Category Theory

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2 The Yoneda Lemma

To have an entire chapter of a course devoted to a lemma might seem excessive; but the Yoneda Lemma is not really a lemma, it is much more an entire way of thinking about categories. It is also not really due to Yoneda, or at least the attribution is questionable, since Yoneda never published it. (Nobuo Yoneda was a Japanese mathematician who became interested in categories around 1950 and wrote two papers on the subject (neither of which contains the lemma); he then turned to computer science, in which he had a distinguished career, and published nothing further in pure mathematics. The origin of the name was explained by Saunders Mac Lane in an obituary notice for Yoneda which he wrote many years later: the two men met at a conference in Paris in about 1950, and struck up a friendship, so that when Mac Lane was due to move on to another conference in London at the end of the week (travelling by train and boat, as one did in those days), Yoneda, who was staying on in Paris for a while, came to the Gare du Nord to say goodbye to him. The two men were standing on the platform chatting about this and that, waiting for the boat train to be ready for boarding, when Yoneda said 'By the way, Saunders, did you know the following?' and proceeded to tell him the statement of the lemma. Mac Lane didn't know it; but he recognized its importance perhaps more acutely than Yoneda (given that Yoneda never bothered to publish it) and began including it in his lectures on category theory in various forums, in which he referred to it as 'Yoneda's Lemma' because he had learned it from Yoneda. Because Mac Lane was very influential, as one of the founders of category theory, the name stuck; but the result is really a 'folk-theorem' which probably occurred to several people independently.)

The Yoneda Lemma is about locally small categories, which we now define:

Definition 2.1 We say a category $\mathcal{C}$ is *locally small* if, for any two objects A and B, the morphisms $A \to B$ in $\mathcal{C}$ are parametrized by a set $\mathcal{C}(A, B)$.

For example, **Set** is locally small (though not small), as indeed are all the categories we have met so far with the exception of functor categories $[\mathcal{C}, \mathcal{D}]$ where $\mathcal{C}$ is not small. Indeed, some authors include local smallness in their definition of a category; we refrained from doing so because we wanted to make the definition independent of set theory, and also because we do occasionally want to refer to non-locally-small functor categories, and it is inconvenient to have to use some name such as 'paracategory' or 'improper category' for them.

If A is an object of a locally small category $\mathcal{C}$, we have a functor $\mathcal{C}(A, -) \colon \mathcal{C} \to \mathbf{Set}$ sending an object B to the set $\mathcal{C}(A, B)$ and a morphism $f \colon B \to C$ to the mapping $g \mapsto fg$. (Associativity of composition in $\mathcal{C}$ ensures that this is functorial.) Similarly, by fixing the second variable, we obtain a functor $\mathcal{C}(-, B) \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. The Yoneda Lemma comes in two parts; we shall state both of them together, but we shall prove the first (easy) part and draw a couple of corollaries from it before attempting to explain the second part.

Lemma 2.2 *Let A be an object of a locally small category $\mathcal{C}$, and $F\colon \mathcal{C} \to \mathbf{Set}$ a functor. Then there is a bijection between natural transformations $\mathcal{C}(A,-) \to F$ and elements of FA. Moreover, this bijection is natural in both F and A.*

Proof (i) Given $\alpha : \mathcal{C}(A,-) \to F$, we define $\Phi(\alpha) = \alpha_A(1_A) \in FA$; given $x \in FA$, we define $\Psi(x)_B(f : A \to B) = (Ff)(x) \in FB$. The fact that $\Psi(x)$ is a natural transformation follows easily from functoriality of F. It is clear that $\Phi\Psi(x) = F(1_A)(x) = x$ for any x; and if α is any natural transformation $\mathcal{C}(A,-) \to F$, then for any $f\colon A \to B$ we have

$$\alpha_B(f) = \alpha_B \mathcal{C}(A,f)(1_A) = (Ff)\alpha_A(1_A) = (Ff)(\Phi(\alpha)) = \Psi\Phi(\alpha)_B(f)$$

using naturality of α at the second step. So Φ and Ψ are inverse bijections. $\square$

