Model Theory

Lectures by Gabriel Conant

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1 Review of First Order Logic

A **language** is a set $\mathcal{L}$ of function symbols, relation symbols, and constant symbols. Additionally, each function/relation symbol has an assigned *arity* $n \geq 1$.

By convention, we view constant symbols as 'function symbols of arity 0'.

An **$\mathcal{L}$-structure** $\mathcal{M}$ consists of:

- a non-empty set M (the **universe** of $\mathcal{M}$)
- for every function symbol f of arity n, a function $f^{\mathcal{M}} : M^n \to M$
- for every relation symbol R of arity n, a subset $R^{\mathcal{M}} \subseteq M^n$
- for every constant symbol c, an element $c^{\mathcal{M}} \in M$ (*i.e.* identified with the unique element in its image)

Syntax: we build formulas using symbols in $\mathcal{L}$ along with

$$\wedge \ \neg \ \forall \ = \ (\ ) \ ,$$

and countably many variable symbols.

$\mathcal{L}$-term: these are our way of creating new functions by composing the ones we already have.

- constant symbols and variables are terms
- if $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is a term

Given a structure $\mathcal{M}$ and a term t, we are going to interpret the term in the structure in exactly the way you might expect. Inductively, define (for appropriate r) $t^{\mathcal{M}} : M^r \to M$ as:

- constant symbol c: $c^{\mathcal{M}}$ (case $r = 0$)
- variable x: identify function ($r = 1$)
- general term $f(t_1, \ldots, t_n)$: $f^{\mathcal{M}}(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}})$

$\mathcal{L}$-formulas: new relations. We have the following *atomic* L-formulas:

- If t_1 and t_2 are terms, then $(t_1 = t_2)$ is a formula
- If R is an n-ary relation symbol and $t_1, \ldots, t_n$ are terms, then $R(t_1, \ldots, t_n)$ is a formula

We can then create more complicated formulas. Given formulae φ and ψ:

- $\neg\varphi$
- $(\varphi \wedge \psi)$
- $\forall x \varphi$ for any variable x

An occurrence of a variable x is **free** in φ if x does not occur in the scope of $\forall x$. Otherwise, the occurrence is **bound**.

For instance, if φ is the statement $\forall x \neg(f(x) = y)$, x is bound and y is free.

Notation: Given a formula φ, we write $\varphi(x_1, \ldots, x_n)$ to denote that $x_1, \ldots, x_n$ are the free variables of φ.

Given a formula $\varphi(x_1, \ldots, x_n)$, a structure $\mathcal{M}$, $a_1, \ldots, a_n \in M$, we define "$\bar{a}$ satisfies φ in $\mathcal{M}$", written $\mathcal{M} \models \varphi(a_1, \ldots, a_n)$, as follows:

- If φ is $(t_1 = t_2)$ then $\mathcal{M} \models \varphi(\bar{a})$ iff $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$

- If φ is $R(t_1, \ldots, t_n)$ then $\mathcal{M} \models \varphi(\bar{a})$ iff $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_n^{\mathcal{M}}(\bar{a}) \in R^{\mathcal{M}}$
- $\mathcal{M} \models (\varphi \wedge \psi)(\bar{a})$ iff $\mathcal{M} \models \varphi(\bar{a})$ and $\mathcal{M} \models \psi(\bar{a})$
- $\mathcal{M} \models \neg\varphi(\bar{a})$ iff $\mathcal{M} \not\models \varphi(\bar{a})$
- Suppose φ is $\forall w \psi(x_1 \ldots, x_n, w)$. Then $M \models \varphi(\bar{a})$ iff for all $b \in M$, $\mathcal{M} \models \psi(\bar{a}, b)$

We emphasise that the focus of this course will not be on the precise definitions and semantics, so much as the meaning of what we are doing. All we seek is a first order logic that works for us, so that we can use it to do interesting things.

Abbreviations: We have *global* abbreviations such as

- $(\varphi \vee \psi)$ is $\neg(\neg\varphi \wedge \neg\psi)$
- $(\varphi \rightarrow \psi)$ is $(\neg\varphi \vee \psi)$
- $(\varphi \leftrightarrow \psi)$ is $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
- $\exists x \varphi$ is $\neg\forall x \neg\varphi$

We note that the last equivalence in a semantic sense hinges on the assumption that universes are non-empty. Since we will be almost exclusively be studying infinite structures, we will not worry about this.

We also have *local* abbreviations, often specific to the language we are studying. For instance, in $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ (the language of ordered rings):

- $x + y$ is $+(x, y)$
- $x < y$ is $< (x, y)$
- $x \leq y$ is $(x < y) \wedge (x = y)$
- $x < y < z$ is $(x < y) \wedge (y < z)$
- x^2 is $x \cdot x$
- nx is $\underbrace{x + x + \cdots + x}_{n \text{ times}}$

An **$\mathcal{L}$-sentence** is an $\mathcal{L}$-formula with no free variables. For instance, $\forall x(f(x) \neq y)$ is *not* a sentence, but $\exists y \forall x(f(x) \neq y)$ *is* a sentence. Sentences can be thought of as actually saying something meaningful.

If φ is a sentence and $\mathcal{M}$ is a structure, then we have the notion of $\mathcal{M} \models \varphi$, "$\mathcal{M}$ satisfies φ" or "$\mathcal{M}$ models φ".

L-theory: An **L-theory** is a set of L-sentences.

Given a theory T, we write $\mathcal{M} \models T$ ("$\mathcal{M}$ is a **model** of T) if $\mathcal{M} \models \varphi$ for all $\varphi \in T$.

T is **satisfiable** if it has a model.

Example: $T = \{\neg\exists x(x = x)\}$ - this sentence claims there are no elements in the universe. In our setting, this is unsatisfiable (though it is technically a matter of opinion).

Similarly, $\exists x(x = x)$ ("The Axiom of Non-Triviality") is always satisfied in any $\mathcal{L}$-structure.

Recall: T is **consistent** if it does not prove a contradiction (*e.g.* $(\varphi \wedge \neg\varphi)$)

A consequence of **Gödel's Completeness Theorem** is that a theory is satisfiable iff it is consistent. This is a very important theorem, though we will mostly be focussing on the model theoretic aspect (satisfiability).