Quantum Information Theory Revision Questions

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1. Define the surprisial of a random variable.
2. Define the Shanon entropy of a discrete random variable X.
3. Define binary entropy.
4. Define a source alphabet.
5. Define a memoryless source, and characterise it with random variables.
6. Define the Shannon entropy of a memoryless source.
7. Generally speaking, why is data compression possible?
8. Give two methods of data encoding and describe their differences.
9. Defina a typical/atypical signal.
10. Define a compression map C^n, and its rate.
11. Define a corresponding decompression map D^n.
12. The triple $\mathcal{C}_n$...?
13. Write down the average probability of error of $\mathcal{C}_n$.
14. Define what it means for $\mathcal{C}_n$ to be reliable.
15. Define the data compression limit for a source.
16. Define the ε-typical set $T_\varepsilon^{(n)}$, and the typical sequences.
17. What is the approximate probability of a typical sequence?
18. Demonstrate that this definition agrees with our intuitive notion of what a typical sequence should be.
19. State the Typical Sequence Theorem.
20. Prove (some bits of) the Typical Sequence Theorem.
21. State a corollary of the TST.
22. State Shannon's Source Coding Theorem.
23. State a lemma bounding the probability of a sufficiently small set of sequences.

24. Sketch a proof.
25. Explain how this implies the converse of Shannon.
26. Define joint entropy.
27. Define conditional entropy.
28. State and prove the chain rule identity relating joint and conditional entropy.
29. Define the relative entropy/Kullback Leibler divergence.
30. State Jensen's inequality.
31. Show that $D(p||q) \geq 0$. When does equality hold?
32. Define the mutual information $I(X : Y)$.
33. **LECTURE 3 DIAGRAM**
34. Define $D(P||Q)$ for a pairt of functions (instead of random variables).
35. Express $H(X), I(X : Y), H(X|Y)$ in terms of the relative entropy.
36. State the chain rule for entropies.
37. Define the conditional mutual information $I(X : Y|Z)$.
38. State the data-processing inequality.
39. Prove the data-processing inequality.
40. Show that $D(p||q) \geq 0$, with equality iff $p = q$.
41. Show that $H(X) \geq 0$, with equality iff X is deterministic.
42. Show that $H(X|Y) \geq 0$, or equivalently that $H(X, Y) \geq H(Y)$.
43. Show that if X takes values in J, then $H(X) \leq \log |J|$
44. Show that $H(X, Y) \leq H(X) + H(Y)$ (subadditivity).
45. Show that the Shannon entropy is concave.
46. Show that $I(X : Y) \geq 0$, with equality iff X and Y are independent.
47. Define a discrete channel.
48. Define a memoryless (discrete) channel.
49. Define a symmetric memoryless channel.
50. Define the memoryless, binary, symmetric channel.
51. What is majority voting?
52. What is the probability of error in this scheme?
53. How much has this scheme improved the reliability of transmission?

54. What is this type of error-correcting code called?
55. **LECTURE 4 DIAGRAM.**
56. Define $\mathcal{C}_n$, an error-correcting code.
57. Define the probability of error $p(\mathcal{C}_n)$.
58. Define an achievable rate for a channel.
59. Define the capacity $C(\mathcal{N})$ of a memoryless channel.
60. State Shannon's Noisy Channel Coding Theorem.
61. State three properties of $C(\mathcal{N})$.
62. Give some intuition behind the proof of SNCCT.
63. Use SNCCT to calculate the capacity of the binary memoryless symmetric channel.
64. Give two examples of physical realisations of qubit.s
65. What are reliably distinguishable states?
66. What is the Hamming space?
67. Define $\mathcal{B}(\mathcal{H})$.
68. Define the Hilbert-Schmidt inner product.
69. Define the Pauli matrices.
70. What is an open system?
71. What is decoherence?
72. State the four postulates of quantum mechanics.
73. What postulates are no longer valid in open systems?
74. How can we view open systems to get around this issue?
75. Describe the density matrix formalism.
76. State and prove two properties of density matrices.
77. What can we conclude from these properties?
78. Give an equivalent, but more abstract, definition of a density matrix.
79. Give an important property of $\mathcal{D}(\mathcal{H})$.
80. Define a pure state.
81. Define a mixed state.
82. Give an equivalent characterisation of pure/mixed states.
83. What is the purity of a state? What range of values can this take?