Corollary 2.3 *For any locally small category $\mathcal{C}$, the assignment $A \mapsto \mathcal{C}(A,-)$ defines a full and faithful functor $\mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathbf{Set}]$.*

Proof Putting $F = \mathcal{C}(B,-)$ in 2.2, we have a bijection from $\mathcal{C}(B,A)$ to the collection of natural transformations $\mathcal{C}(A,-) \to \mathcal{C}(B,-)$. So we have merely to verify that this bijection is functorial (i.e. respects composition); but the natural transformation corresponding to $f : B \to A$ is given by composition with f, so this follows straightforwardly from the associative law in $\mathcal{C}$. $\square$

The full and faithful functor of 2.3 (or, more commonly, the similar functor $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ defined by the functors $\mathcal{C}(-,B)$) is known as the *Yoneda embedding*. The real message of Yoneda is thus that any locally small category may be viewed, up to equivalence, as a full subcategory of a **Set**-valued functor category.

Definition 2.4 We call a functor $F\colon \mathcal{C} \to \mathbf{Set}$ *representable* if it is isomorphic to $\mathcal{C}(A,-)$ for some A. By a *representation* of F, we mean a pair (A,x) where $x \in FA$ is such that $\Psi(x)$ is an isomorphism; we also call x a *universal element* of F.

Corollary 2.5 *('Representations are unique up to unique isomorphism') Suppose (A,x) and (B,y) are both representations of F. Then there is a unique isomorphism $f\colon A \to B$ such that $Ff(x) = y$.*

Proof The equation $Ff(x) = y$ is equivalent to saying that the triangle

$$\begin{tikzcd}
\mathcal{C}(B,-) \arrow[r, "\mathcal{C}(f,-)"] \arrow[dr, "\Psi(y)"'] & \mathcal{C}(A,-) \arrow[d, "\Psi(x)"] \\
 & F
\end{tikzcd}$$

commutes, so f must be the unique isomorphism whose image under the Yoneda embedding is $(\Psi(x))^{-1}\Psi(y)$. $\square$

We are now ready to revert to the second part of 2.2 (the naturality statement).

Proof (ii) To understand it, suppose for the moment that $\mathcal{C}$ is actually small, so that $[\mathcal{C}, \mathbf{Set}]$ is locally small. Then we have two functors $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$: the first is simply given on objects by $(A,F) \mapsto FA$ (and on morphisms by the diagonals of naturality squares), and the second is the composite

$$\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{Y \times 1} [\mathcal{C}\ \mathbf{Set}]^{\mathrm{op}} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{[\mathcal{C}, \mathbf{Set}](-,-)} \mathbf{Set}$$

where Y is the Yoneda embedding and the second part is the functor obtained by allowing both arguments to vary in $[\mathcal{C}, \mathbf{Set}]\,(F, G)$. The assertion is then that Φ and Ψ are natural isomorphisms between these two functors. But if we translate this into elementary terms, it makes sense even without the assumption that $\mathcal{C}$ is small: it says that, if we are given $f\colon A \to A'$ in $\mathcal{C}$ and $\alpha\colon F \to F'$ in $[\mathcal{C}, \mathbf{Set}]$, if $x \in FA$ and x' is its image under the diagonal of the naturality square

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"] & FA' \arrow[d, "\alpha_{A'}"] \\ F'A \arrow[r, "F'f"] & F'A' \end{tikzcd}$$

then $\Psi(x')$ is the composite

$$\mathcal{C}\,(A', -) \xrightarrow{Yf} \mathcal{C}\,(A, -) \xrightarrow{\Psi(x)} F \xrightarrow{\alpha} F' \quad .$$

But this is easy to verify. □

Many familiar functors are representable, and so partake of the uniqueness guaranteed by 2.5.

Examples 2.6 (*a*) The forgetful functor $\mathbf{Gp} \to \mathbf{Set}$ is represented by the free group $\mathbb{Z}$ on one generator; similarly the forgetful functor $\mathbf{Rng} \to \mathbf{Set}$ is represented by the polynomial ring $\mathbb{Z}[X]$, $\mathbf{Top} \to \mathbf{Set}$ is represented by the one-point space, and so on.

