Model Theory

Lectures by Gabriel Conant

Contents

0 Review of First Order Logic

A ***language*** is a set $\mathcal{L}$ of function symbols, relation symbols, and constant symbols. Additionally, each function/relation symbol has an assigned *arity* $n \geq 1$.

By convention, we view constant symbols as 'function symbols of arity 0'.

An ***$\mathcal{L}$-structure*** $\mathcal{M}$ consists of:

- a non-empty set M (the ***universe*** of $\mathcal{M}$)
- for every function symbol f of arity n, a function $f^{\mathcal{M}} : M^n \to M$
- for every relation symbol R of arity n, a subset $R^{\mathcal{M}} \subseteq M^n$
- for every constant symbol c, an element $c^{\mathcal{M}} \in M$ (*i.e.* identified with the unique element in its image)

Syntax: we build formulas using symbols in $\mathcal{L}$ along with

$$\wedge \ \neg \ \forall \ = \ (\ ) \ ,$$

and countably many variable symbols.

$\mathcal{L}$-term: these are our way of creating new functions by composing the ones we already have.

- constant symbols and variables are terms
- if $t_1, \dots, t_n$ are terms and f is an n-ary function symbol, then $f(t_1, \dots, t_n)$ is a term

Given a structure $\mathcal{M}$ and a term t, we are going to interpret the term in the structure in exactly the way you might expect. Inductively, define (for appropriate r) $t^{\mathcal{M}} : M^r \to M$ as:

- constant symbol c: $c^{\mathcal{M}}$ (case $r = 0$)
- variable x: identify function ($r = 1$)
- general term $f(t_1, \dots, t_n)$: $f^{\mathcal{M}}(t_1^{\mathcal{M}}, \dots, t_n^{\mathcal{M}})$

$\mathcal{L}$-formulas: new relations. We have the following *atomic* L-formulas:

- If t_1 and t_2 are terms, then $(t_1 = t_2)$ is a formula
- If R is an n-ary relation symbol and $t_1, \dots, t_n$ are terms, then $R(t_1, \dots, t_n)$ is a formula

We can then create more complicated formulas. Given formulae φ and ψ:

- $\neg\varphi$
- $(\varphi \wedge \psi)$
- $\forall x \varphi$ for any variable x

An occurrence of a variable x is ***free*** in φ if x does not occur in the scope of $\forall x$. Otherwise, the occurrence is ***bound***.

For instance, if φ is the statement $\forall x \neg(f(x) = y)$, x is bound and y is free.

Notation: Given a formula φ, we write $\varphi(x_1, \dots, x_n)$ to denote that $x_1, \dots, x_n$ are the free variables of φ.

Given a formula $\varphi(x_1, \dots, x_n)$, a structure $\mathcal{M}$, $a_1, \dots, a_n \in M$, we define "$\overline{a}$ satisfies φ in $\mathcal{M}$", written $\mathcal{M} \models \varphi(a_1, \dots, a_n)$, as follows:

- If φ is $(t_1 = t_2)$ then $\mathcal{M} \models \varphi(\overline{a})$ iff $t_1^{\mathcal{M}}(\overline{a}) = t_2^{\mathcal{M}}(\overline{a})$

- If φ is $R(t_1, \ldots, t_n)$ then $\mathcal{M} \models \varphi(\overline{a})$ iff $(t_1^{\mathcal{M}}(\overline{a}), \ldots, t_n^{\mathcal{M}}(\overline{a}) \in R^{\mathcal{M}}$
- $\mathcal{M} \models (\varphi \wedge \psi)(\overline{a})$ iff $\mathcal{M} \models \varphi(\overline{a})$ and $\mathcal{M} \models \psi(\overline{a})$
- $\mathcal{M} \models \neg\varphi(\overline{a})$ iff $\mathcal{M} \not\models \varphi(\overline{a})$
- Suppose φ is $\forall w \psi(x_1 \ldots, x_n, w)$. Then $M \models \varphi(\overline{a})$ iff for all $b \in M$, $\mathcal{M} \models \psi(\overline{a}, b)$

We emphasise that the focus of this course will not be on the precise definitions and semantics, so much as the meaning of what we are doing. All we seek is a first order logic that works for us, so that we can use it to do interesting things.

Abbreviations: We have *global* abbreviations such as

- $(\varphi \vee \psi)$ is $\neg(\neg\varphi \wedge \neg\psi)$
- $(\varphi \rightarrow \psi)$ is $(\neg\varphi \vee \psi)$
- $(\varphi \leftrightarrow \psi)$ is $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
- $\exists x\varphi$ is $\neg\forall x\neg\varphi$

We note that the last equivalence in a semantic sense hinges on the assumption that universes are non-empty. Since we will be almost exclusively be studying infinite structures, we will not worry about this.

We also have *local* abbreviations, often specific to the language we are studying. For instance, in $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ (the language of ordered rings):

- $x + y$ is $+(x, y)$
- $x < y$ is $< (x, y)$
- $x \leq y$ is $(x < y) \wedge (x = y)$
- $x < y < z$ is $(x < y) \wedge (y < z)$
- x^2 is $x \cdot x$
- nx is $\underbrace{x + x + \cdots + x}_{n \text{ times}}$

An ***$\mathcal{L}$-sentence*** is an $\mathcal{L}$-formula with no free variables. For instance, $\forall x(f(x) \neq y)$ is *not* a sentence, but $\exists y \forall x(f(x) \neq y)$ *is* a sentence. Sentences can be thought of as actually saying something meaningful.

If φ is a sentence and $\mathcal{M}$ is a structure, then we have the notion of $\mathcal{M} \models \varphi$, "$\mathcal{M}$ satisfies φ" or "$\mathcal{M}$ models φ".

Definition: (L-theory) An ***L-theory*** is a set of L-sentences.

Given a theory T, we write $\mathcal{M} \models T$ ("$\mathcal{M}$ is a ***model*** of T) if $\mathcal{M} \models \varphi$ for all $\varphi \in T$.

T is ***satisfiable*** if it has a model.

Example: $T = \{\neg\exists x(x = x)\}$ - this sentence claims there are no elements in the universe. In our setting, this is unsatisfiable (though it is technically a matter of opinion).

Similarly, $\exists x(x = x)$ ("The Axiom of Non-Triviality") is always satisfied in any $\mathcal{L}$-structure.

Recall: T is ***consistent*** if it does not prove a contradiction (*e.g.* $(\varphi \wedge \neg\varphi)$)

A consequence of ***Gödel's Completeness Theorem*** is that a theory is satisfiable iff it is consistent. This is a very important theorem, though we will mostly be focussing on the model theoretic aspect (satisfiability).

1 Lecture

We now consider a fixed language $\mathcal{L}$.

An $\mathcal{L}$-theory T is ***finitely satisfiable*** if every finite subset of T is satisfiable. This leads us to one of the most important theorems for getting Model Theory off the ground:

Theorem: (Compactness Theorem) *An $\mathcal{L}$-theory T satisfiable iff it is finitely satisfiable*

Another important theorem of Model Theory is the following.

Theorem: (Downward Lowenheim-Skolem Theorem) *Any satisfiable $\mathcal{L}$-theory has a model of cardinality at most $|\mathcal{L}| + \aleph_0$*

The proofs of the above are non-examinable; see Part II notes for details.

Theorem: ((Upward) Lowenheim-Skolem Theorem) *Suppose T is an $\mathcal{L}$-theory with infinite models. Then T has a model of cardinality κ for any $\kappa \geq |\mathcal{L}| + \aleph_0$*

We note that by the 'cardinality' of a structure we mean the cardinality of its universe.

Proof. What we need to do here is build a model of this theory, but do it such that it's not just a model of the theory but that it also has some extra properties of our choosing. This is a common technique in model theory.

We want more elements, so we add more symbols to our language and more sentences claiming various properties about these symbols.

Let $\mathcal{L}^* = \mathcal{L} \cup \{c_i : i < \kappa\}$ where each c_i is a new constant symbol.

Then let $T^* = T \cup \{c_i \neq c_j : i \neq j\}$. Suppose $\Sigma \subseteq T^*$ is finite. Then $\Sigma \subseteq T \cup \{c_i \neq c_j : i, j \in I\}$ for some finite set I.

Let $\mathcal{M} \models T$ be an infinite $\mathcal{L}$-structure. Expand $\mathcal{M}$ to an $\mathcal{L}^*$ structure $\mathcal{M}^*$ by interpreting $c_i^{\mathcal{M}^*}$ as distinct elements for $i \in I$, and interpreting $c_i^{\mathcal{M}^*}$ for $i \notin I$ arbitrarily. Note that this is 'physically' the same structure, all we have changed is its interpretation.

Then $M^* \models \Sigma$, so T is finitely satisfiable. Hence by the Compactness Theorem T^* is satisfiable. Then by DLST, T^* has a model $\mathcal{N}^*$ of cardinality at most $|\mathcal{L}^*| + \aleph_0 = \kappa$. Moreover, every model has cardinality *at least* κ, so $\mathcal{N}^*$ indeed has cardinality κ.

Then let $\mathcal{N}$ be the reduct of $\mathcal{N}^*$ to $\mathcal{L}$ (same universe, different interpretation). Then $\mathcal{N} \models T$ and $|N| = \kappa$. $\square$

Complete Theories

Definition 1.1: (Semantic Entailment) Let T be an $\mathcal{L}$-theory and φ an $\mathcal{L}$-sentence. Then $T \models \varphi$ ('T ***models*** φ, 'T ***implies*** φ') if any model of T is also a model of φ.

Example 1.2:

1) $\{\varphi, \psi\} \models \varphi \wedge \psi$
2) If T is consistent then $T \models \exists x(x = x)$ (also if it's not consistent). So $\emptyset \models \exists x(x = x)$ since we assume all models are non-empty.
3) Let T be the theory of groups in the language of groups $\mathcal{L} = \{*, e\}$.

   Then $T \models \forall x \forall y \forall z\,((x * y = e \wedge x * z = e) \to y = z)$, since in any group inverses are unique.

Definition 1.3: (Complete Theory) An $\mathcal{L}$-theory T si ***complete*** if, for any $\mathcal{L}$-sentence φ, we have $T \models \varphi$ or $T \models \neg\varphi$.

Example 1.4:

1) The theory of groups is not complete. Consider $\forall x \forall y(x * y = y * x)$ - this asserts that the group is abelian. Since there are some groups with this property and some without it, then neither $T \models \varphi$ nor $T \models \neg\varphi$.
2) ZFC is not complete (if it is consistent); consider the Continuum Hypothesis.

Definition 1.5: (Theory of a structure) Let $\mathcal{M}$ be an $\mathcal{L}$-structure. The ***theory of*** $\mathcal{M}$ is

$$\mathrm{Th}(\mathcal{M}) = \mathrm{Th}_{\mathcal{L}}(\mathcal{M}) \coloneqq \{\varphi : \varphi \text{ is an } \mathcal{L}\text{-sentence and } \mathcal{M} \models \varphi\}$$

Note that $\mathrm{Th}(\mathcal{M})$ is complete, since for every φ either $\varphi \in \mathrm{Th}(\mathcal{M})$ or $M \models\!\!/ \varphi$. However, this makes $\mathrm{Th}(\mathcal{M})$ complicated as a set; every sentence or its negation is in the set, including many that are pointless or redundant. We want to look for complete theories that have a much more efficient presentation.

Definition 1.6: (Elementarily Equivalent) Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are ***elementarily equivalent***, written $\mathcal{M} \equiv \mathcal{N}$ if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}((N))$.

Note that $\equiv$ is an equivalence relation on $\mathcal{L}$-structures. To emphasise that this only a discussion of $\mathcal{L}$-structures for a specific language $\mathcal{L}$, we may sometimes write $\equiv_{\mathcal{L}}$.

Exercise: (Sheet 1 Question 2) Let T be an $\mathcal{L}$-theory. TFAE

i) T is complete
ii) For an $\mathcal{L}$-sentence φ, if $T \not\models \varphi$ then $T \models \neg\varphi$. We remark that for a model $\mathcal{M}$, $\mathcal{M} \not\models \varphi \implies \mathcal{M} \models \neg\varphi$, but this is *not* the case for *theories* in general.
iii) Any two models of T are elementarily equivalent.

Example 1.7: Let $\mathcal{L} = \emptyset$ and $T = \{\varphi_n : n \geq 2\}$ where φ_n is

$$\exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} x_i \neq x_j$$

T is then the ***theory of infinite sets***; its models are all of the infinite $\mathcal{L}$-structures. So, as $\mathcal{L}$-structures, $\mathcal{N} \equiv \mathbb{Z} \equiv \mathbb{Q} \equiv \mathbb{R} \equiv \mathbb{C} \equiv \mathcal{P}(\mathbb{C}) \equiv$ any infinite set.

Theorem 1.8: (Vaught's Test) *Let T be an $\mathcal{L}$-theory such that*

a) T has no finite models

b) $\exists \kappa \geq |\mathcal{L}| + \aleph_0$ such that any two models of T of cardinality κ are elementarily equivalent

Then T is complete.

Proof. Suppose T is not complete. Then there is a sentence φ such that $T \cup \{\neg\varphi\}$ is satisfiable, and $T \cup \{\varphi\}$ is satisfiable.

By (a), these theories have infinite models. By Lowenheim-Skolem, these theories have models of size κ. But these are both models of T and hence are elementarily equivalent $\perp$ by (b). $\square$

Showing that two structures are elementarily equivalent is often difficult to do directly, so we need to find other ways around it.

2 Lecture

Let $\mathcal{L}$ be a language.

Definition 2.1: ($\mathcal{L}$-Homomorphism) Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. A function $h : M \to N$ is an ***$\mathcal{L}$-homomorphism*** if

i) for any n-ary function symbol f and $a_1, \ldots, a_n \in M$

$$h(f^{\mathcal{M}}(a_1, \ldots, a_N)) = f^{\mathcal{N}}(h(a_1), h(a_2), \ldots, h(a_n))$$

ii) for any n-ary relation symbol R and $a_1, \ldots, a_n \in M$

$$(a_1, \ldots, a_n) \in R^{\mathcal{M}} \iff (h(a_1), \ldots, h(a_n)) \in R^{\mathcal{N}}$$

iii) for any constant symbol c, $h(c^{\mathcal{M}}) = c^{\mathcal{N}}$.

We write $h : \mathcal{M} \to \mathcal{N}$ for $\mathcal{L}$-homomorphisms h.

If h is also injective, then h is an ***$\mathcal{L}$-embedding***. If h is also bijective, then h is an ***$\mathcal{L}$-isomorphism***.

Theorem 2.2: *Suppose $h : \mathcal{M} \to \mathcal{N}$ is an $\mathcal{L}$-isomorphism. Then for any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$, we have*

$$\mathcal{M} \models \varphi(a_1, \ldots, a_n) \iff \mathcal{N} \models \varphi(h(a_1), \ldots, h(a_n))$$

Proof. Often in situations like this, we will need to induct on the complexity of the formula, with the base case simply being the terms, and then atomic formulae, then all formulae.

<u>Claim</u>: For any $\mathcal{L}$-term $t(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$

$$h(t^{\mathcal{M}}(a_1, \ldots, a_n)) = t^{\mathcal{N}}(h(a_1), \ldots, h(a_n))$$

Proof of claim: induction on terms. If t is a constant symbol c, then $h(t^{\mathcal{M}}) = h(c^{\mathcal{M}}) = h(c^{\mathcal{N}}) = t^{\mathcal{N}}$ since h preserves functions (and thus constant symbols).

If t is a variable x_1, then $h(t^{\mathcal{M}}(a_1)) = h(a_1) = t^{\mathcal{N}}(h(a_1))$ since variables are interpreted as the identity function.

Let f be an m-ary function symbol. Assume the result for terms $t_1, \dots, t_m$ whose free variables are among $x_1, \dots, x_n$. Let t be $f(t_1, \dots, t_m)$. Given $a_1, \dots, a_n \in M$:

$$\begin{aligned} h(t^{\mathcal{M}}(\overline{a}) &= h(f^{\mathcal{M}}(t_1^{\mathcal{M}}(\overline{a}), \dots, t_m^{\mathcal{M}}(\overline{a}))) \\ &= f^{\mathcal{N}}(h(t_1^{\mathcal{M}}(\overline{a})), \dots, h(t_m^{\mathcal{M}}(\overline{a}))) \\ &= f^{\mathcal{N}}(t_1^{\mathcal{N}}(h(\overline{a})), \dots, t_m^{\mathcal{N}}(h(\overline{a}))) \\ &= t^{\mathcal{N}}(h(\overline{a})) \end{aligned}$$

So the claim is proven. Now we prove the theorem by induction on φ.

Base case: φ is atomic.

1) φ is $t_1 = t_2$:

$$\begin{aligned} M \models \varphi(\overline{a}) &\iff t_1^{\mathcal{M}}(\overline{a}) = t_2^{\mathcal{M}}(\overline{a}) \\ &\iff h(t_1^{\mathcal{M}}(\overline{a})) = h(t_2^{\mathcal{M}}(\overline{a})) \text{ (h injective)} \\ &\iff t_1^{\mathcal{N}}(h(\overline{a})) = t_2^{\mathcal{N}}(h(\overline{a})) \text{ (by claim)} \\ &\iff \mathcal{N} \models \varphi(h(\overline{a})) \end{aligned}$$

2) φ is $R(t_1, \dots, t_n)$ (Exercise).

Induction Step: Assume the result for φ and ψ.

Exercise: check $\varphi \wedge \psi$ and $\neg\varphi$.

We will do $\forall x_n \varphi(x_1, \dots, x_n)$, with free variables $x_1, \dots, x_{n-1}$. Fix $a_1, \dots, a_{n-1} \in M$.

$$\begin{aligned} M \models \forall x_n \varphi(a_1, \dots, a_{n-1}, x_n) &\iff \text{for all } b \in M,\ \mathcal{M} \models \varphi(a_1, \dots, a_{n-1}, b) \\ &\iff \text{for all } b \in M,\ \mathcal{N} \models \varphi(h(a_1), \dots, h(a_{n-1}), h(b)) \text{ (induction)} \\ &\iff \text{for all } c \in N,\ \mathcal{N} \models \varphi(h(a_1), \dots, h(a_{n-1}), c) \text{ (h surjective)} \\ &\iff \mathcal{N} \models \forall x_n \varphi(h(a_1), \dots, h(a_{n-1}), x_n) \end{aligned}$$

And so we are done. In particular, $\mathcal{L}$-isomorphisms preserve all formulae. □

Notation: We write $\mathcal{M} \cong \mathcal{N}$ if there is an $\mathcal{L}$-isomorphism $h : \mathcal{M} \to \mathcal{N}$.

Corollary 2.3: *If $\mathcal{M} \cong \mathcal{N}$ then $\mathcal{M} \equiv \mathcal{N}$.*

Note that, as we can see, $\cong$ is stronger than $\equiv$; $\cong$ says that two structures are more or less the same, whereas $\equiv$ only makes an assertion about first order statements satisfied by the models.

Corollary 2.4: *$h : \mathcal{M} \to \mathcal{N}$ is an $\mathcal{L}$-embedding iff for any quantifier-free the conclusion of Theorem 2.2 holds for all quantifier-free formulas $\varphi(x_1, \dots, x_n)$. That is to say, $\mathcal{L}$-embeddings preserve all quantifier-free formulas.*

Proof. ($\implies$) is done by the proof of 2.2; we only used the surjectivity of h for the quantifier step. For ($\impliedby$), see Sheet 1, Question 6. □

An embedding is precisely characterised by preserving quantifier-free formulae. This motivates the question, what about maps that preserve all formulas? We know that isomorphisms will do, but is that all of them? The answer is in fact no, in general.

Definition 2.5: (Elementary L-Embedding) $h : \mathcal{M} \to \mathcal{N}$ is an ***elementary $\mathcal{L}$-embedding*** if for any L-formula $\varphi(\overline{x})$ and $\overline{a}$ from M, $\mathcal{M} \models \varphi(\overline{a})$ iff $\mathcal{N} \models \varphi(h(\overline{a}))$.

Note that isomorphisms are elementary embeddings, but elementary embeddings need not be isomorphisms.

Definition 2.6: (Elementary Substructure) Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures with $M \subseteq N$. Let $h : M \hookrightarrow N$ be the inclusion map. Then $\mathcal{M}$ is a ***substructure*** of $\mathcal{N}$ (respectively, ***elementary substructure***), written $\mathcal{M} \subseteq \mathcal{N}$ (respectively $\mathcal{M} \preceq \mathcal{N}$) if h is an $\mathcal{L}$-embedding (respectively, elementary embedding).

Similarly, $\mathcal{N}$ is an ***extension*** of $\mathcal{M}$ (respectively, ***elementary extension***).

Note: If $\mathcal{M} \preceq \mathcal{N}$ then $M \subseteq N$ and $\mathcal{M} \equiv \mathcal{N}$.

Example 2.7: Let $\mathcal{M} = (2\mathbb{Z}, <)$ and $\mathcal{N} = (\mathbb{Z}, <)$.

