Category Theory

Notes by Peter Johnstone

Michaelmas 2020

3 Adjunctions

We now come to what is perhaps the most important single contribution that category theory has made to mathematics: the concept of adjunction. Unlike the Yoneda Lemma, which as we saw is really a folk-theorem, the notion of adjunction has a definite origin: it was introduced by Daniel Kan, in a paper published in 1958. Others had come close to finding the notion before, but Kan's decisive contribution was to emphasize the duality between left and right adjoints which gives the notion so much of its power.

Definition 3.1 Let $\mathcal{C}$ and $\mathcal{D}$ be categories. By an *adjunction* between $\mathcal{C}$ and $\mathcal{D}$, we mean a pair of functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ together with a bijection between morphisms $FA \to B$ in $\mathcal{D}$ and morhisms $A \to GB$ in $\mathcal{C}$ which is natural in A and B. (If $\mathcal{C}$ and $\mathcal{D}$ are locally small, we may express the naturality by saying that $\mathcal{D}(F-,-)$ and $\mathcal{C}(-,G-)$ are naturally isomorphic functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$; but if we re-express this in elementary terms, we don't need to assume local smallness.) We say F is *left adjoint* to G, and G is *right adjoint* to F, and write $(F \dashv G)$.

The name is of course derived from the notion of adjoint linear operators between inner-product spaces: if H and K are two such operators (in opposite directions), they are adjoint if the inner products $\langle Hx, y\rangle$ and $\langle x, Ky\rangle$ are equal for all x and y. But note that this relationship between H and K is symmetric, whereas that between left and right adjoint functors is not (cf. 3.2(b) below).

Examples 3.2 (a) The free group functor $F : \mathbf{Set} \to \mathbf{Gp}$ is left adjoint to the forgetful functor $U : \mathbf{Gp} \to \mathbf{Set}$, since every function $A \to UG$ extends uniquely to a homomorphism $FA \to G$. The naturality of the bijection in the variable A was built into the way in which we made F into a functor in 1.3(b); naturality in G is obvious.

(b) The forgetful functor $U: \mathbf{Top} \to \mathbf{Set}$ also has a left adjoint D, which equips a set with its discrete topology, since any mapping from a set A to a space X is continuous for the discrete topology on A. But U also has a right adjoint I, which equips A with the 'indiscrete topology' whose only open sets are $\emptyset$ and A; again, any mapping $X \to A$ is continuous for the indiscrete topology on A. Thus, in general, a functor may have left and right adjoints which are not isomorphic.

(c) The functor $\mathrm{ob}: \mathbf{Cat} \to \mathbf{Set}$ similarly has a left adjoint D which sends A to the discrete category with objects A, and a right adjoint I which sends A to the category with objects A and just one morphism $a \to b$ for each pair of elements (a, b). But in this case D has a further left adjoint π_0: $\pi_0\mathcal{C}$, the set of *connected components* of $\mathcal{C}$, is the quotient of $\mathrm{ob}\,\mathcal{C}$ by the smallest equivalence relation which identifies $\mathrm{dom}\, f$ with $\mathrm{cod}\, f$ for every $f \in \mathrm{mor}\,\mathcal{C}$. It is easily seen that any functor $\mathcal{C} \to DA$ must be constant on each connected component of $\mathcal{C}$, and so induces a well-defined function $\pi_0\mathcal{C} \to A$; this establishes the adjunction. In fact it is possible to have strings of adjoint functors of arbitrary (finite or infinite) length; cf. Exercise 3.12 below.

(d) For a fixed set A, we can regard the operation of forming products with A as a functor $(-) \times A\colon \mathbf{Set} \to \mathbf{Set}$. This functor has a right adjoint $\mathbf{Set}\,(A, -)$, since the process known as 'λ-conversion' allows us to regard a function $f : B \times A \to C$ as a function $\lambda f : B \to \mathbf{Set}\,(A, C)$, namely $\lambda f(b)(a) = f(b, a)$. More generally, a category $\mathcal{C}$ in which products of pairs of objects (defined as in 2.6(e)) always exist is said to be *cartesian closed* if $(-) \times A\colon \mathcal{C} \to \mathcal{C}$ has a right adjoint (usually denoted $(-)^A$) for all $A \in \mathrm{ob}\,\mathcal{C}$. (See Exercise 3.13 for an example.)