84. **LECTURE 6 FIGURE AND DISCUSSION.**

85. Show that $\mathcal{D}(\mathcal{H})$ is convex. Show that pure states are extremal points.

86. Define the expectation of an observable A.

87. Show that this is linear, positive and normal.

88. Define the reduced density operator/matrix of a bipartite state ρ_{AB}.

89. Define the partial trace.

90. **LECTURE 6 DETAILED EXPRESSIONS AND DERIVATIONS.**

91. Showt that a reduced density matrix is a valid density matrix.

92. Let $M_{AB} = M_A \otimes I_B$. Show that $\langle M_{AB} \rangle = \mathrm{Tr}(M_A \rho_A)$.

93. Define a separable bipartite pure state, and write down its reduced density operators.

94. Define an entangled bipartite pure state. What form do its reduced density operators have?

95. Define maximally entangeled states.

96. Write down the Bell/EPR states. Who are they named after?

97. What does it mean for a bipartite **mixed** state ρ_{AB} to be separable?

98. What does it mean for a bipartite mixed state to be entangled?

99. Show that separable bipartite states can always be expressed as a convex combination of pure product states.

100. State the Schmidt Decomposition Theorem.

101. Prove the Schmidt Decomposition Theorem.

102. Given an immediate consequence of this theorem for the reduced density matrices of a pure state Ψ_{AB}.

103. Under what condition is the Schmdit decomposition of $|\Psi_{AB}\rangle$ uniquely determined by ρ_A and ρ_B?

104. Define the Schmidt rank of a bipartite pure state $|\Psi_{AB}\rangle$. How is it denoted?

105. A bipartite pure state is entangled iff...?

106. Prove that a BPS is a product state iff its Schimdt number is equal to one.

107. Prove that a BPS is a product state iff its reduced density matrices are pure states.

108. When can we apply Schmidt, more generally?

109. Define the purification of a state ρ_A.

110. Define a reference system for a state ρ_A.

111. Prove that any state ρ_A can be purified.

112. Give a more general purification of ρ_A.

113. How can we express $|\Psi_{AR}\rangle$ with reduced state ρ_A having eigenvectors $\{|i\rangle\}$.

114. State the No-Cloning Theorem.

115. Prove the No-Cloning Theorem.

116. Why does this extend to full generality?

117. Why is the No-Cloning Theorem a Big Problem?

118. Who (independently) devised the first quantum error-correcting codes, and when?

119. Demonstrate how No-Cloning prevents superluminal communication.

120. Define a quantum operation, and give a simple example.

121. Define a linear CPTP map.

122. Why are quantum operations given by linear CPTPs?

123. Define $\mathcal{M}_n^+$.

124. Why is complete positivity a physically reasonable condition? **LECTURE 8/9 DISCUSSION.**

125. Give a map that is positive but not completely positive.

126. State a theorem that describes when a linear operator map is completely positive.

127. Prove this theorem.

128. Define the Choi matrix/state $J(\Lambda)$ for a quantum operation Λ.

129. State (a simplified version of) Stinespring's Dilation Theorem.

130. Therefore, any quantum operation can be composed of which three building blocks?

131. What does "going to the church of the larger Hilbert space" mean? **SCHEMATIC IN LECTURES.**

132. State the Kraus Representation Theorem.

133. Prove one direction of the Kraus Representation Theorem.

134. Show that it is essentially a restatement of Stinespring.

