Topics in Combinatorics Revision Questions

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1. State Jensen's inequality.
2. **Prove that if X is a random variable, then $\mathbb{P}[X \geq EX] > 0$ and $\mathbb{P}[X \leq EX] > 0$**
3. State and prove a proposition about average degrees in bipartite graphs.
4. What is this method called, roughly speaking?
5. State and prove another proposition about average degrees, concerning an inequality instead of an equality.
6. Show that for a planar graph, $E \geq 3F/2$.
7. Define $\partial_s \mathcal{A}$, in context. What is this called?
8. State and prove a lower bound on $|\partial_S \mathcal{A}|$, representing it in two different ways.
9. State Sperner's Theorem.
10. **Prove Sperner's Theorem.**
11. State the Erdos-Ko-Rado Theorem.
12. **Prove the Erdos-Ko-Rado Theorem.**
    (a) Show that, for a random ordering, at most k intervals can be part of an intersecting family.
    (b) What is the expected total number of such intervals belonging to the family?
    (c) In the equality case, how are we able to still have k intervals?
    (d) Construct a helpful cylcic order to show any A of size k is contained in $\mathcal{A}$.
13. Define the crossing number of G.
14. Show that a planar graph with n vertices has at most $3n - 6$ edges.
15. Prove that a graph with n vertices and m edges has crossing number at least $m - 3n$.

16. Let G be a graph drawn in the plane with n vertices and m edges, with $m \geq 6n$. G must have at least how many crossings?
17. **Prove this.**
18. State the Szemeredi-Trotter Theorem
19. **Prove the Szemeredi-Trotter Theorem.**
    (a) What is an appropriate way to convert the points/lines into a graph?
    (b) Put two different bounds on the number of crossings.
20. Give three examples of graphs demonstrating the S-T bound (up to a constant).
21. State Stirling's Formula. [optional]
22. State a gentler upper/lower bound on $n!$.
23. Prove this.
24. State bounds on $2^{-n}\binom{n}{n/2}$.
25. Prove these bounds.
26. Give an example of an event that occurs with probability $2^{-n}\binom{n}{n/2}$.
27. Give a bound on $\binom{n}{m}$, useful for when $m \ll n$.
28. Improve this bound slightly. In what scenario is this often useful?
29. State the quotient of consecutive binomial coefficients.
30. Bound $\sum_{k=0}^{m}\binom{n}{k}$ in the case $m = \alpha n$, $\alpha < 1/2$.
31. State an example of the *concentration of measure phenomenon*.
32. Let $m = (1/2 - \theta)n$ with $0 < \theta \leq 1/2$. Then $2^{-n}\binom{n}{m} \leq \mathrm{e}^{-\theta^2 n/2}$.
33. **Prove that if $X_1, \ldots, X_n$ are independent random variables of mean zero taking values in $[-1, 1]$, and $X = \sum X_i$, then $\mathbb{P}[X \geq \varepsilon n] \leq \mathrm{e}^{-\varepsilon^2 n/4}$.**
    (a) What is the exponential moment?
    (b) Markov...
    (c) Optimise over something?
34. What similar result do we immediately get by switching signs?
35. Let $m = (1/2 - \varepsilon)n$. Show that $2^{-n}\sum_{k=0}^{m}\binom{n}{k} \leq \mathrm{e}^{-\varepsilon^2 n}$.
36. What is a general question regarding well-separated sets.
37. What is the critical threshold for changing behaviour at intersections of size αn, and why?