Then $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M} \equiv \mathcal{N}$, but $\mathcal{M} \not\preceq \mathcal{N}$, for instance $\mathcal{M} \models \neg\exists x(0 < x < 2)$, but this is of course untrue for $\mathcal{N}$.

So the inclusion map might be an embedding, but it is not necessarily elementary.

3 Lecture

<u>Q</u>: Suppose $\mathcal{M} \equiv \mathcal{N}$. Then is it true that $\mathcal{M} \cong \mathcal{N}$?

<u>A</u>: No - *e.g.* theory of infinite sets, any two infinite sets are elementarily equivalent but many are obviously not isomorphic. More generally, if $\mathcal{M}$ is infinite then $\mathrm{Th}(\mathcal{M})$ has models of arbitrarily large size.

So a theory with infinite models *never* has a unique model up to isomorphism, as models of different cardinalities cannot be isomorphic (since an isomorphism contains a bijection).

Definition 3.1: (κ-categorical) An $\mathcal{L}$-theory T is ***κ-categorical*** if it has a unique model of size κ up to isomorphism.

Our main focus for theories here will be those T that have infinite models and $\kappa \geq |\mathcal{L}| + \aleph_0$.

Example 3.2:

1) $\mathrm{Th}(\mathbb{N})$ in $\mathcal{L} = \emptyset$ is κ-categorical for all $\kappa \geq \aleph_0$ (Sheet 1 #3)
2) $\mathrm{Th}(\mathbb{Q}, +)$ is κ-categorical iff $\kappa > \aleph_0$ (related to Sheet 1 #4)
3) $\mathrm{Th}(\mathbb{Q}, <)$ is κ-categorical iff $\kappa = \aleph_0$
4) $\mathrm{Th}(\mathbb{Z}, +)$ is κ-categorical for no κ

Non-Examinable

Theorem: (Morley's Theorem (1965)) *Let T be a complete theory in a countable language. If T is κ-categorical for some $\kappa > \aleph_0$, then it is κ-categorical for all $\kappa > \aleph_0$.*

End of non-examinable section

Definition 3.3: (Theory of Dense Linear Orders) Let $\mathcal{L} = \{<\}$ (binary relation) be the language of partial orders. Define ***DLO (dense linear orders)*** to be the following theory

- $\forall x \neg(x < x)$
- $\forall x \forall y \forall z((x < y \wedge y < z) \to x < z)$ (partial order)
- $\forall x \forall y((x \neq y) \to (x < y \vee y < x))$ (linear order)
- $\forall x \forall y(x < y \to \exists z(x < z < y))$ (dense)
- $\forall x \exists y \exists z(y < x < z)$ (no endpoints)

Note that $(\mathbb{Q}, <) \models$ DLO.

Theorem 3.4: (Cantor, 1895) *DLO is $\aleph_0$-categorical.*

Proof. "Back and Forth Construction".

Fix countable models $\mathcal{M}, \mathcal{N} \models$ DLO. Let $M = \{a_n : n \geq 0\}$ and $N = \{b_n : n \geq 0\}$. With these enumerations we will construct an isomorphism between $\mathcal{M}$ and $\mathcal{N}$.

We will inductively construct a sequence $(h_n)_{n=0}^{\infty}$ of functions such that:

1) $h_n : X_n \to Y_n$ is an order-preserving bijection, where $X_n \subseteq M$ and $Y_n \subseteq N$ are finite
2) $X_n \subseteq X_{n+1}$, $Y_n \subseteq Y_{n+1}$ and $h_n \subseteq h_{n+1}$
3) $a_n \in X_n$ and $b_n \in Y_n$

Once we have done this, we will have a sequence of increasing functions with domains and ranges getting bigger and bigger. We can then let $h = \bigcup_{n=0}^{\infty} h_n$. Then h is an order-preserving bijectionfrom M to N, which in this language is precisely an $\mathcal{L}$-isomorphism.

<u>Base case</u>: Let $X_0 = \{a_0\}$, $Y_0 = \{b_0\}$, and $h_0 = \{(a_0, b_0)\}$; this trivially satisfies all the desired properties.

Now assume we have $h_n : X_n \to Y_n$ as above.

<u>Forth</u>: Construct an order-preserving bijection $h_* : X_* \to Y_*$ extending h_n with $a_{n+1} \in X_*$. Enumerate $X_n = \{x_1, \ldots, x_k\}$ such that $x_1 <^{\mathcal{M}} \cdots <^{\mathcal{M}} x_k$. Let $y_i = h_n(x_i)$. Then $y_1 <^{\mathcal{N}} \cdots <^{\mathcal{N}} y_k$ since h_n is order-preserving.

Define $h_* = h_n \cup \{(a_{n+1}, b)\}$ where $b \in N$ is chosen as follows.

<u>Case 1</u>: $a_{n+1} = x_i$ for some $i \leq k$. Let $b = y_i$.

<u>Case 2</u>: $x_k <^{\mathcal{M}} a_{n+1}$. Choose $b \in N$ such that $y_k <^{\mathcal{N}} b$.

<u>Case 3</u>: $a_{n+1} <^{\mathcal{M}} x_1$. Choose $b \in N$ such that $b <^{\mathcal{N}} y_1$.

Case 4: $x_i <^{\mathcal{M}} a_{n+1} <^{\mathcal{M}} x_{i+1}$ for some $i < k$. Choose $b \in N$ such that $y_i <^{\mathcal{N}} b <^{\mathcal{N}} y_{i+1}$.

Back: Construct order-preserving $h_{n+1} : X_{n+1} \to Y_{n+1}$ extending h_* such that $b_{n+1} \in Y_{n+1}$; details are an exercise (though it is basically the same as the above). □

Corollary 3.5: *DLO is a complete theory.*

Proof. Apply Vaught's Test. Note that DLO clearly has no finite models.

If $\mathcal{M}, \mathcal{N} \models$ DLO are countable then $\mathcal{M} \cong \mathcal{N}$, so $\mathcal{M} \equiv \mathcal{N}$. □

So $(\mathbb{Q}, <) \equiv (\mathbb{R}, <) \equiv$ any dense linear order without endpoints. In particular, any two such orders cannot be distinguished by a first order statement in the language of partial orders.

More Notions: Let $\mathcal{L}$ be a language. Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure. Fix a collection $(\mathcal{M}_i)_{i \in I}$ of substructures of $\mathcal{M}$. Let $\mathcal{N} = \bigcap_{i \in I} M_i$; assume $N \neq \emptyset$ (this will always happen as along as there are some constant symbols, say). Then we have a canonical L-structure $\mathcal{N}$ with universe N, by interpreting the language in the only way that makes sense. That is, $f^{\mathcal{N}} = f^{\mathcal{M}}|_{\mathcal{N}} = f^{\mathcal{M}_i}|_{\mathcal{N}}$, $R^{\mathcal{N}} = R^{\mathcal{M}} \cap N^{\alpha(R)} = R^{\mathcal{M}_i} \cap N^{\alpha(R)}$, $c^{\mathcal{N}} = c^{\mathcal{M}} = c^{\mathcal{M}_i}$.

Note $\mathcal{N} \subseteq \mathcal{M}_i$ for all $i \in I$.

Definition 3.6: (Generated Substructure) Given a structure $\mathcal{M}$ and a non-empty set $A \subseteq M$, the ***substructure of $\mathcal{M}$ generated by*** A is the intersection of all substructures of $\mathcal{M}$ containing A.

Definition 3.7: (Chain of $\mathcal{L}$-structures) Let α be a limit ordinal. A collection $(\mathcal{M}_i)_{i<\alpha}$ of $\mathcal{L}$-structures is a ***chain*** if $\mathcal{M}_i \subseteq \mathcal{M}_j$ for all $i < j$.

If in fact the condition above is strengthened to $\preceq$, then we say it is an ***elementary chain***.

If $(M_i)_{i<\alpha}$ is a chain then we have a well-defined structure $\bigcup_{i<\alpha} \mathcal{M}_i$.

4 ..

5 Lecture

Lecture naming got messed up here

Recall: $(K, +, \cdot, 0, 1)$ is a ***field*** if $(K, +, 0)$ and $(K \backslash \{0\}, \cdot, 1)$ are abelian groups and $\forall x \forall y \forall z ((x \cdot (y + z) = x \cdot y + x \cdot z))$.

K is ***algebraically closed*** if every non-constant polynomial over K has a root in K.

Let $\mathcal{L} = \{+, \cdot, 0, 1\}$, the language of fields.

Definition 5.1: (ACF) The first order $\mathcal{L}$-theory axiomatising algebraically closed fields is known as ***ACF*** - all the above statements can be given as first order $\mathcal{L}$-sentences.

In particular, this contains the field axioms and for every $d \geq 1$ the claim that every degree d polynomial has a root:

$$\forall v_0 \forall v_1 \ldots \forall v_{d-1} \exists x (x^d + v_{d-1} x^{d-1} + \cdots + v_1 x + v_0 = 0)$$

We take this statement for every d, i.e. we have infinitely many.

Remark: ACF is not complete, since it does not specify characteristic - hence different models are distinguishable by a first order property.

Definition 5.2: ($\mathbf{ACF}_0$, $\mathbf{ACF}_p$) For $n \geq 1$, let χ_n be the $\mathcal{L}$-sentence

$$\underbrace{1 + 1 + \cdots + 1}_{n} = 0$$

We then have the theory of algebraically closed fields of characteristic zero, ***ACF***$_0$:

$$\text{ACF}_0 = \text{ACF} \cup \{\neg\chi_n : n \geq 1\}$$

For a prime p, we have $\text{ACF}_p = \text{ACF} \cup \{\chi_p\}$

Theorem 5.3: *ACF_0, ACF_p are κ-categorical for all $\kappa > \aleph_0$.*

Proof. The ***transcendence degree*** of $K \models$ ACF is the cardinality of the largest algebraically independent susbet of K.

For example, $\text{trdeg}(\overline{Q}) = 0$, $\text{trdeg}(\overline{\mathbb{Q}(\pi)}) = 1$, $\text{trdeg}(\mathbb{C}) = 2^{\aleph_0}$, $\text{trdeg}(\overline{\mathbb{Q}(x_i)}_{i<\kappa}) = \kappa$

Facts:

(1) Suppose $K, L \models$ ACF. Then $K \cong L$ iff $\text{trdeg}(K) = \text{trdeg}(L)$, $\text{char}(K) = \text{char}(L)$, and $|K| = |L|$

(2) If $K \models$ ACF and $\kappa =$trdeg(K), then $|K| = \aleph_0 + \kappa$

<u>Conclusion</u>: If $K, L \models \text{ACF}_0$ (or ACF_p) are uncountable and $|K| = |L|$, then $K \cong L$. □

Corollary 5.4: *ACF_0 and ACF_p are complete.*

Proof. Vaught's Test. □

Remark: ACF_0, ACF_p are not $\aleph_0$-categorical.

The countable models are precisely the countable ACF_ps of trdeg n for $n \in \mathcal{N} \cup \{\aleph_0\}$.

Definition 5.5: (Polynomial Map) Let K be a field. A function $\Phi : K^m \to K^n$ is a ***polynomial map*** if

$$\Phi = \big(p_1(x_1, \ldots, x_m), p_2(x_1, \ldots, x_m), \ldots, p_n(x_1, \ldots, x_m)\big)$$

where $p_i \in K[\overline{x}]$ for each i.

Theorem 5.6: (Ax-Grothendieck) *Let $K \models ACF$ and suppose $\Phi : K^n \to K^n$ is an injective polynomial map. Then Φ is surjective.*

Proof. First, suppose that $K = \overline{\mathbb{F}}_p$ for some prime p. Recall that $\overline{\mathbb{F}}_p = \bigcup_k \mathbb{F}_{p^k}$. Fix m such that all coefficients in Φ come from $\mathbb{F}_{p^m}$. Note that $\overline{\mathbb{F}}_p = \bigcup_k \mathbb{F}_{p^{km}}$.

Then for any $k \geq 1$, Φ induces an injective polynomial map from $\mathbb{F}^n_{p^{km}} \to \mathbb{F}^n_{p^{km}}\mathrm{m}$, which therefore is surjective since the sets we are dealing with are finite.

$$\begin{aligned}
\Phi\left(\overline{\mathbb{F}}^n_p\right) &= \Phi\left(\bigcup_k \mathbb{F}^n_{p^{km}}\right) \\
&= \bigcup_k \Phi\left(\mathbb{F}^n_{p^{km}}\right) = \bigcup_k \mathbb{F}^n_{p^{km}} \\
&= \overline{\mathbb{F}}^n_p
\end{aligned}$$

Now, given $n, d \geq 1$, let $\psi_{n,d}$ be the $\mathcal{L}$-sentence which says:

"Every injective polynomial map with n coordinates, each of which is a polynomial in n variables and degree $\leq d$, is surjective."

Exercise: show that this is first order. □

We've shown $\overline{\mathbb{F}}_p \models \psi_{n,d}$ for all primes p and n, d.

So for any prime p, $\mathrm{ACF}_p \models \psi_{n,d}$ for all n, d since ACF_p is complete.

Now consider ACF_0. For contradiction, suppose that there exists soem n, d such that $\mathrm{ACF}_0 \not\models \psi_{n,d}$. Then $\mathrm{ACF}_0 \models \neg\psi_{n,d}$ since ACF_0 is complete. By Compactness, there is a finite set $\Sigma \subseteq \mathrm{ACF}_0$ such that $\sigma \models \neg\psi_{n,d}$. So $\Sigma \subseteq \mathrm{ACF} \cup \{\neg\chi_1, \dots, \neg\chi_m\}$ for some m. Choose a prime $p > m$. Then $\mathrm{ACF}_p \models \Sigma$.

So $\mathrm{ACF}_p \models \neg\psi_{n,d}$, which is a contradiction.

Theorem 5.7: (Lefschetz Principle) *Let φ be an $\mathcal{L}$-sentence. TFAE*

(1) $ACF_0 \models \varphi$ i.e. φ is true in every $K \models ACF$

(2) $ACF_0 \cup \{\varphi\}$ is consistent, i.e. φ is true in some $K \models ACF_0$

(3) There is some $n > 0$ such that $ACF_p \models \varphi$ for all $p > n$ i.e. φ is true in every $K \models ACF$ of sufficiently large characteristic

(4) For all $n > 0$ there exists $p > n$ such that $ACF_p \cup \{\varphi\}$ is consistent, i.e. φ is true in some $K \models ACF$ of arbitrarily large characteristic.

Diagrams & Extensions

Let $\mathcal{M}$ be an $\mathcal{L}$-structure.

Remark 5.8: If $h : \mathcal{M} \to \mathcal{N}$ is an $\mathcal{L}$-embedding then after identifying $a \in M$ with $h(a) \in N$, we can view $\mathcal{M}$ as a substructure of $\mathcal{N}$.

Similarly, if h is an elementary embedding then $\mathcal{M}$ can be viewed as an elementary substructure of $\mathcal{N}$.

Given $A \subseteq M$, let $\mathcal{L}_A = \mathcal{L} \cup \{\underline{a} : a \in A\}$, where $\underline{a}$ is a new constant symbol. We underline it to differentiate it from the element in A.

Then $\mathcal{M}$ is canonically an $\mathcal{L}_A$-structure, with $\underline{a}^{\mathcal{M}} = a$.

Definition 5.9: (Diagram) The ***diagram of*** $\mathcal{M}$, written $D(\mathcal{M})$, is the $\mathcal{L}_M$-theory consisting of all quantifier-free $\mathcal{L}_M$-sentences φ such that $\mathcal{M} \models \varphi$.

Similarly, the ***elementary diagram of*** $\mathcal{M}$, written $\mathrm{Th}_M(\mathcal{M}) \coloneqq \mathrm{Th}_{\mathcal{L}_M}(M)$.

Proposition 5.10: *Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure and $\mathcal{N}^*$ is an $\mathcal{L}_M$-structure such that $\mathcal{N}^* \models D(\mathcal{M})$. Let $\mathcal{N}$ be the reduct of $\mathcal{N}^*$ to $\mathcal{L}$. Define $h : \mathcal{M} \to \mathcal{N}$ such that $h(a) = \underline{a}^{\mathcal{N}^*}$. Then h is an $\mathcal{L}$-embedding.*

Moreover, if $\mathcal{N}^* \models Th_M(\mathcal{M})$, then h is an elementary embedding.

Proof. Use Corollary 3.4. Let $\varphi(x_1, \ldots, x_n)$ be a quantifier-free $\mathcal{L}$-formula, and fix $a_1, \ldots, a_n \in M$. Then $\mathcal{M} \models \varphi(a_1, \ldots, a_n)$ iff $\mathcal{M} \models \varphi(\underline{a_1}, \ldots, \underline{a_n})$ iff $\varphi(\underline{a_1}, \ldots, \underline{a_n}) \in D(\mathcal{M})$ iff $\mathcal{N}^* \models \varphi(\underline{a_1}, \ldots, \underline{a_n})$ iff $\mathcal{N} \models \varphi(h(a_1), \ldots, h(a_n))$.

The "moreover" statement is similar (just drop the quantifier-free claim). □

Application to Groups

Recall that an abelian group G is ***orderable*** if there is a linear order $<$ on G such that for all $x, y, z \in G$, if $x < y$ then $x + z < y + z$.

Note that any orderable abelian group is torsion-free, since $x > 0 \implies nx > 0$ for every n. Similarly for $x < 0$. We now prove the converse:

Theorem 5.11: (Levi 1942) *Any torsion-free abelian group is orderable.*

Proof. Let $\mathcal{L}^0 = \{+, 0\}$ be the language of (abelian) groups. Set $\mathcal{L} = \mathcal{L}^0 \cup \{<\}$, where $<$ is a binary relation symbol. Let σ be the $\mathcal{L}$-sentence

$$\forall x \forall y \forall z (x < y \to x + z < y + z)$$

Now let G be a torsion-free abelian group, viewed as an $\mathcal{L}^0$ structure.

Define the $\mathcal{L}_G$-theory

$$T = \underbrace{\mathcal{D}(G)}_{\mathcal{L}^0_G\text{-theory}} \cup \{\text{axioms for linear order \& abelian groups}\} \cup \{\sigma\}$$

Suppose T has a model $\mathcal{M}$. Then $(M, +^{\mathcal{M}}, 0^{\mathcal{M}}, <^{\mathcal{M}})$ is an ordered abelian group, and $G \subseteq (M, +^{\mathcal{M}}, 0^{\mathcal{M}})$ by Prop 5.10. So G is a subgroup of an ordered abelian group, which is thus orderable. s So all that remains is to show that T has a model.

Fix $\Sigma \subseteq T$ finite. Let $A = \{a \in G : \underline{a}$ appears in some $\mathcal{L}^0_G$-sentence in $\Sigma\}$, and let $H = \langle A \rangle \leq G$. Then $H \cong \mathbb{Z}^n$ for some $n \geq 0$ by the structure theorem for (torsion-free) finitely generated abelian groups. View H as an $\mathcal{L}_A$-structure such that $\underline{a}^H = a$ and $<^H$ is the lexicographic ordering. Then $H \subseteq_{\mathcal{L}^0_A} G$, and so $H \models \varphi$ for any $\varphi \in \mathcal{D}(G)$, using only extra constants from A by Corollary 2.4.

So $H \models \Sigma$. So done by Compactness. □

Quantifier Elimination

Idea: Let T be an $\mathcal{L}$-theory and let $\mathcal{M} \models T$. Then $X \subseteq M^n$ is ***definable*** if there is an $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ such that $X = \{\overline{a} \in M^n : \mathcal{M} \models \varphi(\overline{a})\}$.

Goal: Study definable subsets of models of T.

Unfortunately, quantifiers make this difficult. X itself might be nice, by the projection $Y = \{(a_1, \ldots, a_{n-1}) \in M^{n-1} : (\overline{a}, \overline{b}) \in X$ for some $b \in M\}$ (defined by $\exists x_n \varphi(\overline{x})$) might be complicated.

Definition 5.12: (Quantifier Elimination) An $\mathcal{L}$-theory T has ***quantifier elimination*** if for any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ there is a quantifier-free $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$ such that

$$T \models \forall \overline{x}\big((\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})\big)$$

That is to say, φ and ψ define the same set in any $\mathcal{M} \models T$.

Example 5.13:

(1) $T = \mathrm{Th}(F)$, where F is a field. Let $\varphi(w, x, y, z)$ be the statement "$\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ has an inverse", *i*.e. there exist s, t, u, v forming a matrix that inverts it.

Then $T \models \forall w \forall x \forall y \forall z\big(\varphi(w, x, y, z) \leftrightarrow wz - xy \neq 0\big)$.

