

Infinite Games

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Lecture 1

0 Introduction

Before this course kicks off, we will first discuss a few things that this course is *not* about. Do bear in mind though that this is still an advanced set theory course, building off the content found in IID Logic and Set Theory.

Literature: Some useful literature can be found in ‘The Higher Infinite’ - in particular, Chapter 6 “Determinacy”, sections 27 (Infinite Games) and 28 (AD and Combinatorics).

The term “Infinite Games” can evoke different reactions in different mathematicians. We will get on to formal definitions later, but the games we will consider have the following properties. They are:

- Two-player
- Length ω
- Win-lose
- Perfect information
- Perfect recall

Essentially, they are infinite versions of board games.

A notable theorem from the finite analogue of this theory is that of Zermelo, which is that very finite such game is determined - this is considered rather trivial, and we will see the difference between the application of this to the finite case versus the infinite case.

0.1 Two Players

We have two players: I and II.

Three-player games do not, in general, admit winning strategies. Consider the following example:

I, II, III are playing. Player I is given a gold coin.

Round 1: I can give it to II or to III.

Round 2: Whoever has the coin can give it to I or to II.

The person with the coin wins.

So if II gets the coin, they will keep it and win. Player I can either hand the win to II, or leave it to chance with III. III has no win condition, and II cannot force anyone to give them the coin. So no-one has a winning strategy. However, *coalitions* $\{I, II\}$, $\{I, III\}$, $\{II, III\}$ can each ensure a win.

What we have encountered here is **cooperation**, which fundamentally changes the strategies and payoffs; this phenomenon cannot arise in two-player games, as they players are always competing directly with each other.

0.2 Length ω

In game theory in economics, there is research on potentially infinite games.

Consider the prisoner's dilemma, concisely represented by:

2	2	1	3
3	1	0	0

There is a lot of research on this game as a single-move game, but it can also be thought of as a repeated game, where many rounds are played one after the other. In fact, if you fix a length of the game in advance this will affect the strategy of the players. Economists have deal with games which are, while certainly not infinte, of unknown length, and as such it happens that infinite games can serve as a useful model for this situation From there we can study asymptotic behaviour, or evolutionary phenomena.

The point here is that if we only think of *truly* infinite games, *i.e.* it will take an infinite amount of time to win, then the situation is fundamentally different.

Example: The Prime Factor Game

I	k_0	k_1	k_2	k_3	\dots
II	p_0	p_1	p_2	p_3	\dots

The $k_i \geq 2$ are natural numbers, and the p_i are prime numbers. At the end of the game, we look at $K = \{k_i : i \in \mathbb{N}\}$ and $P = \{p_i : i \in \mathbb{N}\}$. We say that Player II wins if P is the set of all prime factors in K (and no more).

Observe: Player II has a winning strategy.

Let $k_0 = q_0^{\ell_0} \dots q_m^{\ell_m}$. Then play as though $p_0 = q_0$, $p_1 = q_1$, $p_2 = q_2$, exhausting the finitely many prime factors of k_0 before moving on to k_1 . Repeating ad infinitum, it is clear that the set P is precisely what is desired.

However, if we look at how this is going for II after any finite number of moves $N \in \mathbb{N}$, then in most runs of the game, the finite sets $\{k_0, \dots, k_N\}$ and $\{p_0, \dots, p_N\}$ do not look like a win for player II; it will seems as though things are going worse and worse for II.

This critically highlights how the asymptotic behaviour can be drastically different from the outcome of the game after infinite time.

A common objection to this type of material is that you can't play these games, so how could you know who wins?

The above example clearly highlights that even though, of course, the games cannot actually be played, we may still be able to prove that a winning strategy does or does not exist for either player. So we are replacing actually playing the game with thinking about the different strategies for it.

Let's modify the PFG slightly:

I	k_0	k_1	k_2	\dots
II	p_0	p_1	p_2	\dots

Now Player I has a winning strategy instead. Take p_0 , find another prime $q \neq p_0$ and play $k_i = q^{i+1}$. Then $K = \{q^{i+1} : i \in \mathbb{N}\}$. So II wins iff $P = \{q\}$, but $p_0 \in P$. So II loses.

0.3 Win-Lose

There is a related notion here called **zero-sum**; in these games, there is a fixed payoff that is split between the two players. So, for instance, the Prisoner's Dilemma is **not** zero-sum because the total payoff differs between some outcomes. However, the game

1	1	0	2
2	0	1	1

is zero-sum.

Win-lose simply means that the payoff is an indivisible 1. So in our case, payoff functions are characteristic functions of a **payoff set**.

0.4 Perfect Information

Paradigmatic: board games, after which this idea was modelled.

A non-example is *card games*, in which your own hand is only known to you. Unsurprisingly, this scenario is called *imperfect information*.

Consider yet another variant of PFG:

I	k_0	k_1	k_2	k_3	\dots
II	p_0	p_1	p_2	p_3	\dots

Here, player I picks k_i , but does not have to reveal k_i before II has played p_i . Here, although it may happen with probability zero, it is possible for II to beat any set of moves that I makes simply by being lucky, and guessing only prime factors for numbers chosen by I. However, it is clear that it is impossible to ensure that this is the case.

So neither of the two players has a winning strategy in this variant. The study of these imperfect information games is closely related to probability.

0.5 Perfect Recall

This means that both of the players remember everything that has happened before; the opposite of course would be that the players have a finite, bounded memory.

For instance, take PFG with the additional constraint that Player II can only remember the last 1000 moves. Now Player I has a strategy that might win; on the first move, pick a natural number with at least 1001 distinct prime factors. Then II will have no idea what move to make at $N = 1001$; they might guess, and so they can still win, but they have no way to ensure this (and again it will not be very likely).

Imperfect recall is very relevant in applications of infinite games in computer science.

Lecture 2

We fix a set M of moves. In most cases, M will simply be \mathbb{N} - but we will aim to keep this slightly more general for now.

Note that from the perspective of a set theorist, we think of \mathbb{N} as equal to ω , and in particular $n = \{0, 1, \dots, n-1\}$. Moreover, functions are *set-theoretic* functions, *i.e.* sets of ordered pairs with the function property. For instance:

$$M^n = \{s; s : n \rightarrow M\}$$

is the set of functions from the set n to the set M . If $s \in M^n$ and $t \in M^k$, with $k > n$, then $s \subseteq t$ is the same as saying “ s is an initial segment of t ”, or that “ t is an extension of s ”.

Since we are thinking of sequences as functions, we can also write: if $m < n$ and $s \in M^n$, then $s \upharpoonright m \in M^m$.

These are well-known formal definitions of these objects, but they will be used particularly ruthlessly here.

We make another important definition:

$$M^{<\omega} := \bigcup_{n \in \mathbb{N}} M^n$$

This is the set of all finite sequences of elements of M ; these will be called the **positions** of the game. We also have:

$$M^\omega := \{x; x : \mathbb{N} \rightarrow M\}$$

is the set of all **runs** or **plays** of the game *i.e.* the set of all sequences of M of length ω . Note that if $x \in M^\omega$ is a run and $n \in \mathbb{N}$, then

$$x \upharpoonright n : n \rightarrow M$$

is the position that the play producing x was in after n rounds.

The games on M :

I	m_0	m_2	m_4	\dots
II	m_1	m_3	m_5	\dots

We are restricting our attention to games where I,II play in alternation and player I starts. [Remark: more general games can be described by these; see later.]

Then $x(i) := m_i$ is the run produced by the game, and $s := x \upharpoonright n$ is the n^{th} position.

If $x \in M^\omega$, we write

$$\begin{aligned} x_I(i) &:= x(2i) \\ x_{II}(i) &:= x(2i+1) \end{aligned}$$

$x_I, x_{II} \in M^\omega$ correspond to the moves made by players I,II respectively. If $x, y \in M^\omega$, we write $x * y$ (**interleaving**) for the sequence z defined by:

$$z(n) := \begin{cases} x(k) & n = 2k \\ y(k) & n = 2k + 1 \end{cases}$$

Clearly, $x_I * x_{II} = x$.

If $A \subseteq M^\omega$, we call A a **payoff set**. In the game $G(A)$, we say that player I wins a run $x \in M^\omega$ if $x \in A$; otherwise player II wins.

We call any function

$$\sigma : M^{<\omega} \rightarrow M$$

a **strategy**. Note that a strategy looks at the entire game up until that point to decide the next move; this is the perfect recall aspect. You may wonder why we bother defining a strategy for player I at odd length positions, and this is largely for notational convenience; it is a little easier to have this notational overkill.

Note that each strategy in this sense can be thought of as a strategy for I, plus a strategy for II. Let

$$O := \bigcup_{n \text{ odd}} M^n$$

$$E := \bigcup_{n \text{ even}} M^n$$

Then $\sigma \upharpoonright E$ is a strategy for I, and $\sigma \upharpoonright O$ is a strategy for II. So there is redundancy in the notation.