(e) We shall see in 3.8 below that if F and G form an equivalence of categories, then F is both left and right adjoint to G. But a functor can have left and right adjoints which coincide without being part of an equivalence. Let M be the two-element monoid $\{1, e\}$ with $e^2 = e$, so that an M-set may be thought of as a pair (A, e) where $e\colon A \to A$ is an idempotent endomorphism. We have a functor $F\colon \mathbf{Set} \to [M, \mathbf{Set}]$ which equips an arbitrary set with its identity function, and a functor $G\colon [M, \mathbf{Set}] \to \mathbf{Set}$ sending (A, e) to the set of fixed points of e (this is functorial since a morphism $(A, e) \to (B, f)$ must map fixed points of e to fixed points of f). We have $(F \dashv G)$ since any morphism $FA \to (B, e)$ must take values in $G(B, e)$; but we also have $(G \dashv F)$ since a morphism $f\colon (A, e) \to FB$ is uniquely determined by its restriction to $G(A, e)$, as $f(a) = f(ea)$ for all $a \in A$. But G is not faithful, and so cannot be part of an equivalence.

(f) Let $\mathbf{1}$ denote the discrete category with one object. A left adjoint for the unique functor $\mathcal{C} \to \mathbf{1}$, if it exists, picks out an *initial object* of $\mathcal{C}$, i.e. an object admitting a unique morphism to every object of $\mathcal{C}$; similarly, a right adjoint for $\mathcal{C} \to \mathbf{1}$ is a terminal object of $\mathcal{C}$. Once again, examples such as $\mathbf{Gp}$ show that these two functors may coincide even though $\mathcal{C}$ is not equivalent to $\mathbf{1}$.

(g) Adjunctions occur frequently in maps between posets. For example, let $f\colon A \to B$ be a function, and regard the power-sets PA and PB as posets under inclusion. Then, for subsets $A' \subseteq A$ and $B' \subseteq B$, we have $Pf(A') \subseteq B'$ iff $A' \subseteq P^*f(B')$, i.e. $(Pf \dashv P^*f)$.

(h) If X is a topological space and $\mathcal{O}X$ denotes the set of open subsets of X, then the inclusion $\mathcal{O}X \to PX$ has a right adjoint given by the interior operator $(-)^\circ$, since if $U \subseteq X$ is open and A is an arbitrary subset, we have $U \subseteq A$ iff $U \subseteq A^\circ$.

(i) Another source of adjunctions between posets is the notion of *Galois connection.* Suppose given sets A and B and a particular relation $R \subseteq A \times B$; we define mappings $(-)^r\colon PA \to PB$ and $(-)^l\colon PB \to PA$ by $(A')^r = \{b \in B \mid (\forall a \in A')((a, b) \in R\}$ and $(B')^l = \{a \in A \mid (\forall b \in B')((a, b) \in R)\}$. It is clear that these are contravariant functors (i.e. order-reversing mappings), and that we have

$$A' \subseteq (B')^l \Leftrightarrow A' \times B' \subseteq R \Leftrightarrow B' \subseteq (A')^r \;;$$

we say that $(-)^l$ and $(-)^r$ are adjoint to each other *on the right.* (The example which gives rise to the name is that underlying the fundamental theorem of Galois theory, in which A is a field, B is a group of automorphisms of A and R relates elements of A to automorphisms which fix them.)

(j) Contravariant adjunctions also occur frequently between categories which are not posets. The contravariant power-set functor P^* is self-adjoint on the right, since functions $A \to PB$ and $B \to PA$ both correspond naturally to arbitrary relations $R \subseteq A \times B$. Similarly, the dual vector space functor is self-adjoint on the right, since linear maps $V \to W^*$ and $W \to V^*$ both correspond to bilinear forms on $V \times W$.

The next result establishes a useful criterion for the existence of an adjoint. We saw in 3.2(f) that an initial object is a particular case of a left adjoint; but left adjoints can also be thought of as (families of) particular cases of initial objects.

