Model Theory: Sheet 1

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1. Let φ_n be the $\mathcal{L}$-sentence

$$\exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} \neg(x_i = x_j)$$

Then let $T' = T \cup \{\varphi_n : n \in \mathcal{N}\}$. T has arbitrarily large finite models, so T' is finitely satisfiable. Hence T' has a model, which must both be infinite and a model of T. $\qquad\square$

2. (ii) is equivalent to the claim that either $T \models \varphi$ or $T \models \neg\varphi$, which is the definition of completeness as in (i).

(i) $\Longrightarrow$ (iii): Let $\mathcal{M}, \mathcal{N}$ be models of T, and φ and $\mathcal{L}$-sentence. By definition, $T \models \varphi$ implies $\mathcal{M} \models \varphi$ and $\mathcal{N} \models \varphi$. Otherwise, $T \models \neg\varphi$ and so do $\mathcal{M}, \mathcal{N}$, so $\mathcal{M} \equiv \mathcal{N}$.

(iii) $\Longrightarrow$ (i): Suppose T not complete. Then there exists a sentence φ such that both $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ are consistent, and thus have models. But any two models are elementarily equivalent, and no model can satisfy both φ and $\neg\varphi$. $\qquad\square$

3. Let $\mathcal{M}, \mathcal{N}$ be two infinite sets of cardinality κ. Then there exist bijections $f : M \to \kappa$, $g : N \to \kappa$, so we define $h : M \to N$ by $h = g^{-1} \circ f$. Owing to the lack of function symbols, relation symbols or constant symbols in $\mathcal{L}$, h is in fact a homomorphism $h : \mathcal{M} \to \mathcal{N}$, and since it is bijective it is also an isomorphism. So $\mathcal{M} \equiv \mathcal{N}$, and the theory of infinite sets is κ-categorical for any infinite κ.

The theory of infinite sets has no finite models, so by Vaught's Test it is complete. $\qquad\square$

4. (note: heavily AC dependent) We have $\mathcal{L}$ the language of groups: $\mathcal{L} = \{+, 0\}$ (chosen thusly as our groups are going to be abelian).

Let φ_n denote the $\mathcal{L}$-sentence $\forall x(x = 0 \vee nx \neq 0)$, and ψ_m the sentence $\forall x \exists y(x = my)$. Our theory of nontrivial torsion-free divisible abelian groups is then

$$\begin{aligned}
T = \ & \{\forall x \exists y(x + y = y + x = 0)\} \\
& \cup \{\forall x(x + 0 = 0 + x = x)\} \\
& \cup \{\forall x \forall y \forall z((x + y) + z = x + (y + z))\} \\
& \cup \{\forall x \forall y(x + y = y + x)\} \\
& \cup \{\varphi_n : n \in \mathbb{N}\} \\
& \cup \{\psi_n : n \in \mathbb{N}\}
\end{aligned}$$

This theory is complete because it is κ-categorical for all $\kappa \geq \aleph_1$; we need only use $\kappa = \aleph_1$.

First, note that the divisibility property is unique, *i.e.* for $x \in G$ and $n > 0$, there is a unique $y \in G$ such that $x = ny$. Indeed, if $ny = nz$, then $n(y - z) = 0$ and since G is torsion-free we must have $y - z = 0$, so $y = z$. In particular, given any $x \in G$ and $n > 0$ we can express the unique such element as x/n.

We can then add elements in the natural way: $x/n + y/n = (x + y)/n$, $m(x/n) = (mx)/n$, and so $x/n + y/m = (mx + ny)/(mn)$.

Denote by $\langle X \rangle$ the torsion-free divisible abelian group generated by X, *i.e.*

$$\langle X \rangle = \left\{ \frac{a_1 x_1 + \cdots + a_k x_k}{n} : a_i \in \mathbb{Z}, k, n \in \mathbb{N}, \ n > 0, x_i \in X \right\}$$

Let G be a model for T as above, of cardinality κ - this exists by Lowenheim-Skolem, noting that T is satisfiable since $(\mathbb{Q}, +)$ is a model.

Let $(P, <)$ be the partial order defined on $\mathcal{P}G$ by $X \leq Y$ iff $\langle X \rangle \leq \langle Y \rangle$ as groups.

Let $\mathcal{C} = \{C_i : i \in I\}$ be a chain in P. Claim that $C = \bigcup_{i \in I} C_i$ is an upper bound for $\mathcal{C}$. Indeed, it is clear that $\langle C_i \rangle \leq \langle C \rangle$ for each $i \in I$, so this result is immediate.

So by Zorn's Lemma, P has a maximal element S. Then we must have $\langle S \rangle = G$, else we have some $x \in G \backslash \langle S \rangle$, so $\langle S \cup \{x\} \rangle$ is a strictly larger subgroup.