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(2) $T = \mathrm{Th}(\mathbb{R}, +, \cdot, 0, 1)$. $\varphi(x)$ is $\exists y(x = y^2)$. Note that φ defines $\mathbb{R}^{\geq 0}$. We can in fact not write this quantifier-free:

Suppose $\psi(x)$ is quantifier-free. In this case the terms are just polynomials, so $\psi(x)$ is a Boolean combination of polynomial equations. So ψ defines a finite or cofinite subset of $\mathbb{R}$, and in particular cannot define the positive reals. Hence T does not have QE.

We will later see that $\mathrm{Th}(\mathbb{R}, +, \cdot, <, 0, 1)$ *d*oes have QE. Note that $x < y \iff \exists z : (z \neq 0 \wedge y - x = z^2)$. In particular, adding the ordering relation doesn't really add anything new to what we can define in this language.

So $(\mathbb{R}, +, \cdot, 0, 1)$ and $(\mathbb{R}, +, \cdot, <, 0, 1)$ have the same definable sets, which means they are very similar structures. So the important thing to note is that QE is very language-dependent, and while we might not immediately have QE we might be able to just go out and look for it.

We will now discuss some quantifier elimination tests.

Lemma 6.1: *Suppose T is an $\mathcal{L}$-theory such that for any q.f. formula $\varphi(x_1, \ldots, x_m, y)$ there is a q.f. $\psi(x_1, \ldots, x_n)$ such that $T \models \forall \overline{x}\big(\exists y \varphi(\overline{x}, y) \leftrightarrow \psi(\overline{x})\big)$. Then T has QE.*

Proof. Induction on formulas (exercise). □

So we only need to eliminate one quantifier at a time.

Theorem 6.2: *Let T be an $\mathcal{L}$-theory. TFAE:*

i) T has QE

ii) Suppose $\mathcal{M}, \mathcal{N} \models T$ and $\mathcal{A} \subseteq \mathcal{M}, \mathcal{A} \subseteq \mathcal{N}$. Then for any q.f. formula $\varphi(\overline{x}, y)$ and any tuple $\overline{a}$ of parameters from A, if $\mathcal{M} \models \exists y \varphi(\overline{a}, y)$ then $\mathcal{N} \models \exists y \varphi(\overline{a}, y)$.

iii) For any $\mathcal{L}$-structure $\mathcal{A}$, $T \cup \mathcal{D}(\mathcal{A})$ is a complete $\mathcal{L}_A$-theory.

Proof. (i) $\Longrightarrow$ (iii). Assume T has QE. Let $\mathcal{A}$ be an $\mathcal{L}$-structure and suppose $\mathcal{M}, \mathcal{N} \models T \cup \mathcal{D}(\mathcal{A})$. We want to show that $\mathcal{M} \equiv_{\mathcal{L}_A} \mathcal{N}$.

Let σ be an $\mathcal{L}_A$-sentence such that $\mathcal{M} \models \sigma$. WTS $\mathcal{N} \models \sigma$. Write σ as $\varphi(\underline{a}_1, \ldots, \underline{a}_n)$ for some $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in A$. By QE, there is a q.f. $\psi(x_1, \ldots, x_n)$ such that $T \models \forall \overline{x} \big(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})\big)$.

Since $\mathcal{M} \models T$ and $\mathcal{M} \models \varphi(\overline{a})$, we have $\mathcal{M} \models \psi(\overline{a})$. Since $\mathcal{M} \models \mathcal{D}(\mathcal{A})$, we have $\psi(\underline{a}_1, \ldots, \underline{a}_n) \in \mathcal{D}(\mathcal{A})$. But $\mathcal{N}$ models the diagram, so $\mathcal{N} \models \psi(\underline{a}_1^{\mathcal{N}}, \ldots, \underline{a}_n^{\mathcal{N}})$. Since $\mathcal{N} \models T$, $\mathcal{N} \models \varphi(\underline{a}_1, \ldots, \underline{a}_n)$, *i.e.* $\mathcal{N} \models \sigma$.

(iii) $\Longrightarrow$ (ii) Let $\mathcal{M}, \mathcal{N}, \mathcal{A}, \varphi(\overline{x}, y), \overline{a}$ be as in the hypothesis of (ii). Since $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{A} \subseteq \mathcal{N}$, we have that $\mathcal{M}, \mathcal{N} \models T \cup \mathcal{D}(\mathcal{A})$ by Cor 2.4. By (iii), this is a complete theory and so $\mathcal{M} \equiv_{\mathcal{L}_A} \mathcal{N}$. So $\mathcal{M} \models \exists y \varphi(\overline{a}, y)$, which is an $\mathcal{L}_A$-sentence, implies $\mathcal{N} \models \exists y \varphi(\overline{a}, y)$.

(ii) $\Longrightarrow$ (i) By Lemma 6.1, it suffices to fix q.f. $\varphi(\overline{x}, y)$ and find q.f. $\psi(\overline{x})$ such that $T \models \forall \overline{x} \big(\exists y \varphi(\overline{x}, y) \leftrightarrow \psi(\overline{x})\big)$.

Let $\mathcal{L}^* = \mathcal{L} \cup \{c_1, \ldots, c_n\}$ where c_i is a new constant symbol. Let $\Gamma = \{\psi(\overline{c}) : \psi(\overline{x}) \text{ is a q.f. } \mathcal{L}\text{-formula and } T \models \forall \overline{x} \big(\exists y \varphi(\overline{x}, y) \to \psi(\overline{x})\big)\}$.

<u>Claim</u>: $T \cup \Gamma \models \exists y \varphi(\overline{c}, y)$.

First, assume the claim holds. By Compactness, there exists a q.f. $\psi_1(\overline{x}), \ldots, \psi_m(\overline{x})$ such that $T \cup \{\psi_1(\overline{c}), \ldots, \psi_m(\overline{c})\} \models \exists y \varphi(\overline{c}, y)$ and $T \models \forall \overline{x} \big(\exists y \varphi(\overline{x}, y) \to \bigwedge_{i=1}^m \psi_i(\overline{x})\big)$. Let $\psi(\overline{x})$ be $\bigwedge_{i=1}^m \psi_i(\overline{x})$. Then $T \models \big(\psi(\overline{c}) \to \exists y \varphi(\overline{c}, y)\big)$.

So $T \models \forall \overline{x} \big(\psi(\overline{x}) \to \exists y \varphi(\overline{x}, y)\big)$ (exercise "generalisation"). So $T \models \forall \overline{x} \big(\psi(\overline{x}) \leftrightarrow \exists y \varphi(\overline{x}, y)\big)$.

<u>Proof of Claim</u>: Suppose not. There is $\mathcal{N} \models T \cup \Gamma \cup \{\neg \exists y \varphi(\overline{c}, y)\}$. Let $a_i = c_i^{\mathcal{N}}$ and let $\mathcal{A} \subseteq \mathcal{N}$ be the substructure generated by $a_1, \ldots, a_n$. Then $\mathcal{N} \models T$, $\mathcal{A} \subseteq \mathcal{N}$, and $\mathcal{N} \models \neg \exists y \varphi(\overline{a}, y)$.

By ES1 #7, any $b \in A$ is of the form $t^{\mathcal{N}}(\overline{a})$ for some $\mathcal{L}$-term b. So we can view $\mathcal{D}(\mathcal{A})$ as an $\mathcal{L}^*$-theory by replacing $\underline{b}$ with $t(c_1, \ldots, c_n)$. Let $\Sigma \models T \cup \mathcal{D}(\mathcal{A}) \cup \{\exists y \varphi(\overline{c}, y)\}$. If we build $\mathcal{M} \models \Sigma$ then $\mathcal{M} \models T$, $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{M} \models \exists y \varphi(\overline{a}, y)$, contradicting (ii).

So it suffices to show Σ has a model, which we will do by compactness. Suppose this fails. Then by compactness there are q.f. $\psi_1(\overline{x}), \ldots, \psi_m(\overline{x})$ such that $\psi_1(\overline{c}), \ldots, \psi_m(\overline{c}) \in \mathcal{D}(\mathcal{A})$ and

$$T \cup \left\{ \bigwedge_{i=1}^m \psi_i(\overline{c}) \right\} \cup \{\exists y \varphi(\overline{c}, y)\}$$

is unsatisfiable. Let $\psi(\overline{x})$ be $\neg \bigwedge_{i=1}^m \psi_i(\overline{x})$. Then $T \models \big(\exists y \varphi(\overline{c}, y) \to \psi(\overline{c})\big)$. So $T \models \forall \overline{x} \big(\exists y \varphi(\overline{x}, y) \to \psi(\overline{x})\big)$. So $\psi(\overline{c}) \in \Gamma$. So $\mathcal{N} \models \psi(\overline{c})$. since $\mathcal{N} \models \mathcal{D}(\mathcal{A})$, we have $\mathcal{N} \models \neg \psi(\overline{c})$. Contradiction. $\square$

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Remark: Recall Theorem 6.2 (QE test)

1) In condition (iii), we may assume that $\mathcal{A} \subseteq \mathcal{M}$ for some model $\mathcal{M} \models T$. Otherwise, $T \cup \mathcal{D}(\mathcal{A})$ is inconsistent and thus complete

2) In both conditions (ii) and (iii), we may assume that $\mathcal{A}$ is finitely generated

Theorem 7.1: *ACF has quantifier elimination.*

Proof. We apply Theorem 6.2(iii). Fix a finitely-generated $\mathcal{L}$-structure $\mathcal{A}$ (in the language of fields). We want to show $\mathrm{ACF} \cup \mathcal{D}(\mathcal{A})$ is complete. We use Vaught's Test.

Fix $K_1, K_2 \models \mathrm{ACF} \cup \mathcal{D}(\mathcal{A})$ uncountable with $|K_1| = |K_2|$. Then $\mathcal{A}$ is a finitely generated integral domain contained in K_1 and K_2.

So $\mathrm{char}(K_1) = \mathrm{char}(K_2)$. Let F_i be the field of fractions of $\mathcal{A}$ in K_i. There is a field isomorphism $\tau : F_1 \to F_2$ fixing $\mathcal{A}$ pointwise. Since $\mathcal{A}$ is finitely generated, $\mathrm{trdeg}(F_i)$ is finite. So $\mathrm{trdeg}(K_1/F_1) = \mathrm{trdeg}(K_2/F_2)$.

So τ extends to an isomorphism $\tau^* : K_1 \to K_2$ fixing $\mathcal{A}$. □

We now see a very common application of quantifier elimination of ACF.

Definition 7.2: (Constructible Set) Let F be a field. Then $X \subseteq F^n$ is ***constructible*** if it is a Boolean combination of subsets of F^n defined by $p(x_1, \ldots, x_n) = 0$, where $p \in F[x_1, \ldots, x_n]$.

Corollary 7.3: (Chevalley) *If $K \models ACF$ and $X \subseteq K^n$ is constructible, then the projection*

$$Y = \{(a_1, \ldots, a_{n-1}) \in K^{n-1} : (\overline{a}, b) \in X \text{ for some } b \in K\}$$

is constructible.

Compare: Consider $X = \{(x, y) \in \mathbb{R}^2 : x = y^2\}$. Then $Y = \mathbb{R}^{\geq 0}$.

Exercise: think about more examples in the rationals.

Proof. Note that $X \subseteq K^n$ is constructible iff there is a quantifier-free formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ (y_i parameters) and parameters $b_1, \ldots, b_m \in K$ such that X is defined by $\varphi(\overline{x}, \overline{b})$.

Fix quantifier-free formula $\varphi(\overline{x}, \overline{y})$ and $\overline{b}$ such that $\varphi(\overline{x}, \overline{b})$ defines X. Let $\psi(x_1, \ldots, x_{n-1}, \overline{y})$ be $\exists x_n \varphi(\overline{x}, \overline{y})$. Then $\psi(\overline{x}, \overline{b})$ defines Y. Then by QE $\psi(\overline{x}, \overline{y})$ is equivalent to some quantifier-free formula. So Y is constructible. □

Rado Graphs

We work with the language of graphs $\mathcal{L} = \{E\}$, E a binary relation.

Definition 7.4: (Rado Graph) A ***Rado Graph*** is a graph (V, E) such that $V \neq \emptyset$ and for any finite disjoint $X, Y \subseteq V$ there is some $v \in V$ such that $E(v, x)$ for all $x \in X$ and $\neg E(v, y)$ for all $y \in Y$.

Definition 7.5: (RG) We let RG be the theory of Rado graphs in the language of graphs. In particular:

$$\begin{aligned}\mathrm{RG} = &\{\forall x \neg E(x,x), \forall x \forall y (E(x,y) \to E(y,x))\} \\ &\cup \left\{ \forall x_1, \ldots, x_k \forall y_1, \ldots, y_k \left(\bigwedge_{i,j} x_i \neq x_j \to \exists v \left(\bigwedge_{i=1}^{k} E(x_i, v) \wedge \bigwedge_{i=1}^{k} \neg E(y_i, v) \right) \right) : k \geq 1 \right\}\end{aligned}$$

Theorem 7.6: *RG is $\aleph_0$-categorical.*

Proof.

1) RG has a countable model.

   Let $A = (V, E)$ be any finite graph. Set $A_0 = A$. Given A_n, define $V(A_{n+1}) = V(A_n) \cup \{v_{X,Y} : X, Y \subseteq V(A_n) \text{ disjoint}\}$, with new edges $E(v_{X,Y}, x)$ for all $x \in X$ (and no others). So $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$ is a chain of substructures.

   Let $M = \bigcup_{n=0}^{\infty} A_n$. Then $M \models \mathrm{RG}$. Moreover, M is countable since each A_n is finite.

2) Any two countable models are elementarily equivalent.

   Let $\mathcal{M}, \mathcal{N} \models \mathrm{RG}$ countable. We show $\mathcal{M} \cong \mathcal{N}$ via a back and forth argument. Enumerate $M = \{a_n : n \geq 0\}$ and $N = \{b_n : n \geq 0\}$. Let $h_0 : a_0 \mapsto b_0$. Given $h_n : X_n \to Y_n$, extend to include a_{n+1} and b_{n+1}.

   Partition X_n into the neighbourhood of a_{n+1} and its complement. We then partition Y_n by its image under h_n, and use the Rado axioms to find a vertex b connected to $h_n(\Gamma(a_{n+1}))$ and none of its complement. Similarly find appropriate a for b_{n+1}, and extend h_n to include these pairs, giving us h_{n+1}.

Hence RG is $\aleph_0$-categorical. □

Corollary 7.7: *RG is complete*

Proof. Use Vaught's Test. Note that RG has no finite models. □

Claim: If $\mathcal{M} \models \mathrm{RG}$ then every finite graph is an induced subgraph of $\mathcal{M}$.

Proof. The proof of Theorem 7.6 shows this when $\mathcal{M}$ is countable. Then for any $\mathcal{M} \models \mathrm{RG}$ there exists a countable $\mathcal{M}_0$ such that $\mathcal{M}_0 \leq \mathcal{M}$ by DLST (*c*.f. Sheet 1 Question 9). □

Exercise: Suppose $\mathcal{M}, \mathcal{N} \models \mathrm{RG}$ countable and $f : X \to Y$ is a graph isomorphism for some finite $X \subseteq \mathcal{M}$ and $Y \subseteq \mathcal{N}$. Then f extends to an isomorphism from $\mathcal{M}$ to $\mathcal{N}$.

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Theorem 8.1: *RG has QE*

Proof. Option 1: Theorem 6.2(iii). Consider $RG \cup \mathcal{D}(\mathcal{A})$, where $\mathcal{A}$ is a finite graph (see last exercise).

Option 2: Theorem 6.2(ii). Fix $\mathcal{M}, \mathcal{N} \models$ RG and $\mathcal{A} \subseteq \mathcal{M} \cap \mathcal{N}$. Fix a q.f. formula $\varphi(x_1, \dots, x_n, y)$ and $a_1, \dots, a_n \in A$. Assume $\mathcal{M} \models \varphi(\overline{a}, b)$ for some $b \in M$. Want to show that $\mathcal{N} \models \exists \varphi(\overline{a}, y)$.

Write $\varphi(\overline{x}, y)$ as

$$\bigvee_{s=1}^{k} \bigwedge_{t=1}^{\ell_s} \theta_{s,t}(\overline{x}, y)$$

where each $\theta_{s,t}$ is atomic or negated atomic (this is known as *disjunctive normal form*). $\exists s \leq k$ such that

$$\mathcal{M} \models \bigwedge_{t=1}^{\ell_s} \theta_{s,t}(\overline{a}, b)$$

Each $\theta_{s,t}$ is one of: $x_i = x_j, x_i = y, E(x_i, x_j), E(x_i, y)$, or the negation of one of the above. If we have $x_i = y$ appearing then $b = a_i \in A \subseteq N$. So $\mathcal{N} \models \varphi(\overline{a}, b)$ since φ is q.f.. We can assume that no $x_i = y$ appears. Let $X = \{a_i : \mathcal{M} \models E(a_i, b)\}$ and $Y = \{a_i : \mathcal{M} \models \neg E(a_i, b)\}$. X and Y are finite disjoint subsets of $A \subseteq N$. Choose $c \in N$ such that $\mathcal{N} \models E(a_i, c)$ iff $a_i \in X$, and $c \notin \{a_1, \dots, a_n\}$. We do this by finding a new element connected to everything in X and nothing in Y, and ensure that c is not connected to this new vertex either.

Then $\mathcal{N} \models \bigwedge_t \theta_{s,t}(\overline{a}, c)$ (check). So $\mathcal{N} \models \varphi(\overline{a}, c)$. $\square$

Types

Motivation: Given $\mathcal{M}$, we want to understand "potential behaviour" of elements in elementary extensions.

Terminology: Given an $\mathcal{L}$-structure $\mathcal{M}$ and $A \subseteq M$, we call an $\mathcal{L}_A$-formula an ***$\mathcal{L}$-formula with parameters from*** A. We write these as $\varphi(\overline{x}, \overline{a})$ where $\varphi(\overline{x}, \overline{y})$ is an $\mathcal{L}$-formula and $\overline{a}$ is from A. (Identifiy a with $\underline{a}^{\mathcal{M}}$ - this should not cause problems in most cases).

Now suppose $\mathcal{M}$ is an $\mathcal{L}$-structure and $\mathcal{N} \succeq \mathcal{M}$. If $a \in N \backslash M$ then the $\mathcal{L}_N$-formula $x = a$ doescribes the new behvaiour in a trivial way.

OTOH: If $\varphi(x)$ is an $\mathcal{L}$-formula with parameters from $\mathcal{M}$ and $\mathcal{N} \models \varphi(a)$ for some $a \in N$, then $\mathcal{N} \models \exists x \varphi(x)$ so $\mathcal{M} \models \exists \varphi(x)$.

Idea: New behaviour cannot be controlled with one formula at a time.

Notation: Let p be a set of formulas in free variables $x_1, \dots, x_n$. We also write $p(x_1, \dots, x_n)$. Given $\mathcal{M}$ and $a_1, \dots, a_n \in M$, we write $\mathcal{M} \models p(a_1, \dots, a_n)$ if $\mathcal{M} \models \varphi(\overline{a})$ for all $\varphi \in p$. We say "$\overline{a}$ realises p (in $\mathcal{M}$)". Also write $\overline{a} \models p$. We call p ***consistent*** if it is realised in some structure.

Exercise: p is consistent iff every finite subset of p is consistent.

Definition 8.2: (n-type) Let $\mathcal{M}$ be an $\mathcal{L}$-structure and fix $A \subseteq M$. An ***n-type over A w.r.t.*** $\mathcal{M}$ is a set p of $\mathcal{L}$-formulae with parameters from A in free variables $x_1, \dots, x_n$ such that $p \cup \mathrm{Th}_A(\mathcal{M})$ is consistent.

We say p is ***complete*** if, for every $\mathcal{L}_A$-formula $\varphi(x_1, \dots, x_n)$, either $\varphi \in p$ or $\neg\varphi \in p$.

Let $S_n^{\mathcal{M}}(A)$ denote the set of all complete n-types over A w.r.t. $\mathcal{M}$.

Example 8.3: Given $a_1, \dots, a_n \in M$, let $\mathrm{tp}^{\mathcal{M}}(a_1, \dots, a_n/A)$ be the set of all $\mathcal{L}_A$-formulae $\varphi(x_1, \dots, x_n)$ such that $\mathcal{M} \models \varphi(\overline{a})$. Then $\mathrm{tp}^{\mathcal{M}}(\overline{a}/A) \in S_n^{\mathcal{M}}(A)$, and $\overline{a} \models \mathrm{tp}^{\mathcal{M}}(\overline{a}/A)$.

Proposition 8.4: *If $p \in S_n^{\mathcal{M}}(A)$, then there is $\mathcal{N} \succeq \mathcal{M}$ with $|N| \leq |M| + |\mathcal{L}|$, and $\overline{a} \in N^n$ such that $p = tp^{\mathcal{N}}(\overline{a}/A)$.*

Proof. By assumption, $p \cup \mathrm{Th}_A(\mathcal{M})$ is consistent. We want to show that $p \cup \mathrm{Th}_M(\mathcal{M})$ is consistent, which is not *quite* what we have.