135. Is Kraus decomposition unique?

136. State the Choi-Jamilkowski Isomorphism (Theorem).
137. Define the adjoint Λ^* to Λ.
138. **LECTURE 9/10 VERIFY ONE C-J PART.**
139. What suffices to prove that the C-J maps are mutual inverses?
140. Prove this.
141. Prove the other direction of Kraus.
142. State Stinespring's Dilation Theorem (in full generality).
143. Prove Stinespring's Dilation Theorem.
144. Why are standard projective measurements insufficient for us?
145. Describe the Generalised Measurement Postulate.
146. What does POVM stand for, and why?
147. How can projective measurements be viewed as a special case of a generalised measurement?
148. What is the POVM formalism? When is it used?
149. What does a POVM not do?
150. Define a POVM.
151. What is a pure POVM?
152. State Neumark's Theorem.
153. Show that a projective measurement is a special case of a POVM.
154. Give an example (/case study?) of a POVM being useful.
155. Describe how to implement a generalised measurement using an ancilla, unitary dynamics and projective measurements.
156. Define the trace distance $D(\rho, \sigma)$ of two states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$.
157. Define $||A||_1$.
158. Give a helpful decomposition of the difference operator.
159. State and prove an identity relating D and this decomposition.
160. Prove that $D(\rho, \sigma) = \max_{0 \leq P \leq I} \operatorname{Tr}(P(\rho - \sigma))$.
161. Prove that D is a metric on the space of density operators.
162. Prove the monotonicity of D under quantum operations.
163. Relate the trace distance to quantum hypothesis testing with binary POVMs.
164. Which measurement maximises p^*_{success}?

165. The trace distance is the...?
166. Define the fidelity $F(\rho, \sigma)$ for $\rho, \sigma \in \mathcal{B}(\mathcal{H})$.
167. What form does F take when $[\rho, \sigma] = 0$?
168. What form does F take when one of ρ, σ is pure? What about both states pure?
169. Prove that fidelity is invariant under unitary transformation.
170. State Uhlmann's Theorem.
171. Prove that $||A||_1 = \sup_U |\mathrm{Tr}(UA)|$, U unitary.
172. Prove Uhlmann's Theorem.
173. Show that $0 \leq F(\rho, \sigma) \leq 1$, with $= 1$ iff $\rho = \sigma$.
174. Show that $F(\rho, \sigma) = F(\sigma, \rho)$.
175. Prove that $F(\rho_{AB}, \sigma_{AB}) \leq F(\rho_A, \sigma_A)$.
176. Define the entanglement fidelity $F_{\mathrm{e}}(\rho, \Lambda)$.
177. Express $F_{\mathrm{e}}(\rho, \Lambda)$ in terms of fidelity F.
178. Prove that $F_{\mathrm{e}}(\rho, \Lambda) = \sum_k |\mathrm{Tr}(A_k \rho)|^2$, for A_k Kraus operators of Λ.
179. Show that $F_{\mathrm{e}}(\rho, \Lambda) \leq F(\rho, \Lambda(\rho))^2$.
180. Define the von Neumann entropy of a density matrix ρ.
181. What is its classical analogue?
182. State four properties of the von Neumann entropy.
183. Prove concavity.
184. For $\rho \in \mathcal{D}(\mathcal{H})$ and $\sigma \in \mathcal{B}(\mathcal{H})$, define the quantum relative entropy $D(\rho||\sigma)$.
185. State Klein's inequality.
186. Prove Klein's inequality.
187. Now prove $S(\rho) \leq \log d$.
188. State the data-processing inequality.
189. State joint convexity of quantum relative entropy.
190. **LIEB'S CONCAVITY THEOREM DISCUSSION LECTURE 14.**
191. State two properties of $D(\rho||\sigma)$ involving products and unitaries.
192. Define the Heisenberg-Weyl operators.