We then need to use AC again to obtain an independent $S' \subset S$ with $\langle S' \rangle = \langle S \rangle$, i.e. for all $x \in S'$, $x \notin \langle S' \backslash \{x\} \rangle$. Informally, this amounts to going through S and removing unnecessary elements until all remaining elements are necessary. We lose nothing each time we throw away an unnecessary element.

Finally, and this was the whole point, we must have $|S'| = \kappa$. If S' were countable, then so would be G, as we would be able to identify $\langle S' \rangle$ with finite rational sequences, the set of which is countable. We of course cannot have $|S'| > \kappa$ either.

Thus for any G with $|G| = \kappa$, we have $G = \langle S_G \rangle$ for some independent $S_G \subset G$ with $|S_G| = \kappa$. So for any two G, H, any bijection $f : S_G \to S_H$ induces a group isomorphism $\overline{f} : G \to H$ in the natural way (e.g. if $S_G = \{x_i : i < \kappa\}$ and $S_H = \{y_i : i < \kappa\}$, we have a group isomorphism $G \to H$ by 'change any x to a y'). Thus $G \cong H$.

So T is κ-categorical for $\kappa = \aleph_1$, and indeed for any $\kappa \geq \aleph_1$ by a very similar proof (a minor tweak about cardinality is needed at the end).

Hence T is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. The phrasing of the question is highly suggestive; we want to reframe subsets of V as theories, and k-colourings as models.

Let $\mathcal{L} = \{e, c\}$ where e, c are both binary relation symbols, and let T assert that e is an edge relation and c an equivalence relation: $\forall x \neg e(x, x)$, $\forall x \forall y (e(x, y) \iff e(y, x))$, c the usual equivalence relation axioms (symmetry, reflexivity, transitivity).

Also include axioms that assert c is a colouring, i.e. $\forall x \forall y (e(x, y) \implies \neg c(x, y))$.

Moreover, we require that c is a k-colouring: $\forall x_1 \ldots \forall x_{k+1} \bigvee_{i \neq j} c(x_i, x_j)$ - i.e. there are at most k equivalence classes of c.

We denote this whole theory as T. Its models are precisely k-colourable graphs equipped with a k-colouring.

Now onto the specific case of a graph (V, E). Let its vertex set be written $V = \{v_i : i \in I\}$. We augment our language to $\mathcal{L}^* = \mathcal{L} \cup \{v_i : i \in I\}$, and augment our theory to $T^* = T \cup \{v_i \neq v_j : i \neq j\} \cup \{e(v_i, v_j) : v_i v_j \in E\}$.

By assumption, every finite subset of T^* is satisfied by some finite subgraph of (V, E), so by the compactness theorem T^* has a model (V', E', C') with $V \subset V'$, $E \subset E'$ and C' a k-colouring of (V', E'). Thus we may exhibit a k-colouring of (V, E) by simply restricting C'. Hence (V, E) is k-colourable. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6. $\implies$ was done in lectures.

For $\impliedby$, we need to show that $h : \mathcal{M} \hookrightarrow \mathcal{N}$ is an $\mathcal{L}$-embedding.

Firstly, let $\varphi(x_1, \ldots, x_n)$ be $R(x_1, \ldots, x_n)$ for R an n-ary relation symbol in $\mathcal{L}$. Then for any $a_i \in M$, $\mathcal{M} \models \varphi(a_1, \ldots, a_n)$ states $(a_1, \ldots, a_n) \in R^{\mathcal{M}}$, so by assumption we have $(a_1, \ldots, a_n) \in R^M \iff (a_1, \ldots, a_n) \in R^{\mathcal{M}}$, so h preserves relations.

Next, let c be a constant symbol in $\mathcal{L}$, and let $\varphi(x)$ be the L-formula $(x = c)$. We then have $\mathcal{M} \models \varphi(c^{\mathcal{M}})$, and so $\mathcal{N} \models \varphi(c^{\mathcal{M}})$, hence $c^{\mathcal{M}} = c^{\mathcal{N}}$.

For a general n-ary function symbol $f \in \mathcal{L}$, we may adopt a similar approach. Let $\varphi(x_1, \ldots, x_n, y)$ be the statement $f(x_1, \ldots, x_n) = y$. Then we have, for $a_1, \ldots, a_n, b \in M$:

$$\begin{aligned}
f^{\mathcal{M}}(a_1, \ldots, a_n) = b &\iff \mathcal{M} \models \varphi(a_1, \ldots, a_n, b) \\
&\iff \mathcal{N} \models \varphi(a_1, \ldots, a_n, b) \\
&\iff f^{\mathcal{N}}(a_1, \ldots, a_n) = b
\end{aligned}$$

From which we conclude $f^{\mathcal{M}}(a_1, \ldots, a_n) = f^{\mathcal{N}}(a_1, \ldots, a_n)$, hence h is indeed an embedding. $\qquad\square$

7. We probably haven't covered this exactly yet, but I'm assuming the substructure *generated by* A is the smallest substructure $\mathcal{M} \subset \mathcal{N}$ such that $A \subset M$ - the notion of *smallest* is well defined since if $\mathcal{M}, \mathcal{M}'$ are substructures then so is $\mathcal{M} \cap \mathcal{M}'$.