Fix $\Sigma \subseteq p \cup \mathrm{Th}_M(\mathcal{M})$ finite. So $\Sigma \subseteq p \cup \{\varphi_1, \dots, \varphi_t\}$ where φ_i is an $\mathcal{L}_M$-sentence, and $\mathcal{M} \models \varphi_i$. Let φ^* be $\bigwedge_{i=1}^t \varphi_i$. We can write φ^* as $\varphi(\underline{b}_1, \dots, \underline{b}_m)$ where $b_1, \dots, b_m \in \mathcal{M} \backslash A$ and $\varphi(x_1, \dots, x_n)$ is an $\mathcal{L}_A$-formula. Since $\mathcal{M} \models \varphi(b_1, \dots, b_m)$, we have that $\mathcal{M} \models \exists \overline{v} \varphi(v_1, \dots, v_m)$. So $\exists \overline{v} \varphi(\overline{v}) \in \mathrm{Th}_A(\mathcal{M})$. Since $p \cup \mathrm{Th}_A(\mathcal{M})$ is consistent, there is $\mathcal{N} \models \mathrm{Th}_A(\mathcal{M})$ and $\overline{a} \in N^n$ such that $\mathcal{N} \models p(\overline{a})$.

Since $\mathcal{N} \models \exists \overline{v} \varphi(\overline{v})$, there is $\overline{c} \in N^m$ such that $\mathcal{N} \models \varphi(\overline{c})$. Expand $\mathcal{N}$ to an $\mathcal{L}_M$-structure such that $\underline{b}_i^{\mathcal{N}} = c_i$ and $\underline{b}^{\mathcal{N}}$ is arbitrary for $b \in M \backslash (A \cup \{b_1, \dots, b_m\})$. Then $\mathcal{N} \models \varphi(\underline{b}_1, \dots, \underline{b}_m)$, *i.e.* $\mathcal{N} \models \varphi^*$. So $\mathcal{N} \models \Sigma$. □

9 Lecture

Remark 9.1: If $\mathcal{M} \leq \mathcal{N}$ and $A \subseteq M$, then $S_n^{\mathcal{M}}(A) = S_n^{\mathcal{N}}(A)$.

Proof. It is enough to show that $\mathrm{Th}_A(\mathcal{M}) = \mathrm{Th}_A(\mathcal{N})$. If $\varphi(x_1, \dots, x_m)$ is an $\mathcal{L}$-formula $a_1, \dots, a_m \in A$, then $\mathcal{M} \models \varphi(\overline{a})$ iff $\mathcal{N} \models \varphi(\overline{a})$ since $\mathcal{M} \preceq \mathcal{N}$. □

Remark 9.2: p is an n-type over A wrt $\mathcal{M}$ iff for any finite $q \subseteq p$, $\exists \overline{a} \in \mathcal{M}^n$ such that $\overline{a} \models q$.

Proof. ($\implies$): Choose $\mathcal{N} \succeq \mathcal{M}$ realising p. Fix finite $a \subseteq p$. Let $\varphi(\overline{x})$ be the conjunction of all $\mathcal{L}_A$-formulae in q. $\mathcal{N} \models \exists \overline{x} \varphi(\overline{x})$, the $\mathcal{L}_A$-sentence. So $\mathcal{M} \models \exists \overline{x} \varphi(\overline{x})$ since $\mathcal{N} \succeq \mathcal{M}$. □

Example 9.3: Suppose $K \models \mathrm{ACF}$, and $A \subseteq K$. We aim to describe $S_n^K(A)$.

Fix $p \in S_n^K(A)$. By QE, we only need to consider q.f. formulae in p.

<u>Note</u>: $\varphi \wedge \psi \in p$ iff $\varphi, \psi \in p$, and $\neg \varphi \in p$ iff $\varphi \notin p$.

So it suffices to focus on atomic formulae, in variables $x_1, \dots, x_n$ with parameters from A, *i.e.*polynomial equations in $F[\overline{x}]$, where F is the subfield generated by A. Le t $I_p = \{f(\overline{x}) \in F[\overline{x}] : f(\overline{x}) = 0 \text{ is in } p\}$. Then I_p is a prime ideal. In fact, this map $p \mapsto I_p$ is a bijection between $S_n^K(A)$ and the set of prime ideals in $F[\overline{x}]$ (*i.e.* $\mathrm{Spec}(F[\overline{x}])$).

For example, the set of 1-types with all parameters $S_1^K(K) = \{p_a : a \in K\} \cup \{q\}$ where p_a contains $x = a$, and q contains $x \neq a$ for all $a \in K$. In particular, $|S_1^K(K)| = |K|$.

Example 9.4: Let $\mathcal{M} \models \mathrm{RG}$. We will describe $S_1^{\mathcal{M}}(M)$.

For $a \in M$, let $p_a \in S_1^{\mathcal{M}}(M)$ be the unique type containing $x = a$.

Why is this unique? Suppose $x = a$ is in p, q distinct. Choose $\varphi(x)$ such that $\varphi(x) \in p$ and $\neg\varphi(x) \in q$. Then $x = a \wedge \varphi(x)$, $x = a \wedge \neg\varphi(x)$ both consistent. $\perp$

For $V \subseteq M$, set p_V as follows:

$$\begin{aligned} p_V = &\{x \neq a : a \in M\} \\ &\cup \{E(x,a) : a \in V\} \\ &\cup \{\neg E(x,a) : a \in M \backslash V\} \end{aligned}$$

Then p_V is a 1-type wrt $\mathcal{M}$, and by QE this determines a unique, complete 1-type. So we have $S_1^{\mathcal{M}}(M) = \{p_a : a \in M\} \cup \{p_V : V \subseteq M\}$, and $|S_1^{\mathcal{M}}(M)| = 2^{|M|}$ *i.e.*there is a type for every subset of the model.

Note: In general, $|S_n^{\mathcal{M}}(A)| \leq 2^{|A|+|\mathcal{L}|+\aleph_0}$

Type Spaces

Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$. Given $\mathcal{L}_A$-formula $\varphi(x_1, \ldots, d_n)$, define $[\varphi(\overline{x})] = \{p \in S_n^{\mathcal{M}}(A) : \varphi(\overline{x}) \in p\}$.

Basic Properties:

1. $S_n^{\mathcal{M}}(A) = [\bigwedge_{i=1}^n x_i = x_i]$
2. $[\varphi(\overline{x}) \wedge \psi(\overline{x})] = [\varphi(\overline{x})] \cap [\psi(\overline{x})]$
3. $[\neg\varphi(\overline{x})] = S_n^{\mathcal{M}}(A) \backslash [\varphi(\overline{x})]$

We then define a topology on $S_n^{\mathcal{M}}(A)$ by using $[\varphi(\overline{x})]$ for all $\mathcal{L}_A$-formulae $\varphi(\overline{x})$ as a basis of open sets. Here, S is for "Stone"; see: Stone space.

Theorem 9.5: *$S_n^{\mathcal{M}}(A)$ is a totally disconnected compact Hausdorff space.*

Proof. Showing that the topology is well-defined is an exercise (Sheet 2 #7).

Hausdorff: Fix distinct $p, q \in S_n^{\mathcal{M}}(A)$. Find $\varphi(\overline{x})$ such that $\varphi(\overline{x}) \in p$ and $\neg\varphi(\overline{x}) \in q$. Then $p \in [\varphi(\overline{x})]$ and $q \in [\neg\varphi(\overline{x})]$.

Compactness: It suffices to consider open covers consisting of basic open sets. Fix a collection of $\mathcal{L}_A$-formulae $(\varphi_i(\overline{x}))_{i \in I}$ such that $S_n^{\mathcal{M}}(A) = \bigcup_{i=I} [\varphi_i(\overline{x})]$.

Let $\Sigma = \{\neg\varphi_i(\overline{x}) : i \in I\}$. Then $\Sigma \cup \mathrm{Th}_A(\mathcal{M})$ is inconsistent. Otherwise, $\mathcal{N} \models \mathrm{Th}_A(\mathcal{M})$ and $\overline{a} \in N^n$ such that $\overline{a} \models \Sigma$. Let $p = \mathrm{tp}^{\mathcal{N}}(\overline{a}/A)$. Then $p \in S_n^{\mathcal{M}}(A)$ but $p \in [\varphi_i(\overline{x})]$ for all $i \in I$ $\perp$.

Then by the Compactness Theorem, there is some finite $I_0 \subseteq I$ such that $\{\neg\varphi_i(\overline{x}) : i \in I_0\} \cup \mathrm{Th}_A(\mathcal{M})$ is inconsistent. $(*)$

We show $S_n^{\mathcal{M}}(A) = \bigcup_{i \in I_0} [\varphi_i(\overline{x})]$. Fix $p \in S_n^{\mathcal{M}}(A)$. Choose $\mathcal{N} \models \mathrm{Th}_A(\mathcal{M})$ and $\overline{a} \in \mathcal{N}^n$ such that $\overline{a} \models p$. By $(*)$, there exists $i \in I_0$ such that $\mathcal{N} \models \varphi_i(\overline{a})$. So $\varphi_i(\overline{x}) \in p$ (since p is complete). So $p \in [\varphi_i(\overline{x})]$.

Totally Disconnected: A compact Hausdorff space is totally disconnected iff any two distinct points can be separated by clopen sets (not just open sets). Note that in this case the basic open sets are clopen (they are closed because their compliment is open). □

We now have a long-term goal: to analyse countable models of complete theories.

For example, DLO and RG are $\aleph_0$-categorical. For ACF_p, the countable models are K_α for $\alpha \in \mathcal{N} \cup \{\aleph_0\}$ where K_α has transcendence degree α.

10 Lecture

Saturated Models

Definition 10.1: Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure, and let $\kappa \geq |\mathcal{L}| + \aleph_0$. Then $\mathcal{M}$ is ***κ-saturated*** if for any $A \subseteq M$, with $|A| < \kappa$, every type in $S_n^{\mathcal{M}}(A)$ is realised in $\mathcal{M}$ for all $n \geq 1$.

Remark:

a) Restricting to complete types is not important since since any n-type over A wrt $\mathcal{M}$ can be extended to some $p \in S_n^{\mathcal{M}}(A)$ (Sheet 2 #6).

b) (Sheet 2 #8) It suffices to assume $n = 1$ to prove κ-saturation.

c) If $\mathcal{M}$ is κ-saturated then $|M| \geq \kappa$.

   Proof. $\{x \neq a : a \in \mathcal{M}\}$ is a 1-type over M wrt $\mathcal{M}$, and is not realised in $\mathcal{M}$. $\square$

Definition 10.3: (Partial elementary map, κ-homogeneous) Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures, and suppose $A \subseteq M, B \subseteq N$. Then a function $f : A \to B$ is ***partial elementary*** if for any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in A$, $\mathcal{M} \models \varphi(\overline{a})$ iff $\mathcal{N} \models \varphi(f(\overline{a}))$.

Given $\kappa \geq |\mathcal{L}| + \aleph_0$, $\mathcal{M}$ is ***κ-homogeneous*** if, for any $A \subseteq M$ with $|A| < \kappa$, any partial elementary $f : A \to M$, and any $c \in M$, there exists $d \in M$ such that $f \cup \{(c, d)\}$ is partial elementary. That is to say, any partial elementary map can be extended.

Let T be a complete $\mathcal{L}$-theory. Fix $\mathcal{M}, \mathcal{N} \models T$. Then $S_n^{\mathcal{M}}(\emptyset) = S_n^{\mathcal{N}}(\emptyset)$ since $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N}) = T$.

Definition 10.4: $S_n(T) \coloneqq S_n^{\mathcal{M}}(\emptyset)$ for some (equivalently, any) $\mathcal{M} \models T$.

Proposition 10.5: *$\mathcal{M} \models T$ is $\aleph_0$-saturated iff $\mathcal{M}$ is $\aleph_0$-homogeneous and $\mathcal{M}$ realises all types in $S_n(T)$ for all $n \geq 1$.*

Proof. ($\Longrightarrow$) Assume $\mathcal{M} \models T$ is $\aleph_0$-saturated. Then $\mathcal{M}$ realises all types in $S_n(T)$ sicne $\emptyset$ is finite. Fix finite $A \subseteq M$, partial elementary $f : A \to M$, and $c \in M$. Define $p \in S_1(f(A))$ such that $\varphi(x, f(\overline{a})) \in p$ iff $\mathcal{M} \models \varphi(c, \overline{a})$.

<u>Notation</u>: $f(\mathrm{tp}^{\mathcal{M}}(c/A)) = p$. $p \in S_1(f(A))$, *e.g.* p is finitely satisfiable in $\mathcal{M}$: if $\varphi(x, f(\overline{a})) \in p$ then $\mathcal{M} \models \exists x \varphi(x, \overline{a})$, so $\mathcal{M} \models \exists x \varphi(x, f(\overline{a}))$.

Let $d \in M$ realise p. Then $f \cup \{(c, d)\}$ is partial elementary.

($\Longleftarrow$) Fix $a_1, \ldots, a_n \in M$ and $p \in S_1^{\mathcal{M}}(\{a_1, \ldots, a_n\})$. Want to show that $\mathcal{M}$ realises p.

Set $q = \{\varphi(x, y_1, \ldots, y_n) : \varphi(x, \overline{a}) \in p\}$. Then $q \in S_{n+1}(T)$. Let $d, b_1, \ldots, b_n \in M$ such that $(d, \overline{b}) \models q$. Then $\mathrm{tp}^{\mathcal{M}}(\overline{b}) = \mathrm{tp}^{\mathcal{M}}(\overline{a})$. So $f : b_i \to a_i$ for all i is partial elementary.

Let $c \in M$ such that $f \cup \{(d, c)\}$ is partial elementary. Then $\mathrm{tp}^{\mathcal{M}}((c, \overline{a})) = \mathrm{tp}^{\mathcal{M}}((d, \overline{b})) = q$. So $(c, \overline{a}) \models q$, *i.e.* $c \models p$. $\square$

This tells us that if we want to build a saturated model, we at least need to be able to build homogeneous models.

Notation: Given $\mathcal{M}$, $\overline{a}, \overline{b} \in M^n$, write $\overline{a} \equiv^{\mathcal{M}} \overline{b}$ if $\operatorname{tp}^{\mathcal{M}}(\overline{a}) = \operatorname{tp}^{\mathcal{M}}(\overline{b})$. So $\mathcal{M}$ is $\aleph_0$-homogeneous iff whenever $\overline{a} \equiv^{\mathcal{M}} \overline{b}$ and $c \in M$, there exists $d \in M$ such that $(\overline{a}, c) \equiv^{\mathcal{M}} (\overline{b}, d)$.

Lemma 10.6: *For any $\mathcal{M} \models T$, there is $\mathcal{N} \succeq \mathcal{M}$ such that $|N| \leq |M| + |\mathcal{L}|$ and $\mathcal{N}$ is $\aleph_0$-homogeneous.*

Proof. Claim: For any $\mathcal{M} \models T$, there is $\mathcal{N} \succeq \mathcal{M}$ such that $|N| \leq |M| + |\mathcal{L}|$ and $\forall \overline{a}, \overline{b}, c$ from M, such that $\overline{a} \equiv^{\mathcal{M}} \overline{b}$, there exists $d \in N$ such that $(\overline{a}, c) \equiv^{\mathcal{N}} (\overline{b}, d)$.

Proof of Claim: Enumerate all $(\overline{a}, \overline{b}, c)$ as $(\overline{a}_\alpha, \overline{b}_\alpha, c_\alpha)_{\alpha < |M|}$. We build an elementary chain $(M_\alpha)_{\alpha < |M|}$ such that $\mathcal{M}_0 = \mathcal{M}$ and $|\mathcal{M}_\alpha| \leq |M| + |\mathcal{L}|$ for all α.

For α a limit, let $\mathcal{M}_\alpha = \bigcup_{i<\alpha} M_i$. Then $|M_\alpha| \leq |\alpha|(|M| + |\mathcal{L}|) = |M| + |\mathcal{L}|$.

Given M_α, look at $(\overline{a}_\alpha, \overline{b}_\alpha, c_\alpha)$. We have $\overline{a}_\alpha \equiv^{\mathcal{M}} \overline{b}_\alpha$. Let $f_\alpha : \overline{a}_\alpha \to \overline{b}_\alpha$ be partial elementary. Apply Prop to find $\mathcal{M}_{\alpha+1} \geq \mathcal{M}_\alpha$ such that $|M_{\alpha+1}| \leq |M_\alpha| + |\mathcal{L}| \leq |M| + |\mathcal{L}|$, and there exists $d \in M_{\alpha+1}$ realising $f_\alpha(\operatorname{tp}(c_\alpha/\overline{a}_\alpha))$. Then $(\overline{a}_\alpha, c_\alpha) \equiv^{\mathcal{M}} (\overline{b}_\alpha, d)$. Let $\mathcal{N} = \bigcup_{\alpha<|M|} M_\alpha$. Then $|N| \leq |M|(|M| + |\mathcal{L}|) = |M| + |\mathcal{L}|$. □

We now build $\mathcal{M} = \mathcal{N}_0 \leq \mathcal{N}_1 \leq \mathcal{N}_2 \leq \dots$ such that $|N_i| \leq |M| + |\mathcal{L}|$, and $\forall \overline{a}, \overline{b}, c$ from $\mathcal{N}_i$ if $\overline{a} \equiv \overline{b}$ then there exists $d \in N_{i+1}$ such that $(\overline{a}, c) \equiv (\overline{b}, d)$. We do this by iterating the claim. Then let $\mathcal{N} = \bigcup_{i<\aleph_0} \mathcal{N}_i$. Then $|N| \leq |M| + |\mathcal{L}|$.

$\mathcal{N}$ is $\aleph_0$-homogeneous: any $\overline{a}, \overline{b}, c$ from $\mathcal{N}$ all lie in N_i for some i, so we find a solution in $\mathcal{N}_{i+1}$. □

11 Lecture

Recall: $\mathcal{M}$ is κ-saturated $\implies |\mathcal{M}| \geq \kappa$.

Definition 11.1: (Saturated) $\mathcal{M}$ is ***saturated*** if it is $|M|$-saturated.

Let T be a complete consistent theory with infinite models and $\mathcal{L}$ is countable.

Theorem 11.2: *T has a countable, saturated model iff $S_n(T)$ is countable for all $n \geq 1$.*

Proof. ($\implies$) If $\mathcal{M} \models T$ is countable and saturated, then $S_n(T)$ is countable since M^n is countable and $p \mapsto \overline{a} \models p$ is injective.

($\implies$) Enumerate $\bigcup_{n\geq 1} S_n(T) = \{p_1, p_2, p_3, \dots\}$. Fix $\mathcal{M}_0 \models T$ countable. Build a chain $\mathcal{M}_0 \preceq \mathcal{M}_1 \preceq \mathcal{M}_2 \preceq \dots$ such that $\mathcal{M}_i$ realises p_i and is countable (by Prop 8.4).

Let $\mathcal{N} = \bigcup_{n\geq 1} \mathcal{M}_n$. $\mathcal{N} \models T$ is countable. Apply Lemma 10.6 to obtain $\mathcal{M} \succeq \mathcal{N}$ countable and $\aleph_0$-homogeneous. So $\mathcal{M}$ is saturated by Prop 10.5. □

Example 11.3:

(1) ACF_p. Let $F = \mathbb{Q}$ if $p = 0$, and $\mathbb{F}_p$ otherwise. Then $S_n(T) \leftrightarrow \operatorname{Spec}(F[x_1, \dots, x_n])$. So $S_n(T)$ is countable since every ideal in $F[\overline{x}]$ is finitely generated. So ACF_p has a countable saturated model, which is the model of countably infinite transcendence degree $\aleph_0$, $\overline{F[x_1, x_2, \dots]}$. Note that if $K \models \mathrm{ACF}_p$ and $\operatorname{trdeg}(K) = n < \aleph_0$, then the $(n+1)$-type saying "$x_1, \dots, x_{n+1}$ algebraically independent" is not realised in K.

(2) TFDAG (torsion-free divisible abelian groups) has a countable saturated model, which is the $\mathbb{Q}$-vector space of dimension $\aleph_0$.

(3) Let $T = \mathrm{Th}(\mathbb{Z}, +, 0)$. Given $n \geq 1$, let $\delta_n(x)$ be the $\mathcal{L}$-formula $\exists y(x = ny)$. Let $\mathbb{P}$ be the set of primes. Given $X \subseteq \mathbb{P}$, let $q_x = \{\delta_n(x) : n \in X\} \cup \{\neg\delta_n(x) : n \in \mathbb{P}\backslash X\}$. Note q_X is finitely satisfiable in $\mathbb{Z}$. So we can extend it to a complete type; there exists $p_X \in S_1(T)$ such that $q_X \subseteq p_X$ (Sheet 2).

    If $X \neq Y$ then $p_X \neq p_Y$, so $|S_1(T)| = 2^{\aleph_0}$. So T does not have a countable saturated model.

