erdos-Ko-Rado Theorem.
11. **Prove the Erdos-Ko-Rado Theorem.**
    (a) Show that, for a random ordering, at most k intervals can be part of an intersecting family.
    (b) What is the expected total number of such intervals belonging to the family?
    (c) In the equality case, how are we able to still have k intervals?
    (d) Construct a helpful cylcic order to show any A of size k is contained in $\mathcal{A}$.
12. Define the crossing number of G.
13. Show that a planar graph with n vertices has at most $3n - 6$ edges.
14. Prove that a graph with n vertices and m edges has crossing number at least $m - 3n$.
15. Let G be a graph drawn in the plane with n vertices and m edges, with $m \geq 6n$. G must have at least how many crossings?

16. **Prove this.**
17. State the Szemeredi-Trotter Theorem
18. **Prove the Szemeredi-Trotter Theorem.**
    (a) What is an appropriate way to convert the points/lines into a graph?
    (b) Put two different bounds on the number of crossings.
19. Give three examples of graphs demonstrating the S-T bound (up to a constant).
20. State Stirling's Formula. [optional]
21. State a gentler upper/lower bound on $n!$.
22. Prove this.
23. State bounds on $2^{-n}\binom{n}{n/2}$.
24. Prove these bounds.
25. Give an example of an event that occurs with probability $2^{-n}\binom{n}{n/2}$.
26. Give a bound on $\binom{n}{m}$, useful for when $m \ll n$.
27. Improve this bound slightly. In what scenario is this often useful?
28. State the quotient of consecutive binomial coefficients.
29. Bound $\sum_{k=0}^{m}\binom{n}{k}$ in the case $m = \alpha n$, $\alpha < 1/2$.
30. State an example of the *concentration of measure phenomenon.*
31. Let $m = (1/2 - \theta)n$ with $0 < \theta \leq 1/2$. Then $2^{-n}\binom{n}{m} \leq \mathrm{e}^{-\theta^2 n/2}$.
32. **Prove that if $X_1, \ldots, X_n$ are independent random variables of mean zero taking values in $[-1, 1]$, and $X = \sum X_i$, then $\mathbb{P}[X \geq \varepsilon n] \leq \mathrm{e}^{-\varepsilon^2 n/4}$.**
    (a) What is the exponential moment?
    (b) Markov...
    (c) Optimise over something?
33. What similar result do we immediately get by switching signs?
34. Let $m = (1/2 - \varepsilon)n$. Show that $2^{-n}\sum_{k=0}^{m}\binom{n}{k} \leq \mathrm{e}^{-\varepsilon^2 n}$.
35. What is a general question regarding well-separated sets.
36. What is the critical threshold for changing behaviour at intersections of size αn, and why?
37. **Prove that if $\alpha > 1/4$, there can be exponentially many subsets of $[n]$ of size $n/2$ intersecting in no more than αn.**

(a) Let A be a random set of size $n/2$. What is a good probability to estimate?

(b) Then how many bad intersections can we have?

(c) How can we mitigate this?

38. What is the characteristic function of a set?

39. What space does it live in?

40. What is an oftentimes more useful function with which to associate a set?

41. Why is it thusly named?

42. Prove that if $A, B \subset [n]$ have size $n/2$ and f_A, f_B are their balanced functions, then $\langle f_A, f_B \rangle = |A \cap B| - n/4$.

43. Use this to prove that if $A_1, \ldots, A_m \subset [n]$ intersect in at most $(1/4 - \delta)n$, then $m \leq 1 + \delta/4$.

44. State and prove a more general theorem, of which the above is a special case.

45. **Prove that if $x_1, \ldots, x_m$ are non-zero vectors in $\mathbb{R}^n$ such that $\langle x_i, x_j \rangle \leq 0$ for every $i \neq j$ then $m \leq 2n$.**

    (a) Induction on n.

    (b) How can we reduce to $n-1$-dimensional space?

46. **Furthermore, prove that if $m = 2n$ then there is an orthonormal basis $a_1, \ldots, a_n$ such that each x_i is a multiple of some a_j (so we have exactly one positive multiple and one negative multiple of each a_i).**

47. Relate the above back to the problem of finding families of sets with intersections in at most $n/4$.

48. Define a Hadamard matrix.

49. Define the Walsh matrices W_m.

50. State three basic facts about W_m.

51. How do we find the desired set system from the W_m?

52. Define the Paley matrix P_n.

53. Prove that for every prime p and every $d \not\equiv 0 \mod p$ we have that $\sum_{x \in \mathbb{Z}_p} \left(\frac{x}{p}\right)\left(\frac{x+d}{p}\right) = -1$

54. Hence show P_n has orthogonal rows.

55. What is the smallest n, a multiple of 4, for which no Hadamard matrix is known?

56. Prove that if $|A| = n$, then $|A + A| \geq 2n - 1$. When do we have equality?
57. Demonstrate a similar result for $|A.A|$.
58. State a famous conjecture of Erdos and Szemeredi.
59. Define $\rho_A^+(x)$ and $\rho_A^\times(x)$.
60. Define the multiplicative energy and additive energy of A.
61. What is the quotient set of A?
62. State and prove a lemma about multliplicative energy.
63. Prove that $\sum_{i=1}^n |a_i|^2 \geq n^{-1}(\sum_i |a_i|)^2$.
64. Prove it again, in a different way.
65. Show that the multiplicative energy of A is at least $|A|^4/|A.A|$.