(*b*) The contravariant power-set functor $P^*\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is represented by the set $2 = \{0, 1\}$: this is just the familiar correspondence between subsets and characterstic (or indicator) functions. However, the covariant functor $P\colon \mathbf{Set} \to \mathbf{Set}$ is not representable; we may see this by observing that if 1 is the singleton set $\{\emptyset\}$ then $\mathbf{Set}\,(A, 1)$ is also a singleton for any A, whereas $P1$ has two elements.

(*c*) The dual vector space functor is not strictly representable, since its codomain is not $\mathbf{Set}$; but its composite with the forgetful functor $\mathbf{Vect}_k \to \mathbf{Set}$ is representable by the 1-dimensional space k.

(*d*) For a group G, we recall that functors $G \to \mathbf{Set}$ correspond to permutation representations of G. The representable functor corresponding to the unique object of G is the *Cayley representation*, i.e. the action of G on itself by left multiplication.

(*e*) If A and B are objects of a locally small category $\mathcal{C}$, we may form the functor $\mathcal{C}\,(-, A) \times \mathcal{C}\,(-, B)$. This may or may not be representable; if it is, we call the representing object a (categorical) *product* of A and B, and denote it by $A \times B$. The universal element consists of a pair of morphisms $(\pi_1\colon A \times B \to A, \pi_2\colon A \times B \to B)$ (the *product projections*) which have the property that, given any $f\colon C \to A$ and $g\colon C \to B$, there exists a unique $h = (f, g)$ satisfying $\pi_1 h = f$ and $\pi_2 h = g$. (Note that this notion makes good sense even without the assumption that $\mathcal{C}$ is locally small.) Dually, we have the notion of *coproduct* $A + B$ with coprojections $\nu_1\colon A \to A + B$ and $\nu_2\colon B \to A + B$. (It is a fair guess that the person who chose the name 'coproduct' did not have the benefit of a classical education: 'copros' is the Greek word for 'shit', so a 'coproduct' ought to mean a sewer.)

(*f*) Rather similarly, if we are given a parallel pair $f, g\colon A \rightrightarrows B$ in a locally small category $\mathcal{C}$, we may form the subfunctor of $\mathcal{C}\,(-, A)$ whose elements are those $h\colon C \to A$ satisfying $fh = gh$. A representation for this functor, if it exists, is called an *equalizer* of f and g; once again, this notion can be presented in elementary terms which make sense in non-locally-small categories. Note that the universal element $e\colon E \to A$ of an equalizer is necessarily monic, since anything which factors through it does so in just one way; we call a monomorphism *regular* if it occurs as an equalizer of some pair. Dually, we have the notions of *coequalizer* and *regular epimorphism.* Note that if a morphism is both epic and regular monic then it is necessarily an isomorphism, since the parallel pair of which it is an equalizer must already be equal.

Many familiar categories possess (co)products for arbitrary pairs of objects and (co)equalizers for arbitrary parallel pairs of morphisms. For example, in **Set** the usual cartesian product $A \times B$ is a categorical product, and the disjoint union $A \amalg B = A \times \{0\} \cup B \times \{1\}$ serves as a coproduct. It is also the case that in **Set** all injections are regular monomorphisms and all surjections are regular epimorphisms. But it is not always true that all monos and epis are regular: for example, **Top** has products and coproducts constructed much as in **Set**, but a regular mono in **Top** is an injection $X \to Y$ for which X is topologized as a subspace of Y, and similarly regular epis are those surjections for which Y is topologized as a quotient of X.

The message of the Yoneda Lemma is that, if we know all the functors $\mathcal{C}(A, -)$, $A \in \mathrm{ob}\, \mathcal{C}$, we know all there is to know about $\mathcal{C}$. But there are many cases where we don't need to know *all* the $\mathcal{C}(A, -)$, but only some of them: this leads to the notion of what is commonly called a generating family of objects of $\mathcal{C}$. Unfortunately there are two slightly different definitions of this term, and they are not equivalent; some people get round this by using the term 'strong generating family' for the second one, but since neither in fact implies the other this is not very satisfactory. We shall therefore introduce two entirely different names for the two notions — though, for the sake of tradition, we shall denote the class of objects by $\mathcal{G}$.