If σ, τ are strategies, we can play them against each other by interleaving them as $\sigma * \tau \in M^\omega$, which is defined by:

$$(\sigma * \tau)(2n) := \sigma((\sigma * \tau) \upharpoonright 2n)$$

$$(\sigma * \tau)(2n + 1) := \tau((\sigma * \tau) \upharpoonright 2n + 1)$$

We say that σ is **winning for I in** $G(A)$ if $\forall \tau (\sigma * \tau \in A)$, *i.e.* I always wins regardless of II's strategy. Similarly, we say that τ is **winning for II in** $G(A)$ if $\forall \sigma (\sigma * \tau \notin A)$.

We say that a set A is **determined** if one of the two players has a winning strategy in $G(A)$.

Remark:

- Clearly, at most one player can have a winning strategy (otherwise, play them against each other).
- However, it is not obvious (and not true, up to the axiom of choice) that every set is determined.
- In fact, we will see that AC implies that there are non-determined sets, but “every set is determined” (Axiom of Determinacy, AD) is consistent ZF - though this requires more nuance, but we will discuss all of this later.

You may ask: is that really the most general form of games that we want to look at? What if we wanted to include things like ‘forbidden’ moves, or allowing one or two players to make several moves at a time, or having two different move sets? It turns out that we don't need to worry about this:

A set $T \subseteq M^{<\omega}$ is called a **tree** if it is closed under initial segments, *i.e.* if $s \in T$ and $t \subseteq s$ then $t \in T$.

These trees look very much like the trees one might encounter in graph theory/combinatorics; though there are some small differences. In this set-theoretic notion of a tree, each node in the tree contains within it all of the information about the path from the root (*i.e.* the empty set, \emptyset) to it.

If T is a tree on M and $x \in M^\omega$, we say that x is a **branch through** T if for all $n \in \mathbb{N}$, $x \upharpoonright n \in T$. We write $[T]$ for the set of branches through T ; in some literature this is referred to as the **body of** T .

Example: $M^{<\omega}$ is a tree; $[M^{<\omega}] = M^\omega$.

We can think of a tree T as “finitary” rules for a game: if $x \notin [T]$, then there is a least n for which $x \upharpoonright n \notin T$. If n is odd, then player I left the tree, and if n is even then player II left the tree.

Define a game $G(A; T)$ where $A \subseteq [T]$ and T is a tree on M . The game is as usual:

I	m_0	m_2	m_4	\dots
II	m_1	m_3	m_5	\dots

If $x \in A$, then player I wins. If $x \notin [T]$ and the least n for which $x \upharpoonright n \notin T$ is even, then player I wins. In all other cases, player II wins.

Now, even though this looks more general due to the introduction of the tree T , it can be seen that this is in fact a special case of a $G(A)$ game, since we can define:

$$A_T := \{x \in M^\omega; x \in A \text{ or } x \notin [T] \text{ and the least } n \text{ s.t. } x \upharpoonright n \notin T \text{ is even}\}$$

. Then $G(A; T)$ and $G(A_T)$ are *the same game*.

Note that we haven't quite defined what it means to be the same game, but in this particular case it should be rather clear that these two are indeed the same game.

This idea of using trees gives us a lot of flexibility with the move set.

Example 1: Suppose the moves for I are in X , and the moves for II are in Y . We can take $M := X \cup Y$, $A \subseteq M^\omega$, and

$$T := \{s; \forall n \in \mathbb{N}, s(2n) \in X \text{ and } s(2n+1) \in Y\}$$

Then $G(A; T)$ is the game we desire.

Example 2: Suppose I can always make two moves in X , but II can only make one move. We then take $M := X^2 \cup X^1$, and apply the idea of Ex. 1 with $X = X^2$ and $Y = X^1$.

Example 3: If $X \subseteq Y$, then every game $G(A)$ on X can be thought of as a game on Y by $G(A; T)$, where

$$T := X^{<\omega} \subseteq Y^{<\omega}$$

Definition: (Strategic Tree) Let σ be a strategy. We define the *I-strategic tree* and the *II-strategic tree* on M as follows:

$$\begin{aligned} T_\sigma^I &:= \{s \in M^{<\omega}; \forall n (s(2n) = \sigma(s \upharpoonright 2n))\} \\ T_\sigma^{II} &:= \{s \in M^{<\omega}; \forall n (s(2n+1) = \sigma(s \upharpoonright 2n+1))\} \end{aligned}$$

When drawing out these trees, for instance T_σ^I , the layers alternate between making any choice from M (representing II's moves) and making the only choice dictated by σ for I.

II-strategic trees look the same except that we have branching in odd length nodes and no branching in even length nodes.

T is called *strategic* if there is σ such that $T = T_\sigma^I$ or $T = T_\sigma^{II}$.

Observe:

$$\begin{aligned} T_\sigma^I &= \{(\sigma * \tau) \upharpoonright n; \tau \text{ any strategy and } n \in \mathbb{N}\} \\ T_\sigma^{II} &= \{(\tau * \sigma) \upharpoonright n; \tau \text{ any strategy and } n \in \mathbb{N}\} \end{aligned}$$

Therefore:

$$\begin{aligned} [T_\sigma^I] &= \{\sigma * \tau; \tau \text{ any strategy}\} \\ [T_\sigma^{II}] &= \{\tau * \sigma; \tau \text{ any strategy}\} \end{aligned}$$

Proposition:

1. σ is a winning strategy for I in $G(A) \iff [T_\sigma^I] \subseteq A$
2. σ is a winning strategy for II in $G(AS) \iff [T_\sigma^{II}] \cap A = \emptyset \iff [T_\sigma^I] \subseteq M^\omega \setminus A$

Also: A is determined iff either A contains $[T_\sigma^I]$ for some σ or $M^\omega \setminus A$ contains $[T_\sigma^{II}]$ for some σ .

Notation: If $s, t \in M^{<\omega}$, we write st for the concatenation of s and t . This also works if t is infinite; if $x \in M^\omega$ and $s \in M^{<\omega}$, then similarly $sx \in M^\omega$ is the concatenation.

If t is a length 1 sequence, say $t = \langle m \rangle$, we also write sm for $st = s\langle m \rangle$; this is usually unambiguous. For the length of a sequence we write $\ell h(s) = \text{dom}(s)$.

Definition: (Splitting Node, Perfect Tree, Perfect Set)

- 1) If T is a tree and $s \in T$ we say s is a **splitting node** if there are $m \neq m'$ such that both $sm, sm' \in T$.
- 2) T is **perfect** if for each $s \in T$ there is a $t \supseteq s$ such that $t \in T$ and t is splitting in T .

[Remark: every strategic tree is perfect]

- 3) $A \subseteq M^\omega$ is **perfect** if there is a perfect tree T such that $A = [T]$.

Remark: Compare to the topological notion of a **perfect set**: closed without isolated points. We will find out later that, with the right topology on M^ω , these notions will coincide.

Theorem: (Cantor) *Suppose $A \subseteq 2^\omega \equiv \{0, 1\}^\omega$ is perfect and non-empty. Then A has cardinality 2^{\aleph_0} .*

Proof. $A \subseteq 2^\omega$ and $|2^\omega| = 2^{\aleph_0}$, so $|A| \leq 2^{\aleph_0}$. So by Cantor-Schröder-Bernstein, it is enough to show that there is an injection from 2^ω into A .

We define this injection via a function $\varphi : 2^{<\omega} \rightarrow T$, where T is perfect such that $A = [T]$. We will define this by recursion (this is known as a *Cantor scheme*):

$$\varphi(\emptyset) := \emptyset$$

Suppose $\varphi(s) = t \in T$. Since T was perfect, find $u \supseteq t, u \in T$ that is splitting: $u0, u1 \in T$. To ensure this is uniquely defined (to potentially avoid issues with Choice), take the minimal one. Then:

$$\varphi(s0) := u0$$

$$\varphi(s1) := u1$$

This finishes the definition of φ . We then define:

$$\begin{aligned} \hat{\varphi} : 2^\omega &\rightarrow [T] = A \\ \hat{\varphi}(x) &:= \bigcup_{n \in \mathbb{N}} \varphi(x \upharpoonright n) \end{aligned}$$

We need to check some things:

1. $\ell h(\varphi(x \upharpoonright n)) \geq n$
2. $\varphi(x \upharpoonright n) \subseteq \varphi(x \upharpoonright m)$ if $n \leq m$
 $\implies \hat{\varphi} : 2^\omega \rightarrow 2^\omega$
3. $\hat{\varphi}(x) \upharpoonright u \subseteq \varphi(x \upharpoonright k)$ for some k so $\hat{\varphi}(x) \upharpoonright u \in T$, so $\hat{\varphi}(x) \in [T]$.

So we indeed have that $\hat{\varphi} : 2^\omega \rightarrow [T]$.