38. **Prove that if $\alpha > 1/4$, there can be exponentially many subsets of $[n]$ of size $n/2$ intersecting in no more than αn.**
    (a) Let A be a random set of size $n/2$. What is a good probability to estimate?
    (b) Then how many bad intersections can we have?
    (c) How can we mitigate this?
39. What is the characteristic function of a set?
40. What space does it live in?
41. What is an oftentimes more useful function with which to associate a set?
42. Why is it thusly named?
43. Prove that if $A, B \subset [n]$ have size $n/2$ and f_A, f_B are their balanced functions, then $\langle f_A, f_B \rangle = |A \cap B| - n/4$.
44. Use this to prove that if $A_1, \ldots, A_m \subset [n]$ intersect in at most $(1/4 - \delta)n$, then $m \leq 1 + \delta/4$.
45. State and prove a more general theorem, of which the above is a special case.
46. **Prove that if $x_1, \ldots, x_m$ are non-zero vectors in $\mathbb{R}^n$ such that $\langle x_i, x_j \rangle \leq 0$ for every $i \neq j$ then $m \leq 2n$.**
    (a) Induction on n.
    (b) How can we reduce to $n-1$-dimensional space?
47. **Furthermore, prove that if $m = 2n$ then there is an orthonormal basis $a_1, \ldots, a_n$ such that each x_i is a multiple of some a_j (so we have exactly one positive multiple and one negative multiple of each a_i).**
48. Relate the above back to the problem of finding families of sets with intersections in at most $n/4$.
49. Define a Hadamard matrix.
50. Define the Walsh matrices W_m.
51. State three basic facts about W_m.
52. How do we find the desired set system from the W_m?
53. Define the Paley matrix P_n.
54. Prove that for every prime p and every $d \not\equiv 0 \mod p$ we have that $\sum_{x \in \mathbb{Z}_p} \left(\frac{x}{p}\right) \left(\frac{x+d}{p}\right) = -1$
55. Hence show P_n has orthogonal rows.

56. What is the smallest n, a multiple of 4, for which no Hadamard matrix is known?
57. Prove that if $|A| = n$, then $|A + A| \geq 2n - 1$. When do we have equality?
58. Demonstrate a similar result for $|A.A|$.
59. State a famous conjecture of Erdos and Szemeredi.
60. Define $\rho_A^+(x)$ and $\rho_A^\times(x)$.
61. Define the multiplicative energy and additive energy of A.
62. What is the quotient set of A?
63. State and prove a lemma about multliplicative energy (a reformulation).
64. Prove that $\sum_{i=1}^n |a_i|^2 \geq n^{-1}(\sum_i |a_i|)^2$.
65. Prove it again, in a different way.
66. Show that the multiplicative energy of A is at least $|A|^4/|A.A|$.
67. **Prove that $\sum_{m \in A/A} \rho_A^{\div}(m^{-1})^2 \leq 2|A + A|^2 \lceil \log |A| \rceil$.**
    (a) What is dyadic decomposition?
    (b) Think a bit about what $\rho_A^{\div}(m^{-1})^2$ means.
68. Hence, prove Solymosi's Theorem.
69. Define the Kneser Graph $G_{n,k}$.
70. Give a $k + 2$-colouring of $G_{n,k}$.
71. State the Borsuk-Ulam Theorem.
72. State an equivalent formulation of the Borsuk-Ulam Theorem.
73. Prove this equivalence.
74. State yet another variant of Borsuk-Ulam.
75. Show that it is implied by the above.
76. State the mixed version.
77. Show that the open sets version implies the mixed version.
78. **Let $\delta > 0$. Prove that the graph on S^d defined by joining u, v iff $\langle u, v \rangle < -1 + \delta$ has chromatic number at least $d + 2$.**
79. Show that this bound is sharp for sufficiently small δ.
80. If δ is small, what does this say about where a vertex is located relative to its neighbourhood? What can we conclude about odd cycles in this graph?

81. **Prove that the chromatic number of the Kneser Graph $G_{n,k}$ is $k+2$.**
    (a) How might we represent this graph on the sphere?
    (b) What is a good choice of a base set of $2n+k$ points? What key property should they have?
    (c) Find a good choice of $k+2$ open/closed sets to apply B-U to.
82. Define what it means for permutation σ to contain permutation π.
83. Give a construction of a general 2413-avoiding permutaiton.
84. Define what it means in terms of adjacency matrices of bipartite graphs for one to contain/avoid another.
85. State the conjecture solved by Marcus & Tardos.
86. Define an ordered quotient of an $n \times n$ 01-matrix.
87. Let P be a permutation matrix and let A be a P-avoiding 01-matrix. Show that every ordered quotient of A avoids P.
88. Suppose $k^2|n$. Define a block of A.
89. What does it mean for a block to be wide?
90. What does it mean for a block to be tall?
91. Roughly speaking, what is the structure of the following proof?
92. Show that if A does not contain the $k \times k$ P, then for each j the number of wide blocks with columns in C_j is at most $(k-1)\binom{k^2}{k}$ and similarly for each i the number of tall blocks with rows in R_i is at most $(k-1)\binom{k^2}{k}$.
93. Prove that, for P a $k \times k$ permutation matrix, and $k^2|n$, then $f(n)$, the largest number of non-zero entires in any 01-matrix avoiding P, satisfies