Theorem 3.3 *Let $G\colon \mathcal{D} \to \mathcal{C}$ be a functor. For each $A \in \text{ob}\,\mathcal{C}$, let $(A \downarrow G)$ denote the category whose objects are pairs (B, f) with $B \in \text{ob}\,\mathcal{D}$ and $f : A \to GB$ in $\mathcal{C}$ and whose morphisms $(B, f) \to (B', f')$ are morphisms $g\colon B \to B'$ in $\mathcal{D}$ making*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f'"'] & GB \arrow[d, "Gg"] \\ & GB' \end{tikzcd}$$

commute. Then specifying a left adjoint for G is equivalent to specifying an initial object of $(A \downarrow G)$ for each A.

Proof First suppose G has a left adjoint F. Let $\eta_A\colon A \to GFA$ be the morphism corresponding under the adjunction to 1_{FA}; we claim (FA, η_A) is initial in $(A \downarrow G)$. For naturality of the bijection in the second variable ensures that

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & GFA \arrow[d, "Gg"] \\ & GB \end{tikzcd}$$

commutes iff g is the morphism corresponding to f under the adjunction; so there is a unique such g.

Conversely, suppose given an initial object (FA, η_A) of $(A \downarrow G)$ for each A. We use this to define F on objects, and make it into a functor much as we did for the free group construction in 1.3(b): given $f : A \to A'$, Ff is the unique morphism $(FA, \eta_A) \to (FA', \eta_{A'}f)$ in $(A \downarrow G)$. Functoriality as usual comes from uniqueness: given $f' : A' \to A''$, $F(f'f)$ and $(Ff')(Ff)$ are both morphisms $(FA, \eta_A) \to (FA'', \eta_{A''}f'f)$ in $(A \downarrow G)$. The bijection sends a morphism $f\colon A \to GB$ to (the morphism of $\mathcal{D}$ underlying) the unique morphism $(FA, \eta_A) \to (B, f)$ in $(A \downarrow G)$, with inverse sending $g\colon FA \to B$ to $(Gg)\eta_A$; naturality in each variable is easy to verify. $\square$

Corollary 3.4 *(Adjoints are unique up to isomorphism) If F and F' are both left adjoint to G, then there is a canonical natural isomorphism $F \to F'$.*

Proof For each A, (FA, η_A) and $(F'A, \eta'_A)$ are both initial objects of $(A \downarrow G)$, so there is a unique isomorphism between them in this category. This is easily seen to be the A-component of a natural isomorphism $F \to F'$. $\square$

There is of course a converse to 3.4: any functor isomorphic to a left adjoint for G is itself left adjoint to G. We often refer loosely to *the* (rather than a) left adjoint of a given functor, even though it is only defined up to isomorphism.

Lemma 3.5 *(Adjunctions can be composed) Suppose given categories and functors*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \arrow[r, shift left, "H"] & \mathcal{E} \arrow[l, shift left, "K"] \end{tikzcd}$$

with $(F \dashv G)$ and $(H \dashv K)$. Then $(HF \dashv GK)$.

Proof We have bijections between morphisms $HFA \to C$, $FA \to KC$ and $A \to GKC$, both of which are natural in A and C. $\square$

The following consequence of the last two results, though seeemingly rather special, is surprisingly often useful.

Corollary 3.6 *Suppose given a commutative square of categories and functors*

$$\begin{tikzcd} \mathcal{B} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{C} \arrow[d, "H"] \\ \mathcal{D} \arrow[r, "K"] & \mathcal{E} \end{tikzcd}$$

in which all the functors have left adjoints. Then the square of left adjoints commutes up to natural isomorphism.

Proof By 3.5, the two ways round the square of left adjoints are both left adjoint to the diagonal of the original square, so by 3.4 they are isomorphic. □

Given an adjunction $(F \dashv G)$, we saw in the proof of 3.3 that the morphism $\eta_A : A \to GFA$ corresponding to 1_{FA} is the A-component of a natural transformation $\eta : 1_{\mathcal{C}} \to GF$. We call this transformation the *unit* of the adjunction; dually, we have the *counit* $\epsilon \colon FG \to 1_{\mathcal{D}}$, where ϵ_B corresponds to 1_{GB}. The second main result of this chapter characterizes adjunctions in terms of their units and counits.

Theorem 3.7 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. Then specifying an adjunction $(F \dashv G)$ is equivalent to specifying natural transformations $\eta \colon 1_{\mathcal{C}} \to GF$ and $\epsilon \colon FG \to 1_{\mathcal{D}}$ such that the diagrams*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\epsilon_F"] \\ & F \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd}$$

commute. (Here ϵ_F denotes the natural transformation whose value at an object A is ϵ_{FA}, and so on.)

