Model Theory: Example Sheet 3

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1. T is a complete theory in a countabel language $\mathcal{L}$.

   (a) Let $\mathcal{M} \models T$ be atomic and let $f : A \to M$ be partial elementary, A finite, and let $c \in M$.

   $\mathcal{M}$ is atomic, so every realised type in $\mathcal{M}$ is isolated. Let $p = \mathrm{tp}^{\mathcal{M}}(a_1, \dots, a_n, c)$. Then $(\overline{a}, c) \models p$, so p is isolated. Hence there exists some φ such that $\mathcal{M} \models \forall \overline{x}, y(\varphi(\overline{x}, y) \to \psi(\overline{x}, y))$.

   But also $\mathcal{M} \models \varphi(\overline{a}, c)$, so in particular $\mathcal{M} \models \exists y \varphi(\overline{a}, y)$, so $\mathcal{M} \models \exists y \varphi(f(\overline{a}), y)$ by partial elementaryness of f. So there is some d such that $\mathcal{M} \models \varphi(f(\overline{a}, d))$.

   Now take any $\psi(\overline{x}, y)$ with $\mathcal{M} \models \psi(\overline{a}, c)$. We have that $\mathcal{M} \models \varphi(f(\overline{a}), d)$ and so $\mathcal{M} \models \psi(f(\overline{a}), d)$. So done.

   (b) Enumerate $\mathcal{M} = \{a_1, a_2, \dots\}$ and $\mathcal{N} = \{b_1, b_2, \dots\}$. We build a bunch of partial elementary functions and then take their union.

   Idea: $a_1 \models \mathrm{tp}(a_1)$, so there is some $b \in N$ such that $b \models \mathrm{tp}(a_1) = \mathrm{tp}(b)$. Then take b_i where i is least, and consider $(b, b_i) \models \mathrm{tp}(b, b_i)$. There must then exist $c, d \in M$ such that $(c, d) \models \mathrm{tp}(b, b_i)$. But note that for $\varphi(X) \in \mathrm{tp}(b)$, we have $(\varphi(x) \wedge y = y) \in \mathrm{tp}(b, b_i)$, hence $\mathcal{M} \models \varphi(c)$, so $c \models \mathrm{tp}(b) = \mathrm{tp}(a_1)$, and $c \equiv_{\mathrm{tp}} a$, so there is a' such that $(c, d) \equiv_{\mathrm{tp}} a_1, a'$.

   Keeping going, using back and forth argument to ensure that $a_n \in \mathrm{dom} f_n$, $b_n \in \mathrm{Im} f_n$, f_n partial elementary. Then take $f = \bigcup f_n$. This is elementary, since any counterexample must be a counterexample on some finite f_n, which was partial elementary. Moreover, it is surjective and injective (check). So isomorphism.

   Indeed, suppose we have $f_n : \{a_1, \dots, a_{2n}\} \to \{b_1, \dots, b_{2n}\}$ partial elementary, such that $\mathrm{tp}(\overline{a}) = \mathrm{tp}(\overline{b})$ and the a_is (respectively b_is) are pairwise distinct.

   Pick $a_{2n+1} \in A$. Then there exists $\overline{c} \in M$ such that $\mathrm{tp}(a_1, \dots, a_{2n+1}) = \mathrm{tp}(c_1, \dots, c_{2n+1})$. Moreover, must have $\mathrm{tp}(a_1, \dots, a_{2n}) = \mathrm{tp}(b_1 \dots, b_{2n}) = \mathrm{tp}(c_1, \dots, c_{2n})$ so by $\aleph_0$ homogeneity can find b_{2n+1} such that $\mathrm{tp}(b_1, \dots, b_{2n+1}) = \mathrm{tp}(c_1, \dots, c_{2n+1}) = \mathrm{tp}(a_1, \dots, a_{2n+1})$. Then send a_{2n+1} to b_{2n+1}. Keep going.