It remains to show that $\hat{\varphi}$ is an injection:

Suppose $x \neq y$. Find n such that $x \upharpoonright n = y \upharpoonright n$, but $x(n) \neq y(n)$. WLOG, say $x(n) = 0$ and $y(n) = 1$. But then $\varphi(x \upharpoonright n+1) \neq \varphi(y \upharpoonright n+1)$, since the former ends in 0 and the latter in 1. This implies that $\bigcup_{k \in \mathbb{N}} \varphi(x \upharpoonright k) \neq \bigcup_{k \in \mathbb{N}} \varphi(y \upharpoonright k)$, hence $\hat{\varphi}(x) \neq \hat{\varphi}(y)$. \square

Remark: If $|M| \geq 2$ and T is a perfect tree on M , then the same proof shows that $2^{\aleph_0} \leq |[T]|$.

Corollary: If $|M| \geq 2$, then:

- (i) if player I has a winning strategy in $G(A)$, then $|A| \geq 2^{\aleph_0}$
- (ii) if player II has a winning strategy in $G(A)$ then $|M^\omega \setminus A| \geq 2^{\aleph_0}$.

This follows from:

1. strategic trees are perfect
2. perfect sets are large
3. winning strategy means “includes strategic tree”

Note that if $A \subseteq M^\omega$ with $|M| \geq 2$, then either $|A| \geq 2^{\aleph_0}$ or $|M^\omega \setminus A| \geq 2^{\aleph_0}$.

The corollary gives a necessary condition on when a fixed player has a winning strategy, but no non-trivial necessary condition for determinacy.

Sufficient Conditions

Let's do the following as a warmup.

Prove that if A is countable, then player II has a winning strategy in $G(A)$.

Proposition: If $A = \{a_i; i \in \mathbb{N}\}$ is countable, then player II has a winning strategy in $G(A)$.

Proof. In II's round k [that means digit $2k+1$], II takes care of a_k , simply by playing $1 - a_k(2k+1)$ (assume again we are playing on $M = \{0, 1\}$; on anything else just pick something different to $a_k(2k+1)$).

So the strategy τ is:

- ignore everything player I does
- blindly play $1 - a_k(2k+1)$ in your k^{th} move.

Clearly then for any σ ,

$$\begin{aligned} (\sigma * \tau)_\Pi(k) &= (\sigma * \tau)(2k+1) \\ &= 1 - a_k(2k+1) \\ &\neq a_k(2k+1) \end{aligned}$$

So $\sigma * \tau \neq a_k$ for arbitrary k , so $\sigma * \tau \notin A$. Thus τ is winning. □

Lecture 3

Necessary: we have determined some necessary conditions for wins:

- I wins $G(A) \implies |A| = 2^{\aleph_0}$
- II wins $G(A) \implies |\omega^\omega \setminus A| = 2^{\aleph_0}$

Sufficient: we also have the sufficient condition:

- if A is countable, then player II wins.

Note that we write ‘I/II’ wins as shorthand for ‘I/II has a w.s.’

Theorem:

1. If $A \subseteq \omega^\omega$ such that $|A| < 2^{\aleph_0}$, then player II has a winning strategy in $G(A)$.
2. If $A \subseteq \omega^\omega$ such that $|\omega^\omega \setminus A| < 2^{\aleph_0}$ then player I has a winning strategy in $G(A)$.

Proof. The proofs of 1 and 2 are essentially just switching the roles of I,II. So we are just going to prove 1.

Caution: our games are *not* fully symmetric; I is not in the same situation as II. Moving first can sometimes be an advantage, and sometimes a disadvantage. The above claim must thus be checked carefully.

Let $|A| < 2^{\aleph_0}$. Define an equivalence relation \sim on ω^ω by

$$x \sim y \iff x_{II} = y_{II}$$

. So equivalence classes look like this:

$$C_z := \{x; x_{II} = z\}$$

So there is a bijection between the \sim -equivalence classes and ω^ω . In particular, there are 2^{\aleph_0} such equivalence classes. By the pigeonhole principle, we find z such that

$$C_z \cap A = \emptyset$$

. Then define τ as:

- ignore everything player I does
- just play the next digit of z

Formally, we can say $\tau(s) := z(n)$ if $\ell h(s) = 2n+1$, and whatever you like on the even entries. Then if σ is any strategy, we have

$$\begin{aligned} (\sigma * \tau)_{II} &= z \\ \implies \sigma * \tau &\in C_z \\ \implies \sigma * \tau &\notin A \end{aligned}$$

So τ is a w.s. for II. □

This is called a **blindfolded strategy**, since II doesn't care about what I is doing (and doesn't need to know). Formally, this is when there is $z \in \omega^\omega$ such that for player II, $\tau(s) := z(n)$ if $\ell h(s) = 2n+1$, or similarly for player I $\sigma(s) := z(n)$ if $\ell h(s) = 2n$. We normally denote this as τ_z if for player II, and σ_z if for player I.

Consequence: If we have any set A that is not determined, then it must be the case that $|A| = |\omega^\omega \setminus A| = 2^{\aleph_0}$. So we have a bit of an idea that a non-determined set must look a bit symmetric.

Next goal: Find such a non-determined set.

Theorem: (uses AC) *There is a non-determined subset $A \subseteq \omega^\omega$.*

Proof. The idea here is to ensure that A contains no strategic trees, which we do by enumerating them and then distributing the branches between A and its complement.

We did prove in Lecture 3 that if T is a strategic tree, then $||T|| = 2^{\aleph_0}$.

Question: How many strategic trees are there?

Notation: We write $\text{Trees} := \{T; T \text{ is a tree on } \omega\}$, and $\text{STrees} := \{T; T \text{ is a strategic tree on } \omega\}$.

We remark that a tree $T \subseteq \omega^\omega$, which is countable, so this gives an upper bound on the size of Trees . So we have $|\text{STrees}| \leq |\text{Trees}| \leq 2^{\aleph_0}$. Can we also find a lower bound? This is where the blindfolded strats come in...

If $z \neq z' \in \omega^\omega$, then $[T_{\sigma_z}^I] \cap [T_{\sigma_{z'}}^I] = \emptyset$. So $T_{\sigma_z}^I \neq T_{\sigma_{z'}}^I$.

Together (with Cantor-Schröder-Bernstein), we get a bijection between 2^{\aleph_0} and STrees .

Note that so far we haven't used AC, with the mild exception that the notation 2^{\aleph_0} implies the existence of an ordinal in bijection with the powerset of ω , which requires AC. However, the above can be reformulated as saying that we have injections $\omega^\omega \hookrightarrow \dots \hookrightarrow \omega^\omega$, and then use CSB; choiceless.

However, we need a little bit of choice now. We aren't going to use the full AC, but only that the set ω^ω is wellorderable. This implies:

- 2_0^{\aleph} is an ordinal (so we can do transfinite recursion on it)
- there is a choice function $c : \mathcal{P}\omega^\omega \setminus \emptyset \rightarrow \omega^\omega$ (i.e. $c(A) \in A$)

Equipped with this, we do the construction. We had that $|\text{STrees}| = 2^{\aleph_0}$, so write $\text{STrees} = \{T_\alpha; \alpha < 2^{\aleph_0}\}$ and do the following transfinite recursion:

We are going to define sets A_α, B_α in stages for $\alpha < 2^{\aleph_0}$ in such a way that $|A_\alpha| = |B_\alpha| = |\alpha|$ (*). In the end we will see $A_\alpha \cap B_\alpha = \emptyset$.

$\alpha = 0$: $A_0 = B_0 = \emptyset$.

$\alpha = \beta + 1$: Suppose we have A_β, B_β . Consider $T_\beta \in \text{STrees}$. Then $|[T_\beta]| = 2^{\aleph_0}$. By (*), $|A_\beta| = |B_\beta| = |\beta| < 2^{\aleph_0}$. This implies that $|A_\beta \cup B_\beta| < 2^{\aleph_0}$. Thus $[T_\beta] \setminus (A_\beta \cup B_\beta) \neq \emptyset$ (even better, it has 2^{\aleph_0} many elements).

Define:

$$\begin{aligned} a_\beta &:= c([T_\beta] \setminus (A_\beta \cup B_\beta)) \\ b_\beta &:= c([T_\beta] \setminus (A_\beta \cup B_\beta \cup \{a_\beta\})) \end{aligned}$$

Note that the latter input for c is still non-empty since $|[T_\beta] \setminus (A_\beta \cup B_\beta)| = 2^{\aleph_0}$.

Then let $A_\alpha := A_\beta \cup \{a_\beta\}$, and similarly $B_\alpha := B_\beta \cup \{b_\beta\}$. Moreover, $|A_\alpha| = |\beta + 1| = |\alpha| = |B_\alpha|$, satisfying IH.

α is a limit: For all $\beta < \alpha$, A_β, B_β are defined and satisfy (*). So we let:

$$\begin{aligned} A_\alpha &:= \bigcup_{\beta < \alpha} A_\beta \\ B_\alpha &:= \bigcup_{\beta < \alpha} B_\beta \end{aligned}$$

Obviously, $|A_\alpha| = |\alpha| = |B_\alpha|$ since they are each unions of things of the right size (there is an increasing sequence of bijections that converges to what we want, formally), so (*) is still satisfied.