Unsurprisingly, the commutative diagrams in the statement of 3.7 are called the *triangular identities* for η and ϵ.

Proof First suppose given an adjunction $(F \dashv G)$. We have already seen how to construct η and ϵ; also, naturality of the bijection in the first variable ensures that $(\epsilon_{FA})(F\eta_A)$ is the morphism corresponding to $(1_{GFA})(\eta_A)$, i.e. it is 1_{FA}. The other identity is dual.

Conversely, suppose given η and ϵ satisfying the triangular identities. Given $f \colon A \to GB$, we define $\Phi(f)$ to be the composite $\epsilon_B(Ff) : FA \to FGB \to B$, and given $g : FA \to B$ we define $\Psi(g)$ to be $(Gg)\eta_A$. Naturality of η and ϵ ensures that these mappings are both natural in A and B. And we have

$$\Psi\Phi(f) = (G\epsilon_B)(GFf)\eta_A = (G\epsilon_B)(\eta_{GB})f = f$$

by naturality of η and the second triangular identity; the verification that $\Phi\Psi(g) = g$ is dual. □

You will recall that, when I introduced the notion of equivalence of categories in 1.9, I made the natural isomorphisms α and β point in opposite directions $1_{\mathcal{C}} \to GF$ and $FG \to 1_{\mathcal{D}}$. The reason is now clear: I wanted them to look like the unit and counit of an adjunction. But at that point I said nothing about the triangular identities; so are they always satisfied in an equivalence? The answer is no; but if they fail, it is merely because we have made the wrong choice of α and β:

Lemma 3.8 *('Every equivalence is an adjoint equivalence') Suppose given an equivalence (F, G, α, β) as in* 1.9*. Then there exist natural isomorphisms $\alpha' : 1_{\mathcal{C}} \to GF$ and $\beta' : FG \to 1_{\mathcal{D}}$ which satisfy the triangular identities. In particular, F is both left and right adjoint to G.*

Proof In fact the statement gives us more 'wriggle room' than we need: it is possible to keep either α or β unchanged, and modify the other one to satisfy the identities. For definiteness, let us take $\alpha' = \alpha$ and set β' to be the composite

$$FG \xrightarrow{(FG\beta)^{-1}} FGFG \xrightarrow{(F\alpha_G)^{-1}} FG \xrightarrow{\beta} 1_{\mathcal{D}} \ .$$

Note, incidentally, that $FG\beta = \beta_{FG}$, since the square

$$\begin{tikzcd} FGFG \arrow[r, "FG\beta"] \arrow[d, "\beta_{FG}"] & FG \arrow[d, "\beta"] \\ FG \arrow[r, "\beta"] & 1_{\mathcal{D}} \end{tikzcd}$$

commutes by naturality and β is monic; so we could equivalently have taken the first component of β' to be $(\beta_{FG})^{-1}$. Similarly, we have $GF\alpha = \alpha_{GF}$. Because of the asymmetry we have introduced by changing β but not α, the verifications of the two triangular identities are not dual to each other, so we have to do both of them; but they follow from the commutative diagrams

$$\begin{tikzcd} F \arrow[r, "F\alpha"] \arrow[dr, "1_F"'] & FGF \arrow[r, "(\beta_{FGF})^{-1}"] \arrow[d, "(F\alpha)^{-1}"] & FGFGF \arrow[d, "(F\alpha_{GF})^{-1} = (FGF\alpha)^{-1}"] \\ & F \arrow[r, "(\beta_F)^{-1}"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\beta_F"] \\ & & F \end{tikzcd}$$

and

$$\begin{tikzcd}
G \arrow[r, "\alpha_G"] \arrow[dr, "1_G"'] & GFG \arrow[r, "(GFG\beta)^{-1}"] \arrow[d, "(\alpha_G)^{-1}"] & GFGFG \arrow[d, "(GF\alpha_G)^{-1} = (\alpha_{GFG})^{-1}"] \\
& G \arrow[r, "(G\beta)^{-1}"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\beta"] \\
& & G
\end{tikzcd}$$

in which the squares commute by naturality of β^{-1} and α^{-1} respectively. So by 3.7 we have $(F \dashv G)$; but $(\beta')^{-1}$ and $(\alpha')^{-1}$ also satisfy the triangular identities, and so form the unit and counit of an adjunction $(G \dashv F)$. □

Incidentally, we could also have extracted a proof of 3.8 from Lemma 1.12, since the α and β constructed in the second half of that proof do satisfy the triangular identities.

