Local Fields: Sheet 1

Otto Pyper

1. (a) We show firstly that the addition and multiplication operations are well defined. Let $(a_n) \sim (x_n)$ and $(b_n) \sim (y_n)$. Then $(a_n) + (b_n) = (a_n + b_n) \sim (x_n + y_n)$ since $a_n - x_n + b_n - y_n \to 0$. Similarly, $(a_n - x_n) \cdot b_n + x_n(b_n - y_n) \to 0$, so $a_n b_n - x_n y_n \to 0$ and $(x_n y_n) \sim (a_n b_n)$.

The equivalence class $[(1)_{n=1}^{\infty}]$ is then clearly a multiplicative identity, $[(0)]$ the additive identity, and for non-zero $[(x_n)]$ only finitely many $x_i = 0$, so wlog we choose representatives with no zero terms. $[(x_n^{-1})]$ is then a multiplicative inverse, and since $a_n^{-1} x_n^{-1}(x_n - a_n) \to 0$ (else one of a_n, x_n vanishes in the limit, so $\sim [(0)]$ $\perp$) we have that $(a_n^{-1}) \sim (x_n^{-1})$. Likewise for additive inverses. Distributive properties follow similarly.

(b) The reverse triangle inequality says that $||x_m| - |x_n|| \leq |x_m - x_n|$, and (x_m) is Cauchy so the right hand side vanishes for $m, n \to \infty$. Hence $|x_m|$ is Cauchy also, and since it has a real sequence it thus converges to some limit.

We can then define $|(x_n)| = \lim_{n\to\infty} |x_n|$. This is an absolute value since $|(x_n)| = 0 \iff \lim |x_n| = 0 \iff x_n \to 0$ since $|\cdot|$ is an absolute value on K. But the last condition is equivalent to $(x_n) \sim (0)$.

$|(x_n)(y_n)| = |(x_n y_n)| = \lim |x_n y_n| = \lim |x_n||y_n| = \lim |x_n| \lim |y_n| = |(x_n)||(y_n)|$ as $|x_n|, |y_n|$ are real sequences.

The triangle inequality also holds: $|(x_n) + (y_n)| = \lim |x_n + y_n| \leq \lim(|x_n| + |y_n|) = |(x_n)| + |(y_n)|$.

(c) The discretely valued case is fairly trivial, but we can do both in general. Let $y \in \mathbb{R}\backslash v(K^\times)$, and suppose that $(v(x_n))_{n=1}^{\infty}$ converges to y. We show that (x_n) cannot be Cauchy.

currently some significant confusion over the definition of a valuation/valued field - our definition of valuation appears to always be non-archimedean, so need this be the case for $|\cdot|$ on K? Also a potential typo with the direction of the inequality...

We first remark that for $a, b \in K$, if $v(a) > v(b)$ then $v(b) \geq \min(v(b-a), v(a)) \implies v(a) > v(b) \geq v(b-a)$, and similarly if $v(b) > v(a)$ then $v(a) \geq v(a-b) = v(b-a)$. Hence $v(a-b) \leq \min(v(a), v(b))$ for $a \neq b$.

Now note that $v(x_n)$ is not eventually constant, else we have $v(x_N) = y$ for large enough N. So we can find infinitely many $m_i > n_i$, with $n_i \to \infty$, such that $v(x_{m_i} - x_{n_i}) \leq \min(v(x_{m_i}), v(y_{n_i})) \leq y + \varepsilon_i \leq y + 1$ for large enough i. But then $|x_{m_i} - x_{n_i}| \geq \alpha^{y+1} > 0$ for infinitely many i. Hence (x_n) is not Cauchy.

I think this works, but I can't claim to have a good intuition for why...

2. Write $x = \prod_{i=0}^{k} p_i^{e_i}$ with $e_i \in \mathbb{Z}$, $p_0 = -1$. Then for $i > 0$, $|x|_{p_i} = p_i^{-e_i}$, and for $\alpha \notin \{\infty, p_1, p_2, \ldots, p_k\}$ we have $|x|_\alpha = 1$. Thus

$$\begin{aligned}
\prod_{\alpha} |x|_\alpha &= |x|_\infty \prod_{i=1}^{k} |x|_{p_i} \\
&= \left(\prod_{i=1}^{k} p_i^{e_i}\right)\left(\prod_{i=1}^{k} p_i^{-e_i}\right) \\
&= 1
\end{aligned}$$

3. Let $S_n = \sum_{i=0}^{n} (-c)^i$. Then $(1+c)S_n = 1 + (-c)^{n+1}$, so $(1+c)S_n - 1 = (-c)^{n+1} \to 0$ since $|c|_p < 1$. Hence $\lim S_n = 1 - c + c^2 - c^3 + \cdots = (1+c)^{-1}$.

