Ramsey Theory

Lectures by Imre Leader

0 Introduction

Ramsey Theory is all about the following question:

Can we find some order in (enough) disorder?

In a sense, the entire course is about answering this question in different settings.

Chapter 1: Monochromatic systems. Abstract and concrete.

Chapter 2: Partition Regular Equations. More concrete; looking at the naturals with addition and multiplication, and asking about order/disorder there.

Chapter 3: Infinite Ramsey Theory. Abstract; taken to a limit (countable only though).

Prerequisites: None (other than some basic concepts of topology *e.g.* open/closed/compact sets).

Literature: In theory, this course is self-contained. But if you would like a different viewpoint/some reinforcement, consider:

- Bollobas, “Combinatorics”, C.U.P. 1986 (For Ch. 3)
- Graham, Rothschild, Spencer, “Ramsey Theory”, Wiley 1990 (For Ch. 1&2)

Example Sheets: this a 16 lecture course, so there are 3 sheets and 3 classes.

1 Chapter 1

1.1 Monochromatic Systems

This is the introductory chapter from which everything else will flow.

In this course, we take $\mathbb{N} = \{1, 2, 3, \dots\}$, and write $[n] = \{1, 2, \dots, n\}$. For a set X and $r \in \mathbb{N}$, we write $X^{(r)} = \{A \subseteq X : |A| = r\}$; this is the collection of all r-sets in X.

Queston: Suppose that we are given a 2-colouring of $\mathbb{N}^{(2)}$, *i.e.* a $c : \mathbb{N}^{(2)} \to \{1, 2\}$. Can we find an infinite subset $M \subseteq \mathbb{N}$ such that M is monochrome, *i.e.* c is constant on $M^{(2)}$.

Let's try some examples to get a feel for it.

Examples:

1. Colour ij red if $i + j$ is even, and blue if $i + j$ is odd.

   Here we can do it, rather easily - take $M = \{2, 4, 6, 8, \dots\}$, or any subset thereof, or all the odd numbers, *etc.*...

2. Colour ij red if $\max\{n : 2^n | i + j\}$ even, and blue otherwise.

   This is also a yes, *e.g.* $M = \{4^0, 4^1, 4^2, \dots\}$.

3. Colour ij red if $i + j$ has an even number of (distinct) prime factors, and blue otherwise.

   This is also a yes, but it is not obvious how...

Theorem 1.1: (Ramsey's Theorem) *Let c be a 2-colouring of $\mathbb{N}^{(2)}$. Then there exists an infinite monochromatic $M \subseteq \mathbb{N}$.*

Proof. Pick any $a_1 \in \mathbb{N}$. There are infinitely many edges out of a_1, so infinitely many have the same colour; say all edges from a_1 to infinite set B_1 have colour c_1.

Now within B_2, we take a point a_2 and find an finite $B_2 \subseteq B_1 \backslash \{a_2\}$ such that all edges from a_2 to B_2 are the same colour - these may be red or blue.

Repeat this process again, within B_3, and repeat ad infinitum.

This gives us an infinite sequence $(a_i)_{i=1}^{\infty}$ of points, and infinite sequence of colours c_i such that edge a_i to a_j with $i < j$ has colour c_i.

Take a constant subsequence of c_i, say $c_i : i \in I$. Then $M = \{a_i : i \in I\}$ is monochromatic. $\square$

Remarks:

1. This is sometimes called a '2-pass' proof, because we had to do the whole induction, and then go over it again to finish it off.

2. In example 3, no such example is known!

3. The same proof shows that whenever $\mathbb{N}^{(2)}$ is k-coloured (*i.e.* $c : \mathbb{N}^{(2)} \to [k]$), there still exists an infinite monochromatic set.

   Alternatively, we could deduce this from Theorem 1 plus induction. We could view the colours as '1' and '2 or 3 or ... or k' and apply Theorem 1; if we get an infinite 1-set, then done, and if we get an infinite '2 or ... or k'-set then done by induction on k.

4. An infinite monochromatic set is ***more*** than having arbitrarily large finite monochromatic sets, *e.g.* take disjoint blue K_2, K_3, K_4, and so on, then connect everything remaining with red edges.

While this (of course) does not contradict Ramsey's Theorem, we clearly have arbitrarily large blue sets (the K_ns), but there is no *infinite* blue set.

Example: Any sequence $x_1, x_2, \dots$ in $\mathbb{R}$ (or any totally ordered set) has a monotone subsequence.

Indeed, 2-colour $\mathbb{N}^{(2)}$ by giving ij $(i < j)$ colour *up* if $x_i < x_j$, and *down* if $x_i > x_j$. Then apply Theorem 1.

