Ramsey Theory: Example Sheet 2

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1. Just work it through.

2. Given a finite colouring c on $\mathbb{N}$, induce a finite colouring d on $\mathbb{N}^{(2)}$ by $d(ij) = c(j-i)$ for $i < j$. Then use Ramsey to find big (infinite) set on which d is constant. Then find i, j, k, ℓ with d constant, so $c(j-i) = c(k-j) = c(k-i) = c(\ell-i) = c(\ell-j) = c(\ell-k)$ *etc.*, so done.

3. You can try pretty much anything and it'll work, can be written out precisely using careful notation.

4. PR over $\mathbb{N}$ obviously implies PR over $\mathbb{Z}$. For the other direction, suppose we're PR over $\mathbb{Z}$ but there's a bad colouring c for $\mathbb{N}$. Construct a bad colouring c' for $\mathbb{Z}$ by $c'(m) = (c(m), 1)$ for $m \in \mathbb{N}$ and $c'(-m) = (c(m), 2)$ for $m \in \mathbb{N}$, where the colours are ordered pairs. Then by PRness find a mono solution, but this must all have the same sign, so its also solved in c.

5. aaaaaaaaaaaaaaah

6. Hindman: Find $x_1, x_2, \dots$ with finite sums monochromatic. Let $f_1 = x_1$, and define f_n inductively such that the least non-zero term in the binary expansion of f_{n+1} occurs later than the last non-zero term in the binary expansion of f_n. We achieve this by looking at the infinite set of things other than the f_is, and looking at their ends. If infinitely many end in 0, move on to the next entry, if only finitely many do then get a new infinite set of things ending in 0 by pairing up things ending in 1 and adding them. Keeping repeating this process, killing off all the terms in the binary expansion until we're clear of all previous f_is. Everything is a finite sum, so we're good.

Then go back to the original colouring of finite subsets; we've been identifying finite subsets of $\mathbb{N}$ with elements of $\mathbb{N}$ by looking at binary expansion; finite sums is nearly what we want, and indeed with disjointness in the above sense we get disjointness in exactly the way we want it.

Without Hindman: [from class:

If we can give them all different sizes $a_1, \dots, a_m$, and then finite sums on the sizes ensures we don't run into any problems with unions; then use Ramsey to find a nice subset where things are actually coloured by size.]

7. No; can have A, A^c both PR. For instance, we can let A and A^c contain copies of $\lambda[n]$ for arbitrarily large n (something like $1, [2, 4], 5, 10, 15, [16, 32, 48, 64], \dots$); then both clearly PR, and obviously the intersection is not PR.

8. Suppose $x_1, x_2, \dots$ is a convergent subsequence, converging to $\mathcal{U}$. Let $A \subseteq \{x_1, x_2, \dots\}$ as numbers, say $A = \{x_{n_1}, x_{n_2}, \dots\}$. This is also a convergent subsequence, converging to $\mathcal{U}$. Then $A \in \mathcal{U}$, since if $A^c \in \mathcal{U}$ then $\mathcal{U} \in C_{A^c}$, so by definition of convergence there exists $x_{n_i} \in A$ also in C_{A^c}, so $A^c \in \tilde{x}_{n_i}$, and in particular, $x_{n_i} \in A^c$. But then $A = \{x_1, x_3, x_5, \dots\}$ and $B = \{x_2, x_4, x_6, \dots\}$ are both in $\mathcal{U}$, so $\emptyset \in \mathcal{U}$, contradiction.

This then means the topology cannot be induced by a metric; if it were, say d, then pick some non-principal $\mathcal{U}$, and consider $B_{1/n}(\mathcal{U})$ for each $n \in \mathbb{N}$. By density, we can always find some $\tilde{m}$ in this open ball, larger and larger m, creating a convergent subsequence of $1, 2, 3, \dots$.

9. Suppose $S = \{s_1, s_2, \dots\}$ is a countable dense subset.

First note that for $A \subseteq \mathbb{N}$ finite, $\mathcal{U}$ ultrafilter, $A \in \mathcal{U} \implies \mathcal{U}$ principal, since A is the countable union of singletons, and if none of the singletons is in $\mathcal{U}$ then the intersection of their complements is, so A itself isnt; contradiction.

Now we find $A \subseteq \mathbb{N}$ infinite with infinite complement, such that $A \notin s_i$ for any i. Then $C_A \cap S = \emptyset$.

To find this set, start with any such $A_1 \notin s_1$, which exists by non-principality. Then look at s_2 and snip A_1 in half (each half infinite with infinite complement); we can't have both halves in s_2, since then their intersection is in s_2 and is empty. Call one of the halves that's not there A_2. So we have $A_1 \supseteq A_2$.

Continue, get $A_1 \supseteq A_2 \supseteq A_3 \dots$ all infinite with infinite complement such that $A_i \notin s_i$. Then define $A = \{a_1, a_2, \dots\}$ with $a_i \in A_i \backslash A_{i+1}$. Now claim A is what we want.

Indeed, $A \notin s_n$; if it were, then since $B_n = \{a_n, a_{n+1}, \dots\} \subseteq A_n$ we have that $B_n \notin s_n$, hence $B_n^c \in s_n$, hence $B_n^c \cap A \in s_n$. But $B_n^c \cap A = \{a_1, a_2, \dots, a_{n-1}\}$, which is finite so s_n principal. Contradiction.

So A lies in no s_i, A is clearly infinite, and $A \subseteq A_1$ so A^c infintie also. So done.

10. If two ultrafilters $\mathcal{U}$ and $\mathcal{V}$ are distinct then we can separate them with open neighbourhoods C_A and C_B.

So Let $\mathcal{U} \in C_{A_i}$ and $\mathcal{U}_i \in C_{B_i}$ such that $\mathcal{U}$ and $\mathcal{U}_i$ are separated. Then $\bigcap C_{A_i} = C_{\bigcap A_i} = C_A$ is disjoint from $\bigcup C_{B_i}$, and in particular $\mathcal{U}_i \notin C_A$ for all i, but $\mathcal{U} \in C_A$. Hence we have found $A \in \mathcal{U}$ such that $A \notin \mathcal{U}_i$ for all i.

If we have infinitely many $\mathcal{U}_i$, then just take $\mathcal{U}_i = \tilde{i}$; any $A \in \mathcal{U}$ is also in any $\mathcal{U}_i$ for which $i \in A$.

If instead each $\mathcal{U}_i$ is non-principal, we have seen already that the $\mathcal{U}_i$s do not form a dense subset and as such there is some C_A disjoint from the set of $\mathcal{U}_i$, so it's certainly possible for there to be $A \in \mathcal{U}$ with $A \notin \mathcal{U}_i$ for all i, and indeed it seems likely that this is always the case.

We are looking for an open set containing $\mathcal{U}$ that contains no $\mathcal{U}_i$; if no such set exists then we must have $\mathcal{U}_i \to \mathcal{U}$, and indeed in this case no such set will exist since there is some $\mathcal{U}_i$ in every open set containing $\mathcal{U}$.

Suppose the $\mathcal{U}_i$ have a subsequence converging to $\mathcal{U}$. Then for any $A \in \mathcal{U}$, there exists some $\mathcal{U}_i \in C_A$ by the definition of convergence, hence $A \in \mathcal{U}_i$ for some i. So in this case the claim fails.

If the $\mathcal{U}_i$ do not have a subsequence converging to $\mathcal{U}$,