As above we have $|(1+c)S_n - 1|_p = |c|^{n+1}$, so we might desire that $|c| = 5^{-1}$ and $n \geq 9$, such that $4a = (1+c)S_n = 1 + (-c)^{n+1}$ has a solution. This requires that n is even and $c \equiv 1 \mod 4$, so we take $c = 5$ and $n = 10$. Then $a = (1 - 5^{11})/4$ will suffice.

not my solution that follows, but probably the inteded one:

We can say $4^{-1} = -1 - 5 - 5^2 - \dots$¡ so let $a = -1 - 5 - \dots - 5^9$, and write $-1/4 = a + x$ with $|x|_5 = 5^{-10}$. Then $|4a - 1|_5 = |4x|_5 = 5^{-10}$.

4. (a) $v_a((t-a)^m f_1/g_1 \cdot (t-a)^n f_2/g_2) = v_a((t-a)^{m+n} f_1 f_2/g_1 g_2) = m + n$ since a is not a root of any f_i or g_i. For the (equivalent) of the ultrametric inequality, wlog $m \geq n$. Then

$$\begin{aligned} v_a\left((t-a)^m \frac{f_1}{g_1} + (t-a)^n \frac{f_2}{g_2}\right) &= v_a\left((t-a)^n \left[(t-a)^{m-n}\frac{f_1}{g_1} + \frac{f_2}{g_2}\right]\right) \\ &= n + v_a\left(\frac{g_1 f_2 + f_1 g_2 (t-a)^{m-n}}{g_1 g_2}\right) \end{aligned}$$

Now the denominator does not have a as a root, so $v_a(\cdot) \geq 0$, so done.

$v_\infty(f_1/g_1 \cdot f_2/g_2) = v_\infty(f_1 f_2/g_1 g_2) = \deg g_1 + \deg g_2 - \deg f_1 - \deg f_2$. Again, similarly to the above:

$$\begin{aligned} v_\infty\left(\frac{f_1}{g_1} + \frac{f_2}{g_2}\right) &= v_\infty\left(\frac{f_1 g_2 + g_1 f_2}{g_1 g_2}\right) \\ &\geq \deg g_1 + \deg g_2 - \max(\deg f_1 g_2, \deg g_1 f_2) \\ &= \min(\deg g_1 - \deg f_1, \deg g_2 - \deg f_2) \end{aligned}$$

(b) The existence of multiplicative inverses are the only non-trivial part of showing $k((t))$ is a field. Suppose we have $f(t) = \sum_{i=n}^\infty a_i t^i$, with $a_n \neq 0$. Then $t^{-n} a_n^{-1} f(t) = 1 + g(t)$, where $g(t) = \sum_{i=1}^\infty b_i t^i$ with $b_i = a_{i+n}/a_n$. We invert this in the expected way:

$$\begin{aligned} (1+g)^{-1} &= 1 - g + g^2 - \dots \\ &= \sum_{j=0}^{\infty} c_j t^j \end{aligned}$$

for some appropriately complicated c_j. The important thing to note is that each c_j is a well-defined, finite sum of combinations of $b_i : i \leq j$, since g^m contributes no terms when $m > j$. This definition genuinely is the inverse of $1+g$; $(1+g)(1-g+g^2-\dots+(-g)^m) = 1 + (-g)^{m+1}$, so given $m > 0$, the coefficient of t^m in $(1+g)(1-g+g^2-\dots)$, which is the same as the coefficient of t^m in $1+(-g)^{m+1}$, is zero, hence the formal Laurent series for $(1+g)(1-g+g^2-\dots)$ is simply 1.

We then have that $a_n t^n f^{-1} = \sum_{j=0}^\infty c_j t^j$, so $f^{-1} = \sum_{j=0}^\infty c_j a_n^{-1} t^{j-n} = \sum_{i=-n}^\infty c_{i+n} a_n^{-1} t^i \in k((t))$. So $k((t))$ is a field.