Lecture 2

What if we coloured $\mathbb{N}^{(r)}$, say for $r = 3, 4, \dots$. Given a 2-colouring $c : \mathbb{N}^{(r)} \to \{1, 2\}$, must there be an infinite monochromatic set?

For instance, $r = 3$: colour ijk $(i < j < k)$ red if $i | j + k$, and blue otherwise. Here we can do this easily - just take $M = \{2^0, 2^1, 2^2, \dots\}$.

Theorem 1.2: (Ramsey for r-sets) *Let $r \in \mathbb{N}$. Then whenever $\mathbb{N}^{(r)}$ is 2-coloured, there exists an infinite monochromatic set.*

Proof. In the previous proof, when we picked a_1 and looked at the lines out of it to other points, we in fact used the $r = 1$ case on the colouring induced on the singletons in the neighbourhood of a_1. Armed with these ideas, the proof here will act in exactly the same way.

$r = 1$ is trivial (just infinite pigeonhole), or if you prefer $r = 2$ is Theorem 1. We apply induction on r. Thus suppose the result holds for $r - 1$. Given $c : \mathbb{N}^{(r)} \to \{1, 2\}$, we look at the induced colouring.

Pick $a_1 \in \mathbb{N}$, and look at $(\mathbb{N} \backslash \{a_1\})^{(r-1)}$. This has an induced colouring d given by $d(F) = c(F \cup \{a_1\})$. Now by induction there is an infinite $B_1 \subseteq \mathbb{N} \backslash \{a_1\}$ such that all the $r - 1$-sets have the same colour according to d, *i.e.* $c(F \cup \{a_1\}) = c_1$ for all $F \subseteq B^{(r-1)}$, for some colour $c_1 \in [r]$.

Repeating, we have $a_1 \in B_1$, and infinite $B_2 \subseteq B_1 \backslash \{a_2\}$ such that all $F \cup \{a_2\}$, $F \in B_2^{(r-1)}$, have the same colour c_2.

We keep going to infinity, giving us an infinite sequence of distinct points $a_1, a_2, \dots$ and colours $c_1, c_2, \dots$ such that $c(a_{i_1}, a_{i_2}, \dots, a_{i_r}) = c_{i_1}$ $(i_1 < i_2 < \cdots < i_r)$. There is then an infinite index set I such that $c_i : i \in I$ is constant, and so $M = \{a_i : i \in I\}$ is monochrome. $\square$

Example: We saw from Theorem 1 that given points $(1, x_1), (2, x_2), \dots$ in $\mathbb{R}^2$, there exists a subsequence such that the induced (piecewise-linear) function is monotone.

Functions can have other properties that are stricter; for instance, we could ask for the function to be convex/concave - in fact we can ensure that this is the case.

On the surface, this seems really hard - but not for us, with r-set Ramsey.

Indeed, just 2-colour $\mathbb{N}^{(3)}$ by giving ijk colour *convex* if they form a convex shape, and *concave* (otherwise) if they from a concave shape (any three points must fall into one of the two categories).

We get an infinite monochrome subsequence; the induced function is either convex or concave for any of the three points; and so the overall function is convex/concave.

How long does the proof take in the $r = 3$ case? Well, to find each a_i we need to do an infinite two-pass proof (of the $r = 2$ case). So this happens an infinite number of times, and then theres another pass at the end. Essentially, it takes a very long time.

Surprisingly, the infinite version of Ramsey *implies* the finite version.

Theorem 1.3: (Finite Ramsey) *For all m, r there exists an n such that whenever $[n]^{(r)}$ is 2-coloured, there exists a monochromatic m-set.*

Proof. Suppose not. We will show that there ia a 2-colouring of $\mathbb{N}^{(r)}$ without a monochromatic m-set, (massively) contradicting Theorem 2.

For each $n \geq r$, have a 2-colouring c_n of $[n]^{(r)}$ with no monochromatic m-set. [We want to take their union to get a bad colouring of the whole of $\mathbb{N}^{(r)}$, but the problem is that the colours aren't necessarily *nested*, *i.e.* any two agree where both defined.]

There are only finitely many ways to 2-colour $[r]^{(r)}$ (two, in fact), so infinitely many of the c_n agree on $[r]^{(r)}$: say $c_n \upharpoonright [r]^{(r)} = d_r$ for all $n \in A_1$, some $d_n : [r]^{(r)} : \{1, 2\}$.