Definition 2.7 Let $\mathcal{G}$ be a class of objects of a locally small category $\mathcal{C}$.

(a) We say $\mathcal{G}$ is a *separating family* for $\mathcal{C}$ if the functors $\mathcal{C}(G, -)$, $G \in \mathcal{G}$, are collectively faithful; i.e., if $f, g\colon A \rightrightarrows B$ satisfy $fh = gh$ whenever $\mathrm{dom}\, h \in \mathcal{G}$, then $f = g$.

(b) We say $\mathcal{G}$ is a *detecting family* if the $\mathcal{C}(G, -)$ collectively reflect isomorphisms; i.e., if $f\colon A \to B$ is such that every $h\colon G \to B$ with $G \in \mathcal{G}$ factors uniquely through f, then f is an isomorphism.

If $\mathcal{G}$ is a singleton $\{G\}$, we call the object G a *separator* or a *detector*, as appropriate.

The two notions, though not equivalent, are closely related:

Lemma 2.8 (i) *If $\mathcal{C}$ has equalizers, then any detecting family in $\mathcal{C}$ is also separating.*

(ii) *If $\mathcal{C}$ is balanced, then any separating family is also detecting.*

Proof (i) Suppose $\mathcal{G}$ is detecting, and let (f, g) satisfy the hypothesis of 2.7(a). Then any $h\colon G \to A$ with $G \in \mathcal{G}$ factors uniquely through the equalizer of f and g, so the latter is an isomorphism. Hence $f = g$.

(ii) Suppose $\mathcal{G}$ is separating, and let f satisfy the hypothesis of 2.7(b). If $fg = fh$, then any $k\colon G \to \mathrm{dom}\, g$ with $G \in \mathcal{G}$ satisfies $gk = hk$, since both are factorizations of fgk through f, and hence $g = h$; i.e., f is monic. Again, if $lf = mf$, then any $n\colon G \to B$ with $G \in \mathcal{G}$ satisfies $ln = mn$, since it factors through f, so $l = m$; i.e., f is epic. Since $\mathcal{C}$ is balanced, f is an isomorphism. □

Examples 2.9 (*a*) In algebraic categories such as **Gp** and **Rng**, the objects representing the forgetful functor to **Set** (cf. 2.6(*a*)) are both separators and detectors.

(*b*) The Yoneda Lemma itself can be interpreted as saying that, for any small $\mathcal{C}$, the set $\{\mathcal{C}(A,-) \mid A \in \text{ob}\,\mathcal{C}\}$ is both separating and detecting in $[\mathcal{C}, \mathbf{Set}]$ (recall from 1.8 that pointwise isomorphisms in $[\mathcal{C}, \mathbf{Set}]$ are isomorphisms).

(*c*) In **Top**, the singleton space 1 is a separator; but **Top** has no detecting set. To see this, note that for any infinite cardinal κ we can find two topologies on a set of cardinality κ (namely, the discrete topology and the topology whose closed sets are the whole space and those of cardinality less than κ) which agree on any subspace of cardinality less than κ; so, given any set $\mathcal{G}$ of spaces, if we take κ strictly larger than the cardinalities of all the members of $\mathcal{G}$, it will not be able to detect that the identity mapping between the two spaces of cardinality κ is not a homeomorphism.

(*d*) To give a counterexample to the converse implication, we shall need to quote two hard theorems from algebraic topology. Let $\mathcal{C}$ be the category of connected CW-complexes with basepoints (a CW-complex is just a 'nice space' built up in a standard way by glueing together balls in Euclidean spaces) and basepoint-preserving homotopy classes of maps between them. A famous theorem of Henry Whitehead shows that in this category 'every weak homotopy equivalence is a homotopy equivalence', i.e. if $f\colon X \to Y$ induces isomorphisms of homotopy groups in all dimensions, then it is an isomorphism in $\mathcal{C}$. And the homotopy group functors (or rather their composites with the forgetful functor $\mathbf{Gp} \to \mathbf{Set}$) are represented by the spheres S^n; so this is tantamount to saying that $\{S^n \mid n \geq 1\}$ is a detecting set for $\mathcal{C}$. But another theorem due to Peter Freyd shows that there is *no* faithful functor $\mathcal{C} \to \mathbf{Set}$; hence $\mathcal{C}$ cannot have a separating set of objects.

