Infinite Games Revision Questions

Otto Pyper

1. Which five properties do the games we consider have?
2. Define $M^{<\omega}$, M^{ω}.
3. Given $x \in M^{\omega}$, define x_{I} and x_{II}.
4. Define the interleaving of $x, y \in M^{\omega}$.
5. What is $A \subseteq M^{\omega}$ called?
6. What is a strategy?
7. Given two strategies σ, τ define $\sigma * \tau$.
8. Define a winning strategy.
9. Define a determined set.
10. Define a tree on M.
11. Given a tree T, define a branch through T.
12. Define $[T]$. What is it called?
13. Define $G(A; T)$ and represent it as $G(B)$ for some B.
14. Define a I/II-strategic tree.
15. Define a strategic tree in general.
16. What is $[T^{\mathrm{I}}_{\sigma}]$?
17. σ is a winning strategy for player I in $G(A) \iff$ ...?
18. A is determined iff...?
19. For $s \in M^{<\omega}$, what is $\ell h(s)$?
20. Define a splitting node.
21. Define a perfect tree.
22. Define a perfect set.
23. State Cantor's Theorem about perfect, non-empty subsets of 2^{ω}.
24. Prove Cantor's Theorem.
25. State and prove two corollaries of Cantor's Theorem for $|M| \geq 2$.
26. Prove that if A is countable then player II has a winning strategy in $G(A)$.
27. Prove that if $|A| < 2^{\aleph_0}$, then player II has a winning strategy in $G(A)$ (and a similar result for player I).

28. Define a blindfolded strategy.
29. Prove (in ZFC) that there is a non-determined subset $A \subseteq \omega^\omega$.
30. Define a quasistrategy.
31. Define a (winning) quasistrategic tree.
32. Define a quasidetermined set.
33. What condition on M allows the construction of strategies from quasistrategies?
34. Define a closed set.
35. Represent Zermelo's finite games with closed payoff sets.
36. State the Gale-Stewart Theorem.
37. Prove the Gale-Stewart Theorem.
38. Define Baire space.
39. What are the open balls in this metric?
40. Define Cantor space.
41. What is alternative characterisation of these topologies?
42. Show that Cantor space is compact, but that Baire space is (very) disconnected.
43. Given A in Baire space, define T_A.
44. Prove that $[T_A]$ is the closure of A.
45. State and prove the tree representation theorem for closed sets.
46. Basic open sets are...? Spaces with this property are called...?
47. Singletons are...?
48. Prove that these spaces are Hausdorff.
49. A function f on ω^ω is continuous iff what?
50. Prove this.
51. What is the general rule of thumb for determining whether or not f is continuous?
52. Show that $(\omega^\omega)^2$ and ω^ω are homeomorphic.
53. Baire space is homeomorphic to...? How do we thus sometimes refer to elements of Baire space?
54. What is $\mathrm{AC}_X(Y)$?
55. Define the Borel Hierarchy.
56. Define a G_δ space.
57. Give some spaces that are G_δ.
58. Prove that if X has a countable, clopen topology base then X is G_δ.
59. When does the Borel Hierarchy terminate if:
    (a) X is discrete?
    (b) singletons are closed and X is countable?

60. Prove that for arbitrary X, $\mathbf{\Delta}_{\aleph_1} = \mathbf{\Sigma}_{\aleph_1} = \mathbf{\Pi}_{\aleph_1}$
61. (In ZFC) what is the height of the Borel Hierarchy for Cantor space/Baire space/$\mathbb{R}$?
62. What technique does the proof of the above use?
63. Define a pointclass.
64. Define the dual pointclass, and the ambiguous pointclass.
65. What does it mean for a pointclass to be boldface? [Why is this silly?]
66. What does it mean for Γ to be closed under continuous images?
67. Define what it means for a set U to be X-universal for $\Gamma(Y)$.
68. Prove that if U is X-universal for $\Gamma(X)$ and Γ is boldface, then $\Gamma(X) \neq \breve{\Gamma}(X)$.
69. Prove that for every $\alpha < \aleph_1$, $\mathbf{\Sigma}^0_\alpha$ has an ω^ω-universal set.
70. Prove that if $U \subseteq X \times X$ is X-universal for $\Gamma(X)$, then $X \times X \backslash U$ is X-universal for $\breve{\Gamma}(X)$.
71. Let $\lambda < \omega_1$. Suppose that for each $\alpha < \lambda$ there is an ω^ω-universal set U_α for $\mathbf{\Pi}^0_\alpha(\omega^\omega)$. Then there is an ω^ω-universal set for $\mathbf{\Sigma}^0_\lambda$.
72. Deduce the Borel Hierarchy Theorem.
73. Where did we use/need AC in the above proof?
74. What does $\mathrm{Det}(\Gamma)$ mean?
75. Show that in general the class of determined sets is not closed under complementation.
76. Who proved the following, and when?
    (a) $\mathrm{Det}(\mathbf{\Sigma}^0_2)$
    (b) $\mathrm{Det}(\mathbf{\Sigma}^0_3)$
    (c) $\mathrm{Det}(\mathbf{\Sigma}^0_4)$
77. What did who prove about $\mathrm{Det}(\mathbf{\Sigma}^0_5)$?
78. This paved the way for who to prove what, and when?
79. Prove (in ZFC) that $|\mathcal{B}| = 2^{\aleph_0} < 2^{2^{\aleph_0}}$.
80. What is the Feferman-Levy Model $\mathcal{M}$?
81. Use it show that we need choice in the above proof.
82. What is the famous mistake of Henri Lebesgue?
83. Define a projection.
84. Define $\exists^{\omega^\omega}\Gamma$.
85. Define what it means for Γ to be closed under projections.
86. Define the projective hierarchy.
87. Prove that the projective hierarchy does not collapse.
88. Prove that every Borel set is $\mathbf{\Sigma}^1_1$.
89. Deduce Suslin's Theorem.
90. State the Continuum Hypothesis, and an equivalent formulation of it under ZFC.