There are only finitely many ways to 2-colour $[r+1]^{(r)}$, so infintely many of the $c_n : n \in A_1$ agree on $[r+1]^{r+1}$: say $c_n \upharpoonright [r+1]^{(r)} = d_{r+1}$ for all $n \in A_2$, for some $d_{r+1} : [r+1]^{(r)} \to \{1, 2\}$.

Continue inductively. We obtain colourings $d_n : [n]^{(r)} \to \{1, 2\}$ for each $n \geq r$ such that

1) d_n has no monochromatic m-set, since $d_n = c_n' \upharpoonright [n]^{(r)}$ for some $n' \geq n$
2) $d_{n'} \upharpoonright [n]^{(r)} = d_n$ for all $n' \geq n$ by construction of the d_ns

Now put $c(F) = d_n(F)$, any $n \geq \max F$ (for each r-set F). We can say *any*, because all the colourings agree.

This is well-defined (by 2)), and has no monochromatic m-set (by 1)). Massive contradiction. □

Remarks: This is called a 'compactness' argument, similar to the proof of Bolzano-Weierstrass in IA Analysis. What we are essentially doing is showing that if the space $\{1, 2\}^{\mathbb{N}}$ of 2-colourings, with the product topology (*i.e.* induced from the metric $d(f, g) = 1/(\min\{n : f(n) \neq g(n)\})$) is (sequentially) compact.

<u>Note</u>: This proof also gives *no* information on how large $n = n(m, r)$ can be (such proofs do exists though, *c.f.* IID Graph Theory).

What if we colour $\mathbb{N}^{(2)}$ with infinitely many colours, *i.e.* have $c : \mathbb{N}^{(2)} \to X$, some set X?

Of course, we do not get an infinite monochromatic set, since *e.g.* we can just colour each edge with a unique colour. But we can ask a slightly different question...

Do we always get an infinte ste m such that $c \upharpoonright m^{(2)}$ is either constant or injective?

Sadly, the answer is still no. We can achieve this by colouring ij $(i < j)$ with colour i. This clearly is neither injective nor constant on any infinite subset.

Another option here is that we can colour ij with colour j $(i < j)$ - then each colour class is finite, instead of infinite. As it happens, at least one of these four situations must arise:

Theorem 1.4: *Let c be a colouring of $\mathbb{N}^{(2)}$ with an arbitrary set of colours. Then there exists an infinite $M \subseteq \mathbb{N}$ such that one of the following holds:*

i) *c is constant on $M^{(2)}$*

ii) *c is injective on $M^{(2)}$*

iii) $\forall ij, kl \in M^{(2)}$, $c(ij) = c(kl) \iff i = k$

iv) $\forall ij, kl \in M^{(2)}$, $c(ij) = c(kl) \iff j = l$

[It is worth remarking that this trivially implies Theorem 1: if there are only finitely many colours, then cases ii), iii), iv) cannot happen.]

Proof. There is a superbly nice idea here. The act of comparing pairs of edges is the same as asking a question about the 4-set of the vertices, and we can use Ramsey's Theorem for $r = 4$ to help us here.

Define a 2-colouring of $\mathbb{N}^{(4)}$ by giving $ijkl$ colour *same* if $c(ij) = c(kl)$, and *dif* if $c(ij) \neq c(kl)$. Note that, as ever, the notation implies $i < j < k < l$.

By Ramsey for 4-sets, ther exists an infinite monochromatic M_1 for this colouring. If the colour is *same*, then we have case i). Indeed, given $ij, kl \in M_1^{(2)}$, choose $m < n$ in M_1 with $m > i, j, k, l$. Then $c(ij) = c(mn)$, and $c(kl) = c(mn)$ (this deals with the case $j = k$, and other anomalies). So we may otherwise assume that M_1 has colour diff.

Now, 2-colour $M_1^{(4)}$ by giving $ijkl$ colour *same* if $c(jk) = c(il)$, and *diff* if not.

Again by *Ramsey*-4, we have infinite $M_2 \subseteq \mathcal{M}_1$ monochromatic for this colouring. Note that we cannot have M_2 be the colour *same*, as otherwise pick $i < j < k < l < m < n \in M_2$. We would then have $c(jk) = c(in) = c(lm)$, contradicting $j, k, l, m \in M_1$. Thus M_2 is colour *diff*.

The final type of edge pairs in a 4-set could be interlocking:

2-colour $M_2^{(4)}$ by giving $ijkl$ colour *same* if $c(ik) = c(jl)$, and *diff* if not. Again we obtain an infinite monochromatic $M_3 \subseteq M_2$. Once again, we cannot have M_3 colour *same*, else we choose $i < j < k < l < m < n \in M_3$, and we have $c(im) = c(kn) = c(jl)$, contradicing the fact that $i, j, l, m \in M_2$. Thus M_3 is colour *diff*.