There is an important link between properties of the counit of an adjunction and properties of the right adjoint (and, dually, between properties of the unit and of the left adjoint).

Lemma 3.9 *Let $(F \dashv G)$ be an adjunction with counit ϵ. Then*

(i) *ϵ is pointwise epic iff G is faithful.*

(ii) *ϵ is an isomorphism iff G is full and faithful.*

Proof (i) Given $g \colon B \to B'$ in $\mathcal{D}$, Gg corresponds under the adjunction to the composite $g\epsilon_B$; so ϵ_B is epic iff G is injective on morphisms with domain B and specified codomain. This holds for all B iff G is faithful.

(ii) Similarly, ϵ_B is an isomorphism iff G acts bijectively on morphisms with domain B and specified codomain, and this holds for all B iff G is full and faithful. □

Definition 3.10 By a *reflection* we mean an adjunction satisfying the conditions of 3.9(ii). By a *reflective subcategory* of $\mathcal{C}$ we mean a full subcategory $\mathcal{D}$ for which the inclusion $\mathcal{D} \to \mathcal{C}$ has a left adjoint.

By 1.12, any reflection is equivalent to the inclusion of a reflective subcategory (i.e., it induces an equivalence between $\mathcal{D}$ and the full subcategory of $\mathcal{C}$ on objects in the image of G, and the latter is reflective). There are many familiar examples of reflective and coreflective subcategories in algebra and topology:

Examples 3.11 (*a*) The category **AbGp** is reflective in **Gp**; the left adjoint of the inclusion sends a group G to its *abelianization* G/G', where G' is the (normal) subgroup of G generated by all commutators. Clearly, any homomorphism from G to an abelian group A maps all commutators to the identity, so it induces a well-defined homomorphism $G/G' \to A$; this establishes the adjunction.

(*b*) Recall that a group is said to be *torsion* if all its elements have finite order, and *torsion-free* if its only element of finite order is the identity. In an abelian group A the elements of finite order form a subgroup A_t; and the mapping $A \mapsto A_t$ defines a right adjoint to the inclusion $\mathbf{tAbGp} \to \mathbf{AbGp}$, where **tAbGp** is the full subcategry of torsion abelian groups, since any homomorphism from a torsion group to A must take values in A_t; so this subcategory is coreflective. Also, the category **tfAbGp** of torsion-free abelian groups is reflective in **AbGp**, with left adjoint sending A to the quotient A/A_t.

(c) A famous example from topology: let **KHaus** $\subseteq$ **Top** be the full subcategory of compact Hausdorff spaces. This subcategory is reflective, the left adjoint to the inclusion being the *Stone–Čech compactification* functor β. This compactification was discovered independently by Marshall Stone and by Eduard Čech in papers published in 1937; interestingly, the constructions they gave were substantially different, but since each of them had proved that their construction had the universal property of a left adjoint, it was immediately obvious that their spaces must be homeomorphic — it wasn't necessary for anyone to write a paper proving this. We shall give a construction of β in the next chapter, as an application of the Special Adjoint Functor Theorem (see 4.17 below).

(d) A subset A of a topological space X is called *sequentially closed* if it is closed under limits of convergent sequences, i.e. if $x_n \in A$ for all n and $x_n \to x_\infty$ as $n \to \infty$ implies $x_\infty \in A$. Clearly, any closed set is sequentially closed; we say X is a *sequential space* if the converse holds. The full subcategory **Seq** of sequential spaces is coreflective in **Top**: the coreflection sends an arbitrary space X to the space X_s with the same underlying set retopologized by declaring all sequentially closed sets to be closed. This does not change the notion of convergent sequence, so X_s is sequential; and the identity mapping $X_s \to X$ is the counit of the adjunction.

(e) The full subcategory **Preord** of small preorders is reflective in **Cat**: the reflection sends a category $\mathcal{C}$ to the quotient $\mathcal{C}/\simeq$ (defined as in 1.3(d)) where $\simeq$ is the largest possible congruence on $\mathcal{C}$, i.e. the equivalence relation which identifies all parallel pairs of morphisms.