Proposition 11.4: *If $\mathcal{M}, \mathcal{N} \models T$ are countable and saturated, then $\mathcal{M} \cong \mathcal{N}$.*

Proof. (Sketch). Enumerate $\mathcal{M} = \{a_n : n \geq 1\}, \mathcal{N} = \{b_n : n \geq 1\}$. Build partial elemenatary maps $f_0 \subseteq f_1 \subseteq \dots$ such that $a_n \in \mathrm{dom}(f_n)$, $b_n \in \mathrm{Im}(f_n)$, $\mathrm{dom}(f_n)$ is finite.

Let $f_0 = \emptyset$. Note that this is partial elementary since $\mathcal{M} \equiv \mathcal{N}$.

Given f_n, let $d \in N$ realise $f_n\left(\mathrm{tp}\left(a_{n+1}/\mathrm{dom}(f_n)\right)\right)$. Now let $c \in M$ realise $f_*^{-1}(\mathrm{tp}(b_{n+1}/\mathrm{Im}(f_n) \cup \{d\}))$, where $f_* = f_n \cup \{(a_{n+1}, d)\}$. Let $f_{n+1} = f_* \cup \{(c, b_{n+1})\}$.

Then let $f = \bigcup f_n$. Then by construction f is an $\mathcal{L}$-isomorphism from $\mathcal{M}$ to $\mathcal{N}$. □

Omitting Types

Let $\mathcal{M}$ be an $\mathcal{L}$-structure.

Definition 11.5: (Isolated type) $p \in S_n^{\mathcal{M}}(A)$ is ***isolated*** if it is an isolated point wrt the Stone space topology, *i.e.* $\{p\}$ is open.

Example: If $a \in A \subseteq M$, then $\mathrm{tp}^{\mathcal{M}}(a/A)$ is isolated since $\{\mathrm{tp}^{\mathcal{M}}(a/A)\} = [x = a]$.

Proposition 11.6: *—Given $p \in S_n^{\mathcal{M}}(A)$, TFAE:*

i) p is isolated

*ii) $\{p\} = [\varphi(\overline{x})]$ for some $\mathcal{L}_A$-formula $\varphi(\overline{x})$ (we say $\varphi(\overline{x})$ **isolates** p)*

iii) There is an $\mathcal{L}_A$-formula $\varphi(\overline{x}) \in p$ such that for any $\mathcal{L}_A$-formula $\psi(\overline{x})$, if $\psi(\overline{x}) \in p$ then $Th_A(\mathcal{M}) \models \forall\overline{x}(\varphi(\overline{x}) \to \psi(\overline{x}))$

Proof. (i) $\iff$ (ii) follows by definition of the basis for the topology.

(ii) $\implies$ (iii). Assume $\varphi(\overline{x})$ isolates p. Fix an $\mathcal{L}_A$-formula $\psi(\overline{x}) \in p$. WTS $\mathcal{M} \models \forall\overline{x}(\varphi(\overline{x}) \to \psi(\overline{x}))$. Suppose $\overline{a} \in M^n$ such that $\mathcal{M} \models \varphi(\overline{a})$. Then $\mathrm{tp}^{\mathcal{M}}(\overline{a}/A) \in [\varphi(\overline{x})]$, so $p = \mathrm{tp}^{\mathcal{M}}(\overline{a}/A)$. So $\mathcal{M} \models \psi(\overline{a})$.

(iii) $\implies$ (ii). Assume (iii). Then for all $\mathcal{L}_A$-formulae $\psi(\overline{x}) \in p$, we have $\varphi(\overline{x}) \subseteq [\psi(\overline{x})]$ since any $q \in [\varphi(\overline{x})]$ is realised by $\overline{a} \in N^n$ in some $\mathcal{N} \models \mathrm{Th}_A(\mathcal{M})$. So $\mathcal{N} \models \psi(\overline{a})$. So $q \in [\psi(\overline{x})]$. If $q \in [\varphi(\overline{x})]$ then $p \subseteq q$. So $[\varphi(\overline{x})] = \{p\}$. □

12 Lecture

Let T be a complete, consistent theory.

Proposition 12.1: *If $p \in S_n(T)$ is isolated then p is realised in any $\mathcal{M} \models T$.*

Proof. Fix $p \in S_n(T)$ isolated by $\varphi(\overline{x}) \in p$. Fix $\mathcal{M} \models T$. By Prop 8.4, there is $\mathcal{N} \succeq \mathcal{M}$ realising p. So $\mathcal{N} \models \exists\overline{x}\varphi(\overline{x})$. So $\mathcal{M} \models \exists\overline{x}\varphi(\overline{x})$. Fix $\overline{a} \in M^n$ such that $\mathcal{M} \models \varphi(\overline{a})$. We then show $\overline{a} \models p$.

Fix $\psi(\overline{x}) \in p$. Then $T \models \forall\overline{x}(\varphi(\overline{x}) \to \psi(\overline{x}))$. So $\mathcal{M} \models \psi(\overline{a})$. $\square$

Theorem: (Omitting Types Theorem) *Assume $\mathcal{L}$ is countable, and $p \in S_n(T)$ is non-isolated. Then there is countable $\mathcal{M} \models T$ such that p is not realised in $\mathcal{M}$ (i.e. $\mathcal{M}$ **omits** p)*

This is a relatively complicated argument.

Proof. (Henkin construction; non-examinable) Let $\mathcal{L}^* = \mathcal{L} \cup C$, where C is a countably infinite set of new constant symbols.

An $\mathcal{L}^*$-theory T^* has the ***witness property*** if for any $\mathcal{L}^*$-formula $\varphi(x)$ there is a constant symbol $c \in C$ such that $T^* \models (\exists x\varphi(x) \to \varphi(c))$.

<u>Fact</u>: (Part II) Suppose T^* is a complete, satisfiable $\mathcal{L}^*$-theory with the witness property.

Define $\sim$ on C such that $c \sim d$ iff $T^* \models c = d$. Let $M = C/\sim$ and define an $\mathcal{L}^*$-structure $\mathcal{M}$ on M such that

$$\begin{cases} c^{\mathcal{M}} = [c] \text{ ($\sim$-equivalence class)} \\ f^{\mathcal{M}}([c_1], \dots, [c_n]) = [d] \text{ iff } T^* \models f(c_1, \dots, c_n) = d \\ R^{\mathcal{M}} = \{([c_1], \dots, [c_n]) \in M^n : T^* \models R(c_1, \dots, c_n)\} \end{cases}$$

Then $\mathcal{M}$ is a well-defined $\mathcal{L}^*$ structure and $\mathcal{M} \models T^*$ - this requires checking. In particular, for any $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ and $c_1, \dots, c_n \in C$, $\mathcal{M} \models \varphi([c_1], \dots, [c_n])$ iff $T^* \models \varphi(c_1, \dots, c_n)$, the $\mathcal{L}^*$-sentence.

We call $\mathcal{M}$ the ***Henkin model*** of T^*.

Fix $p \in S_N(T)$ non-isolated.

<u>Goal</u>: Build a complete, satisfiable L^*-theory $T^* \supseteq T$ with the witness property such that $\forall c_1, \dots, c_n \in C$ there is $\psi(\overline{x}) \in p$ such that $T^* \models \neg\psi(c_1, \dots, c_n)$.

Then given such a T^*, the Henkin model omits p since it denies some formula from p on every tuple of (equivalence classes of) constants.

Enumerate all $\mathcal{L}^*$-sentences $\varphi_0, \varphi_1, \dots$ and also enumerate $C^n = \{\overline{c}_0, \overline{c}_1, \dots\}$. We build a satisfiable $\mathcal{L}^*$-theory $T^* = T \cup \{\theta_0, \theta_1, \dots\}$ such that

0) $\models \theta_i \to \theta_j$ for all $i > j$ (this is for convenience)
1) Either $\models \theta_{3i+1} \to \varphi_i$ or $\models \theta_{3i+1} \to \neg\varphi_i$ (completeness)
2) If φ_i is $\exists v\psi(v)$ for some ψ and $\models \theta_{3i+1} \to \varphi_i$, then $\models \theta_{3i+2} \to \psi(c)$ for some $c \in C$

   (witness property: $T^* \models (\exists v\psi(v) \to \psi(c))$ as $\mathcal{N} \models T^*$ and $\mathcal{N} \models \exists v\psi(v)$, and $\mathcal{N} \models \varphi_i$ so $\mathcal{N} \models \psi(c)$)

3) $\models \theta_{3i+3} \to \neg\psi(\overline{c}_i)$ for some $\psi(\overline{x}) \in p$ (omit p)

We construct this model inductively. Let θ_0 be $\forall v(v = v)$ (θ_0 does nothing). Now suppose we have $\theta_0, \ldots, \theta_m$ as above.

Case 1: $m + 1 = 3i + 1$. If $T \cup \{\theta_m, \varphi_i\}$ is satisfiable then θ_{m+1} is $\theta_m \wedge \varphi_i$. Otherwise, let θ_{m+1} be $\theta_m \wedge \neg\varphi_i$. Then $T \cup \{\theta_{m+1}\}$ is satisfiable, since in either case we're adding the conjunction of two axioms. This relies on the inductive hypothesis that $T \cup \{\theta_m\}$ is satisfiable.

Case 2: $m + 1 = 3i + 2$. Suppose φ_i is $\exists v\psi(v)$ for some ψ, and $\models \theta_m \to \varphi_i$ (if this fails we simply let θ_{m+1} be θ_m). Choose $c \in C$ not used in θ_m. Let θ_{m+1} be $\theta_m \wedge \psi(c)$. This satisfies the above hypotheses. Moreover, $T \cup \{\theta_{n+1}\}$ is satisfiable: Let $\mathcal{N} \models T \cup \{\theta_m\}$. Then $\mathcal{N} \models \varphi_i$. Choose $a \in N$ such that $\mathcal{N} \models \psi(a)$. Re-interpret $c^{\mathcal{N}} = a$. Then $\mathcal{N} \models T \cup \{\theta_{m+1}\}$.

Case 3: $m+1 = 3i+3$. Let $\overline{c}_i = (c_1, \ldots, c_n)$. WLOG assume $x_1, \ldots, x_n$ are not used in θ_m. We build an $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ from θ_m as follows:

- replace c_t by x_t for all $t \leq n$.
- then replace any $c \in C \backslash \{c_1, \ldots, c_n\}$ by a new variable v_c and add $\exists v_c$ to the front.

Then $\varphi(\overline{x})$ does not isoalte p. By Prop 11.6, $\exists\psi(\overline{x}) \in p$ such that $\not\models \forall\overline{x}(\varphi(\overline{x}) \to \psi(\overline{x}))$.

Let θ_{m+1} be $\theta_m \wedge \neg\psi(c_1, \ldots, c_n)$. $T \cup \{\theta_{m+1}\}$ is satisfiable: Choose $\mathcal{N} \models T$ such that $\mathcal{N} \not\models \forall\overline{x}(\varphi(\overline{x}) \to \psi(\overline{x}))$. Pick $\overline{a} \in N^n$ such that $\mathcal{N} \models \varphi(\overline{a}) \wedge \neg\psi(\overline{a})$. Make $\mathcal{N}$ an $\mathcal{L}^*$-structure:

Interpret $c_t^{\mathcal{N}}$ as a_t. If $c \in C \backslash \{c_1, \ldots, c_n\}$, then $c^{\mathcal{N}}$ is a witness to $\exists v_c$ in $\mathcal{N} \models \varphi(\overline{a})$. Then $\mathcal{N} \models \theta_m$ and $\mathcal{N} \models \neg\psi(c_1, \ldots, c_t)$. So $\mathcal{N} \models \theta_{m+1}$. $\square$

13 Prime & Atomic Models

T is a complete, consistent $\mathcal{L}$-theory with infinite models.

Definition 13.1:

1) $\mathcal{M}$ is ***atomic*** if every n-type over $\emptyset$ realized in $\mathcal{M}$ is isolated

2) $\mathcal{M}$ is ***prime*** if for any $\mathcal{N} \models T$ there is an elementary embedding $\mathcal{M} \hookrightarrow \mathcal{N}$

Example: $K \models \mathrm{ACF}_0$. Then $\overline{\mathbb{Q}} \subseteq K$. So $\overline{\mathbb{Q}} \preceq K$ by QE.

Theorem 13.2: *Assume $\mathcal{L}$ is countable. Then $\mathcal{M} \models T$ is prime iff it is countable and atomic.*

So up to issues of cardinality, we can think of prime and atomic as the same thing.

Proof. $\Longrightarrow$: Assume $\mathcal{M} \models T$ is prime. Then $\mathcal{M}$ is countable since T has a countable model (by DLST), into which $\mathcal{M}$ embeds. Suppose $p \in S_n(T)$ is non-isolated. By OTT there is some $\mathcal{N} \models T$ omitting p. Since $\mathcal{M} \preceq \mathcal{N}$, $\mathcal{M}$ omits p. So $\mathcal{M}$ is atomic.

$\Longleftarrow$: Assume $\mathcal{M} \models T$ is countable and atomic. Fix $\mathcal{N} \models T$. WTS $\mathcal{M} \preceq \mathcal{N}$. Enumerate $M = \{a_n : n \geq 1\}$. We build partial elementary $f_0 \subseteq f_1 \subseteq f_2 \ldots$ from M to N such that $a_n \in \mathrm{dom}(f_n)$ and $\mathrm{dom}(f_n)$ is finite. Then $f = \bigcup f_n$ is an elementary embedding from $\mathcal{M}$ to $\mathcal{N}$.

We start as before with $f_0 = \emptyset$, which is partial elementary since $\mathcal{M} \equiv \mathcal{N}$. Now suppose we have f_n. Let $\varphi(x_1, \ldots, x_{n+1})$ be a n $\mathcal{L}$-formula isolating $\mathrm{tp}^{\mathcal{M}}(a_1, \ldots, a_{n+1})$, which exists since $\mathcal{M}$ is atomic. $\mathcal{M} \models \exists x_{n+1}\varphi(a_1, \ldots, a_n, x_{n+1})$, so $\mathcal{N} \models \exists x_{n+1}\varphi(f_n(a_1), \ldots, f_n(a_n), x_{n+1})$. Pick $b \in N$ such that

$\mathcal{N} \models \varphi(f(a_1), \dots, f_n(a_n), b)$. Fix an $\mathcal{L}$-formula $\psi(x_1, \dots, x_{n+1})$ such that $\mathcal{M} \models \psi(a_1, \dots, a_{n+1})$. WTS $\mathcal{N} \models \psi(f_n(a_1), \dots, f_n(a_n), b)$.

By Prop 11.6, $T \models \forall x_1 \dots \forall x_{n+1}(\varphi(\overline{x}) \to \psi(\overline{x}))$. So $\mathcal{N} \models \psi(f_n(a_1), \dots, f_n(a_n), b)$. So $f_{n+1} = f_n \cup \{(a_{n+1}, b)\}$ is partial elementary. □

Theorem 13.3: *Assume $\mathcal{L}$ is countable. TFAE:*

i) T has a prime model

ii) T has an atomic model

iii) For all $n \geq 1$, the isolated types in $S_n(T)$ are dense.

Proof. We have (i) $\iff$ (ii) by Theorem 13.2 (and Sheet 1 #9).

(ii) $\implies$ (iii): Let $\mathcal{M} \models T$ be atomic. Fix $n \geq 1$, and an $\mathcal{L}$-formula $\varphi(\overline{x})$ such that $[\varphi(\overline{x})] \neq \emptyset$. WTS $[\varphi(\overline{x})]$ contains an isolated type. Note $\mathcal{M} \models \exists \overline{x} \varphi(\overline{x})$. Choose $\overline{a} \in M^n$ such that $\mathcal{M} \models \varphi(\overline{a})$. Then $\text{tp}^{\mathcal{M}}(\overline{a})$ is isolated (since $\mathcal{M}$ is atomic) and it is in $[\varphi(\overline{x})]$.

(iii) $\implies$ (ii): [Henkin construction, non-examinable] Let $\mathcal{L}^* = \mathcal{L} \cup \{c_1, c_2, \dots\}$. Let $\varphi_0, \varphi_1, \dots$ enumerate all $\mathcal{L}^*$-sentences. We build $T^* = T \cup \{\theta_0, \theta_1, \dots\}$ such that T^* is complete, satisfiable, has the witness property, and such that the Henkin model of T^* is atomic (as an $\mathcal{L}$-structure). This is similar to the OTT.

Let θ_0 be $\forall x(x = x)$. Suppose we have $\theta_0, \theta_1, \dots, \theta_m$.

The cases $m + 1 \in \{3i + 1, 3i + 2\}$ are identical to the proof of OTT.

Case $m + 1 = 3i + 3$: Choose $n \geq i$ such that all new constants used in θ_n are in $\{c_1, \dots, c_n\}$. Let $\psi(x_1, \dots, x_n)$ be an $\mathcal{L}$-formula such that θ_m is $\psi(c_1, \dots, c_n)$. By induction, $T \cup \{\theta_m\}$ is consistent. So $T \cup \{\psi(x_1, \dots, x_n)\}$ is consistent, and so $[\psi(\overline{x})] \neq \emptyset$. Then by (iii), there is some isolated type $p \in [\psi(\overline{x})]$. Let $\varphi(\overline{x})$ isolate p. Let θ_{m+1} be $\theta_m \wedge \varphi(c_1, \dots, c_n)$.

$T \cup \{\theta_{m+1}\}$ is consistent: choose $\mathcal{N} \models T$ with $\overline{a} \in N^n$ realising p. Expand $\mathcal{N}$ to an $\mathcal{L}^*$-structure such that $c_i^{\mathcal{N}} = a_i$ (for $i \leq n$). Then $\mathcal{N} \models T \cup \{\theta_{m+1}\}$.

Now let $\mathcal{M} \models T^*$ be the Henkin model. WTS $\mathcal{M}$ is atomic (as an $\mathcal{L}$-structure). For arbitrarily large n, we have $\varphi(x_1, \dots, x_n)$ isolating $p \in S_n(T)$ such that $T^* \models \varphi(c_1, \dots, c_n)$. So $\text{tp}^{\mathcal{M}}(c_1^{\mathcal{M}}, \dots, c_n^{\mathcal{M}})$ is isolated for all $n \geq 1$.

For any tuple $\overline{a}$ from $\mathcal{M}$, WTS $\text{tp}^{\mathcal{M}}(\overline{a})$ is isolated. WLOG the coordinates of the tuple are distinct, *i.e.* (a, b, c) isolated by $\psi(x_1, x_2, x_3)$, (a, a, b, c) isolated by $\psi(x_1, x_3, x_4) \wedge x_2 = x_3$.

So $\overline{a}$ is a sub-tuple of $([c_1], \dots, [c_n])$ for some n.

General fact: Given any $\mathcal{M}$ and $a_1, \dots, a_n \in M$, if $\text{tp}^{\mathcal{M}}(a_1, \dots, a_n)$ is isolated by $\varphi(x_1, \dots, x_n)$, then for all $\emptyset \neq I \subseteq \{1, \dots, n\}$, $\text{tp}^{\mathcal{M}}((a_i)_{i \in I})$ is isolated by $(\exists x_i)_{i \notin I} \varphi(x_1, \dots, x_n)$. □

14 Lecture

T is a complete theory in a countable language with infinite models.

Recall: For any $n \geq 1$, $|S_n(T)| \leq 2^{\aleph_0}$.

Lemma 14.1: *For any $n \geq 1$, if $|S_n(T)| < 2^{\aleph_0}$ then $S_n(T)$ is countable and the isolated types are dense.*

Note that this is a purely topological result, as is seen in the proof.

Proof. $S_n(T)$ is a second countable, totally disconnected, compact, Hausdorff space. Let X be any such space. We show that if X is uncountable *or* the isolated points are not dense, then $|X| \geq 2^{\aleph_0}$.

Let $\mathcal{B}$ be a countable basis for X consisting of clopen sets, and assume $\mathcal{B}$ is closed under intersections and complements (is a *Boolean algebra*)

Case 1: X is uncountable.

Claim: If $U \in \mathcal{B}$ and $|U| > \aleph_0$ then $\exists V \in \mathcal{B}$ such that $|U \cap V|, |U \backslash V| > \aleph_0$.

Proof of claim: Suppose not. Let $\mathcal{C} = \{V \in \mathcal{B} : |U \cap V| > \aleph_0\}$. Fix $V_1, V_2 \in \mathcal{C}$. Set $W = V_1 \cap V_2$. If $W \notin \mathcal{C}$ then $|U \backslash W| > \aleph_0$. Note $U \backslash W = (U \backslash V_1) \cup (U \backslash V_2)$, so WLOG $|U \backslash V_1| > \aleph_0$, which is a contradiction since $V_1 \in \mathcal{C}$. So $\mathcal{C}$ is a collection of non-empty closed sets, and $\mathcal{C}$ is closed under intersections. Since X is compact, there is some $p \in X$ such that $p \in V$ for all $V \in \mathcal{C}$. We then show that

$$U = \{p\} \cup \bigcup_{V \in \mathcal{B} \backslash \mathcal{C}} U \cap V$$

Then U is a countable union of countable sets, and hence countable.