So now we know that any two edges in M_3 have different colours if they are not adjacent; we now deal with the adjacent case.

2-colour $M_3^{(3)}$ by giving ijk colour *same* if $c(ij) = c(jk)$, and *diff* if not. Have an infinite monochromatic $M_4 \subseteq M_3$. We cannot have M_4 colour *same*, else pick $i < j < k < l \in M_4$, and we have $c(ij) = c(jk) = c(kl)$, contradicting the above. So M_4 is colour *diff*.

There are other ways that edge pairs can be adjacent, and these account for the various cases in the Theorem.

2-colour $M_4^{(3)}$ by giving ijk colour *left-same* if $c(ij) = c(ik)$, and *left-diff* if not. We obtain infinite monochromatic $M_5 \subseteq M_4$.

Finally, 2-colour $M_5^{(3)}$ by giving ijk colour *right-same* if $c(jk) = c(ik)$, and *right-diff* otherwise.

We obtain infinite monochromatic $M_6 \subseteq M_5$.

If M_6 is *left-diff* and *right-diff*: All pairs of edges have a different colour; this is case 2.

If M_6 is *left-same* and *right-diff*: If edges meet at the left they are the same; at the right they are different; this is case 3.

If M_6 is *left-diff* and *right-same*: Similarly, this is case 4.

If M_6 is *left-same* and *right-same*: Pick $i < j < k \in M_6$. Then $c(ij) = c(ik) = c(jk)$, contradiction. $\square$

Remarks:

1. We could use just one colouring, by colouring a 4-set $ijkl$ with the partition of $[4]^{(2)}$ induced by c on $\{i, j, k, l\}$. The number of colours is then the number such partitions, which is $\binom{4}{2} = 6$. We didn't do this because it seems a bit magical and out of the blue, and slightly obscures what's going on. There are also lots of symbols, making it a bit unpleasant.

2. Similarly, if $c : \mathbb{N}^{(r)} \to X$ is an arbitrary colouring, we get an infinite $M \subseteq \mathbb{N}$ and a set $I \subseteq [r]$ such that $\forall x_1, \ldots, x_r, y_1, \ldots, y_r \in M^{(r)}$ we have $c(x_1, \ldots, x_r) = c(y_1, \ldots, y_r) \iff x_i = y_i \ \forall i \in I$, where I is an index set. These 2^r coloursing are called the ***canonical*** colourings of the r-sets.

   For instance, $= 2$:

   - Case i) corresponds to $I = \emptyset$
   - Case ii) corersponds to $I = \{1, 2\}$
   - Case iii) corresponds to $I = \{1\}$
   - Case iv) corresponds to $I = \{2\}$

1.2 Van der Waerden's Theorem

If we 2-colour $\mathbb{N}$, can we find 3 consecutive poitns of the same colour?

Answer: of course not; just colour $\mathbb{N}$ alternately.

What about 3 equally spaced points, *i.e.* $(a, a+d, a+2d)$? This is not obviously false. If it were true though, that would be nice - we would have found some order amongst the disorder. Could we find more, perhaps 4 points, or even m?

Aim: For every m, whenever $\mathbb{N}$ is 2-coloured there exists a monochromatic artihmetic progression of length m.

Just to be clear, by length we mean the number of terms *i.e.* a sequence $\{a, a+2, \dots, a+(m-1)d\}$ has length m.

This is Van der Waerden's Theorem, and it is very hard to solve.

By our usual compactness argument, this is the same as:

Aim': $(\forall m)(\exists N)$ such that whenever $[n]$ is 2-coloured, there exists a monochromatic AP of length m.

Indeed, if this is false then there is an m such that for every $n \geq m$ there is a colouring c_n of $[n]$ with no monochromatic AP of length m. We want to combine these into one big colouring of $\mathbb{N}$, but we can't yet.

But infinitely many c_n agree on $[m]$, and, of those, infinitely many agree on $[m+1]$, and so on.... Put together those (nested) restrictions to obtain a 2-colouring of $\mathbb{N}$ with no monochromatic AP of length m. Contradiction.

In proving this (Aim'), one key idea is to generalise: we in fact show that $\forall m, k\ \exists n$ such that whenever $[n]$ is k-coloured there exists a monochromatic AP of length m.

Note: proving a *stronger* result might be *easier*, *e.g.* in a proof by induction.