   (c) prime iff countable and atomic, implies countable and $\aleph_0$-homogeneous. So just need to show the type realising thing.

   Let $p \in S_n(T)$. Then if $\mathcal{M}$ realises p, p must be isolated (since $\mathcal{M}$ atomic) and so p is realised in all structures, including $\mathcal{N}$. Similarly the other way. So $\mathcal{M} \cong \mathcal{N}$.

2. If models $\mathcal{M}, \mathcal{N}$ disagree on some $\mathcal{L}$-sentence, φ, then they disagree on some $\mathcal{L}^0$-sentence φ where $\mathcal{L}^0$ finite. They both still model T_k^0, so we need only show T_k^0 complete.

   But this is the theory of DLO plus partition into dense subsets (plus identifying finitely many points in one class), and this is $\aleph_0$-categorical; given two countable models, first identify the c_is. Then each section between some c_is (or on either side) is just a copy of DLO plus partition into dense subsets (without any identified points), and this theory is complete by a back and forth argument.

Then it is clear to see that $I(T, \aleph_0) = k+2$, since we have $k+2$ options for the limit properties of the c_ns. Either they are unbounded above, or bounded but with no limit, or they have a limit $c \in U_i$, where $1 \leq i \leq k$. Any two models with different limit properties can clearly not be isomorphic, and any two models with the same limit properties are isomorphic since after identifying the c_is and the limits (if they exist) we can just do another back-and-forth type argument on the remaining sections of $\mathbb{Q}$.

3. Let $B \subseteq M$, $|B| < \kappa$ and let p be a complete n-type in the language $\mathcal{L}_A$ with respect to the $\mathcal{L}_A$-structure $\mathcal{M}^*$ with parameters from B. Then $p \cup \mathrm{Th}_B(\mathcal{M}^*)$ is consistent, *i.e.* p is a collection of $\mathcal{L}_{A\cup B}$ sentences consistent with all $\mathcal{L}_{A\cup B}$ sentences modelled by $\mathcal{M}$. In particular, $p \cup \mathrm{Th}_{A\cup B}(\mathcal{M})$ is consistent, so $p \in S_n^{\mathcal{M}}(A \cup B)$, and $|A \cup B| < \kappa$ (as long as $\kappa \geq \aleph_0$) so this type is realised by κ-saturation.

4. Similar to 1(b). Let $\kappa = |M| = |N|$, $M = \{a_\alpha : \alpha < \kappa\}$ and $\mathcal{N} = \{b_\alpha : \alpha < \kappa\}$ and define nested $f_\alpha : \alpha < \kappa$ such that $a_\alpha \in \mathrm{dom} f_\alpha$, $b_\alpha \in \mathrm{Im} f_\alpha$.

   Indeed, given $f_\alpha : \alpha < \gamma < \kappa$ a limit, define $f_\gamma = \bigcup f_\alpha$, whose domain still has cardinality less than κ because κ is uncountable and initial.

   Given f_α, write $A_\alpha = \mathrm{dom} f_\alpha$, $B_\alpha = \mathrm{Im} f_\alpha$ and consider $p = \mathrm{tp}^{\mathcal{M}}(a_\alpha / A_\alpha)$. Let q be the set of formulae obtained by applying f to every parameter in the formulae in p. This gives a set of formulae with parameters from B_α, and is complete and consistent since f_α is partial elementary. Then by saturation, and the fact that $|B_\alpha| < \kappa$, q is realised by some b. So (a_α, b) is our new point.

   Same again backwards, pick b_α and consider $\mathrm{tp}^{\mathcal{N}}(b_\alpha / B_\alpha \cup \{b\})$. This is a set of formulae with parameters in the range of $f_\alpha \cup \{(a_\alpha, b)\}$, so undo the function on the formulae to obtain a new consistent type, which is thus realised in $\mathcal{M}$.

   Repeat, then union all the f_αs together.

5.