Now we let

$$\begin{aligned} A &:= \bigcup_{\alpha < 2^{\aleph_0}} A_\alpha \\ B &:= \bigcup_{\alpha < 2^{\aleph_0}} B_\alpha \end{aligned}$$

Note 1: $A \cap B = \emptyset$; if not, then $a_\alpha = b_\beta$ for some α, β . WLOG $\alpha \leq \beta$. This then contradicts the choice of b_β .

Note 2: $|A| = 2^{\aleph_0} = |B|$. This is good, since it was a necessary condition for A and B being non-determined, as we found earlier.

Claim: A is not determined (and similarly B).

Proof of Claim: Suppose A is determined. Then there is a strategic tree T_α , $\alpha < 2^{\aleph_0}$, such that either $[T_\alpha] \subseteq A$ or $[T_\alpha] \cap A = \emptyset$. But consider $a_\alpha, b_\alpha \in [T_\alpha]$. We have $a_\alpha \in A$, $b_\alpha \in B$, i.e. $b_\alpha \notin A$. This rules out both cases, so contradiction. This proves the theorem. \square

You may notice that this is the very same diagonalisation argument that crops up all over the place.

Discussion: we used AC to produce a non-determined set. [Usually, AC implies the existence of pathologies, e.g. a non-Lebesgue measurable set, or the Banach-Tarski decomposition of the unit ball. AC does not produce a constructive method for the pathological objects, since the construction depends on the choice function.]

Motto (hope): If a set A is ‘nice’ or ‘simple’, then it is not pathological.

Similarly here; though we have just proved that non-determined sets may exist, the ones we have found are all pathological. We might ask if non-determined sets only arise this way. Hence:

Goal: Make ‘nice’ and ‘simple’ precise, and prove that nice sets are determined.

Lecture 5

Definition: (Quasistrategy) A function

$$\sigma : M^{<\omega} \rightarrow \mathcal{P}(M) \setminus \{\emptyset\}$$

is called a **quasi-strategy**. Strategies can be identified as a special case of quasistrategies:

$$(\forall s) |\sigma(s)| = 1$$

which clearly induced a strategy as we have defined it.

We also have a notion of **quasistrategic trees**:

$$\begin{aligned} Q_\sigma^I &:= \{s \in M^{<\omega}; (\forall n) s(2n) \in \sigma(s \upharpoonright 2n)\} \\ Q_\sigma^{II} &:= \{s \in M^{<\omega}; (\forall n) s(2n+1) \in \sigma(s \upharpoonright 2n+1)\} \end{aligned}$$

A quasistrategy is **winning for I** in $G(A)$ if $[Q_\sigma^I] \subseteq A$, and **winning for II** in $G(A)$ if $[Q_\sigma^{II}] \cap A = \emptyset$.

As before, at worst one of the two players can have a winning quasistrategy.

Definition: (Quasidetermined Set) A set $A \subseteq M^\omega$ is called **quasidetermined** if either I or II has a winning qs in $G(A)$.

Lemma: If M is wellorderable, then every quasistrategic tree on M contains a strategic tree on M .

[Consequence: if M is wellorderable and $A \subseteq M^\omega$ is quasidetermined, then it is determined.]

Proof. Let σ be a quasistrategy $\sigma : M^{<\omega} \rightarrow \mathcal{P}(M) \setminus \{\emptyset\}$. If M is wellorderable, then there is a choice function $c : \mathcal{P}(M) \setminus \{\emptyset\} \rightarrow M$ (i.e. such that $c(A) \in A$ for each A).

Then define $\sigma^* := c \circ \sigma : M^{<\omega} \rightarrow M$. By construction, $T_{\sigma^*}^I \subseteq Q_\sigma^I$ and $T_{\sigma^*}^{II} \subseteq Q_\sigma^{II}$. \square

Note that we needed a bit of choice here - that M is wellorderable.

Compare with our proof of the existence of a non-determined set from Lecture 4 (we used a wellordering of M^ω to get $A \subseteq M^\omega$ non-determined) to this, where we just use a wellordering of M . This requires significantly less choice, and it is often the case that M comes equipped with a wellordering but M^ω does not, e.g. $M = \mathbb{N}$.

Definition: (Closed Set) A set $A \subseteq M^\omega$ is called **closed** if there is a tree T on M such that $A = [T]$.

Remarks:

1. This is actually the notion of being closed in a topological space; we will see this space in the next lecture.
2. Zermelo's finite games can be represented by closed payoff sets.

Finite game of length n : let's say $f : M^n \rightarrow \{I, II\}$ is the function labelling the leaves according to which player wins.

Let $A := \{x \in M^\omega; f(x \upharpoonright n) = I\}$. Then the finite game is just $G(A)$. This is just playing an infinite game, but having decided the outcome already after n moves.

We can also define:

$$T := \{s \in M^{<\omega}; f(s \upharpoonright n) = I \text{ and } \ell h(s) \geq n \text{ or } \ell h(s) < n \text{ and there is a } t \supseteq s \text{ s.t. } f(t) = I\}$$

Clearly, $[T] = A$. Thus all finite games are closed games.

Theorem: (Gale-Stewart) All closed sets $A \subseteq M^\omega$ are quasidetermined.

Proof. If $A = [T]$, this means that if $x \notin A$, then $x \notin [T] = \{x \in M^{<\omega}; (\forall n)x \upharpoonright n \in T\}$. This implies there is some n such that $x \upharpoonright n \notin T$. These are positions that are won for sure by player II.

Define a partial function

$$\ell : M^{<\omega} \rightarrow \{I, II\}$$

by $\ell(s) = II \iff s \notin T$. Apply the following recursion rules to the partial labellings:

We extend ℓ to $\ell^+ \supseteq \ell$ according to the following rules. If $\ell(s)$ is undefined:

- and $\ell h(s)$ is even [I moves] and $\forall m \in M, \ell(sm) = II$, then $\ell^+(s) := II$
- and $\ell h(s)$ is odd [II moves] and $\exists m \in M, \ell(sm) = II$, then $\ell^+(s) = II$

Our transfinite recursion is then in the normal way:

- $\ell_0 := \ell$
- $\ell_{\alpha+1} := (\ell_\alpha)^+$
- $\ell_\lambda := \bigcup_{\alpha < \lambda} \ell_\alpha$

It's easy to construct examples of truly transfinite processes like this [needs $|M| \geq \aleph_0$].

Claim: this process terminates at some ordinal α , i.e. $\ell_\alpha = \ell_{\alpha+1} = (\ell_\alpha)^+$.

Proof of Claim: for $s \in M^\omega$, we can define an *age function*:

$$\text{age}(s) := \begin{cases} \text{least } \beta \text{ s.t. } \ell_\beta(s) \text{ is defined} \\ 0 & \text{if } \ell_\beta(s) \text{ is never defined} \end{cases}$$

So if the process never terminates, then age is a surjection from the set $M^{<\omega}$ onto the (proper) class Ord . This contradicts the Axiom of Replacement.

Let α be this termination point. We can then define:

$$\hat{\ell}(s) := \begin{cases} \text{II} & \text{if } s \in \text{dom}(\ell_\alpha) \\ \text{I} & \text{otherwise} \end{cases}$$

Then $\hat{\ell} \supseteq \ell_\alpha$, effectively just taking ℓ_α and filling everything else in with Is. So $\hat{\ell}$ is a total function.

Claim: If $\hat{\ell}(\emptyset) = \text{I}$, then player I has a winning quasistrategy; otherwise, $\hat{\ell}(\emptyset) = \text{II}$ and player II has a winning q.s.

Given this, the proof is complete.

Subclaim 1: If $\hat{\ell}(\emptyset) = \text{I}$, define

$$Q_{\text{I}} := \{s \in \omega^{<\omega}; \hat{\ell}(s) = \text{I}\}$$

and if $\hat{\ell}(\emptyset) = \text{II}$, define Q_{II} similarly.

Then $Q_{\text{I}}/Q_{\text{II}}$ is a I/II-quasistrategic tree.

[Need to show:

1. if $\hat{\ell}(s) = \text{I}$ and $\ell h(s)$ is even then there is m such that $\hat{\ell}(sm) = \text{I}$
2. if $\hat{\ell}(s) = \text{I}$ and $\ell h(s)$ is odd then for all m , $\hat{\ell}(sm) = \text{I}$
3. if $\hat{\ell}(s) = \text{II}$ and $\ell h(s)$ is even then for all m , $\hat{\ell}(sm) = \text{II}$
4. if $\hat{\ell}(s) = \text{II}$ and $\ell h(s)$ is odd then there is m such that $\hat{\ell}(sm) = \text{II}$

We had $\ell_\alpha = \ell_{\alpha+1} = (\ell_\alpha)^+$, so 3 and 4 follow immediately from the recursion definition of ℓ^+ .

Similarly, if $\hat{\ell}(s) = \text{I}$ and $\ell h(s)$ is even then $s \notin \text{dom}(\ell_\alpha)$. This implies there is an m such that $sm \notin \text{dom}(\ell_\alpha)$, so $\hat{\ell}(sm) = \text{I}$. 2 is similar.]

Subclaim 2: If $\hat{\ell}(\emptyset) = \text{I}$, then Q_{I} is a w.q.s for I.