Another key idea: given APs $A_1, \dots, A_r$, each of length m - so $A_i = \{a_i, a_i + d_i, \dots, a_i + (m-1)d_i\}$ - we say they are ***focused*** at f if $a_i + md_i = f$ for each i. *E.g.* $\{1, 4\}$ and $\{5, 6\}$ are focused at 7.

If in addition each A_i is monochromatic (for a given colouring), with no two the same colour, we say they are ***colour-focused***. Why do we care?

In a k-colouring, if we have APs $A_1, \dots, A_k$, each of length $m-1$, that are colour-focused, then we actually have a monochromatic AP of length m, by asking "what colour is the focus?"

Write $W(m, k)$ for the least n (if it exists) such that whenever n is k-coloured, there exists a monochromatic AP of length m.

Proposition 1.5: $\forall k, \exists n$ *such that whenever* $[n]$ *is* k*-coloured there exists a monochromatic AP of length* 3.

Note: This will be contained in Theorem 6; it is included here for clarity.

Proof. Claim: $\forall r \leq k, \exists n$ such that whenever $[n]$ is k-coloured, we have either:

- a monochromatic AP of length 3, or
- r colour-focused APs of length 2

Given this: put $r = k$ and look at the focus. Whatever colour it is, we get a monochromatic AP of length 3.

Proof of Claim: Induction on r. $r = 1$ is trivial (can take $n = k+1$).

Given n suitable for $r-1$, we'll show that $(k^{2n}+1) \cdot 2n$ is suitable for r. So given a k-colouring of $[(k^{2n}+1)2n]$, with no mono AP of length 3:

Break up $[(k^{2n}+1)2k]$ into blocks of length $2n$, say $B_1, B_2, \dots, B_{k^{2n}+1}$, where $B_i = [(i-1)2n+1, i2n]$ (square brackets denote interval with endpoints).

Inside any interval of length $2n$, we have $r-1$ colour-focused APs of length 2 (by choice of n), together with their focus (as length $= 2n$).

Now, the number of ways to k-colour a block is k^{2n}, and since we have $k^{2n}+1$ it must be the case that some two are identically coloured, say B_s and B_{s+t}. Inside B_s, we have $r-1$ colour-focused APs of length 2, say $\{a_1, a_1+d_1\}, \dots, \{a_{r-1}, a_{r-1}+d_{r-1}\}$ focused at f.

But now the APs $\{a_1, a_1+d_1+2nt\}, \dots, \{a_{r-1}, a_{r-1}+d_{r-1}+2nt\}$ are colour-focused at $f+4nt$, and $\{f, f+2nt\}$ is also focused there, giving r colour-focused APs of length 2. So we have finished the induction, so proved the claim, so finished the proof. □

Remarks:

1. The idea of looking at the number of ways to colour a block is called a ***product argument***.
2. The proof shows that
$$W(3,k) \leq \left. k^{k^{\cdot^{\cdot^{k^{4k}}}}} \right\} k$$
So *e.g.* $W(3,3) \leq 3^{3^{12}}$. This is a 'tower-type' bound.

We are now better-equipped to tackle the full theorem:

Theorem 1.6: (Van der Waerden's Theorem) *$\forall m, k, \exists n$ such that whenever $[n]$ is k-coloured there exists a monochromatic AP of length m.*

Proof. We induct on m. $m = 1$ is trival (or $m = 2$ is pigeonhole, or $m = 3$ is Prop 1.5).

So we may assume that that $W(m-1, k)$ exists for every k.

Claim: for every $r \leq k$, there exists n such that whenever $[n]$ is k-coloured, we have either:

- a monochromatic AP of length m, or
- r colour-focused APs of length $m-1$

[Given this, put $r = k$ and look at the focus to get a mono AP of length m.]

Proof of Claim: Induction on r: $r = 1$ (take $n = W(m-1,k)$). Given n suitable for $r-1$, we'll show that $W(m-1, k^{2n}) \cdot 2n$ is suitable for r.

So, we are given a k-colouring of $[W(k-1, k^{2n})2n]$, with no mono AP of length m:

Break up $[W(m-1, k^{2n})]$ into blocks of length $2n$, say $B_1, B_2, \dots, B_{W(m-1,k^{2n})}$, where $B_i = [(i-1)2n+1, i2n]$.

As in the proof before, we need three equally spaced identical blocks - but this is much harder to get.

The number of ways to k-colour a block is k^{2n}, so since we have $W(m-1, k^{2n})$ *blocks*, we must have (by definition of $W(m-1, k^{2n})$) some $m-1$ equally spaced blockes that are coloured identically. Say $B_s, B_{s+t}, \dots, B_{s+(m-2)t}$.