The last topic we shall mention in this chapter is that of projectivity. It is immediate from the definition that functors of the form $\mathcal{C}(A,-)$ preserve monomorphisms; but they do not preserve epimorphisms in general.

Definition 2.10 We say an object P is *projective* if $\mathcal{C}(P,-)$ preserves epimorphisms; i.e. if, given any diagram

$$\begin{tikzcd} & P \arrow[d, "f"] \\ Q \arrow[r, "g", two heads] & R \end{tikzcd}$$

with g epic, there exists $h\colon P \to Q$ with $gh = f$. Dually, we say P is *injective* if it is projective in $\mathcal{C}^{\text{op}}$.

If P satisfies this condition for g in some particular class $\mathcal{E}$ of epimorphisms, we call it $\mathcal{E}$-*projective*; for example, if $\mathcal{E}$ is the class of regular epimorphisms we speak of *regular projectives*. In what follows we shall consider the class of pointwise epimorphisms in $[\mathcal{C}, \mathbf{Set}]$ (that is, natural transformations α such that α_A is surjective for all $A \in \text{ob}\,\mathcal{C}$); these are in fact exactly the epimorphisms in this category, but we shall not be able to prove this until chapter 4 (see Remark 4.8, and cf. Exercise 2.18 below). Once we have done that, we shall be able to delete the word 'pointwise' from the statements of the next two results.

Corollary 2.11 *For any locally small category $\mathcal{C}$, the functors $\mathcal{C}(A,-)$ are pointwise projective in* $[\mathcal{C}, \mathbf{Set}]$.

Proof This is another immediate consequence of Yoneda: if $P = \mathcal{C}(A,-)$ in the diagram of 2.10, then g_A is surjective, so $\Phi(f)$ is the image of some $\Phi(h) \in QA$. $\square$

As a consequence of 2.11, we may deduce that when $\mathcal{C}$ is small the category $[\mathcal{C}, \mathbf{Set}]$ has 'enough projectives', in the following sense:

Proposition 2.12 *Let $\mathcal{C}$ be a small category. For any functor $F : \mathcal{C} \to \mathbf{Set}$, there is a pointwise epimorphism $P \twoheadrightarrow F$ in $[\mathcal{C}, \mathbf{Set}]$ with P pointwise projective.*

Proof Given F, take P to be the (pointwise) disjoint union $\coprod_{(A,x)} \mathcal{C}(A, -)$, where the union is over all pairs (A, x) with $A \in \mathrm{ob}\, \mathcal{C}$ and $x \in FA$. Clearly, specifying a morphism $P \to G$ is equivalent to specifying a family of morphisms $\mathcal{C}(A, -) \to G$ for each (A, x) in the indexing family (that is, P is an infinitary version of the notion of coproduct introduced in 2.6(e)). It is easy to see that, in any category, a coproduct of $\mathcal{E}$-projective objects is $\mathcal{E}$-projective; so P is pointwise projective. But the morphism $P \to F$ whose (A, x)th component is $\Psi(x) \colon \mathcal{C}(A, -) \to F$ is pointwise epic, since x is in the image of $\Psi(x)_A$. □

We shall explore the projectives in $[\mathcal{C}, \mathbf{Set}]$ further in Exercise 2.17 below.

Exercises on Chapter 2

Exercise 2.13 By an *automorphism* of a category $\mathcal{C}$, we of course mean a functor $F : \mathcal{C} \to \mathcal{C}$ with a (2-sided) inverse. We say an automorphism F is *inner* if it is naturally isomorphic to the identity functor. [To see the justification for this name, think about the case when $\mathcal{C}$ is a group.]

(i) Show that the inner automorphisms of $\mathcal{C}$ form a normal subgroup of the group of all automorphisms of $\mathcal{C}$. [Don't worry about whether these groups are sets or proper classes!]