Exercises on chapter 3

Exercise 3.12 Let $\mathcal{C}$ be a category with initial and terminal objects which are not isomorphic (e.g. $\mathcal{C} =$ **Set**), and let $\mathbf{n}$ denote an n-element totally ordered set (so that functors $\mathbf{n} \to \mathcal{C}$ may be identified with composable strings of $n-1$ morphisms of $\mathcal{C}$). Show that there are functors $F_0, F_1, \ldots, F_n \colon [\mathbf{n-1}, \mathcal{C}] \to [\mathbf{n}, \mathcal{C}]$ and $G_1, G_2, \ldots, G_n \colon [\mathbf{n}, \mathcal{C}] \to [\mathbf{n-1}, \mathcal{C}]$ which form an adjoint string of length $2n+1$: that is,

$$(F_0 \dashv G_1 \dashv F_1 \dashv G_2 \dashv \cdots \dashv G_n \dashv F_n) \ .$$

Show also that this string is maximal, i.e. that F_0 has no left adjoint and F_n has no right adjoint. [*Hint*: a functor with a left adjoint preserves the terminal object, if it exists.] Can you find a maximal string of adjoint functors of arbitrary even length?

Exercise 3.13 Let $\mathcal{C}$ be a small category. Show that the functor category $[\mathcal{C}, \mathbf{Set}]$ is cartesian closed, as defined in 3.2(d). [*Hint*: products in $[\mathcal{C}, \mathbf{Set}]$ are defined pointwise, i.e. $(F \times G)(A) = FA \times GA$ for all A; use the Yoneda lemma to work out what $G^F(A)$ ought to be.]

Exercise 3.14 Let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$ be functors, and suppose we are given natural transformations $\alpha \colon 1_{\mathcal{C}} \to GF$, $\beta \colon FG \to 1_{\mathcal{D}}$ such that the composite $(G\beta)(\alpha_G) \colon G \to GFG \to G$ is the identity. Show that the composite $(\beta_F)(F\alpha) \colon F \to F$ is idempotent, and deduce that if all idempotents in $\mathcal{D}$ split then G has a left adjoint. [*Hint*: if idempotents split in $\mathcal{D}$, then they also split in the functor category $[\mathcal{C}, \mathcal{D}]$.] By taking $\mathcal{C}$ to be the discrete category with one object and choosing $\mathcal{D}$ suitably, show that this conclusion may fail if $\mathcal{D}$ contains non-split idempotents.

Exercise 3.15 Let $(F\colon \mathcal{C} \to \mathcal{D} \dashv G\colon \mathcal{D} \to \mathcal{C})$ be an adjunction with unit η and counit ϵ. Show that the following conditions are equivalent:

(i) $F\eta$ is an isomorphism (i.e. $F\eta_A$ is an isomorphism for all objects A).
(ii) ϵ_F is an isomorphism.
(iii) $G\epsilon_F$ is an isomorphism.
(iv) $GF\eta = \eta_{GF}$.
(v) $GF\eta_G = \eta_{GFG}$.
(vi)–(x) The duals of conditions (i)–(v).

[*Hint*: if you take the conditions in the cyclic order indicated, all the implications are trivial except for (v) $\Rightarrow$ (vi) and its dual (x) $\Rightarrow$ (i).] An adjunction with the equivalent properties of this question is said to be *idempotent*; note that any adjunction between preorders is idempotent, since condition (iv) is automatic in this case.

Exercise 3.16 Let $(F\colon \mathcal{C} \to \mathcal{D} \dashv G\colon \mathcal{D} \to \mathcal{C})$ be an adjunction with unit η and counit ϵ; let $\mathrm{Fix}(GF) \subseteq \mathcal{C}$ (respectively $\mathrm{Fix}(FG) \subseteq \mathcal{D}$) be the full subcategory on objects such that η_A (respectively ϵ_B) is an isomorphism.

(i) Show that F and G restrict to an equivalence between $\mathrm{Fix}(GF)$ and $\mathrm{Fix}(FG)$.
(ii) If $(F \dashv G)$ is idempotent, show that $\mathrm{Fix}(GF)$ is reflective in $\mathcal{C}$.
(iii) Deduce that an adjunction is idempotent iff it can be factored as a reflection followed by a coreflection.
(iv) What do you get if you apply the factorization of (iii) to the idempotent adjunctin of 3.2(g)?