To do this, we fix $q \in U$ such that $q \neq p$. There is $V \in \mathcal{B}$ such that $q \in V$ and $p \notin V$. So $V \in \mathcal{B} \backslash \mathcal{C}$. So $q \in U \cap V$. $\square$

Notation: 2^ω is the set of seqeuences of $0, 1$ indexed by $\mathbb{N}$. $2^{<\omega}$ is the set of finite sequences of $0, 1$. We have a partial order on $2^\omega \cup 2^{<\omega}$ given by proper initial segment.

We build a collection $\{U_\sigma\}_{\sigma \in 2^{<\omega}}$ such that $\forall \sigma \in 2^{<\omega}$, $U_\sigma \in \mathcal{B}$, $|U_\sigma| > \aleph_0$, $U_\sigma = U_{\sigma 0} \cup U_{\sigma 1}$, and $U_{\sigma 0} \cap U_{\sigma 1} = \emptyset$. Let $U_\emptyset = X$. Given U_σ, let $V \in \mathcal{B}$ be as in the Claim. Let $U_{\sigma 0} = U_\sigma \cap V$ and $U_{\sigma 1} = U_\sigma \backslash V$.

Now, for any $\alpha \in 2^\omega$, there is $p_\alpha \in \bigcap_{i \geq 0} U_{\alpha \restriction i}$, where $\alpha \restriction i$ is the infinite sequence α cut off after the first i entries. By construction, $\alpha \neq \beta \implies p_\alpha \neq p_\beta$. So $|X| \geq 2^{\aleph_0}$.

Case 2: The isolated points in X are not dense.

We will build $\{U_\sigma\}_{\sigma \in 2^{<\omega}}$ as above, but just iwth $U_\sigma \neq \emptyset$.

Let $U_\emptyset$ be a non-empty clopen set with no isolated points. Suppose we have U_σ. U_σ has no isolated points, so there exist distinct $p, q \in U_\sigma$. Partition U_σ into $U_{\sigma 0}$ and $U_{\sigma 1}$ with $p \in U_{\sigma 0}$ and $q \in U_{\sigma 1}$ by Hausdorffness. As before, $|X| \geq 2^{\aleph_0}$. $\square$

Theorem 14.2:

a) Suppose $|S_n(T)| < 2^{\aleph_0}$ for all n. Then T has a prime model and a countable saturated model.

b) If T has a countable model, then T has a prime model.

Proof. (a): Apply Lemma 14.1, Theorem 13.3 and Theorem 11.2.

(b): Apply Theorem 11.2, Lemma 14.1, Theorem 13.3. $\square$

Fact: $\mathrm{Th}(\mathbb{Z}, +, 0)$ has no countable saturated model (Ex 11.3(c)), and no prime model (Baldwin, Blass, Glass, Kuecker 1972). This is (essentially) because the type of 1 is not isolated; there is no way to pin down what 1 really is.

On the other hand, $\mathrm{Th}(\mathbb{Z}, +, 0, 1)$ then there is a prime model and no countable saturated model.

Definition 14.3: For $\kappa \geq \aleph_0$, let $I(T, \kappa)$ be the number of models of T of size κ up to isomorphism.

Remark: $1 \leq I(T, \kappa) \leq 2^\kappa$, where the upper bound is given by the number of $\mathcal{L}$-strucutres of size κ, which are determined (essentially) by picking subsets of a set of size κ (relations, graphs of functions).

[Recall Morley's Theorem: If $I(T, \kappa) = 1$ for some $\kappa > \aleph_0$, then $I(T, \kappa) = 1$ for all $\kappa > \aleph_0$.]

Proposition 14.4: *If $I(T, \aleph_0) < 2^{\aleph_0}$, then $S_n(T)$ is countable for all $n \geq 1$ (and so T has a prime model and a countable, saturated model).*

Proof. Assume $I(T, \kappa) = \kappa < 2^{\aleph_0}$. Let $(\mathcal{M}_i)_{i<\kappa}$ be all countable models of T. Fix n. Let X_i be the set of $p \in S_n(T)$ realised in $\mathcal{M}_i$. Each X_i is countable and $S_n(T) = \bigcup_{i<\kappa} X_i$. So $|S_n(T)| \leq \kappa < 2^{\aleph_0}$. So $S_n(T)$ is countable by Lemma 14.1. $\square$

Example: $T = \mathrm{ACF}_p$. $I(T, \aleph_0) = \aleph_0$. Also $T = \mathrm{TFDAG}$.

Vaught's Conjecture (1961): If $I(T, \aleph_0) < 2^{\aleph_0}$, then $I(T, \aleph_0) \leq \aleph_0$.

This is, of course, trivial if one assumes CH - but it is an open problem of ZFC, and one of the oldest in model theory.

Morley (1970): If $I(T, \aleph_0) < 2^{\aleph_0}$ then $I(T, \aleph_0) \leq \aleph_1$.

15 Lecture

Examples of $I(T, \aleph_0)$:
$\underline{I(T, \aleph_0) = 2^{\aleph_0}}$:

1. $T = \mathrm{Th}(\mathbb{Z}, +, 0)$ $[|S_1(T)| = 2^{\aleph_0} \implies I(T, \aleph_0) = 2^{\aleph_0}]$
2. $T = \mathrm{Th}(\mathbb{Z}, <)$. In this case, $S_n(T)$ is countable for all n (via Sheet 2 #5)

   Given a linear order $\mathcal{A}$, let $\mathcal{M}_\mathcal{A} = \mathbb{Z} \cdot \mathcal{A}$ (*i.e.* replace each point in $\mathcal{A}$ with a copy of $\mathbb{Z}$). Then $\mathcal{M}_\mathcal{A} \models T$. Now $A \ncong \mathcal{B} \implies \mathcal{M}_\mathcal{A} \ncong \mathcal{M}_\mathcal{B}$. By Cantor, # of countable linear orders is $2^{\aleph_0}$, and so $I(T, \aleph_0) = 2^{\aleph_0}$.

$\underline{I(T, \aleph_0) = \aleph_0}$: ACF_p, TFDAG.

$\underline{I(T, \aleph_0) = 1}$: (*i.e.* T is $\aleph_0$-categorical) DLO, RG, InfSets.

Remark: If T is $\aleph_0$-categorical then its unique countable model is saturated and prime by Prop 14.4.

Theorem 15.2: (Ryll-Narzewski/Enegler/Svenonius 1959) *Let T be a complete theory in a countable language with infinite models. TFAE:*

i) *T is $\aleph_0$-categorical*

ii) *$\forall n \geq 1$, every type in $S_n(T)$ is isolated.*

iii) $\forall n \geq 1$, *$S_n(T)$ is finite.*

iv) $\forall n \geq 1$, *the number of $\mathcal{L}$-formulae in $x_1, \dots, x_n$ is finite, up to equivalence in T.*

Proof. (i) $\implies$ (ii): Every type over $\emptyset$ is realised in the unique countable model, which is an atomic model (Remark 15.1).

(ii) $\implies$ (iii): Suppose X is a compact space, and every point is isolated. Then $(\{p\})_{p \in X}$ is an open cover, which has a finite subcover; hence X itself is finite.

(iii) $\implies$ (ii): If X is Hausdorff and finite then all points are isolated.

(ii)/(iii) $\implies$ (iv): Fix $n \geq 1$. Let $S_n(T) = \{[p_1, \dots, p_k\}$ and let $\varphi_i(\overline{x})$ isolate p_i. Then for any $\mathcal{L}$-formula $\psi(\overline{x})$, we have that

$$T \models \forall \overline{x} \left(\psi(\overline{x}) \leftrightarrow \bigvee_{\psi \in p_i} \varphi_i(\overline{x}) \right)$$

by Prop 11.6.

(iv) $\implies$ (ii): Fix $n \geq 1$. Let $\varphi_1(\overline{x}), \dots, \varphi_k(\overline{x})$ represent all $\mathcal{L}$-formulae in $x_1, \dots, x_n$. Then $p \in S_n(T)$ is isolated by

$$\bigwedge_{\varphi_i \in p} \varphi_i(\overline{x}) \wedge \bigwedge_{\varphi_i \notin p} \neg\varphi_i(\overline{x})$$

(ii) $\implies$ (i): If (ii) holds, then every model of T is atomic. So every model of T is $\aleph_0$-homogeneous (Sheet 3 #1a). Moreover, every model of T realises all types in $S_n(T)$ by Prop 12.1. So every countable model of T is saturated by Prop 10.5. So T is $\aleph_0$-categorical by Prop 11.4. $\square$

Corollary 15.3: *Let G be an infinite group, and $T = Th(G)$ (in the language of groups) is $\aleph_0$-categorical. Then G has finite exponent.*

Proof. Want to show there is some $n \in \mathbb{N}$ such that $g^n = 1$ for all $g \in G$. Suppose not.

Case 1: G is torsion-free. WLOG G is countable. Then $T \cup \{x^n \neq 1_G : n \geq 1\}$ is finitely satisfiable, so by DLST it has a countable model $H \ncong G$ $\perp$.

Case 2: There is some $g \in G$ of infinite order. For $k \geq 1$, let $p_k = \mathrm{tp}(g, g^k) \in S_2(T)$. If $k < \ell$ then p_k contains $x_2 = x_1^k$, but p_ℓ does not. So $S_2(T)$ is infinite. Contradiction. $\square$

Fact: Any abelian group of finite exponent has an $\aleph_0$-categorical compelte theory.

Corollary 15.4: *Suppose T is a complete $\aleph_0$-categorical $\mathcal{L}$-theory, with $\mathcal{L}$ countable. Then, for any $\mathcal{L}_0 \subseteq \mathcal{L}$, $T \restriction \mathcal{L}_0 = \{\varphi \in T : \varphi$ is an $\mathcal{L}$-sentence$\}$ is still $\aleph_0$-categorical.*

Proof. Apply Theorem 15.2(iv). If there are only finitely many $\mathcal{L}$-formulae modulo T, then there's only finitely many $\mathcal{L}_0$-formulae modulo the restriction of T to $\mathcal{L}_0$ - and these characterise $\aleph_0$-categoricity. $\square$

Example 15.5: $[I(T,\aleph_0)=3]$ Let $\mathcal{L}=\{<,c_0,c_1,c_2,\dots\}$.
Let $T=\text{DLO}\cup\{c_n<c_{n+1}:n\geq 0\}$.

Claim: T is complete.

Proof. (Vaught's Test) Fix countable $\mathcal{M},\mathcal{N}\models T$. Want to show that $\mathcal{M}\equiv\mathcal{N}$. It suffices to show that the reducts to any finite sublanguage are isomorphic.

Note: $\text{DLO}\cup\{c_0<c_1<\dots<c_n\}$ is $\aleph_0$-categorical (*e.g.* as in proof of Sheet 2 #4).

Claim: $I(T,\aleph_0)=3$. Proof of Claim: $\mathcal{M}_1$ is $(\mathbb{Q},<)$ with $c_n^{\mathcal{M}_1}=n$ (no upper bound for the constant c_ns). $\mathcal{M}_2$ is $(\mathbb{Q},<)$ with $\sqrt{2}-1/n<c_n^{\mathcal{M}_2}<\sqrt{2}$ (upper bound exists, but no supremum). $\mathcal{M}_3$ is $(\mathbb{Q},<)$ with $c_n^{\mathcal{M}_3}=1-1/n$ (supremum exists).

These are three countable models of the theory, and no two of them are isomorphic since the upper bound properties must be preserved by isomorphism.

If $\mathcal{M}\models T$, then $\mathcal{M}\cong\mathcal{M}_i$ for some i, depending on which sup properties it has.

This can be modified to obtain $I(T,\aleph_0)=k$ for all $k\geq 3$ (Sheet 3 #2). $\square$

16 Lecture

Theorem 16.1: (Vaught 1959) *Suppose T is a complete $\mathcal{L}$-theory, with $\mathcal{L}$-countable. Then $I(T,\aleph_0)\neq 2$.*

Proof. Assume for contradiction that $I(T,\aleph_0)=2$. By Prop 14.4, T has a prime model $\mathcal{M}$ and a countable, saturated model $\mathcal{N}$. By Theorem 15.2, there exists some non-isolated $p\in S_n(T)$ for some $n\geq 1$. So $\mathcal{M}$ omits p, and there exists $\overline{a}\in N^n$ realising p. Let $T^*=\text{Th}_{\overline{a}}(\mathcal{N})$. Then $\mathcal{N}$ is still saturated as an $\mathcal{L}_{\overline{a}}$-structure (Sheet 3 #3). So T^* has a prime model $\mathcal{B}$ by Theorem 14.2(b).

Let $\mathcal{A}=\mathcal{B}\restriction\mathcal{L}\models T$. So $\mathcal{A}$ realises p. So $\mathcal{A}\not\cong\mathcal{M}$, and hence $\mathcal{A}\cong\mathcal{N}$. So then $\mathcal{B}=\mathcal{N}$ (Sheet 3 #3). So the prime model of T^* is saturated. So T^* is $\aleph_0$-categorical by Theorem 15.2, so T is $\aleph_0$-categorical by Corollary 15.4. $\perp$ $\square$

Uncountably Saturated Models

Theorem 16.2: *For any infinite $\mathcal{M}$ and any $\kappa\geq|\mathcal{L}|+\aleph_0$, there is an $\mathcal{N}\succeq\mathcal{M}$ such that $\mathcal{N}$ is κ^+-saturated and $|N|\leq|M|^\kappa$.*

Proof. Fix $\kappa\geq|\mathcal{L}|+\aleph_0$. Notation: $X\subseteq_\kappa Y$ means $X\subseteq Y$ and $|X|\leq\kappa$.

Claim: For any $\mathcal{M}$, there is $\mathcal{N}\succeq\mathcal{M}$ such that $|N|\leq|M|^\kappa$ and $\mathcal{N}$ realises all types in $S_1^{\mathcal{M}}(A)$ for all $A\subseteq_\kappa M$.

Proof of Claim: # subsets of $\mathcal{M}$ of size $\leq\kappa$ is $|M|^\kappa$ and if $|A|\leq\kappa$, then $\S_1^{\mathcal{M}}(A)|\leq 2^\kappa\leq|M|^\kappa$. Enumerate all such types as $(p_\alpha)_{\alpha<|M|^\kappa}$ (α ordinal). Build an elementary chain $(\mathcal{M}_\alpha)_{\alpha<|M|^\kappa}$ such that $\mathcal{M}_0=\mathcal{M}$, for limit α $\mathcal{M}_\alpha=\bigcup_{i<\alpha}\mathcal{M}_i$, and $\mathcal{M}_{\alpha+1}\succeq\mathcal{M}_\alpha$ realises p_α such that $\mathcal{M}_{\alpha+1}\leq|M_\alpha|+|\mathcal{L}|$ (by Prop 8.4). Let $\mathcal{N}=\bigcup_\alpha\mathcal{M}_\alpha$. Then $|N|\leq|M|^\kappa$ and $\mathcal{N}$ realises all p_α. $\square$

Fix $\mathcal{M}$. Now build elementary chain $(\mathcal{N}_\alpha)_{\alpha<\kappa^+}$ such that $|N_\alpha|\leq|M|^\kappa$, and:

1. $\mathcal{N}_0=\mathcal{M}$

2. α limit, $\mathcal{N}_\alpha = \bigcup_{i<\alpha} \mathcal{N}_i$
3. Given $\alpha < \kappa^+$, let $\mathcal{N}_{\alpha+1} \succeq \mathcal{N}_\alpha$ such that $|N_{\alpha+1}| \leq |N_\alpha|^\kappa$ and $\mathcal{N}_{\alpha+1}$ realises all types over all sets $A \subseteq_\kappa N_\alpha$ (by the Claim).

Then let $\mathcal{N} = \bigcup_{\alpha<\kappa^+} \mathcal{N}_\alpha$. By induction on α, $|N| \leq |M|^\kappa$. $\mathcal{N}$ is κ^+-saturated since $A \subseteq_\kappa N \implies A \subseteq_\kappa \mathcal{N}_\alpha$ for some $\alpha < \kappa^+$. □

Let T be an $\mathcal{L}$-theory with infinite models.

Corollary 16.3: *If $\kappa \geq |\mathcal{L}| + \aleph_0$ and $2^\kappa = \kappa^+$, then T has a saturated model of size κ.*

Proof. By 16.2, T has a κ^+-saturated model of size $(|\mathcal{L}| + \aleph_0)^\kappa = 2^\kappa = \kappa^+$. □

Fact: If $\kappa > |\mathcal{L}| + \aleph_0$ is regular and $2^\lambda \leq \kappa$ for all $\lambda < \kappa$, then T has a saturated model of size κ.

Basic Facts:

1) If $\mathcal{M} \equiv \mathcal{N}$, $|M| = |N|$ and $\mathcal{M}, \mathcal{N}$ are saturated, then $\mathcal{M} \cong \mathcal{N}$ (Sheet 3 #4).
2) Suppose $\kappa > |\mathcal{L}| + \aleph_0$. Then $\mathcal{M}$ is κ-saturated iff $\mathcal{M}$ is κ-homogeneous and for all $\mathcal{N} \equiv \mathcal{M}$, if $|N| < \kappa$ then $\mathcal{N}$ elementarily embeds into $\mathcal{M}$ (we say $\mathcal{M}$ is ***κ-universal***).

Stability

Let T be a complete theory with infinite models.

Definition 16.4: Given $\kappa \geq |\mathcal{L}| + \aleph_0$, we say T is ***κ-stable*** if for any $\mathcal{M} \models T$, $|M| = \kappa$, we have $|S_1(M)| = \kappa$. T is ***stable*** if it is κ-stable for some $\kappa \geq |\mathcal{L}| + \aleph_0$.

Example 16.5:

1. ACF_p, TFADG are κ-stable for all $\kappa \geq \aleph_0$ (see Example 9.3).
2. $T = \mathrm{Th}(\mathbb{Z}, +, 0, 1, (\equiv_n)_{n\geq 2})$, where $x \equiv_n y \iff \exists z(x - y = nz)$. T has QE.

   Fix $\mathcal{M} \models T$. Given $f : \{\text{primes}\} \to \mathbb{N}$ such that $0 \leq f(n) < n$, we have a complete 1-type $p_f = \{x \neq a : a \in \mathcal{M}\} \cup \{x \equiv_n f(n) : n \text{ is prime}\} \in S_1(M)$.

   By QE, $S_1(M) = \{\mathrm{tp}(a/M) : a \in M\} \cup \{p_f\}_f$. So $|S_1(M)| = |M| + 2^{\aleph_0}$. Thus T is κ-stable iff $\kappa \geq 2^{\aleph_0}$.
3. If $\mathcal{M} \models \mathrm{RG}$ then $|S_1(M)| = 2^{|M|}$ (Example 9.4). So RG is not κ-stable for any κ.

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Again T is complete with infinite models.

Recall: A cardinal κ is ***regular*** if every unbounded subset of κ has size κ (*i.e.* cofinality κ). For instance, $\aleph_0$, $\aleph_1$, $2^{\aleph_0}$ are regular. $\aleph_\omega$ is not regular, as $\{\aleph_n : n \geq 1\}$ is unbounded.

Theorem 17.1: *Suppose T is κ-stable and κ is regular. Then T has a saturated model of size κ.*

Proof. Step 1: If $\mathcal{M} \models T$, $|M| = \kappa$, then $\exists \mathcal{N} \succeq \mathcal{M}$ such that $|N| = \kappa$ and $\mathcal{N}$ realises all 1-types over $\mathcal{M}$.

Proof of Step 1: Enumerate $S_1(M) = \{p_\alpha : \alpha < \kappa\}$ by κ-stability. Build chain.

Step 2: Build $(\mathcal{M}_\alpha)_{\alpha<\kappa}$ such that $|M_0| = \kappa$. $\mathcal{M}_{\alpha+1} \geq \mathcal{M}_\alpha$ realises all 1-types over $\mathcal{M}_\alpha$ and $|\mathcal{M}_{\alpha+1}| = \kappa$. Let $\mathcal{M} = \bigcup_{\alpha<\kappa} \mathcal{M}_\alpha$. Then $|M| = \kappa$. If $A \subseteq M$, $|A| < \kappa$, then $A \subseteq \mathcal{M}_\alpha$ for some $\alpha < \kappa$ since κ is regular. So $\mathcal{M}$ realises all 1-types over A. ☐

Theorem 17.2: *Suppose $\mathcal{L}$ is countable and T is $\aleph_0$-stable. Then T is κ-stable for all $\kappa \geq \aleph_0$.*

Proof. Fix $\kappa \geq \aleph_0$. Assume T is not κ-stable, so we have $\mathcal{M} \models T$, $|M| = \kappa$ such that $|S_1(\mathcal{M})| > \kappa$. Then there exists an $\mathcal{L}_M$-formula $\varphi(x)$ such that $|[\varphi(x)]| > \kappa$ (*e.g.* $[x = x] = S_1(M)$).