[Need to show that $[Q_{\text{I}}] \subseteq A$. So fix some $x \in [Q_{\text{I}}]$. So for all x ,

$$\begin{aligned} x \upharpoonright u \in Q_{\text{I}} &\implies \hat{\ell}(x \upharpoonright u) = \text{I} \\ &\implies \hat{\ell}(x \upharpoonright u) \neq \text{II} \\ &\implies \ell(x \upharpoonright u) \notin \text{II} \\ &\implies x \upharpoonright u \in T \end{aligned}$$

So $x \in A$.]

Remark: : What we have done so far hasn't really needed trees/closed sets. Instead, we could let

$$\ell : M^{<\omega} \rightarrow \{\text{II}\}$$

be any partial labelling. Do the transfinite recursion $\ell_{\alpha+1} := (\ell_\alpha)^+$, $\ell_0 := \ell, \dots$, find fixed point ℓ_α ; define $\hat{\ell}$, define Q_{I} . Then Q_{I} is a q.s. that avoids all $s \in \text{dom}(\ell)$.

For player II, it is not always the case that Q_{II} is winning [c.f. Example 12 on ES#1] since you could stay on labels II without ever leaving the tree. So, we need to work slightly harder and find a sub-quasistrategy that is in fact winning.

Recall the age function:

$$\begin{aligned} \text{age} : Q_{\text{II}} &\rightarrow \alpha + 1 \\ \text{age}(s) &:= \text{the least } \beta \text{ such that } s \in \text{dom}(\ell_\beta) \end{aligned}$$

This means:

$$\begin{aligned} \text{age}(s) = 0 &\iff s \in \text{dom}(\ell_0) = \text{dom}(\ell) \\ &\iff s \notin T \end{aligned}$$

Subclaim 3: If $\hat{\ell}(s) = \text{II}$ and $\ell h(s)$ is even, then if $\text{age}(s) = 0$ or for all m , $\text{age}(sm) < \text{age}(s)$. If $\hat{\ell}(s) = \text{II}$ and $\ell h(s)$ is odd then either $\text{age}(s) = 0$ or there is m such that $\text{age}(sm) < \text{age}(s)$.

[This follows directly from the recursive construction step.]

Now define $\hat{Q}_{\text{II}} \subseteq Q_{\text{II}}$ by:

- $\emptyset \in \hat{Q}_{\text{II}}$
- $sm \in \hat{Q}_{\text{II}} : \iff \text{age}(sm) = 0 \text{ or } sm \in Q_{\text{II}} \text{ and } \text{age}(sm) < \text{age}(s)$

Informally, playing by \hat{Q}_{II} means: “play into positions labelled II reducing the age if you can”.

If $\hat{\ell}(\emptyset) = \text{II}$, then by Subclaim 3 \hat{Q}_{II} is a q.s.

Subclaim 4: If $\hat{\ell}(\emptyset) = \text{II}$, then \hat{Q}_{II} is winning for II: $[\hat{Q}_{\text{II}}] \cap A = \emptyset$.

[Suppose $x \in [\hat{Q}_{\text{II}}]$. Consider $a_n := \text{age}(x \upharpoonright n)$. This is a decreasing sequence of ordinals until it hits 0 by construction of \hat{Q}_{II} . Since no infinite, strictly decreasing sequence of ordinals exists, we find k such that $a_k = \text{age}(x \upharpoonright k) = 0$. This implies $x \upharpoonright k \notin T$, hence $x \notin A$.]

This finishes the claim, and hence concludes the entire proof. \square

Remark: If M is wellorderable (e.g. $M = \mathbb{N}$), then GST says that every closed subset of M^ω is determined, and also that every complement of a closed set is determined. [Follows directly from the proof.]

Recall our motto/hope from the end of lecture 2: if a set is *nice* or *simple*, then it is determined. If by ‘nice’ we mean closed, then this is just GST.

Next goal: Define a topology on $M^{<\omega}$.

Let’s focus on the case $M = \mathbb{N}$, or even $M = 2 = \{0, 1\}$.

Definition: (Baire Space) If $x, y \in \omega^\omega$, we can define

$$d(x, y) := \begin{cases} 0 & \text{if } x = y \\ 2^{-m} & \text{if } x \upharpoonright m = y \upharpoonright m \text{ and } x(m) \neq y(m) \end{cases}$$

which is a metric on ω^ω .

What are the open balls for this metric? Let $\varepsilon = 2^{-n}$:

$$\begin{aligned} B_\varepsilon(x) &= \{y \in \omega^\omega : d(x, y) < \varepsilon\} \\ &= \{y; y \upharpoonright (n+1) = x \upharpoonright (n+1)\} \end{aligned}$$

In particular, the open balls are determined by finite sequences.

If $s \in \omega^\omega$, write:

$$[s] := \{x \in \omega^\omega; s \subseteq x\}$$

So $B_{2^{-n}}(x) = [x \upharpoonright (n+1)]$. Thus the topology of the metric space is generated by the basic open sets $\{[s]; s \in \omega^\omega\}$.

This topological space on ω^ω is called **Baire space**. If we restrict to 2^ω , then it is called **Cantor space**.

If you think of ω^ω as

$$\prod_{i \in \omega} Y_i$$

with $Y_i = \omega$, and 2^ω as

$$\prod_{i \in \omega} Y_i$$

with $Y_i = 2 = \{0, 1\}$, then Baire space is just the product topology on $\prod_{i \in \omega} X_i$ with the discrete topology on ω , and Cantor space is the product topology on $\prod_{i \in \omega} Y_i$ with the discrete topology on 2.

Tychonoff implies that Cantor space is compact, but Baire space is not. Indeed, the latter can even be seen with Tychonoff:

$$\omega^\omega = \bigcup_{m \in \omega} [< m >]$$

Since $[< m >] = \{x; x(0) = m\}$, this union is disjoint so this open cover clearly has no finite subcover, and tells us that Baire space is (very) disconnected.

Lecture 7

Next time, we show that $A = [T] \subseteq \omega^\omega \iff A$ is closed in Baire space.

We now study the Baire space and Cantor space in detail. Firstly, consider convergence:

$$\begin{aligned} x_n \rightarrow x &\iff \forall \varepsilon \exists N \forall x > N \ d(x_n, x) < \varepsilon \\ &\iff \forall m \exists N \forall n > N \ d(x_n, x) < 2^{-m} \\ &\iff \forall m \exists N \forall n > N \ x_n \upharpoonright m = x \upharpoonright m \end{aligned}$$

If $A \subseteq \omega^\omega$, we can define

$$T_A := \{x \upharpoonright n; x \in A, n \in \mathbb{N}\}$$

Observe: $A \subseteq [T_A]$, since $x \in A \implies x \upharpoonright n \in T_A$ for all $n \implies x \in [T_A]$.

Proposition: $[T_A]$ is the closure of A , i.e. $\{x; \exists (x_n) \text{ with } x_n \in A \text{ and } x_n \rightarrow x\}$

Proof. Suppose $x_n \in A$ and $x_n \rightarrow x$. By our characterisation of convergence, this means that $x \upharpoonright k = x_n \upharpoonright k$ for some n , so $x \upharpoonright k \in T_A$. Since k was arbitrary, we have that $x \in [T_A]$.

Conversely, suppose that $x \in [T_A]$. For every $k \in \mathbb{N}$, $x \upharpoonright k \in T_A$, so there is some $x_k \in A$ such that $x \upharpoonright k = x_k \upharpoonright k$. Then again by the characterisation of convergence, we have that $x_k \rightarrow x$. So x is in the closure of A . \square

Corollary: $A \subseteq \omega^\omega$ is closed $\iff (\exists T)(A = [T])$ (we know that $T := T_A$ does it).

This is sometimes known as the **tree representation theorem for closed sets**.

Some more topological properties:

Basic open sets are $[s] = [T_s]$, where $T_s := \{t; s \subseteq t \text{ of } t \subseteq s\}$. So basic open sets are closed; we call these **clopen**. Spaces like this are called **zero-dimensional**

If $x \in \omega^\omega$, then $\{x\} = [T_x]$, with $T_x := \{x \upharpoonright n; n \in \mathbb{N}\}$. So singletons are closed, and not open.

We can easily see that this set is **Hausdorff**: if $x \neq y$, find n such that $x \upharpoonright n \neq y \upharpoonright n$. Then $x \in [x \upharpoonright n]$, $y \in [y \upharpoonright n]$, but $[x \upharpoonright n] \cap [y \upharpoonright n] = \emptyset$.

Continuous Functions: [proof on ES#2] If $g : \omega^{<\omega} \rightarrow \omega^{<\omega}$ such that:

1. g is order-preserving, i.e. $s \subseteq t \rightarrow g(s) \subseteq g(t)$
2. g is “unbounded”, i.e. if $x \in \omega^\omega$, then $\ell h(g(x \upharpoonright n)) \rightarrow \infty$

then define

$$\hat{g}(x) := \bigcup_{n \in \mathbb{N}} g(x \upharpoonright n)$$

which is a function ω^ω .

