# Quantum Information Theory: Sheet 2

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**Exercise 1**.

1. For any $v$ we have $\langle v|Av\rangle \geq 0$, and in particular real. So $\langle v|Av\rangle = (\langle v|Av\rangle)^* = \langle Av|v\rangle = \langle v|A^\dagger v\rangle$. Hence for all $v$ we have $\langle v|(A - A^\dagger)v\rangle = 0$. $(A - A^\dagger)$ is skew-Hermitian and hence normal, hence diagonalisable. So in some basis it is diagonal, and using the above relation we see that all the elements on the diagonal are zero. So $(A - A^\dagger)$ is zero in this basis and hence every basis. Thus $A = A^\dagger$.

2. $\mathbb{F}|ij\rangle = |ji\rangle$, which defines its action on an orthonormal basis for $\mathcal{H}_A \otimes \mathcal{H}_B$. Hence we can represent $\mathbb{F}$ as:
$$\mathbb{F} = \sum_{i,j} |ji\rangle\langle ij|$$

   Let $v = \sum_{i,j} a_{ij}|ij\rangle$ be an eigenvector with eigenvalue $\lambda$. Then $\mathbb{F}v = \lambda v = \sum_{i,j} a_{ij}|ji\rangle = \sum_{i,j} a_{ji}|ij\rangle = \sum_{i,j} \lambda a_{ij}|ij\rangle$. So we must have $a_{ji} = \lambda a_{ij}$ for each $i, j$, and so $a_{ji} = \lambda^2 a_{ji}$. Since $v$ is non-zero, there must be some non-zero $a_{ij}$. Hence $\lambda^2 = 1$, and $\lambda = \pm 1$.

   To calculate their multiplicities we can consier the degrees of freedom of the vector elements. Let $\lambda = 1$, and consider the matrix given by $A_{ij} = a_{ij}$. Since $a_{ij} = a_{ji}$, this must be symmetric, and the diagonal is unconstratined. So there are $d + (d-1)d/2 = d(d+1)/2$ degrees of freedom, hence this is the multiplicity.

   So the multiplicity of $\lambda = -1$ is $d(d-1)/2$, which we can also see by remarking that in this case the diagonal elements $a_{ii}$ must all be zero, and then the upper right of the matrix is determined entirely by the lower left, which has $d(d-1)/2$ free elements.

   Slightly more rigorously, the 'degrees of freedom' correspond to basis vectors in the eigenspace.

   We form the operator $\Omega = |\Omega\rangle\langle\Omega| = \sum_{i,j} |j\rangle\langle i| \otimes |j\rangle\langle i|$, and what we want is $\mathbb{F} = \sum_{i,j}$