In particular, we need only show that there exists some substructure of $\mathcal{N}$ satisfying the given property.

The outline of the argument is to find a set $A^* \subset N$ closed under all function symbols in $\mathcal{L}$, with relations induced by $\mathcal{N}$. Then this is a substructure containing A.

Define $A_0 = A \cup \{c^{\mathcal{N}} : c \in \mathcal{L} \text{ is constant}\}$. Then given A_n, define $A_{n+1} = A_n \cup \{f^{\mathcal{N}}(a_1 \ldots, a_{\alpha(f)}) : f \in \mathcal{L} \text{ a function symbol}, a_i \in A_n\}$.

We then define $A^* = \bigcup_{n=0}^{\infty} A_n$. Then A^* is closed under all functions $f^{\mathcal{N}}$, since for $a_1, \ldots, a_k \in A^*$ we must have $a_1, \ldots, a_k \in A_m$ for some m, hence $f^{\mathcal{N}}(a_1, \ldots, a_m) \in A_{m+1} \subset A^*$. Hence restricting $R^{\mathcal{N}}$ to $A^{\alpha(R)}$ gives an $\mathcal{L}$-structure $\mathcal{B}$ with $A \subset B$.

We now show that $\mathcal{B}$ satisfies the desired property; we do this by induction. For the base case, consider $b \in A_0$. Then either b is a constant (so done), or $b = x^{\mathcal{N}}(a_i) = x^{\mathcal{B}}(a_i)$ for some $a_i \in A$, x a variable.

Now suppose $b \in A_n$. Then $b = f^{\mathcal{N}}(x_1, \ldots, x_k) = f^{\mathcal{B}}(x_1, \ldots, x_k)$, with $x_i \in A_{n-1}$. Hence by induction, $b = f^{\mathcal{B}}(t_1^{\mathcal{B}}(a_1^1, a_2^1, \ldots, a_{n_1}^1), \ldots) = t^{\mathcal{B}}(a_1, \ldots, a_m)$ for an $\mathcal{L}$-term t and some $a_i \in A$.

Then since $A \subset B$, this result also holds for $\langle A \rangle \subset \mathcal{B}$ (these structures should be the same, but that isn't technically important). $\qquad\square$

8. (a) The universe of $\mathcal{N}$ is given by $N = \bigcup_{i < \alpha} M_i$. For each function symbol $f \in \mathcal{L}$, we interpret $f^{\mathcal{N}} = \bigcup_{i < \alpha} f^{\mathcal{M}_i}$ (as set-theoretic functions), and similarly $R^{\mathcal{N}} = \bigcup_{i < \alpha} R^{\mathcal{M}_i}$. Equivalently, for $x_j \in N$ we can define $f^{\mathcal{N}}(x_1, \ldots, x_n) = f^{\mathcal{M}_i}(x_1, \ldots, x_n)$ for some sufficiently large i; this is well-defined since such an i must exist, and all such $\mathcal{M}_i$ agree on the value $f^{\mathcal{M}_i}(x_1, \ldots, x_n)$. Similarly for the relation symbols R.

So $\mathcal{N}$ is a well-defined $\mathcal{L}$-structure. From the above it is also clear that $\mathcal{M}_i \subset \mathcal{N}$, by definition of $\mathcal{M}_i$; $f^{\mathcal{M}_i}(x_1, \ldots, x_n) =: f^{\mathcal{N}}(x_1, \ldots, x_n)$, and similarly for the relation symbols.

(b) Since $\mathcal{M}_i \subset \mathcal{N}$, we have that $\mathcal{M}_i$ and $\mathcal{N}$ agree on all quantifier-free formulae as in Q6. So we need only show this holds for quantified formulae.

Suppose that for all i and $a_1, \ldots, a_n \in \mathcal{M}_i$, $\mathcal{M}_i$ and $\mathcal{N}$ agree on $\varphi(a_1, \ldots, a_n)$. Consider the formula ψ given by $\forall x_n \varphi(x_1, \ldots, x_n)$, and suppose that $\mathcal{M}_i \models \psi(a_1, \ldots, a_{n-1})$. Then for every $b \in M_i$, we have $\mathcal{M}_i \models \varphi(a_1, \ldots, a_{n-1}, b)$. Let $d \in \mathcal{N}$. Then $d \in M_j$ for some j, and since $\mathcal{M}_i \preceq \mathcal{M}_j$ we have $\mathcal{M}_j \models \varphi(a_1, \ldots, a_{n-1}, d)$. By induction on φ, $\mathcal{M}_j \models \varphi(a_1, \ldots, a_{n-1}, d) \iff \mathcal{N} \models \varphi(a_1, \ldots, a_{n-1}, d)$. Hence for all $d \in \mathcal{N}$, $\mathcal{N} \models \varphi(a_1, \ldots, a_{n-1}, d)$. Therefore $\mathcal{N} \models \psi(a_1, \ldots, a_{n-1})$.