Proposition: $f : \omega^\omega$ is continuous iff there is $g : \omega^{<\omega} \rightarrow \omega^{<\omega}$ with 1 & 2 satisfied such that $f = \hat{g}$

The **rule of thumb** here is that f is continuous iff in order to determine $f(x)(k)$ you only need $x \upharpoonright n$ for some finite n .

Now consider functions from $(\omega^\omega)^2$ to ω^ω and vice versa and ω^ω to ω^ω :

- $x \mapsto x_I$
- $x \mapsto x_{II}$
- $(x, y) \mapsto x * y$
- $x \mapsto (x_I, x_{II})$

These are all continuous. Moreover, we see that $(\omega^\omega)^2$ and (ω^ω) are homeomorphic by $(x, y) \mapsto x * y$ with inverse $x \mapsto (x_I, x_{II})$. This is unusual, and this phenomenon may explain the name *zero-dimensional*. [Similarly, $(\omega^\omega)^k \cong (\omega^\omega)^\ell$ for any $k, \ell > 0$].

Theorem: *Baire space is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$.*

[The proof uses continued fractions: if $x \in \mathbb{R} \setminus \mathbb{Q}$, then there is a sequence $a_i \in \mathbb{Z}^\omega$ such that $x = [0; a_0, a_1, a_2, \dots]$.]

So while topologically quite different from \mathbb{R} , Baire space is set-theoretically very close to \mathbb{R} : many set theoretic properties/proofs depend only on cardinality, and we have only removed a countable (dense) subset.

Example: ES#1 (4) has choice principles $AC_X(Y)$. These are invariant under replacing X or Y with X', Y' such that X is in bijection with X' and Y is in bijection with Y' .

In particular,

$$\begin{aligned} AC_\omega(\mathbb{R}) &\iff AC_\omega(\omega^\omega) \\ &\iff AC_\omega(2^\omega) \\ &\iff AC_\omega(X) \end{aligned}$$

where X is any set in bijection with \mathbb{R} .

In set theory, we often refer to elements of 2^ω or ω^ω as “reals” and abuse the notation by sometimes writing $\mathbb{R} := \omega^\omega$.

We now repeat our motto/hope: if A is “simple”, then A is determined.

To make precise what ‘simple’ means, we need a complexity hierarchy on Baire space:

Definition: (Borel Hierarchy) Let X be any topological space. We then define **[boldface] sigma zero one** as

$$\Sigma_0^1 := \{A \subseteq X; A \text{ is open in } X\}$$

Then if Σ_α^0 is defined, we define

$$\Pi_\alpha^0 := \{X \setminus A; A \in \Sigma_\alpha^0\}$$

If α is an ordinal and for all $\gamma < \alpha$, Π_γ^0 is defined, then

$$\Sigma_\alpha^0 := \{A; \exists (A_n) \text{ such that } \forall n \ A_n \in \bigcup_{\gamma < \alpha} \Pi_\gamma^0 \text{ and } A = \bigcup_{n \in \mathbb{N}} A_n\}$$

We also have

$$\Delta_\alpha^0 := \Sigma_\alpha^0 \cap \Pi_\alpha^0$$

So we get the Σ s from countable unions, and the Π s from complements.

Properties: By definition, $\Delta_\alpha^0 \subseteq \Sigma_\alpha^0, \Pi_\alpha^0$.

Moreover, by definition $\alpha \leq \beta \implies \Sigma_\alpha^0 \subseteq \Sigma_\beta^0$. This also gives us that $\Pi_\alpha^0 \subseteq \Pi_\beta^0$.

To see the full structure of the hierarchy, we need to show that for $\alpha < \beta$, $\Pi_\alpha^0 \subseteq \Sigma_\beta^0$ (equivalently, $\Sigma_\alpha^0 \subseteq \Pi_\beta^0$). This follows from the definition of Σ_β^0 by letting $A_n := A$, so $\bigcup_{n \in \mathbb{N}} A_n = A$.

Question: When does the Borel hierarchy terminate? This will, of course, depend on the topological space.

Definition: (G_δ space) A topological space is called a G_δ **space** if $\Pi_1^0 \subseteq \Pi_2^0$. That is to say, ‘every closed set is a G_δ set’.

Note: every metric space is a G_δ space.

For Cantor and Baire space, we proved that there is a countable topology base of clopen sets, and this implies being in G_δ .

Proposition: If X has a countable, clopen topology base then X is G_δ .

Proof. Let $F \subseteq X$ be closed, and let $G = X \setminus F$; G is open. For every $x \in G$, find B_x in the topology base such that $x \in B_x \subseteq G$. Since B_x is clopen, $X \setminus B_x$ is also open.

Now consider $\{B_x; x \in G\}$. This is countable, since the basis is countable, so write it as $\{B_n; n \in \mathbb{N}\}$. Then $F = \bigcap_{n \in \mathbb{N}} X \setminus B_n \in \Pi_2^0$.

Note that since countability implies wellorderability, no choice is needed. □

This is slightly irrelevant to what we wanted to do, but it felt like it was worth covering.

In principle, the Borel Hierarchy is defined on arbitrary spaces, but it of course cannot go on forever; it must terminate on some ordinal, and this may differ between spaces.

Question: By the Axiom of Replacement, the Borel Hierarchy terminates at some ordinal α [i.e., $\Sigma_\alpha^0 = \Pi_\alpha^0$]; what can we say about α ?

The upper bound that we get by just using Replacement is pretty bad.

Observations:

1. If X is discrete, then every subset of X is clopen, so

$$\Delta_1^0 = \Sigma_1^0 = \Pi_1^0$$

2. If singletons are closed and X is countable, then

$$\Delta_2^0 = \Sigma_2^0 = \Pi_2^0$$

[If $A = \{x; x \in A\}$ is countable, then $A = \bigcup_{x \in A} \{x\}$]

We can obtain a better upper bound than by the cardinality of X .

Proposition (ZFC): For arbitrary X , $\Delta_{\aleph_1}^0 = \Sigma_{\aleph_1}^0 = \Pi_{\aleph_1}^0$.

Proof. It is enough to show that

$$\Sigma_{\aleph_1}^0 = \bigcup_{\alpha < \aleph_1} \Pi_\alpha^0$$

The \supseteq inclusion is clear. For the other direction, consider the following.

If $A \in \Sigma_{\aleph_1}^0$, there are A_n such that $A = \bigcup_{n \in \mathbb{N}} A_n$ and $\alpha_n < \aleph_1$ such that $A_n \in \Pi_{\alpha_n}^0$. Since \aleph_1 is a regular cardinal, every countable subset $A \subseteq \aleph_1$ is bounded, i.e. there is $\beta < \aleph_1$ such that $A \subseteq \beta$.

So we then look at the countable subset $\{\alpha_n; n \in \mathbb{N}\} \subseteq \aleph_1$, and find for it a countable bound $\beta < \aleph_1$. Then $\{\alpha_n; n \in \mathbb{N}\} \subseteq \beta$. Then for all n , $A_n \in \bigcup_{\alpha < \beta} \Pi_\alpha^0$. Hence $A \in \Sigma_{\beta+1}^0 \subseteq \Pi_{\beta+2}^0$. But $\beta + 2$ is still countable, so A is in the union as in the claim.

The AC required was in showing that \aleph_1 is regular. □

Hence the height of the Borel hierarchy (in ZFC) is $1 \leq \beta \leq \aleph_1$.

Theorem: (ZFC) If X is Cantor space, Baire space or \mathbb{R} , then the height of the Borel hierarchy is \aleph_1 .

[This is not just the case for these spaces; in general, this holds if X is an uncountable Polish space.]

This means that if $\alpha < \aleph_1$, then $\Sigma_\alpha^0 \neq \Pi_\alpha^0$. The proof of the theorem uses the **method of universal sets**.

Definition: (Pointclass) A **pointclass** is an operation that assigns to each topological space X a set of subsets of X .

Examples:

- “open” / Σ_1^0
- “closed” / Π_1^0
- $\Sigma_\alpha^0 / \Pi_\alpha^0 / \Delta_\alpha^0$

If Γ is a pointclass, we define $\check{\Gamma}$ by

$$\check{\Gamma}(X) := \{X \setminus A; A \in \Gamma(X)\}$$

called the **dual pointclass** of Γ , pronounced “Gamma dual”, and Δ_Γ by

$$\Delta_\Gamma(X) := \Gamma(X) \cap \check{\Gamma}(X)$$

called the **ambiguous pointclass** of Γ .

For example, Π_α^0 is $\check{\Sigma}_\alpha^0$, and Δ_α^0 is $\Delta_{\Sigma_\alpha^0}$ (and $\Delta_{\Pi_\alpha^0}$).