$$f(n) \leq 2k^2n(k-1)\binom{k^2}{k} + (k-1)^2 f(n/k^2)$$

    (a) How many 1s can we see in wide blocks or tall blocks?
    (b) Where does the $f(n/k^2)$ term come from?
94. Now prove the theorem of Marcus & Tardos.
95. For the C in question, how does it vary with k?
96. State the Stanley-Wilf conjecture.
97. Prove it, using M-T.
98. State the Khinchin axioms for entropy.
    (a) State axiom 0.

(b) State axiom 1.

(c) State axiom 2.

(d) State axiom 3.

(e) State axiom 4.

(f) State axiom 5.

99. Prove that if X, Y are independent, then $H[Y|X] = H[Y]$ and $H[X, Y] = H[X] + H[Y]$.

100. Prove that if X takes just one value, then $H[X] = 0$.

101. Let $A \subset B$, X uniform on A, Y uniform on B. Then $H[X] \leq H[Y]$, with equality iff $A = B$.

102. Let X be a random variable and $Y = f(X)$, some function X. Then $H[Y] \leq H[X]$.

103. Prove that $H[X] \geq 0$ for every discrete random variable X taking values on a finite set A.

104. Prove that if X takes at least two values with non-zero probability, then $H[X] > 0$.

105. State and prove the chain rule for entropy.

106. Describe the process of expressing the P3 bound using entropy.

107. Show that, in the distribution defined in lectures, the individual edges of the random P3 are uniformly distributed.

108. Show that a bipartite graph with density α must contain at least $\alpha^3|A|^2|B|^2$ labelled P3s, and show this is sharp.

109. State Sidorenko's Conjecture.

110. Write down the formula for entropy.

111. What is the entropy of a uniformly random variable?

112. Prove that the formula for entropy satisfies the Khinchin axioms (in particular, maximality and additivity).

113. Prove that the Khinchin axioms uniquely determine the formula.

(a) Prove that if X is uniform on a set of size 2^k, then $H[X] = k$.

(b) Prove that if X is uniformly distributed on a set of size n, then $H[X] = \log n$.

(c) Now show that $H[X] = \sum_{x \in A} p_a \log(1/p_a)$.

114. Define the permanent of a square matrix A.

115. What is the complexity class of the problem of calculating the permanent?

116. If A is a bipartite adjacency matrix, what does $\mathrm{per}(A)$ represent?

117. State Bregman's Theorem.

118. Demonstrate that the bound given is sharp.

119. **Prove the theorem.**

    (a) Define a random variable whose entropy we aim to bound.

    (b) Write down the upper bound that we are aiming for.

    (c) Bound $H[\sigma(x_1)]$.

    (d) Bound $H[\sigma(x_k)|\sigma(x_1),\ldots,\sigma(x_{k-1})]$.

    (e) For any fixed σ, find the distribution of $d^{\sigma}_{k-1}(x)$.

    (f) Take expectations and complete the proof.

120. Let X, Y be discrete random variables. Prove that $H[X,Y] \leq H[X] + H[Y]$.

    (a) Prove this using the formula.

    (b) Prove this using the axioms.

        i. First uniform.

        ii. Then rational.

        iii. Then real.

121. Deduce a more general subadditivity formula.

122. State Shearer's Lemma.

123. Prove Shearer's Lemma.

124. Let G be a graph with m edges and t triangles. Prove that $t \leq (2m)^{3/2}/6$.

125. Prove that a $\triangle$-intersecting family $\mathcal{G}$ of graphs on vertex set $[n]$ has size at most $2^{\binom{n}{k}}/4$.

126. State the theorem arising from Dvir's solution to the Kakeya problem for finite fields.

127. Let $A \subset \mathbb{F}_p^n$ be a set of size $\binom{n+d}{d}$. Then there exists a non-zero polynomial $P(x_1,\ldots,x_n)$ of degree d that vanishes on A.

    (a) What is another way of representing a polynomial of degree d in variables $x_1,\ldots,x_n$?

    (b) What is an equivalent notion now to vanishing on A?