Claim: For any $\mathcal{L}_M$-formula $\varphi(x)$ if $|[\varphi(x)]| > \kappa$ then there exists an $\mathcal{L}_M$-formula $\psi(x)$ such that $|[\varphi \wedge \psi]| > \kappa$, $|[\varphi \wedge \neg\psi]| > \kappa$.

Proof of Claim: Adapt the claim from the proof of Lemma 14.1.

Now build $\{\varphi_\sigma\}_{\sigma\in 2^{<\omega}}$ such that $[\varphi_\sigma]$ is partitioned $[\varphi_{\sigma 0}] \sqcup [\varphi_{\sigma 1}]$ and $[\varphi_\sigma] \neq \emptyset$. Let $\mathcal{N} \preceq \mathcal{M}$ be such that $|N| = \aleph_0$ (by DLST) and $\mathcal{N}$ contains all parameters used in all φ_σs (possible since everything is countable).

For $\alpha \in 2^\omega$, find $p_\alpha \in S_1(N)$ such that $\varphi_{\alpha\restriction i} \in p_\alpha$ for all $i > 0$. This gives us $|S_1(N)| = 2^{\aleph_0}$, so T is not $\aleph_0$-stable. ☐

Corollary 17.3: *If T is $\aleph_0$-stable then T has a saturated model of size κ for all regular $\kappa \geq \aleph_0$.*

Fact: $\aleph_0$-stable theories have saturated models of all infinite cardinalities - but this is harder to prove.

Definition 17.4: Fix $\mathcal{M} \models T$, and an $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_m, y_1, \ldots, y_n)$. Then $p \in S_m(M)$ is ***definable wrt*** $\varphi(\overline{x}, \overline{y})$ if there exists an $\mathcal{L}_M$-formula $\psi(y_1, \ldots, y_n)$ such that for all $\overline{b} \in M^n$:

$$\varphi(\overline{x}, \overline{b}) \in p \iff \mathcal{M} \models \psi$$

We say $p \in S_m(M)$ is ***definable*** if it is definable wrt any $\mathcal{L}$-formula $\varphi(\overline{x}, \overline{y})$ (any $\overline{y}$).

Example 17.5:

1) $p = \mathrm{tp}(a/M)$ where $a \in M$. Given $\varphi(x, \overline{y})$, let $\psi(\overline{y})$ be $\varphi(a, \overline{y})$ (this is an $\mathcal{L}_M$-formula). So realised types are fairly trivially definable.
2) T is DLO. $\mathcal{M}$ is $\mathbb{Q}$. Choose $p \in S_1(\mathbb{Q})$ such that $x < b$ is in p iff $\sqrt{2} < b$. Let $\varphi(x, y)$ be $x < y$. Then $\{b \in \mathbb{Q} : \varphi(x, b) \in p\} = (-\infty, \sqrt{2}) \cap \mathbb{Q}$. By QE, any definable subset of $\mathbb{Q}$ is a finite Boolean combination of intervals with endpoints in $\mathbb{Q}$.

Notation: Let x be a tuple of variables. M^x denotes $M^{|x|}$.

Definition 17.6: Let $\varphi(x, y)$ be an $\mathcal{L}$-formula (x, y tuples). Then $\varphi(x, y)$ has the ***order property wrt*** T if there exists $\mathcal{M} \models T$ and $(a_i)_{i \geq 0}$, $(b_i)_{i \geq 0}$ such that for all $i \geq 0$ we have $a_i \in M^x, b_i \in M^y$, and, for all i, j, $\mathcal{M} \models \varphi(a_i, b_j)$ iff $i \leq j$.

E.g. In $\mathbb{Q}$, $x \leq y$ has the order property (wrt DLO). Let $a_i = b_i = i$.

Fundamental Theorem of Stability (Shelah 1976): *TFAE:*

1) T is stable.

2) For any $\mathcal{M} \models T$, any $p \in S_n(M)$ is definable.

3) No $\mathcal{L}$-formula has the order property wrt T.

For now we only show $2 \implies 1$.

Proof. FTS2 $\implies$ FTS1: Assume 2. Fix $\kappa \geq |\mathcal{L}| + \aleph_0$ such that $\kappa^{|\mathcal{L}|+\aleph_0} = \kappa$ (*e.g.* $\kappa = 2^{|\mathcal{L}|+\aleph_0}$). We show T is κ-stable.

Fix $\mathcal{M} \models T$, $|M| = \kappa$. Let $X = \{\mathcal{L}\text{-formulae } \varphi(x, \overline{y})\}$ and $Y = \{\text{all } \mathcal{L}_M\text{-formulae } \psi(\overline{y})\}$ (any $\overline{y}$). Given $p \in S_1(M)$, define $F_p : X \to Y$ such that $F_p(\varphi(x, \overline{y}))$ witnesses that p is definable wrt $\varphi(x, \overline{y})$. Then $p \mapsto F_p$ is an injective function (exercise) from $S_1(M)$ to Y^X. So $|S_1(M)| \leq |Y^X| = \kappa^{|\mathcal{L}|+\aleph_0} = \kappa$. $\square$

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T complete, infinite models.

Proof. FTS1 $\implies$ FTS3: Suppose $\varphi(x, y)$ has the order proprety with respect to T (x, y tuples). Fix $\kappa \geq |\mathcal{L}| + \aleph_0$. WTS T is not κ-stable.

Claim: There is a linear order $(I, <)$ such that $|I| > \kappa$ and there is $J \subseteq I$ such that $|J| = \kappa$ and J is dense.

Proof of Claim: Fix minimal $\lambda \leq \kappa$ such that $2^\lambda < \kappa$. Let $I = \mathbb{Q}^\lambda$, and the order given lexicographically: given distinct $f, g \in I$, set $f < g$ iff $f(\alpha) < g(\alpha)$ where $\alpha < \lambda$ is least such that $f(\alpha) \neq g(\alpha)$.

Let $J = \{f \in I : \exists \alpha < \lambda \text{ such that } f(x) = 0 \text{ for all } x \geq \alpha\}$. Then $|J| \leq \max_{\alpha<\lambda} 2^\alpha \leq \kappa$. $\square$

Let I be as in the claim. Consider

$$T \cup \{\varphi(x_i, y_j) : i, j \in I, i \leq j\} \cup \{\neg\varphi(x_i, y_j) : i, j \in I, i > j\}$$

This is finitely satisfiable since $\varphi(x, y)$ has the order property wrt T. So $\exists \mathcal{N} \models T$ and $(a_i)_{i \in I}$ and $(b_i)_{i \in I}, a_i \ni N^d, b_i \in N^y$ and $\mathcal{N} \models \varphi(a_i, b_j)$ iff $i \leq j$. By DLST $\exists \mathcal{M} \preceq \mathcal{N}$ such that $|M| = \kappa$ and $b_i \in M^y$ for all $i \in J$. We show $|S_x(M)| \geq |I| > \kappa$, and thus T is not κ-stable (ES3 #5).

For $i \in I$, let $p_i = \text{tp}(a_i/M)$. Fix $i, j \in I$ such that $i < j$. $\exists k \in J$ such that $i < k < j$. So $\varphi(x, b_k) \in p_i$ and $\varphi(x, b_k) \notin p_j$. So $p_i \neq p_j$. $\square$

Lastly, we have FTS3 $\implies$ FTS2: Fix an $\mathcal{L}$-formula $\varphi(x, y)$ such that φ does not have the order property wrt T. Now fix $\mathcal{M} \models T$. We show that any $p \in S_x(M)$ is definable wrt $\varphi(x, y)$.

This proof is long and topological.

Let $X = S_x(M)$ and $Y = S_y(M)$. Let $A = \{\text{tp}(a/M) : a \in M^x\} \subseteq X$. A is dense in X: Given an $\mathcal{L}_M$-formula $\psi(x)$, if $[\psi(x)] \neq \emptyset$ then $\mathcal{M} \models \exists x \psi(x)$, so $\exists a \in M^x$ such that $\mathcal{M} \models \psi(a)$, so $\text{tp}(a/M) \in [\psi(x)] \cap A$. Also let $B = \{\text{tp}(b/M) : b \in M^y\}$. Then B is dense in Y by the same argument.

Identify A with M^x and B with M^y. Let $2 = \{0, 1\}$ (as a discrete space). Define $f : A \times B \to 2$ such that $f(a, b) = 1$ iff $\mathcal{M} \models \varphi(a, b)$.

<u>Notation</u>: Given $a \in A$, $b \in B$, let $f_b : A \to 2$ and $f^a : B \to 2$ be the corresponding fiber functions (*e.g.* $f_b(a) = f(a, b)$).

Given $b \in B$, let $\hat{f}_b : X \to 2$ such that for $p \in X = S_x(M)$, $\hat{f}_b(p) = 1$ iff $\varphi(x, b) \in p$. Now claim that $\hat{f}_b$ extends f_b: given $a \in A$, $\hat{f}_b(a) = 1$ iff $\varphi(x, b) \in \text{tp}(a/M)$ iff $\mathcal{M} \models \varphi(a, b)$ iff $f_b(a) = 1$ by definition. Moreover, $\hat{f}_b$ is continuous: $\hat{f}_b^{-1}(\{1\}) = [\varphi(x, b)]$, and $\hat{f}_b^{-1}(\{0\}) = [\neg\varphi(x, b)]$ and these are both clopen sets.

Similarly, given $a \in A$, $\hat{f}^a : Y \to 2$ such that $\hat{f}^a(q) = 1$ iff $\varphi(a, y) \in q$.

This brings us to...

<u>Main Claim</u>: There is a separately continuous function $F : X \times Y \to 2$ extending f. By separately continuous, we mean that if one coordinate is fixed the resulting function is continuous.

For now, we suppose that this is true. Fix $p \in S_x(M) = X$. WTS p is definable wrt $\varphi(x, y)$. $F^p : Y \to 2$ is continuous. Set $D = (F^p)^{-1}(\{1\})$. D is clopen in $Y = S_y(M)$. So $D = [\psi(y)]$ for some $\mathcal{L}_M$-formula $\psi(y)$ (ES2#7). Fix $b \in M^y = B$.

$$
\begin{aligned}
\varphi(x, b) \in p &\iff \hat{f}_b(p) = 1 \\
&\iff F_b(p) = 1 \quad F_b, \hat{f}_b \text{ are continuous extensions of } f_b \text{ and } A \text{ is dense} \\
&\iff F^p(b) = 1 \\
&\iff \text{tp}(b/M) \in D = [\psi(y)] \\
&\iff M \models \psi(b)
\end{aligned}
$$

<u>Proof of Main Claim</u>

Goal: show that $\forall p \in X, q \in Y$ there exist open neighbourhood U_{pq} of p and V_{pq} of q such that for all $a \in A \cap U_{pq}$ and $b \in B \cap V_{pq}$, $\hat{f}_b(p) = \hat{f}^a(q) = t_{pq}$ (definition of t_{pq}). $(*)$

Suppose this fails. Fix $p \in X, q \in Y$ such that for all open neighbourhoods U containing p, V containing q, there exists $a \in A \cap U$ and $b \in B \cap V$ such that $\hat{f}_b(p) \neq \hat{f}^a(q)$. We build $(a_n)_{n \geq 0}$ from $A = M^x$ and $(b_n)_{n \geq 0}$ from $B = M^y$ such that for all n:

1. $\hat{f}_{b_n}(p) \neq \hat{f}^{a_n}(q)$
2. $\forall i < n$, $f(a_n, b_i) = \hat{f}_{b_i}(p)$
3. $\forall i < n$, $f(a_i, b_n) = \hat{f}^{a_i}(q)$

Suppose we have this. Pass to subsequences so that $(\hat{f}_{b_n}(p))_n$ and $(\hat{f}^{a_n}(q))_n$ are constant.

<u>Case 1</u>: For all n, $\hat{f}_{b_n}(p) = 0$ and $\hat{f}^{a_n}(q) = 1$

Given this, define $F : X \times Y \to \{0, 1\}$ such that $F(p, q) = t_{pq}$.

F extends f: for $a \in A, b \in B$, $a \in A \cap U_{ab} \implies t_{ab} = \hat{f}^a(b) = f(a, b)$.

F is separately continuous: *e.g.* $p \in X, t \in 2$:

$$(F^p)^{-1}(\{t\}) = \bigcup \{V_{pq} : q \in Y, t_{pq} = t\}$$

So $\mathcal{M} \models \varphi(a_i, b_j)$ for all $i < j$ and $\mathcal{M} \models \neg\varphi(a_i, b_j)$ for all $i > j$.

WLOG $(f(a_n, b_n))_n$ is constant (by passing to a further subsequence). If it is 1, then $\mathcal{M} \models \varphi(a_i, b_j) \iff i \leq j$. If it is 0, let $b'_i = b_{i+1}$. Then $\mathcal{M} \models \varphi(a_i, b'_j) \iff i \leq j$.

Case 2: For all n, $\hat{f}_{b_n}(p) = 1$ and $\hat{f}^{a_n} = 0$.

By a similar argument to above (passing to subsequences when necessary), we get WLOG that $\mathcal{M} \models \varphi(a_i, b_j) \iff i \geq j$. But it turns out this is enough - fix $k \geq 1$. For $i, j \leq k$, let $a''_i = a_{k-i}$ and $b''_i = b_{k-i}$. Then $\mathcal{M} \models \varphi(a''_i, b''_j) \iff i \leq j$. This does not give us an infinite instance of the order property, but rather arbitrarily long finite sequences; it is a consequence of compactness that there then exists an infinite sequence (*c.f.* ES3 #6).

So all we have left is to build the sequence with the above properties, which we do inductively:

Pick a_0, b_0 such that $\hat{f}_{b_0}(p) \neq \hat{f}^{a_0}(q)$ (choose $U = X, V = Y$ in condition $\neg(*)$).

Suppose we have $(a_i)_{i<n}$ and $(b_i)_{i<n}$. Pick a_n, b_n as follows. Let $U = \bigcap_{i<n}\{u \in X : \hat{f}_{b_i}(u) = \hat{f}_{b_i}(p)\}$. This is an open neighbourhood of p. Similarly, let $V = \bigcap_{i<n}\{v \in Y : \hat{f}^{a_i}(v) = \hat{f}^{a_i}(q)\}$.

Now by $\neg(*)$ there exists $a_n \in A \cap U$ and $b_n \in B \cap V$ such that $\hat{f}_{b_n}(p) \neq \hat{f}^{a_n}(q)$.

So for all $i < n$, $f(a_n, b_i) = \hat{f}_{b_i}(a_n) = \hat{f}_{b_i}(p)$, and $f(a_i, b_n) = \hat{f}^{a_i}(b_n) = \hat{f}^{a_i}(q)$.

This concludes the proof of the main claim, and hence the proof of the entire theorem. $\square$

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This was not Shelah's original proof; that uses more combinatorics and involves colouring binary tress; this is a more recent proof.

Remark 19.1: The previous proof did not use Hausdorffness of X or Y. We could instead repeate the previous proof with a coarser topology on Y generated by $[\varphi(a, y)]$ for all $a \in M^x$. Then given $p \in S_x(M)$, we get a φ-definition $\psi(y)$ to be a Boolean combination of $\varphi(a, y)$ for $a \in M^x$.

Remark 19.2: Assume that T is complete with infinite models, and $\mathcal{L}$ is countable.

1) $\aleph_0$-stable $\implies$ κ-stable for all $\kappa \geq \aleph_0$
2) stable $\implies$ κ-stable for all $\kappa^{\aleph_0} = \kappa$.

Shelah 1976: Let $\Sigma(T) = \{\kappa : T \text{ is } \kappa\text{-stable}\}$. Then $\Sigma(T)$ is one of :

- $\{\kappa \geq \aleph_0\}$ (sometimes known as ω-stable). This includes ACF_p, TFDAG, InfSets.
- $\{\kappa \geq 2^{\aleph_0}\}$. Examples include $\mathrm{Th}(\mathbb{Z}, +, 0)$.
- $\{\kappa^{\aleph_0} = \kappa\}$. For instance, $\mathrm{Th}(\mathbb{Z}^\omega, +, 0)$, or any separably closed field that is not ACF.
- $\emptyset$. These theories include RG, DLO.

The first two cases together are known as *superstable*, the third is *strictly stable*, the first three are *stable* and the fourth is *unstable*.

Motto: Stable algebraic structures are "nice".

Example: (fields) $T = \mathrm{Th}(F)$ where F is a field.

[Macintyre 1971/Cherlin-Shelah 1980] If T is superstable then $F \models \mathrm{ACF}$.

[Macintyre-Shelah-Wood 1975] If F is separably closed then T is stable.

Stable Fields Conjecture: If T is stable then F is separably closed.

Open problem: Is $\mathrm{Th}(\mathbb{C}(t))$ stable? This is often a test question for the above.

20 Lecture

Stable Groups

Definition 20.1: (Expansion of a Group) Let $\mathcal{L}$ be a language. An $\mathcal{L}$-structure G is an ***expansion of a group*** if $\mathcal{L}$ contains the language of groups and the reduct of G to the group language is a group.

We conflate the structure G with its universe for notational convenience.

Examples:

1. If G is an abelian group then $\mathrm{Th}(G, +, 0)$ is stable (folklore - no attributed author).
2. If G is a free group then $\mathrm{Th}(G, \cdot, 1)$ is stable (Sela 2006)
3. If $P = \{2^n : n \geq 1\}$ then $\mathrm{Th}(\mathbb{Z}, +, 0, P)$, where P is a unary relation that determines whether or not something is a power of two, is stable (Mousa-Scanlon 2007).
4. Let G be an algebraic group over some $K \models \mathrm{ACF}$. Consider the expansion of G by relation symbols for all subsets of G^n $(n \geq 1)$ definable in the field language. Then $\mathrm{Th}(G)$ is ω-stable.

Let $T = \mathrm{Th}(G)$ where G is an expansion of a group. Given an $\mathcal{L}_G$-formula $\varphi(x)$ (note we could have multiple variables instead) let $\varphi(G) = \{a \in G : G \models \varphi(a)\}$. Recall that $X \subseteq G$ is ***definable*** if $X = \varphi(G)$ for some $\mathcal{L}_G$-formula $\varphi(x)$.

Definition 20.2: Let $\varphi(x, y_1, \ldots, y_n)$ be an $\mathcal{L}$-formula. Define H_φ to be the collection of all finite-index subgroups of G of the form $\varphi(G, \bar{b})$ for some $\bar{b} \in G^n$. Let $G^0(\varphi) = \bigcap_{H \in H_\varphi} H$, which we write as $\bigcap H_\varphi$. If $H_\varphi = \emptyset$ then $G^0(\varphi) = G$.

Example 20.3: $G = (\mathbb{Z}, +, \cdot, 0, 1)$. $\varphi(x, y)$ is $\exists z(x = y \cdot z)$ (colloquially, y divides x). Then $\varphi(\mathbb{Z}, m) = m\mathbb{Z}$. So $G^0(\varphi) = \{0\}$.

Theorem 20.4: (Baldwin-Saxl 1976) *Assume $T = Th(G)$ is stable. Then for any $\mathcal{L}$-formula $\varphi(x, y_1, \ldots, y_n)$, there is a finite $\mathcal{F} \subseteq H_\varphi$ such that $G^0(\varphi) = \bigcap \mathcal{F}$. Moreover, $G^0(\varphi)$ is definable by an $\mathcal{L}$-formula.*

Proof. Fix $\varphi(x, \overline{y})$. WLOG $H_\varphi \neq \emptyset$.

Claim 1: There exists $m \geq 1$ such that for all finite $H \subseteq H_\varphi$ then there exists $\mathcal{F} \subseteq H$, $|\mathcal{F}| = m$, such that $\bigcap H = \bigcap \mathcal{F}$.

Proof of Claim 1: Suppose not. Fix $m \geq 1$. Then there exists finite $H \subseteq H_\varphi$ such that for all $\mathcal{F} \subseteq H$, $|\mathcal{F}| = m$, we have $\bigcap H \subsetneq \bigcap \mathcal{F}$. After thinning H, we may assume $|H| > m$ and if $\mathcal{F} \subsetneq H$ then $\bigcap H \subsetneq \bigcap \mathcal{F}$. Let $H = \{H_1, \ldots, H_k\}$ $(k > m)$.

For $1 \leq i \leq k$, choose $g_i \in \left(\bigcap_{j \neq i} H_j\right) \setminus H_i$. Given $I \subseteq \{1, \ldots, k\}$, let $g_I = \prod_{i \in I} g_i$. Then $g_I \in H_j$ if and only if $j \notin I$:

$$j \notin I \implies g_i \in H_j \ \forall i \ni I \implies g_I \in H_j$$
$$j \in I \implies g_{I\setminus\{j\}} \in H_j \text{ and } g_j \notin H_j \implies g_I \notin H_j$$

Choose $\overline{b}_j \in G^n$ such that $H_j = \varphi(G, \overline{b}_j)$. Let $a_i = g_{<i} \coloneqq \prod_{j<i} g_j$. Then $G \models \varphi(a_i, \overline{b}_j)$ iff $g_{<i} \in H_j$ iff $i \leq j$. Since k can be chosen arbitarily large, we get the order property for $\varphi(x, \overline{y})$ by ES3 #6. $\square$

Fix $m \geq 1$ as in Claim 1. Let $\psi(x, \overline{y}_1, \ldots, \overline{y}_n)$ be $\bigwedge_{i=1}^m \varphi(x, \overline{y}_i)$.

