Topics in Combinatorics: Sheet 1

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5. Colour each member of X red with probability p and blue with probability $1-p$. Let X_i be the indicator of the event that A_i is not monochrome, and let $Y = \sum_{i=1}^{r} X_i$.

$\mathbb{P}[X_i = 1] = 1 - (1/2)^m - (1-1/2)^m = 1 - 2^{1-m}$, since $X_i = 1$ iff the elements of A_i are not all red and not all blue. Hence:

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{i=1}^{r} \mathbb{P}[X_i = 1] \\
&= r(1-2^{1-m}) = r - 2^{1-m}r
\end{aligned}$$

So if $r < 2^{m-1}$ then $2^{1-m}r < 1$, and hence $\mathbb{E}[Y] > r-1$, so there exists some colouring for which $Y = r$, so there exists a colouring of X for which every A_i contains at least one red element and at least one blue element.

Let $R(m)$ be the least r such that there exist sets $A_1, \ldots, A_r$ of size m such that for every red-blue colouring there is some i with A_i monochrome. We can bound $R(m)$ recursively.

Given $R(m)$, construct $R(m)(m+1)+1$ sets of size $m+1$ by extending the $A_1, \ldots, A_{R(m)}$ m-sets.

Pick a set of $B = \{b_i : 1 \leq i \leq m+1\}$ such that $B \cap A_i = \emptyset$ for all i, and then take the sets $C_{i,j} = A_i \cup \{b_j\}$, altogether along with B. Then in any red-blue colouring there must be some i such that $A_i \subset C_{i,j}$ is monochrome, say blue. So then we either have a monochrome set, or every b_j has been coloured red - in which case B is monochrome.

Therefore $R(m+1) \leq R(m)(m+1)+1$. Thus:

$$\begin{aligned}
R(m) &\leq 1 + R(m-1)m \\
&\leq 1 + m + R(m-2)(m-1)m \\
&\leq \sum_{j=0}^{k} \frac{m!}{(m-j)!} + R(m-(k+1))(m-k)(m-(k-1))\ldots(m) \\
&\leq \sum_{j=0}^{m-2} + R(1)m! = \sum_{j=0}^{m-1} \frac{m!}{(m-j)!} = m! \sum_{j=0}^{m-1} \frac{1}{(m-j)!} \\
&= m! \sum_{\ell=1}^{m} \frac{1}{\ell!} \leq m!(-1 + \sum_{\ell=0}^{\infty} \frac{1}{\ell!}) \\
&\leq m!(e-1)
\end{aligned}$$

So all in all we have $2^{m-1} \leq R(m) \leq m!(e-1)$.

8. $\mathcal{A} \subset \mathcal{P}[n]$, such that $A, B \in \mathcal{A} \implies |A \Delta B| \neq 2$.

Construct a graph $G = (V, E)$ where $V = \mathcal{P}[n]$ and $AB \in E$ iff $|A\Delta B| = 2$. Set systems satisfying the given condition are then precisely the independent sets of vertices in G.

Note that each A has exactly $\binom{n}{2}$ neighbours; we can see this since there are $\binom{n}{2}$ possibilities for the symmetric difference $A\Delta B$, which will determine B completely.

Q7 then immediately shows the existence of a set system $\mathcal{A}$ of size $\left(\binom{n}{2} + 1\right)^{-1} 2^n$.

For the upper bound, consider for a given set A what is the largest possible size of an independent subset of $\Gamma(A)$.

Partition $\Gamma(A)$ into classes $C_1, \ldots, C_n$, where $B \in \Gamma(A)$ lies in C_i if $i \in A\Delta B$. Each $B \in \Gamma(A)$ then lies in exactly two C_i, corresponding to the elements of $A\Delta B$.

In general we have that $B_1 \Delta B_2 = (A\Delta B_1)\Delta(A\Delta B_2)$, so if $B_1 \neq B_2 \in C_k$ then $|B_1 \Delta B_2| = 2$, so they are not independent.

Hence an independent subset of $\Gamma(A)$ may only have at most one member from each C_i, and since each B lies in two classes, we can have at most $\lfloor n/2 \rfloor$ - otherwise two lie in the same class.

Now, given such a set system $\mathcal{A}$, let $X = \mathcal{A}$, $Y = \mathcal{P}[n] \backslash A$, and count the edges between X and Y.

Every $x \in X$ has exactly $\binom{n}{2}$ edges into Y, and for each $y \in Y$ there are at most $\lfloor n/2 \rfloor$ edges into X, as we saw above.

Hence $\binom{n}{2}|X| \leq \lfloor n/2 \rfloor |Y|$. In particular:

$$
\begin{aligned}
|\mathcal{A}| &\leq \frac{\lfloor n/2 \rfloor (2^n - |\mathcal{A}|)}{\binom{n}{2}} \\
\therefore |\mathcal{A}| &\leq \frac{\lfloor n/2 \rfloor 2^n}{\binom{n}{2} + \lfloor n/2 \rfloor}
\end{aligned}
$$

This gives $|\mathcal{A}| \leq 2^n/(n+1)$ for n odd, and $|\mathcal{A}| \leq 2^n/n$ for n even. Hence in all cases we have $|\mathcal{A}| \leq n^{-1} 2^n$.