193. Prove that
$$\frac{1}{d^2}\sum_{k,m=0}^{d-1} W_{k,m} A W_{k,m}^\dagger = (\mathrm{Tr}A)I/d$$

194. Define the quantum joint entropy.
195. Define the quantum conditional entropy.
196. Define the quantum mutual information.
197. Express all of the above quantities in terms of the relative entropy.
198. What property of classical entropy is not upheld in the quantum case?
199. Demonstrate the above through an example.
200. Prove additivity of S.
201. State subadditivity of S.
202. Prove subadditivity of S. [or outline a proof] To what classical property is this analogous?
203. Prove equality of entropies of subsystems of a bipartite pure state.
204. State and prove the triangle inequality for S.
205. Prove that $S(\sum_i p_i \rho_i) = H(p) + \sum_i p_i S(\rho_i)$.
206. State strong subadditivity. How is it proved?
207. **LECTURE 15 CHECK FOR PROOF OF SSA.**
208. State three consequences of SSA. [prove them?]
209. Prove the last of these.
210. Define a quantum information source.
211. Given an equivalent formulation of the above.
212. What is a (quantum) compression scheme?
213. What is a corresponding (quantum) decompression scheme?
214. What property must both $\mathcal{C}^{(n)}$ and $\mathcal{D}^{(n)}$ have?
215. Define the *rate* of a compression-decompression scheme.
216. What does it mean for a compression scheme $(\mathcal{C}^{(n)}, \mathcal{D}^{(n)})$ to have rate R?
217. What quantity is being compressed in a quantum compression scheme?
218. What key difference between classical and quantum compression makes state reconstruction difficult?
219. Define the ensemble average fidelity.

220. Define what it means for a compression-decompression scheme to be *reliable*.

221. What is a typical subspace?

222. Give an expression for the density matrix of a memoryless/iid quantum channel.

223. Write down the spectral decomposition for the density matrix $\rho^{(n)}$ of a memoryless source.

224. What is the von Neumann entropy of such a source?

225. Define $T_\varepsilon^(n)$.

226. Give a bound on the probability of a (classical) typical sequence $(i_1 \dots i_n)$.

227. Define $\mathcal{T}^{(n)}$. What is it called?

228. State the Typical Subspace Theorem.

229. What is the probability of the typical subspace $\mathcal{T}_\varepsilon^{(n)}$?

230. State Schumacher's Theorem.

231. **Prove Schumacher's Theorem.** [This might be asking a bit much lmao.]

232. What is a quantum channel?

233. Describe the Bloch sphere representation of a qubit.

234. Give four examples of single qubit channels.

235. Write down the Bit-flip channel and its Kraus operators.

236. How does it act on the Bloch sphere?

237. Describe the class or random unitary channels/mixing-enhancing channels.

238. Describe the Phase flip channel and its Kraus operators.

239. How does it act on the Bloch sphere?

240. Describe the Depolarising channel and its Kraus operators.

241. How does it act on the Bloch sphere?

242. Give an alternative characterisation of the Depoloarising channel.

243. Prove this equivalence.

244. Generalise the depolarising channel to $d > 2$ dimensions.

245. Describe the context of an amplitude damping channel.

246. Write down the unitary transformation describing the evolution of a 2-level atom.

247. Write down the Kraus operators for the channel.

248. Now write down the CPTP map. Is it unital?

249. Describe the evolution of the atom over time (repeated applications of the channel). What is its limiting state?

250. How then does this map change the input state? Why is this surprising?

251. But why is the above not as surprising as it initially seems?

252. **LECTURE 18 DIAGRAM**

253. Define the accessible information of an ensemble $\{p_x, \rho_x\}$.

254. **State the Holevo Bound.**

255. Define the Holveo χ quantity.

256. For an ensemble $\mathcal{E}$ of pure states, what does $\chi(\mathcal{E})$ reduce to? Why?

257. **Prove the Holveo Bound.**
    (a) Define the enlarged Hilbert space representation.
    (b) What is the initial state of the quantum system AQB?
    (c) Define the action Λ of the quantum operation of measuring and recording.
    (d) Show that Λ is indeed a quantum operation.
    (e) Express the state $\rho_{A'Q'B'}$ after application of Λ.
    (f) Prove the Holveo bound in terms of mutual information.
    (g) Demonstrate that this is in fact the Holevo bound.