Closure Properties of Pointclasses

	YES	NO
closed under finite unions	$\Sigma_\alpha^0, \Pi_\alpha^0, \Delta_\alpha^0$	
closed under finite intersections	$\Sigma_\alpha^0, \Pi_\alpha^0, \Delta_\alpha^0$	
closed under countable unions	Σ_α^0	$\Pi_\alpha^0, \Delta_\alpha^0$
closed under countable intersections	Π_α^0	$\Sigma_\alpha^0, \Delta_\alpha^0$
closed under complements	Δ_α^0	$\Sigma_\alpha^0, \Pi_\alpha^0$

Here, ‘no’ means ‘if the space is Baire space and α is countable then no’ (*i.e.* correctly chosen α and space X).

These properties are entirely local; we have another closure property that links the meaning of pointclasses in different spaces.

We say that Γ is **closed under continuous pre-images** if whenever $f : X \rightarrow Y$ is continuous and $A \in \Gamma(Y)$, then $f^{-1}[A] \in \Gamma(X)$. This property is also called **boldface**; this is silly because the notion has been named after the notation used. A simple inductive argument shows that this is consistent with the ‘boldface’ notation we have been using for the Borel classes $\Sigma_\alpha^0, \Pi_\alpha^0, \Delta_\alpha^0$.

Similarly, Γ is **closed under continuous images** if whenever $f : X \rightarrow Y$ is continuous and $A \in \Gamma(X)$ then $f[A] \in \Gamma(Y)$.

We’re going to talk more about continuous images in Lectures 9 & 10.

By restricting our focus to boldface pointclasses, we eliminate any pathological issues whereby pointclasses among different spaces do not correspond in any way.

Definition: Let X, Y be topological spaces, and Γ a pointclass. A set $U \subseteq X \times Y$ is called **X -universal for $\Gamma(Y)$** if:

1. $U \in \Gamma(X \times Y)$ (under the product topology)
2. For every $A \in \Gamma(Y)$, there is some $x \in X$ such that $U_x = A$, where $U_x = \{y \in Y; (x, y) \in U\}$ is the ‘section of U at x ’.

Lemma: Suppose U is X -universal for $\Gamma(X)$ and Γ is boldface. Then $\Gamma(X) \neq \check{\Gamma}(X)$ (**non-selfdual**).

Proof. Consider $X \times X \setminus U \in \check{\Gamma}(X \times X)$, and consider $x \mapsto (x, x)$ the diagonal map $X \rightarrow X \times X$, which is continuous. Let $D := \{x; (x, x) \notin U\} \in \check{\Gamma}(X)$.

Assume that $\Gamma(X) = \check{\Gamma}(X)$. Then find $d \in X$ such that $D = U_d$. Then

$$\begin{aligned} d \in D &\iff (d, d) \in U \\ &\iff d \in U_d \\ &\iff d \notin D \end{aligned}$$

which is a contradiction. \square

This is once again a standard diagonalisation proof that one often encounters in Logic & Set Theory, *e.g.* uncountability of the reals, the halting problem, *etc...*

In Lecture 9, will prove that for $\alpha < \aleph_1$, Σ_α^0 has an ω^ω -universal set. Then by the Lemma, this implies our theorem.

We will use AC fairly liberally, but keeping track of our uses of it when we are done.

Proof. (Proof of Theorem).

Proof by induction:

- Induction base Σ_1^0
- Complementation step $\Sigma \rightarrow \Pi$
- Countable union step $\Pi \rightarrow \Sigma$

This will be done in three lemmas.

Lemma 1: *There is an ω^ω -universal set for $\Sigma_1^0(\omega^\omega)$*

Proof of Lemma 1: an open set is an arbitrary union of basic open sets. Since there are only countably many basic open sets, we can say this is in fact a countable union. So we can write

$$P = \bigcup_{i \in I} [s_i]$$

where I is a countable index set.

Let $\{s_i; i \in \mathbb{N}\}$ be your favourite enumeration of $\omega^{<\omega}$. Thus $\{[s_i]; i \in \mathbb{N}\}$ is an enumeration of the basic open sets. Then define

$$U := \{(x, y); \exists i \in \mathbb{N}, (x(i) \neq 0) \wedge (s_i \subseteq y)\}$$

This is universal for $\Sigma_1^0(\omega^\omega)$:

1. U is open: if $(x, y) \in U$ then there is $i \in \mathbb{N}$ with $x(i) \neq 0$ and $s_i \subseteq y$. Define $t = x \upharpoonright (i+1)$. Then $[t, s_i] \subseteq U$. Note that we haven't exactly defined basic open sets on Baire space squared, but the notation is intuitive/obvious.
2. If P is open, then let

$$x(i) := \begin{cases} 1 & \text{if } [s_i] \subseteq P \\ 0 & \text{if } [s_i] \not\subseteq P \end{cases}$$

Then $P = \bigcup_{x(i) \neq 0} [s_i]$. Thus $P = U_x$.

Wo we are done. \square

Lemma 2: *If $U \subseteq X \times X$ is X -universal for $\Gamma(X)$, then $X \times X \setminus U$ is X -universal for $\check{\Gamma}(X)$.*

Proof of Lemma 2: Obvious.

Lemma 3: *Let $\lambda < \omega_1$. Suppose that for each $\alpha < \lambda$, there is an ω^ω -universal set U_α for $\Pi_\alpha^0(\omega^\omega)$. Then there is an ω^ω -universal set for Σ_λ^0 .*

Proof of Lemma 3:

If λ is a successor ordinal, i.e. $\lambda = \mu + 1$, then let $\alpha_n := \mu$ for all n . If λ is a limit, pick a sequence α_n such that $\bigcup \alpha_n = \lambda$.

Observe that if $A = \bigcup_{n \in \mathbb{N}} A_n$, where $A_n \in \bigcup_{\alpha < \lambda} \Pi_\alpha^0$, then I find (by “postponing if necessary”) a sequence $A'_n \in \Pi_{\alpha_n}^0$ such that

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} A'_n \quad (*)$$

To simplify notation, write $U_n := U_{\alpha_n}$.

Encoding countable sequences of ω^ω by an element of ω^ω :

Take your favourite bijection $[\cdot, \cdot] : \omega \times \omega \rightarrow \omega$. If $x \in \omega^\omega$ and $n \in \mathbb{N}$, define $(x)_n \in \omega^\omega$ by $(x)_n(m) := x([n, m])$. Then $x \mapsto ((x)_n; n \in \mathbb{N})$ is a bijection between ω^ω and $(\omega^\omega)^\omega$.

Note also that the map $x \mapsto (x)_n$ is continuous, because we only require a finite amount of information to determine what x at n is, and by our earlier characterisation this means we have continuity.

Now define

$$U := \{(x, y); \exists n ((x)_n, y) \in U_n\}$$

Claim: U is ω^ω -universal for Σ_λ^0 .

- $\overline{U}_n := \{(x, y); ((x)_n, y) \in U_n\}$ is the pre-image of U_n under the continuous map $x, y \mapsto ((x)_n, y)$, so it's in $\Pi_{\alpha_n}^0$ [closure of the Borel classes under continuous pre-images]. But $U = \bigcup_{n \in \mathbb{N}} \overline{U}_n$, so U is σ_λ^0 .
- Let A be σ_λ^0 . By $(*)$, we find $A_n \in \Pi_{\alpha_n}^0$ such that $A = \bigcup_{n \in \mathbb{N}} A_n$. By universality of U_n , we find x_n such that $A_n = (U_n)_{x_n}$. Now fold up the sequence $(x_n; n \in \omega)$ into a single element of ω^ω such that $(x)_n = x_n$ for all n . We do this by defining $x([n, m]) := x_n(m)$.

Here's the situation:

1. $A_n = (U_n)_{x_n}$
2. $(x)_n = x_n$
3. $A = \bigcup_{n \in \mathbb{N}} A_n$
4. $U = \{(x, y); \exists n ((x)_n, y) \in U_n\}$

Claim: $U_x = A$:

$$\begin{aligned} y \in A &\stackrel{3}{\iff} \exists n y \in A_n \\ &\stackrel{1}{\iff} \exists n y \in (U_n)_{x_n} \\ &\iff \exists n (x_n, y) \in U_n \\ &\stackrel{2}{\iff} \exists n ((x)_n, y) \in U_n \\ &\stackrel{4}{\iff} (x, y) \in U \\ &\stackrel{y}{\iff} y \in U_x \end{aligned}$$

Corollary: Borel Hierarchy Theorem

Proof. This is an inductive proof using L1 - L3. [Recursive definition of a ω^ω -universal set for Σ_λ^0 for each $\lambda < \omega_1$.] \square

Remark: Let's check how much choice we used.

Lemma 1 is a ZF theorem.

Lemma 2 is a ZF theorem.

Lemma 3: $\lambda \mapsto$ pick α_n such that $\bigcup \alpha_n = \lambda$; if λ fixed, no choice is needed. However, after declaring that there *exists* a universal set U_n for each n , we then picked a concrete example of one for each n to play around with. So this needed choice. Perhaps it would have been more prudent to simply formulate Lemma 3 by just assuming that there exists a choice function *only where we need it* for the universal sets. Then we don't need to use the more general Axiom of Choice.

We also made another choice in the presentation of A_n ; if $A \in \Sigma_\lambda^0$, then

$$S_A := \{(A_n)_n; A = \bigcup_{n \in \mathbb{N}} A_n\} \neq \emptyset$$

for every such A , so we use a choice function for this family too.