Now, inside B_s we have $r-1$ colour-focused APs of length $m-1$ (by definition of n), together with their focus (as length $= 2n$): say $A_1, \dots, A_{r-1}$ focused at f, where A_i has first term a_i and common difference d_i. Then the APs $A_1', \dots, A_{r-1}'$, where A_i' has first term a_i and common difference $d_i + 2nt$ are colour-focused at $f + (m-1)2n$, and also $\{f, f+2nt, f+2(2nt), \dots, f+(m-2)2nt$ is monochromatic of a different colour to the A_i's. This gives r colour-focused APs of length $m-1$.

This completes the induction, which proves the claim, which completes the outer induction on m and hence concludes the proof. □

Lecture 5 Note: for these proofs, focus on the pictures (even though I haven't drawn them here)!

The Ackermann/Grzegorczyk Hierarchy

Definition: The ***Ackermann*** or ***Grzegorcyzk Hierarchy*** is the sequence of funcgtions $f_1, f_2, \dots$ (each $\mathbb{N} \to \mathbb{N}$) given by:

- $f_1(x) = 2x$
- $f_{n+1}(x) = f_n^{(x)}(1) = \underbrace{f_n(f_n(\dots f_n}_{x \text{ times}}(1)\dots))$

Let's explore these a bit.

- $f_2(x) = 2^x$
- $f_3(x) = \left.2^{2^{\cdot^{\cdot^{2}}}}\right\} x$
- $f_4(1) = 2,\ f_4(2) = 2^2 = 4,\ f_4(3) = 2^{2^{2^2}} = 65536,\ f_4(4) = \left.2^{2^{\cdot^{\cdot^{2}}}}\right\} 65536$ *etc...*

We say $f : \mathbb{N} \to \mathbb{N}$ is of ***type*** n if there exist $c, d > 0$ such that $f_n(cx) \le f(x) \le f_n(dx)$ for all x. So our upper bound for $W(3,k)$ was a function of k of type 3 (note that even though it is a tower of ks, not 2s, the height of the tower is far more significant). Our bound on $W(m,k)$ (m fixed) is of type m.

In fact, our bound on $W(m) = W(m,2)$, as a function of m, grows faster than every f_n! [This means that $W(m)$ is *not* primitive recursive.] This is often a feature of such 'double inductions', and for a long time it was thought that perhaps $W(m)$ really does grow this fast.

However, Shelah (1987) found a proof of VdW using induction only on m. His proof gives that $W(m,k) \le f_4(m+k)$. This isn't bad, but f_4 is still a pretty big function. Graham offered \$1000 for a bound on $W(m)$ that was $f_3(m)$. Gowers (1998) showed $W(m) \le 2^{2^{2^{2^{2^{m+9}}}}}$ - 'almost' of type 2 - a huge improvement on type 3.

What about a lower bound? It is known that $W(m) \ge \frac{2^m}{8m}$; this is comparatively extremely small.

Corollary 1.7: *Whenever $\mathbb{N}$ is finitely coloured, some colour class arbitrarily long APs.* $\square$

Remark: We *cannot* guarantee an infinite AP - *e.g.* R-BB-RRR-BBBB....

Alternatively, list all infinite APs as $A_1, A_2, \dots$ (which we can do since they are countable). Pick distinct $x_1, y_1 \in A_1$ and make x_1 red and y_1 blue. Now pick distinct $x_2, y_2 \in A_2 \backslash \{x_1, y_1\}$ and make x_2 red, y_2 blue. Continue. Then we wipe out all the APs.

The idea here is that there simply aren't enough APs.

Theorem 1.8: (Strengthened Van der Waerden Theorem) *For all m, whenever $\mathbb{N}$ is finitely coloured there exists an AP of length m that, together with its common difference, is monochromatic.*

Proof. We will prove this with VdW itself (this is the sign of a good theorem).

We induct on k, the number of colours. $k = 1$ is trivial.

Given n suitable for $k-1$ (*i.e.* n such that whenever $[n]$ is $k-1$-coloured, there exists a monochromatic AP+CD of length m), we'll show $W(n(m-1)+1, k)$ suitable for k.

Given a k-colouring of $[W(n(m-1)+1,k)]$, we have a mono AP of length $n(m-1)+1$; say $\{a, a+d, \dots, a+2d, \dots, a+n(m-1)d\}$ is red.

Now if d is red then we are done by $\{a, a+d, \ldots, a+(m-1)d\} \cup \{d\}$ is monochromatic. Similarly, if any rd, $1 \leq r \leq n$ is red, then we are done since $\{a, a+rd, \ldots, a+(m-1)rd\} \cup \{rd\}$ is monochromatic.