258. Show that the Holevo χ quantity is non-negative.

259. Express $\chi(\mathcal{E})$ using the relative entropy.

260. Can a quantum operation increase χ?

261. Show that the von Neumann entropy is not monotonic under quantum operations.

262. Define a memoryless quantum channel.

263. Describe four conditions affecting the type of relevant capactiy of a quantum channel.

264. **LECTURE 19 DIAGRAM**

265. Define the rate of information transmission (for classical info through a quantum channel).

266. Define what it means for the transmission of classical information through Λ to be reliable.

267. Define an achievable rate.
268. Define the capacity of a quantum channel (in this context).
269. Define the product-state classical capacity of a quantum channel.
270. State the Holevo-Schumacher-Westmoreland (HSW) Theorem.
271. Define a classical-quantum (c-q) state.
272. Prove that the maximisation in the Holveo capacity can be restricted to pure state ensembles.
273. Prove that the Holveo capacity is superadditive.
274. Use the HSW theorem to find the product state capacity of the qubit depolarising channel.
275. Prove that an arbitrary quantum channel Λ can be used to transmit classical information, provided the channel is not simply a constant *i.e.* $\Lambda(\rho)$ is not identical for all ρ.
276. Show in a separate way that it suffices to consider pure state ensembles.
277. Use the Holevo bound to justify that by transmitting n qubits to Bob, Alice can send at most n bits of classical information to him.
278. Prove the converse of the HSW theorem.
279. What question does the HSW theorem naturally lead to?
280. State the additivity conjecture of the Holveo capacity.
281. Define $C_{\text{classical}}(\Lambda)$, and give an expression for it.
282. Explain how the additivity conjecture relates to this capacity.
283. Who gave a counterexample to the additivity conjecture, and when? What is the consequence of this?
284. Define the coherent information of a bipartite quantum state.
285. Express this in terms of conditional von Neumann entropy.
286. In some sense, what is this analgous to in the classical case?
287. Let $\rho_{RQ} = \frac{1}{2}\sum_{i=0}^{1} |i\rangle\langle i|_R \otimes |i\rangle\langle i|_Q$. Evaluate $I(R\rangle Q))_\rho$ and $I(R:Q)_\rho$.
288. Now let $|\psi\rangle_{RQE}$ be a purification of ρ_{RQ}.
    (a) Show that $I(R\rangle Q) = S(Q)_\psi - S(E)_\psi$.
    (b) Show that $-S(R|Q)_\rho = I(R\rangle Q)_\rho = S(R|E)_\psi$.
289. Show that $|S(R\rangle Q)_\rho| \leq \log\dim(\mathcal{H}_R)$. When do we have equality?
290. Define the coherent information of a quantum channel Λ with respect to the input state $\rho = \rho_Q$.

291. Show that $I_c(\Lambda, \rho) \leq S(\rho_Q)$. When do we have equality, and why?
292. When do we have $I_c(\lambda, \rho) = S(\rho)$, in general?
293. State the quantum data processing inequality.
294. Prove the quantum data processing inequality.
    (a) What are the two key tools in this proof?
295. Define $Q^{(1)}(\Lambda)$.
296. Define $Q^{(n)}(\Lambda)$.
297. State a theorem of Lloyd, Shor and Devetak.
298. What does LOCC stand for?
299. Define what LOCC is.
300. Describe a LOCC transformation.
301. Define $\text{LOCC}^{\rightarrow}$, $\text{LOCC}^{\leftarrow}$, and $\text{LOCC}^{\leftrightarrow}$.
302. Define what it means for a state ρ_{AB} to be distillable.
303. Define what it means to be LOCC-distillable.
304. ρ_{AB} is distillable iff...?
305. Or equivalently...?
306. Describe the superactivation phenomenon.
307. q[Unsure if this business is really examinable, on the whole.]