128. In particular this is true when?

129. State three sufficient conditions (involving a set A, a degree d, and a polynomial f) for the existence of a non-zero degree d polynomial that vanishes everywhere on $\mathbb{F}_p^n$.

130. Let f be a non-zero polynomial on $\mathbb{F}_p^n$ of degree less than p. Show that f is not identically zero.

131. Now prove Dvir's result.

132. Let f be a non-zero polynomial of degree at most d on $\mathbb{F}_p^n$. Prove that f has at most dp^{n-1} roots. [Schwarz-Zippel Lemma.]

133. Name a real-world application of the Schwarz-Zippel Lemma.

134. State Alon's combinatorial Nullstellensatz.

135. Prove Alon's combinatorial Nullstellensatz.

136. State the Cauchy-Davenport Theorem.

137. Prove the Cauchy-Davenport Theorem.

138. Define $A\dot{+}B$.

139. State a result of da Silva and Hamidoune, which is a variant of C-D T.

140. Prove this.

141. Show that for when $|A| = |B|$ this result is sharp.

142. State a theorem due to Roy Meshulam.

143. State a closely related theorem due to Roth.

144. Define the rank of $f : X \times Y \to \mathbb{F}$.

145. Define the slice rank of $f : X \times Y \times Z \to \mathbb{F}$.

146. Let X be a finite set, let $A \subset X$, $\mathbb{F}$ a field, $f : X^3 \to \mathbb{F}$ a function such that $f(x, y, z) \neq 0$ iff $x = y = z$ and $x \in A$. Show that the slice rank of f is A.

147. Connect the slice rank to the cap-set problem.

148. Show that the slice rank of the polynomial $P(x, y, z) = \prod_{i=1}^{n}(1-(x_i+y_i+z_i)^2)$ is at most $3M$, where M is the number of 012-sequences of length n that sum to at most $2n/3$.

149. Obtain an upper bound on the above M.

150. Define the Hamming cube Q^n.

151. State the sensitivty conjecture (-adjacent problem) that Hao Huang proved.

152. Define a helpful class of matrices A_n: $n \in \mathbb{N}$.

153. Show that A_n is symmetric.

154. Show that the rows of A_n are orthogonal.

155. What are the possible entries of A_n?

156. How many non-zero entries are there in each row/column of A_n?

157. If the entries are indexed with 01 sequences, what is the value of $(A_n)_{xy}$ in terms of x and y?

158. How does A_n then relate to Q^n?

159. Describe the assignment of signs to the adjacency matrix of Q^n inductively.

160. What is A_n^2? Prove this.

161. Prove Huang's Theorem.

162. Suppose $X \subset \mathbb{R}^n$ and the members of X are all pairwise the same distance. How large can X be?

163. Let $a_1, \ldots, a_m \in \mathbb{R}^n$ be such that the number of distinct distances $d(a_i, a_j)$ with $i \neq j$ is at most 2. Then $m \leq (n+1)(n+4)/2$.

164. Suppose $\mathcal{A}$ is a family of subsets of $[n]$, all of even size, all with even intersections. How big can $|\mathcal{A}|$ be?

165. Now what happens if we demand the intersections to be of odd size instead?

166. In particular, let $\mathcal{A}$ be a family of subsets of $[n]$ all of even size, such that any two distinct sets have odd intersection. Prove that $|\mathcal{A}| \leq n$ if n odd, and $|\mathcal{A}| \leq n-1$ if n is even.

167. Now prove the even-even case is $|\mathcal{A}| \leq 2^{\lfloor n/2 \rfloor}$.

168. Let $\mathcal{A}$ be a family of subsets of $[n]$ such that the size of every $A \in \mathcal{A}$ is a multiple of p, but no two distinct sets in $\mathcal{A}$ have intersection of size a multiple of p. Show that $|\mathcal{A}| \leq \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{p-1}$.

169. Let p be an odd prime and let $n = 4p$. Show that the largest measure of a set X of unit vectors in $\mathbb{R}^n$ that contains no pair of orthogonal vectors is exponentially small.

170. State Borsuk's (disproved) conjecture.

171. Let $n = 4p$ for a prime p. Show that $\mathbb{R}^{n^2}$ contains a set X of size $\binom{n}{2p}$ such that every subset of X of smaller diameter has size at most $\sum_{m=0}^{p-1} \binom{n}{m}$.