91. Define what it means for $A \subseteq \omega^\omega$ to have the perfect set property.
92. State the Cantor-Bendixson Theorem.
93. Sketch a proof.
94. Why was this proof important?
95. Define PSP(Γ), and re-state Cantor-Bendixson with this notation.
96. Define PSP and state an observation involving it.
97. State a theorem of Bernstein.
98. How can it be proven?
99. State a theorem of Haudorff. How will we prove this?
100. Prove that if Γ is boldface, then $\mathrm{Det}(\Gamma) \implies \mathrm{PSP}(\Gamma)$.
    (a) Define the asymmetric game $G^*(A)$.
    (b) If $A \in \Gamma$ and $\mathrm{Det}(\Gamma)$ then $G^*(A)$ is determined.
    (c) If player I was a winning strategy in $G^*(A)$ then A contains a perfect subset.
    (d) If player II has a winning strategy, then A is countable.
        i. Define a τ-decisive for x position.
        ii. If τ is winning for II, then for each $x \in A$ ther is a τ-decisive position p for x
        iii. Every position p is τ-decisive for at most one $x \in 2^\omega$.
101. Deduce PSP(Borel), and some further corollaries.
102. State a theorem of Godel and Addison.
103. What is Godel's Constructible Universe? How is it denoted? Why?
104. State explicitly the definition of $\mathbf{\Sigma}^1_1$, and reformulate it in terms of T_x (giving the definition).
105. Define an illfounded/wellfounded tree. With a bit of AC, what is this equivalent to?
106. Hence state the tree representation of analytic and co-analytic sets.
107. Describe how to code a tree on ω or $\omega \times \omega$ as elements of Baire space.
108. Define WF.
109. Define the rank function, and the height function.
110. Prove that $|| \cdot || : \mathrm{WF} \to \omega_1$ is a surjection.
111. Define WF_α, $\mathrm{WF}_{<\alpha}$, $\mathrm{WF}_{\leq\alpha}$.
112. WF can be thought of as [...] in [...] many levels.
113. Prove that WF is $\mathbf{\Pi}^1_1$.
114. What is the general proof technique here?
115. Show that $\mathrm{WF}_{<\alpha}, \mathrm{WF}_{\leq\alpha}, \mathrm{WF}_\alpha$ are $\mathbf{\Pi}^1_1$.
116. Show further that $\mathrm{WF}_{\leq\alpha}$ is also $\mathbf{\Sigma}^1_1$.
117. Deduce that $\mathrm{WF}_\alpha, \mathrm{WF}_{\leq\alpha}, \mathrm{WF}_{<\alpha}$ are all $\mathbf{\Delta}^1_1$.