Thus none of $\{d, 2d, \ldots, nd\}$ is red, *i.e.* $\{d, 2d, \ldots, nd\}$ is $(k-1)$-coloured. So, by definition of n, we are done. □

What kind of bounds do we get here? Since we are iterating on the left hand side, this actually grows vastly more quickly than anything we have considered before; it's so mind bogglingly large that it essentially isn't worth thinking about.

Remarks:

1. From now on, we don't care aboud bounds.
2. The case $m = 2$ is *Schur's Theorem*: Whenever $\mathbb{N}$ is finitely coloured, there exists monochromatic x, y, z with $x + y = z$.
3. Can also prove Schur direcly from Ramsey. Indeed, given a k-colouring c of $\mathbb{N}$, induce a colouring of $[n]^{(2)}$ (n large enough) by $d(ij) = c(j-i)$.

   By Ramsey, we can find a monochromatic triangle, say ijk. So $c(j-i) = c(k-j) = c(j-i)$, and $(j-i) + (k-j) = (k-i)$, so done.

1.3 The Hales-Jewett Theorem

You may have noticed that we haven't taken full advantage of the structure of the natural numbers - all we needed was some idea of equal spacing, and we didn't care so much about the additive structure. It turns out that we have a theorem that describes the same phenomenon but with all the clutter removed.

Definition: (Combinatorial Line) Let X be a finite set. A subset L of X^n ('the n-dimensional cube on alphabet X') is called a ***line*** or ***combinatorial line*** if there exists non-empty $I \subseteq [n]$ and $a_i \in X$, each $i \in [n] - I$ such that

$$L = \{(x_1, \ldots, x_n) \in X^n : x_i = a_i \ \forall i \notin I, x_i = x_j \ \forall i, j \in I\}$$

We call I the set of ***active coordinates***

For example, in $[3]^{(2)}$, lines can be:

- vertical lines, where $I = \{2\}$
- horizontal lines, where $I = \{1\}$
- one diagonal line, where $I = \{1, 2\}$ (the main diagonal only)

In $[3]^3$, we could have

- $\{(1, 2, 1), (1, 2, 2), (1, 2, 3)\}$ with $I = \{3\}$, or
- $\{(1, 3, 1), (2, 3, 2), (3, 3, 3)\}$ with $I = \{1, 3\}$, *etc...*

Note that the definition of 'line' is invariant under reorderings of x - so *e.g.* $\{1, 3), (2, 2), (3, 1)\}$ is *not* a line in $[3]^2$.

Theorem 1.9: (The Hales-Jewett Theorem) *For all m, k there exists n such that whenever $[m]^n$ is k-coloured, there exists a monochromatic line.*

Remarks:

1. The least such n (if it exists) is denoted $\mathrm{HJ}(m, k)$.

2. A game of m-in-a-row Noughts & Crosses, played in enough dimensions, cannot end in a draw! [Exercise: show that it is a first-player win (optional).]

3. HJ $\implies$ VdW immediately: just map X^n linearly to $\mathbb{N}$ (note: perhaps requries positive coefficients).

   Indeed, given a k-colouring c of $\mathbb{N}$, induce a k-colouring d of $[m]^n$ (n large) by $d((x_1, \dots, x_n)) = c(x_1 + \dots + x_n)$. By assumption we have a monochromatic line L in $[m]^n$, which corresponds to a monochromatic AP of length m in $\mathbb{N}$ (with common difference = # active coords of L).

   Thus we can view HJ as an 'abstract version' of VdW.

Before we begin, we also need a few definitions:

For a line L in $[m]^{(n)}$, write L^- for its first point and L^+ for its last point (in the ordering on $[m]^{(n)}$) given by $x \le y$ if $x_i \le y_i$ for all i)

Say lines $L_1, \dots, L_r$ are ***focused*** at f if $L_i^+ = f$ for all i, and they are ***colour-focused*** (for a given colouring) if in addition each $L_i - \{L_i^+\}$ is monochromatic, no two of the same colour.

Proof. Induction on m. $m = 1$ is trivial.

Given $m > 1$, we may assume HJ$(m-1, k)$ exists for all k.

<u>Claim</u>: for all $r \le k$, there exists an n such that whenever $[m]^{(n)}$ is k-coloured, there exists either:

- a monochromatic line, or
- r colour-focused lines

[Then done: put $r = k$ and look a the focus.]

<u>Proof of Claim</u>: Induction on r.

$r = 1$ is done - take $n = \text{HJ}(m-1, k)$.