Thus $\mathcal{M}_i \preceq \mathcal{N}$ for all $i < \alpha$. $\qquad\square$

9. We first remark that if $B \subset M$, then $\mathcal{B}$, the substructure generated by B, has size $|B| + \aleph_0$, since it is an $\aleph_0$-sized union of things of size at most $\aleph_0 + |B|$.

So we let $A_0 = A$, and $\mathcal{A}_0$ the substructure of $\mathcal{M}$ generated by A_0. Then for every $\varphi(x, y_1, \ldots, y_n)$ such that $\mathcal{M} \models \exists x \varphi(x, a_1, \ldots, a_n)$ for $a_i \in \mathcal{A}_0$, find a constant in m in $\mathcal{M}$ such that $\mathcal{M} \models \varphi(m, a_1, \ldots, a_n)$

and add these constants to the universe of $\mathcal{A}_0$ to obtain a set A_1, and induced substructure $\mathcal{A}_1$, with A still contained in $\mathcal{A}_1$.

The issue now is that there may be more statements $\exists x \varphi(x, a_1, \ldots, a_n)$ that $\mathcal{M}$ satisfies with elements a_i in $\mathcal{A}_1$ but not $\mathcal{A}_0$. So we repeat this process again, to obtain a chain of substructures $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2, \ldots$. Let $\mathcal{N}$ be the union of these substructures.

Since each $\mathcal{A}_i$ has size $\leq |A| + |\mathcal{L}| + \aleph_0$, it must be the case that $\mathcal{N} \leq |A| + |\mathcal{L}| + \aleph_0$, since multiplication by $\aleph_0$ does not change this quantity.

Moreover, for every $a_1, \ldots, a_n \in \mathcal{N}$ we have that $\mathcal{M} \models \exists x \varphi(x, a_1, \ldots, a_n) \implies \mathcal{N} \models \exists x \varphi(x, a_1, \ldots, a_n)$ since all the a_i lie in some $\mathcal{A}_n$, and hence φ has a witness in $\mathcal{A}_{n+1} \subset \mathcal{N}$.

So $\mathcal{M}$ and $\mathcal{N}$ agree on all quantifier-free formulae, and formulae of the form $\exists \varphi(x, a_1, \ldots, a_n)$ for $a_i \in N$. Now claim they agree on all $\mathcal{L}$-sentences and hence $\mathcal{N} \preceq \mathcal{M}$.

We proceed by induction on formulae. The base cases of terms/quantifier-free formulae is immediate since $\mathcal{N} \subset \mathcal{M}$. We assume $\mathcal{M}, \mathcal{N}$ agree on φ, ψ.

$\mathcal{M} \models \neg\varphi \iff \mathcal{M} \not\models \varphi \iff \mathcal{N} \not\models \varphi \iff \mathcal{N} \models \neg\varphi$, and $\mathcal{M} \models \varphi \wedge \psi \iff \mathcal{M} \models \varphi$ and $\mathcal{M} \models \psi \iff \mathcal{N} \models \varphi, \psi \iff \mathcal{N} \models \varphi \wedge \psi$.

Lastly, consider $\psi = \forall x_n \varphi(x_1, \ldots, x_n)$. Then $\neg\psi$ is $\exists x_n \neg\varphi(x_1, \ldots, x_{n-1}, x_n)$, and for any fixed $a_i \in N$, $\mathcal{N}, \mathcal{M}$ agree on $\exists x_n \neg\varphi(a_1, \ldots, a_{n-1}, x_n)$. Hence for any fixed $a_i \in N$, $\mathcal{N} \models \forall x_n \varphi(a_1, \ldots, a_{n-1}, x_n) \iff \mathcal{M} \models \forall x_n \varphi(a_1, \ldots, a_{n-1}, x_n)$.

So for any $\mathcal{L}$-sentence φ and $a_i \in N$, $\mathcal{M}$ and $\mathcal{N}$ agree on $\varphi(a_1, \ldots, a_n)$, and hence the inclusion map $\mathcal{N} \hookrightarrow \mathcal{M}$ is an elementary embedding. So $\mathcal{N} \preceq \mathcal{M}$ and $|N| \leq |A| + |\mathcal{L}| + \aleph_0$.