Moreover, this contains $k(t)$ in the sense that there is a field homomorphism $k(t) \hookrightarrow k((t))$ given by $f/g \mapsto fg^{-1}$, where $f, g \in k[t] \subset k((t))$. This preserves the slightly different definitions of the t-adic valuation in these fields, since if $f(0), g(0) \neq 0$ then $f(0)g(0)^{-1}$ is both defined and non-zero, so the Laurent series for fg^{-1} begins with the t^0 term.

(c) Let C be the set of all Cauchy sequences in $k(t)$, and define $K = C/\sim$ as in 1. It is now convenient to view $k(t)$ as a subfield of $k((t))$ in the above way.

Suppose that (f_n) is Cauchy, and define (f_n^m) to be the sequence in k of the coefficient of t^m in f_n. Claim (f_n^m) is eventually constant for each $m \in \mathbb{Z}$: suppose not. Then we find an increasing sequence of $n_i \in \mathbb{N}$ such that $f_{n_{i+1}}^m \neq f_{n_i}^m$. Then for all i, $v(f_{n_{i+1}} - f_{n_i}) \leq m$, which cannot be the case since (f_n) is Cauchy.

Moreover, these coefficients must be the same for $(f_n) \sim (g_n)$, otherwise $v(f_n - g_n)$ is bounded. We also note that for all $m \in \mathbb{Z}$ there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies f_n and f_N agree on all coefficients of terms $t^a : a \leq m$, and since $f_N \in k((t))$ this means that there is a least $m \in \mathbb{Z}$ such that $\lim f_n^m \neq 0$.

This allows us to identify equivalence classes $[(f_n)]$ with elements $\sum_{i=m}^{\infty}(\lim f_n^i)t^i$ of $k((t))$. All that remains to be shown is that this identification (henceforth φ) is a surjective field homomorphism.

We first show surjectivity. Given $f = \sum_{i=n}^{\infty} a_i t^i$, the sequence $f_m = \sum_{i=n}^{m} a_i t^i$ is clearly Cauchy with $\varphi([(f_n)]) = f$.

φ preserves addition, since $\varphi((f_n)+(g_n)) = \varphi((f_n+g_n)) = \sum_{i=-\infty}^{\infty}(\lim(f_n^i+g_n^i))t^i = \sum_{i=-\infty}^{\infty}(\lim f_n^i)t^i + \sum_{i=-\infty}^{\infty}(\lim g_n^i)t^i = \varphi((f_n)) + \varphi((g_n))$.

To see that φ preserves multiplication, write $f = \varphi((f_n))$, $g = \varphi((g_n))$ and consider only the coefficient of t^m in $\varphi((f_n) \cdot (g_n)), m \in \mathbb{Z}$. Choose $N \in \mathbb{N}$ such that for all $n \geq N$, both $(f_n), (g_n)$ are constant in all coefficients $\ell \leq m - \min(v(f), v(g))$, *i.e.* constant in all coefficients that determine the coefficient of t^m in the product. Then for all $M \geq N$, the products $f_M g_M$ and $fg = \varphi((f_n))\varphi((g_n))$ agree on the coefficient of t^m, which must then be constant for all such M. Hence $\varphi((f_n)(g_n))$ and fg agree on all coefficients $t^m : m \in \mathbb{Z}$, and so they are equal.

Hence φ is a field isomorphism, and so $k((t))$ is indeed the completion of $k(t)$ under v_0.

5. (a) We first show that $\mathbb{Z}[1/p]$ is dense in $\mathbb{Q}$; let $x = p^n a/b$. $b \not\equiv 0 \mod p$, so $\exists c \in \mathbb{Z}$ such that $x = p^n ac/(bc)$ with $bc \equiv 1 \mod p$. Write $bc = 1 + m$, with $|m|_p < 1$. We can then expand $1/(bc)$ as in 3. to obtain $x = p^n ac(1 - m + m^2 - \dots)$. The sequence $x_k = p^n ac(1 - m + \cdots + (-m)^k) \in \mathbb{Z}[1/p]$ then converges to x.

So $\mathbb{Z}[1/p]$ is dense in $\mathbb{Q}$, which is in turn dense in $\mathbb{Q}_p$. So the closure of $\mathbb{Z}[1/p]$ contains $\mathbb{Q}$ and is closed, hence it contains the closure of $\mathbb{Q}$, which is $\mathbb{Q}_p$. So the closure of $\mathbb{Z}[1/p]$ is $\mathbb{Q}_p$, and thus $\mathbb{Z}[1/p]$ is dense in $\mathbb{Q}_p$.

(b)