118. Prove that $\mathbf{\Delta}^1_1$ =Borel.
119. Write WF as a union of [...].
120. Let Γ be boldface. Define what it means for A to be Γ-hard, and Γ-complete.
121. Prove that WF is $\mathbf{\Pi}^1_1$-complete.
122. Show that WF is not $\mathbf{\Sigma}^1_1$.
123. Deduce that every $\mathbf{\Pi}^1_1$ set is an ω_1-union of Borel sets.
124. State the Weak CH for $\mathbf{\Pi}^1_1$ sets.
125. Prove the Weak CH for $\mathbf{\Pi}^1_1$ sets.
126. State the Boundedness Lemma.
127. Prove the Boundedness Lemma.
128. Define a set of unique codes.
129. Prove that if C is an SUC it cannot have PSP.
130. Prove that if there is a $\mathbf{\Delta}^1_n$ wellorder of ω^ω, then there is a $\mathbf{\Pi}^1_n$ set without PSP.
131. Give an outline of the corresponding notions between large cardinals, determinacy, and definable wellorders.
132. Roughly describe what it means if Φ is an LCP.
133. Use two of Godel's theorems to contextuatlise this notion.
134. Define what it means if $\Phi C < \Psi C$ and if ΦC, ΨC are equiconsistent.
135. Define a (strong) limit.
136. Define regular.
137. Define an inaccessible cardinal.
138. State the GCH.
139. Show that IC is a large cardinal axiom.
140. Prove that if κ is inaccessible, then $V_\kappa \models$ ZFC.
141. Define an inner model.
    (a) Define an inner model.
    (b) What does it mean for φ to define an inner model?
    (c) Define a canonical model family.
    (d) Define what it means for a canonical model family to be $\mathbf{\Delta}^1_n$-wellordered.
142. State and prove some (basic) correspondences between models and inner models of ZFC.
143. State (and prove?) another two transfers between M and V.
144. So what is $\aleph_1^M$?
145. for μ a canonical model family, we have that [...].
146. Suppose there is a $\mathbf{\Delta}^1_n$-wellordered canonical model family and Det($\mathbf{\Pi}^1_n$). Then there is an inner model of ZFC + IC.

(a) Define a weak set of unique codes.

(b) Show $M \models \aleph_1^V$ inaccessible.

147. Define a filter on κ.
148. Define an ultrafilter.
149. Define λ-complete.
150. If $\lambda = \aleph_1$, what is this completness often called?
151. Define principal/non-principal ultrafilters.
152. How many ultrafilters are there on κ (in ZFC)? How many of these are principal?
153. Define a measurable cardinal.
154. Prove (in ZFC) that every measurable cardinal is inaccessible. [Why do we need AC here?]
155. Define Erdos-Rado Arrow Notation, and all the terms associated with it.
156. Define a weakly compact cardinal.
157. State two facts about weak compactness of cardinals.
158. Prove that every weakly compact cardinal is inaccessible.
159. Define $n - c-$homogeneity and some more arrow notation.
160. State Rowbottom's Theorem.
161. Prove Rowbottom's Theorem.
162. Prove (in ZFC) that $\aleph_1$ is not weakly compact.
163. Define the diagonal intersection of a family of subsets of κ.
164. Define a normal ultrafilter.
165. Prove that if U is an ultrafilter on κ such that all elements of U have size κ and U is normal, then U is κ-complete.
166. Fact (ZFC): If κ is measurable, then there is a [...] on κ.
167. Prove (hence) that measurable cardinals are weakly compact.
168. Define $p[T]$ for T a tree on $\kappa \times \omega$.
169. Define a κ-Suslin set $A \subseteq \omega^\omega$.
170. So being $\aleph_0$-Suslin is equivalent to...?
171. Define the Auxiliary Game.
172. State Schoenfield's Theorem.
173. Prove Schoenfield's Theorem.
    (a) Define the Kleene-Brouwer order.
    (b) Define the Schoenfield tree.
174. State a 1969/70 theorem of Martin.
175. Define what it means for κ to satisfy Rowbottom's Theorem.
176. State Martin's Theorem.

177. Prove Martin's Theorem.
178. What aspect of H did we fail to use in this proof? We don't need the full strength of what?
179. State a ZFC Theorem: $\text{Det}(\mathbf{\Sigma}^1_1) \iff$ ...?
180. Which fragment of AC have we used liberally so far? In particular...?
181. Prove that AD $\implies$ $\text{AC}_\omega(\omega^\omega)$.
182. How does this proof generalise?
183. State the uniformisation principle.
184. U is a non-principal ultrafilter iff it extends which filter?
185. Define the image filter.
186. Shwo that if there is an ultrafilter U on X that is not $\aleph_1$-complete, then there is a non-principal ultrafilter on $\mathbb{N}$.
187. Corollary: ZF + there is no non-principal ultrafilter on $\mathbb{N}$ $\implies$ ...?
188. Prove that AD $\implies$ there is no non-principal ultrafilter on $\mathbb{N}$.
    (a) What is the technique here called?
189. What does it mean if x is definable from y?
190. State six properties of $\leq_D$.
191. Define a preorder.
192. Prove that if $x, y \in \omega^\omega$ and z is such that $x \leq_D z$ and $y \leq_D z$ then $x * y \leq_D z$.
193. Define the Turing Join of x and y.
194. Define the cone of x.
195. Define the cone filter F_D.
196. Define $\equiv_D$-invariant.
197. Describe the relationship between strategies and $\mathcal{D}_D$.
198. State Martin's Lemma.
199. State a corollary of Martin's Lemma (involving the Martin Measure).
200. State Solovay's Theorem.
201. Prove Solovay's Theorem.