But we are still not done; while we didn't need choice for λ fixed in Lemma 3, we have to use Lemma 3 infinitely many times in the proof of the corollary; so we need choice to pick for each limit $\lambda < \omega_1$ a sequence α_n in order to apply L3.

Back to games.

Notation: if Γ is a pointclass, write $\text{Det}(\Gamma)$ for " $\forall A \in \Gamma(\omega^\omega)$, A is determined".

G-S proved $\text{Det}(\Pi_1^0)$. The proof also implies $\text{Det}(\Sigma_1^0)$.

ES#1 (11): in general, the class of determined sets is not closed under complementation.

ES#1 (10): if a pointclass Γ is closed under the operation $A \mapsto \{mx; x \in A\}$, then $\text{Det}(\Gamma) \implies \text{Det}(\check{\Gamma})$. Note that Borel pointclasses are closed under this operation.

Wolfe (1955) proved $\text{Det}(\Sigma_2^0)$; Davis (1964) proved $\text{Det}(\Sigma_3^0)$; Paris (1972) proved $\text{Det}(\Sigma_4^0)$.

Friedman proved that you cannot prove $\text{Det}(\Sigma_5^0)$ without using iterations of the power set axiom. This paved the way for Martin (1975) to prove that *all* Borel sets are determined.

The Borel sets, from the point of view of the ordinary analyst, look extremely complicated already. So while they are not all the sets, you might expect that all 'reasonable' (for some definition of reasonable) sets *are* Borel, so a proof that every Borel set has some property is in practice equivalent to proving that all sets have that property. But this might not be the case, which begs that question: what *is* the size of the set of Borel sets?

Denote the set of all Borel sets by \mathcal{B} . Clearly then $|\mathcal{B}| \leq 2^{2^{\aleph_0}}$, but we can show that it is in fact much smaller:

Theorem: (*ZFC*)

$$|\mathcal{B}| = 2^{\aleph_0} < 2^{2^{\aleph_0}}$$

Proof. Since $\{x\} \in \mathcal{B}$ for every $x \in \omega^\omega$, $x \mapsto \{x\}$ is an injectino from ω^ω into \mathcal{B} . So $2^{\aleph_0} \leq |\mathcal{B}|$.

Upper Bound: Proof by induction.

We prove that $|\Sigma_\alpha^0| = 2^{\aleph_0}$ for all α . This then implies the Theorem:

$$|\mathcal{B}| = \left| \bigcup_{\alpha < \omega_1} \Sigma_\alpha^0 \right| \leq \aleph_1 \cdot 2^{\aleph_0} = 2^{\aleph_0}$$

Base case: for Σ_1^0 , every open set is of the form

$$\bigcup_{i \in I} [s_i]$$

where $I \subseteq \mathbb{N}$ and s_i is our enumeration of $\omega^{<\omega}$. So $I \mapsto \bigcup_{i \in I} [s_i]$ is a surjection from $\mathcal{P}\mathbb{N}$ onto Σ_1^0 . Hence $|\Sigma_1^0| \leq 2^{\aleph_0}$.

Going from $\Sigma \rightarrow \Pi$ is easy, since $A \mapsto \omega^\omega \setminus A$ is a bijection between Σ_α^0 and Π_α^0 , hence $|\Sigma_\alpha^0| = |\Pi_\alpha^0|$.

From $\Pi \rightarrow \Sigma$: Suppose for each $\alpha < \lambda$, we have $|\Pi_\alpha^0| \leq 2^{\aleph_0}$. Using AC, pick surjections $s_\alpha : \omega^\omega \twoheadrightarrow \Pi_\alpha^0$. Define surjection S by

$$\begin{aligned} S : \lambda^\omega \times \omega^\omega &\rightarrow \Sigma_\lambda^0 \\ ((\alpha_i; i \in \omega), x) &\mapsto \bigcup_{i \in \mathbb{N}} S_{\alpha_i}((x)_i) \end{aligned}$$

This clearly is a surjection. Since λ is countable, $|\lambda^\omega| = 2^{\aleph_0}$. So $|\lambda^\omega \times \omega^\omega| = 2^{\aleph_0}$. \square

Remark: We used AC in this proof, and that is not avoidable; consider the *Feferman-Levy Model* \mathcal{M} , which models the theory ‘ZF + “ \mathbb{R} is a countable union of countable sets”’.

In the FL model, $\mathbb{R} = \bigcup_{n \in \mathbb{N}} A_n$. So if $X \subseteq \mathbb{R}$, then $X = \bigcup_{n \in \mathbb{N}} A_n \cap X$. Since $A_n \cap X \subseteq A_n$, this is still countable. Hence, in this model, every subset of the reals is a countable union of countable sets. Now, all countable sets are Borel since they are a countable union of closed sets. But the Borel sets are closed under countable unions, so this means that *every* set is Borel. In particular, $|\mathcal{B}| = \mathcal{P}\mathbb{R}$. So we needed AC after all.

So the Borel sets are in fact only a very small part of the collection of all sets in Baire space. What else is out there?

The famous mistake of Henri Lebesgue.

Measures are defined on σ -algebras \mathcal{A} . In general, $\mathcal{A} \neq \mathcal{P}\mathbb{R}$. [Involves AC; Vitali set is a non-Lebesgue-measurable set that can be constructed AC.]

The smallest σ -algebra, namely the Borel σ -algebra (the smallest one containing the open sets), is the minimal setting for measures, and Lebesgue made an argument for this:

Lebesgue: “All sets we care about are Borel.”

He believed that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and if $A \subseteq \mathbb{R}$ is Borel, then $f[A]$ is Borel. However, this is in fact FALSE.

The mistake was spotted in 1917, when when Suslin proved that there are non-Borel sets which are continuous images of Borel sets - these are called *analytic sets*. We can in fact prove this. We don’t even need the continuous functions to be complicated at all (projection).

Definition: (Projection) Let $A \subseteq X^{n+1}$. Then we call

$$B = \{(x_1, \dots, x_n); \exists x \in X (x, x_1, \dots, x_n) \in A\}$$

the *projection* of A . Write $pA = B$.

The map

$$\pi : (x, x_1, \dots, x_n) \mapsto (x_1, \dots, x_n)$$

is clearly continuous, so $pA = \pi[A]$, and thus projections are indeed special cases of continuous images.

Definition: Let Γ be a pointclass. We define $\exists^{\omega^\omega} \Gamma$ a pointclass by:

$$\exists^{\omega^\omega} \Gamma(X) := \{pA; A \in \Gamma(\omega^\omega \times X)\}$$

We say Γ is **closed under projections** if $\exists^{\omega^\omega} \Gamma \subseteq \Gamma$.

Suslin's Theorem now says: "the pointclass BOREL is not closed under projections".

Definition: (The Projective Hierarchy) We start by defining $\Pi_0^1(\omega^\omega) = \Pi_1^0(\omega^\omega)$. [Note that if $X \neq \omega^\omega$, this won't necessarily be the right starting point; we might want to use Π_2^0 instead.]

We then define

$$\begin{aligned}\Sigma_{n+1}^1 &:= \exists^{\omega^\omega} \Pi_n^1 \\ \Pi_{n+1}^1 &:= \check{\Sigma}_{n+1}^1\end{aligned}$$

Σ_1^1 is called the **analytic sets**, and Π_1^1 is called the **co-analytic sets**.

We also define

$$\Delta_n^1(X) := \Sigma_n^1(X) \cap \Pi_n^1(X)$$

Lebesgue believed that all of these were contained in \mathcal{B} ; he was wrong. We already have the technique to prove this, which is the technique of universal sets.

Theorem: *The projective hierarchy does not collapse.*

Proof. Using the technique of universal sets. we know that $\Pi_0^1 = \Pi_1^0$ has a universal set. We know that if Σ_n^1 has a universal set, then Π_n^1 has a universal set. So all that is left to show is that if V is universal for Π_n^1 , then we can find U universal for Σ_{n+1}^1 .

Let $V \subseteq \omega^\omega \times \omega^\omega \times (\omega^\omega)^k$ be universal for Π_n^1 . Define

$$U := \{(0, \vec{x}) \in \omega^\omega \times (\omega^\omega)^k; \exists v \in \omega^\omega (u, v, \vec{x}) \in V\}$$

It is clear that $U \in \Sigma_{n+1}^1(\omega^\omega \times (\omega^\omega)^k)$, and the same x that is the code for the set A [in V] is the code for pA in U . \square

Proposition: *Every Borel set is Σ_1^1*

Proof. (ES#2) \square

Corollary: *Suslin's Theorem: there is a Σ_1^1 set that is not Borel.*

Question: Does $\text{Det}(\Sigma_1^1)$ hold? Perhaps Σ_2^1 , or Σ_3^1 etc...? These questions are much more interesting set-theoretically. These are not ZFC theorems, but are closely connected to:

1. Large Cardinal Axioms
2. Definability of wellorders of ω^ω

These are the two topics that will cover the remainder of this course.