Claim 2: H_ψ contains a minimal element.

Proof of Claim 2: Suppose not. There is $H_0 > H_1 > H_2 > \ldots$ with $H_i \in H_\psi$. Choose $g_i \in H_i \backslash H_{i+1}$. Then $g_i \in H_j$ iff $j \leq i$. So ψ has the order property wrt T. $\square$

Let H be a minimal element of H_ψ. Note that $H = \bigcap \mathcal{F}$ where $\mathcal{F} \subseteq H_\varphi$, $|\mathcal{F}| = m$.

Claim 3: $H = G^0(\varphi)$.

Proof of Claim 3: $G^0(\varphi) \leq H$ be definition of $G^0(\varphi)$. Fix $K \in H_\varphi$. WTS $H \leq K$.

By Claim 1, $H \cap K \in H_\psi$. By minimality of H, $H = H \cap K$, *i.e.* $H \leq K$. $\square$

Claim 4: $G^0(\varphi)$ is definable by an $\mathcal{L}$-formula.

Proof of Claim 4: Let $k = [G : G^0(\varphi)]$. Then $a \in G^0(\varphi)$ iff for all $\overline{b} \in g^n$, if $\varphi(G, \overline{b})$ is a subgroup of G of index at most k, then $\varphi(a, \overline{b})$.

This is expressible by an $\mathcal{L}$-formula (all the above statements are first-order, exercise; the key is that the index is now uniformly bounded. $\square$

This concludes the proof of the whole theorem. $\square$

21 Lecture

Setting: G is an expansion of a group. $T = \mathrm{Th}(G)$. Assume T is stable.

Definition 21.1: Let G^0 be the intersection of all definable finite index subgroups of G.

Note: $G^0 = \bigcap_\varphi G^0(\varphi)$.

Example 21.2:

1) $\mathrm{Th}(\mathbb{Z}, +, 0) = T$. If $G \models T$ then $G^0 = \bigcap_{n \geq 1} nG$. So if $G = \mathbb{Z}$, $G^0 = \{0\}$. If G is $\aleph_0$-saturated, then G^0 is nontrivial.
2) G is an algebraic group over some ACF. G^0 is the connected component of G wrt the Zariski topology.

Remark 21.3: G^0 is a normal subgroup.

Goal: Another description of G^0.

To work towards this goal, we will look at a more general notion of being a coset of a finite index subgroup.

Definition 21.4: (bi-generic) $X \subseteq G$ is ***bi-generic*** if $\exists a_1, \dots, a_n, b_1, \dots, b_n \in G$ such that $G = \bigcup_{i=1}^n a_i X b_i$.

So cosets are special types of bi-generic subsets.

Lemma 21.5: *If $X \subseteq G$ is definable, then either X of $G \backslash X$ is bi-generic.*

Proof. Suppose not. We build $(a_i)_{i\geq 1}$ and $(b_i)_{i\geq 1}$ such that $a_i b_j \in X$ iff $i \leq j$. [So if $\varphi(x)$ defines X then $\varphi(x \cdot y)$ has the order property. Note that this glosses over the fact that φ might have parameters, but this is easy to deal with. Let $a_0 \in X$ and $b_0 = 1$.

Choose $a_n \notin \bigcup_{i<n} X b_i^{-1}$, which exists since X is not bi-generic. In particular, this says that $a_n b_i \notin X\ \forall i < n$, and choose $b_n \notin \bigcup_{i\leq n} a_i^{-1}(G \backslash X)$ - this says that $a_i b_n \in X\ \forall i \leq n$. These two statements gives us the order property up to n. $\square$

Example 21.6: $\mathrm{Th}(\mathbb{Z}, +, <, 0)$ is unstable. $\mathbb{N}$ and $\mathbb{Z} \backslash \mathbb{N}$ are definable and not bi-generic.

Lemma 21.7: *If $X, Y \subseteq G$ are definable and $X \cup Y$ is bi-generic, then X or Y is bi-generic.*

Proof. Assume $G = \bigcup_{i=1}^n a_i(X \cup Y)b_i = \underbrace{\bigcup_{i=1}^n a_i X b_i}_{A} \cup \underbrace{\bigcup_{i=1}^n a_i Y b_i}_{B}$. Note that A and B are both definable also.

If A is bi-generic, then so is X.

Suppose A is not bi-generic. Then $G \backslash A$ is bi-generic by 21.5. So B is bi-generic since $G \backslash A \subseteq B$). So Y is bi-generic. $\square$

Definition 21.8: $p \in S_1(G)$ is bi-generic if every $X \in p$ is bi-generic.

<u>Convention</u>: Identify $\mathcal{L}_G$-formula $\varphi(x)$ with $\varphi(G)$.

Proposition 21.9: *There is a bi-generic type $p \in S_1(G)$.*

Proof. Let $q = \{\neg X : X \subseteq G$ is definable and not bi-generic$\}$. Then q is finitely satisfiable in G: Fix $X_1, \dots, X_n \subseteq G$ definable, not bi-generic. Then $X_1 \cup \dots \cup X_n$ is not bi-generic by 21.7. So $\neg X_1 \cap \dots \cap \neg X_n = \neg(X_1 \cup \dots \cup X_n) \neq \emptyset$.

Now extend q to some $p \in S_1(G)$. If $X \in p$ then $\neg X \notin q$, so X is bi-generic. $\square$

Definition 21.10: Given $p \in S_1(G)$ and $g \in G$, define $\mathrm{gp} = \{gX : X \in p\}$.

<u>ES4</u>: $\mathrm{gp} \in S_1(G)$.

Definition 21.11: Given $p \in S_1(G)$ and $X \subseteq G$ definable, let

$$H_X^p = \{g \in G : \forall a \in G, aX \in p \iff aX \in gp\}$$

Theorem 21.12: *If $p \in S_1(G)$ and $X \subseteq G$ is definable, then H_X^p is a definable subgroup of G. Moreover, if p is bi-generic then H_X^p has finite index.*

Proof. Fix $p \in S_1(G)$ and definable $X \in G$. Exercise: H_X^p is a subgroup. Since p is definable, there exists $\psi(y)$ such that for all $a \in G$, $aX \in p \iff G \models \psi(a)$. [$\psi(y)$ is the definition for p wrt the formula $\varphi(x, y)$ given by "$y^{-1} \cdot x \in X$"]

So $g \in H_X^p \iff G \models \forall a(\psi(a) \leftrightarrow \psi(g^{-1}a))$, which is a formula in one free variable g and hence the subgroup H_X^p is definable.

Given $q, r \in S_1(G)$, write $q \sim r$ iff: $\forall a, b \in G, aXb \in q \iff aXb \in r$; this is easily shown to be an equivalence relation.

Main Claim: There are only finitely many bi-generic types in $S_1(G)$ mod $\sim$.

For now, we assume this claim. Now assume p is bi-generic. Suppose $\exists g_1, g_2, \dots$ in G such that $\forall i \neq j, g_i^{-1}g_j \notin H_X^p$. So for all $i \neq j$, there exists $a \in G$ such that $aX \in p \iff aX \notin g_i^{-1}g_jp$, *i.e.* $g_iaX \in g_ip \iff g_iaX \notin g_jp$. In particular then, for all $i \neq j$ we have $g_ip \not\sim g_jp$. But each g_ip is bi-generic [ES4], and by the claim we have only finitely many such types; contradicton. □

Corollary 21.13: *If $p \in S_1(G)$ is bi-generic, then*

$$G^0 = \bigcap_{X \text{ def. } \subseteq G} H_X^p$$

So $G^0 = \{g \in G : gp = p\} =: \mathrm{Stab}(p)$.

Proof. $G^0 \leq H_X^p$ for all definable $X \subseteq G$, so it must also lie within their intersection. For the other direction, it suffices to fix definable finite index normal subgroup $K \lhd G$, and show $\bigcap_{X \subseteq G} H_X^p \leq K$.

Check $H_K^p = K$. □

22 Lecture

Lemma 22.1: *There aer only finitely many bi-generic types in $S_1(G)$ modulo $\sim$.*

Proof. Non-examinable.

Let $\theta(x)$ be a formula defining X. WLOG $\theta(x)$ is an $\mathcal{L}$-formula (add constants). Let $\varphi(x; y, z)$ be $\theta(y^{-1} \cdot x \cdot z^{-1})$. Note $\varphi(x; a, b)$ defines aXb.

Given $M \models T$ and $p, q \in S_1(M)$, write $p \sim q$ if $\forall a, b \in M$, $\varphi(x; a, b) \in p$ iff $\varphi(x; a, b) \in q$. Given a set of q of $\mathcal{L}_M$-formulae in free variable x, we define $R(q) \in \mathbb{N} \cup \{-1, \infty\}$:

- $R(q) \geq 0$ iff q is a type w.r.t. M
- $R(q) \geq n + 1$ iff $\exists \mathcal{N} \succeq \mathcal{M}$ and $p_i \in S_1(N)$ for $i \geq 1$, such that $p_i \supseteq q$, $R(p_i) \geq n$, and $p_i \not\sim p_j$ for all $i \neq j$.

Note: If $p \subseteq q$ then $R(q) \leq R(p)$.

Strategy:

1. $R(x = x) < \infty$
2. $\#\{\text{max rank } p \in S_1(G)\}/\sim$ is finite
3. Any bi-generic $p \in S_1(G)$ has maximum rank

Write $\varphi(x; y, z)$ as $\varphi(x; v)$ (where $v = (y, z)$ is a notational convenience). Given $\mathcal{M} \models T$ and $q(x)$ and a cardinal λ (possibly finite), define:

$$\begin{aligned}\Gamma(q, \lambda) = & \bigcup_{\sigma \in \mathbb{N}^\lambda} q(x_\sigma) \\ & \cup \left\{\varphi(x_\sigma, v_{s,i,j}) \longleftrightarrow \neg\varphi(x_\tau, v_{s,i,j}) : \sigma, \tau \in \mathbb{N}^\lambda, s \in \mathbb{N}^{<\lambda}, i, j \in \mathbb{N}, i \neq j, si \leq \sigma, sj \leq \tau\right\}\end{aligned}$$

Claim 1: $R(q) \geq n$ iff $\Gamma(q, n)$ is consistent.

Proof of Claim 1: See notes [long and technical]. $\square$

Claim 2: $R(x = x) = n < \infty$ (for some n).

Proof of Claim 2: Suppose $R(x = x) = \infty$. By Claim 1, $\Gamma(x = x, n)$ is consistent for all $n \geq 1$. Then by Compactness, $\Gamma(x = x, \lambda)$ is consistent for any λ. Fix $n \geq |\mathcal{L}| + \aleph_0$. WTS: T is not κ-stable. Indeed, choose minimal λ such that $\kappa < 2^\lambda$. Let $(a_\sigma, b_{s,i,j}) \models \Gamma(x = x, \lambda)$ in some $\mathcal{N} \models T$. Choose $\mathcal{M} \preceq \mathcal{N}$ such that $b_{s,i,j} \in M$ for all s, i, j, and $|M| \leq |\mathbb{N}^{<\lambda}| + \aleph_0 \leq \kappa$. Let $p_\sigma = \text{tp}(a_\sigma/M)$. If $\sigma \neq \tau$ then $\exists s, i, j$ such that $i \neq j$, $si \leq \sigma, sj \leq \tau$ and so $\varphi(x, b_{s,i,j}) \in p_\sigma$ iff $\neg\varphi(x, b_{s,i,j}) \notin p_\tau$. So $|S_1(M)| \geq |\mathbb{N}^\lambda| = 2^\lambda > \kappa$. $\square$

Claim 3: There are only finitely many rank n types in $S_1(G)$, modulo $\sim$.

Proof of Claim 3: Otherwise $R(x = x) \geq n + 1$. $\square$

Claim 4: any $\mathcal{M} \models T$ and any $p(x)$ over M, there exists finite $q \subseteq p$ such that $R(p) = R(q)$.

Proof of Claim 4: Suppose $R(p) = n$. Then $\Gamma(p, n+1)$ is inconsistent by Claim 3. So there exists some finite $q \subseteq p$ such that $\Gamma(q, n+1)$ is inconsistent. So $R(q) \leq n$. So $n = R(p) \leq R(q) \leq n$. $\square$

Claim 5: Given a $\mathcal{M} \models T$ and $\mathcal{L}_M$-formulas $\psi_1(x), \psi_2(x)$, we have $R(\psi_1(x) \wedge \psi_2(x)) = \max\{R(\psi_1(x)), R(\psi_2(x))\}$.

Proof of Claim 5: "$\geq$" is obvious from the definition; it is enough to show $\leq$. By pigeonhole and since the types p_i in the definition of the rank are complete. $\square$

Note that up to this point we have in fact not used the group structure of G.

Claim 6: Given $\mathcal{M} \models T$, $a, b \in M$, and $q(x)$, we have $R(aqb) = R(q)$.

Proof of Claim 6: ETS $\geq$ (can then get the other direction by multiplying by inverses). We proceed by induction.

Suppose $R(q) \geq n + 1$. Fix $\mathcal{N} \succeq \mathcal{M}$ and $p_i \in S_1(N)$ witnessing it. Let $q_i = ap_ib$. Then $q_i \supseteq aqb$ and $R(q_i) \geq n$ by induction.

Fix $i \neq j$. There exists $c, d \in N$ such that $\varphi(x; c, d) \in p_i$ iff $\neg\varphi(x, c, d) \in p_j$. So $\varphi(x; ac, db) \in q_i$ iff $\neg\varphi(x, ac, db) \in q_j$. So $q_i \not\sim q_j$. So $R(aqb) \geq n + 1$. $\square$

Claim 7: If $p \in S_1(G)$ is bi-generic, then $R(p) = n$.

Proof of Claim 7: By Claim 4, there exists $Y \in p$ such that $R(Y) = R(p)$. So Y is bi-generic. Since $R(G) = R(x = x) = n$, there exists $a, b \in G$ such that $R(aYb) = n$ by Claim 5. So $R(Y) = n$ by Claim 6. So $R(p) = n$. $\square$

Claims 3 and 7 yield the result. □

There is more to say about the "structure" of the collection of bi-generic types in $S_1(G)$.

23 Lecture

Definition: (Amenable Group) A group G is ***amenable*** if there is a left-invariant, finitely additive probability measure on the subsets of G.

Examples: Some examples of amenable groups are:

- finite groups
- solvable groups
- finitely generated groups of "subexponential growth"

Some examples of *non*-amenable groups are:

- nonabelian free groups
- $\mathrm{SL}_3(\mathbb{R})$ (*c.f.* Banach-Tarski)

Let $\mathcal{L}$ be a language and let G be an $\mathcal{L}$-structure expanding a group.

Definition: (Definably Amenable) G is ***definably amenable*** if there is a left-invariant, finitely additive probability measure on the definable subsets of G.

The study of amenable groups has really taken off in Model Theory in the last 15 or so years.

Theorem 23.1: (Newelski-Petrykowski, 2006) *If $Th(G)$ is stable then G is definably amenable.*

Proof. By Prop 21.9, there is a bi-generic type $p \in S_1(G)$. Given a definable set X, let

$$H_X = H_X^p = \{g \in G : \ \forall a \in G(aX \in p \longleftrightarrow aX \in gp)\}$$

and let

$$D_X = \{g \in G : X \in gp\}$$

By Theorem 21.12, H_X is a definable finite-index subgroup of G.

<u>Claim</u>: D_X is a union of left cosets of H_X.

<u>Proof of Claim</u>: Fix $a \in D_X$. Fix $g \in aH_X$. WTS $g \in D_x$, *i.e.* $X \in gp$. We have $X \in ap$ and $a^{-1}g \in H_X$. So $a^{-1}X \in p$, so $a^{-1}X \in a^{-1}gp$ and hence $X \in gp$. □

<u>Notation</u>: Given $H \leq G$ and $D \subseteq G$ a union of left cosets of H, let $|D/H|$ be the number of left cosets of H contained in D (the index).

Given a definable set $X \subseteq G$, let

$$\mu(X) = \frac{|D_X/H_X|}{|G/H_X|} \in [0,1]$$

which can be thought of as just the proportion of cosets of H_X that are in the set D_X. This in fact works as a measure, so all that remains is to check this is the case.

$\underline{\mu \text{ is left-invariant}}$: Fix definable X and $c \in G$. Then $H_{cX} = \{g \in G : \forall a \in G(acX \in p \longleftrightarrow acX \in gp)\} = H_X$. Also, $g \in D_{cX} \iff cX \in gp \iff X \in c^{-1}gp \iff c^{-1}g \in D_X \iff g \in cD_X$. So $D_{cX} = cD_X$. Thus $\mu(cX) = |D_{cX}/H_{cX}|/|G/H_{cX}| = |cD_X/H_X|/|G/H_X| = |D_X/H_X|/|G/H_X| = \mu(X)$. $\square$

$\underline{\mu(G) = 1}$: $H_G = G = D_G$. $\square$

$\underline{\mu \text{ is finitely additive}}$: Fix disjoint, definable subsets $X, Y \subseteq G$. WTS $\mu(X \cup Y) = \mu(X) + \mu(Y)$.

Claim: $D_{X\cup Y} = D_X \cup D_Y$

Proof of Claim: $X \cup Y \in gp \iff X \in gp$ or $Y \in gp$. $\square$

Claim: $D_X \cap D_Y = D_{X\cap Y} = D_\emptyset = \emptyset$.

Claim: $H_X \cap H_Y \leq H_{X\cup Y}$.

Proof of Claim: Fix $g \in H_X \cap H_Y$. Fix any $a \in G$. $a(X \cup Y) \in p \iff aX \in p$ or $Y \in p \iff aX \in gp$ or $aY \in gp \iff a(X \cup Y) \in gp$. $\square$

Note: Suppose $K \leq H \leq G$ and $[G : H] = n$, $[H : K] = m$. Suppose $D \subseteq G$ is a union of ℓ cosets of H. Then D is a union of ℓm cosets of K, and $|D/K|/|G/K| = \ell m/nm = \ell/n = |D/H|/|G/H|$.

Now let $K = H_X \cap H_Y$. Then:

$$\begin{aligned}
\mu(X \cup Y) &= \frac{|X_{X\cup Y}/H_{X\cup Y}|}{|G/H_{X\cup Y}|} = \frac{|D_{X\cup Y}/K|}{|G/K|} \\
&= \frac{|D_X \cup D_Y/K|}{|G/K|} = \frac{|D_X/K| + |D_Y/K|}{|G/K|} \\
&= \frac{|D_X/K|}{|G/K|} + \frac{|D_Y/K|}{|G/K|} = \frac{|D_X/H_X|}{|G/H_X|} + \frac{|D_Y/H_Y|}{|G/H_Y|} \\
&= \mu(X) + \mu(Y)
\end{aligned}$$

and this concludes the proof. $\square$

Remark:

1. $\mu(X) = \mu(D_X)$. In fact, one can show that $\mu(X \Delta D_X) = 0$, which is a stronger statement. Moreover, μ is the unique left-invariant, finitely additive probability measure on definable subsets of G.

   E.g. $(\mathbb{Z}, +, <, 0)$, an unstable group. For all $r \in [0, 1]$ there exists μ such that $\mu(\mathbb{N}) = r$. On the other hand, if $X \subseteq \mathbb{Z}$ is definable in a stable expansion of $(\mathbb{Z}, +)$, then there is a set D which is a union of cosets of a nontrivial subgroup of $\mathbb{Z}$, such that $X \triangle D$ has upper Banach density 0.

2. G stable group. Let $\Gamma = \{p \in S_1(G) : p \text{ is bi-generic}\}$. Then Γ is closed. Fix $p, q \in \Gamma$. Let $G_1 \succeq G$ and $b \in G_1$, $b \models q$. Let $p_1 \in S_1(G_1)$ be the "definable extension" of p, *i.e.* $\varphi(x, c) \in p$, iff $G_1 \in \psi(c)$ where $\psi(y)$ is the φ-definition of p.

   Now repeat. Let $G_2 \succeq G_1$ and $a \in G_2$, $a \models p_1$. Then let $p * q = \mathrm{tp}(ab/G) \in S_1(G)$.

   Fact: $(\Gamma, *)$ is a compact Hausdorff group.

   If G is sufficiently saturated, then $\Gamma \cong G/G^0$. Fix $p \in \Gamma$. For $X \subseteq G$ definable, $\mu(X) = \mathrm{Haar}(\{aG^0 : X \in ap\})$. This is well-defined, since $G^0 = \mathrm{Stab}(p)$.