Given n suitable for $m-1$, we'll show $n + \text{HJ}(m-1, k^{m^n}) =: n + n'$ is suitable for r.

So, given a k-colouring c of $[m]^{n+n'}$ with no monochromatic line: View $[m]^{n+n'}$ as $[m]^n \times [m]^{n'}$. There are k^{m^n} ways to colour $[m]^n \times [m]^{n'}$. At each point of $[m]^{n'}$ we have one of k^{m^n} "patterns".

So by choice of n', we have a line L in $[m]^{n'}$ such that for all $a \in [m]^n$, for all $b, b' \in L - \{L^+\}$ we have $c(a, b) = c(a, b') =: c'(a)$, say. By definition of n, c' has $r-1$ colour-focused lines, say $L_1, \dots, L_{r-1}$ with active coordinate sets $J_1, \dots, J_{r-1}$ respectively, focused at f. Let L have active coord set I. Then the lines $L_1', \dots, L_{r-1}'$, where L_i' starts at (L_i^-, L^-) and has active coord set $J_i \cup I$, are colour-focused at (f, L^+). Also the line starting (f, L^-) with active coord set I is monochromatic (apart from final point), of a different colour to the L_i's, giving r colour-focused lines.

This completes the induction, hence the proof of the claim, has the whole proof. $\square$

What is a line though? If you think about it, it's just a one-dimensional subspace. This begs the question, can we perhaps get a monochromatic *two*-dimesional subspace?

A ***d-parameter set*** or ***d-dimensional subspace*** of X^n is a set $S \subseteq X^n$ such that there exist disjoint, non-empty $I_1, \dots, I_d \subseteq [n]$ and $a_i \in X$, each $i \in [n] - (I_1 \cup \dots \cup I_d)$ such that

$$S = \{x \in X^n : x_i = a_i \ \forall i \notin I_1 \cup \dots \cup I_d, \text{ and } x_i = x_j \ \forall i, j \text{ with } i, j \in I_k \text{ for some } k\}$$

For instance, in X^3:

- $\{(x, y, 2) : x, y \in X\}$ is a 2-parameter set: $I_1 = \{1\}$, $I_2 = \{2\}$

- $\{(x,y,x) : x, y \in X\}$ is a 2-parameter set: $I_1 = \{3\}$, $I_2 = \{2\}$

This is in fact true.

Theorem 1.10: (Extended Hales-Jewett Theorem) *For all m, k, d ther exists m such that whenever $[m]^n$ is k-coloured, there exists a monochromatic d-parameter set.*

Proof. View X^{dn} (the dn-dimensional cube over alphabet X) as $(X^d)^n$, the n-dimensional cube on alphabet X^d. Clearly any line in $(X^d)^n$ (alphabet X^d) corresponds to a d-parameter set in X^{dn} (alphabet X). So we are done - we can take $n = d \cdot \mathrm{HJ}(m^d, k)$. □

Let S be a finite subset of $\mathbb{N}^d$. A ***homothetic copy*** of A is any set of the form $a + \lambda S$, where $a \in \mathbb{N}^d$ and $\lambda \in S$.

For instance, in $\mathbb{N}^1$, a homothetic copy of $\{1, 2, \ldots, m\}$ is precisely an AP of length m.

Can we guarantee a homothetic copy of S that is monochromatic? For instance, in $\mathbb{N}^2$, can we find a monochromatic square?

Theorem 1.11: (Gallai's Theorem) *Let S be a finite subset of $\mathbb{N}^d$. Then, whenever $\mathbb{N}^d$ is fiinitely coloured, there exists a monochromatic homothetic copy of S.*

Proof. Let $S = \{S(1), \ldots, S(m)\}$. Given a k-colouring c of $\mathbb{N}^d$, induce a k-colouring c' of $[m]^n$ (n large) by: $c'(x_1, x_2, \ldots, x_d) = c(S(x_1) + \cdots + S(x_d))$.

Then we have a monochromatic line L for c' (n large), say with active coord set I. But now, taking $S(x_1) + \cdots + S(x_d)$ for each $x \in L$, we have a monochromatic, homothetic copy of S (with $\lambda = |I|$). □

Remarks:

1. Suppose $S = \{(0,0), (1,0), (0,1), (1,1)\} \subseteq \mathbb{N}^2$. Then we get a monochromatic square. Could we instead have applied Extended HJ, $d = 2$, on alphabet of size 2?

   The answer is *no*: this would only give a monochromatic rectangle.

2. Or, can prove Gallai by product arguments and focusing (don't try it, symbol overload, but similar argument).

Lecture 7