(ii) If 1 is a terminal object of $\mathcal{C}$ (i.e. an object such that for any A there is a unique morphism $A \to 1$), show that $F(1)$ is also a terminal object (and hence isomorphic to 1) for any automorphism F of $\mathcal{C}$. Deduce that, for any automorphism F of $\mathbf{Set}$, there is a *unique* natural isomorphism from the identity to F. [*Hint*: Yoneda!]

(iii) Let $\mathcal{C}$ be a full subcategory of $\mathbf{Top}$ containing the singleton space 1 and the *Sierpiński space* S, i.e. the two-point space $\{0, 1\}$ in which $\{1\}$ is open but $\{0\}$ is not. Show that S is, up to isomorphism, the unique object of $\mathcal{C}$ which has precisely three endomorphisms, and deduce that $FS \cong S$ for any automorphism F of $\mathcal{C}$. Show also that there is a unique natural isomorphism $\alpha \colon U \to UF$, where $U \colon \mathcal{C} \to \mathbf{Set}$ is the forgetful functor. By considering naturality squares of the form

$$\begin{tikzcd} UX \arrow[r, "\alpha_X"] \arrow[d, "Uf"] & UFX \arrow[d, "UFf"] \\ US \arrow[r, "\alpha_S"] & UFS \end{tikzcd}$$

where f is continuous, deduce that, if $\mathcal{C}$ also contains a space in which not every union of closed sets is closed, then α_S is continuous. Hence show that F is (uniquely) naturally isomorphic to the identity functor.

(iv) If $\mathcal{C}$ is the category of finite topological spaces, show that the group of inner automorphisms of $\mathcal{C}$ has index 2 in the group of all automorphisms.

Exercise 2.14 Recall the notion of *idempotent* from Exercise 1.17. Show that an idempotent $e \colon A \to A$ splits iff the pair $(e, 1_A)$ has an equalizer, iff the same pair has a coequalizer. Deduce that a split monomorphism (i.e., a morphism having a left inverse) is regular monic.

Exercise 2.15 A monomorphism $f\colon A \to B$ in a category is said to be *strong* if, for every commutative square

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, two heads, "g"'] & A \arrow[d, tail, "f"] \\ D \arrow[r, "k"'] & B \end{tikzcd}$$

with g epic, there exists a (necessarily unique) $t\colon D \to A$ such that $ft = k$ and $tg = h$. Show that every regular monomorphism is strong, but that in the finite category $\mathcal{C}$ whose objects and non-identity morphisms are represented by the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "l"'] & B \arrow[d, shift right, "h"'] \arrow[d, shift left, "k"] & C \arrow[l, "g"'] \arrow[dl, "m"] \\ & D & \end{tikzcd}$$

the morphism f is strong monic but not regular monic.

Exercise 2.16 Let $(f\colon A \to B, g\colon B \to C)$ be a composable pair of morphisms.

(i) If both f and g are monic (resp. strong monic, split monic), show that the composite gf is too.

(ii) If gf is monic (resp. strong monic, split monic), show that f is too.

(iii) If gf is regular monic and g is monic, show that f is regular monic.

(iv) Let $\mathcal{C}$ be the full subcategory of **AbGp** whose objects are groups having no elements of order 4 (though they may have elements of order 2). Show that multiplication by 2 is a regular monomorphism $\mathbb{Z} \to \mathbb{Z}$ in $\mathcal{C}$, but that its composite with itself is not. [*Hint*: first show that equalizers in $\mathcal{C}$ coincide with those in **AbGp**.] In the same category, find a pair of morphisms (f, g) such that gf is regular monic but f is not.

Exercise 2.17 For the purposes of this exercise, you may assume the result that epimorphisms in $[\mathcal{C}, \mathbf{Set}]$ coincide with pointwise epimorphisms; cf. the remark before Corollary 2.11.

(i) Let $\mathcal{C}$ be a small category and $F : \mathcal{C} \to \mathbf{Set}$ a functor. F is said to be *irreducible* if, whenever we are given a family of functors $(G_i \mid i \in I)$ and an epimorphism $\alpha\colon \coprod_{i\in I} G_i \twoheadrightarrow F$ (where $\coprod$ denotes coproduct in $[\mathcal{C}, \mathbf{Set}]$ — cf. 2.12), there exists $i \in I$ such that the restriction of α to G_i is still epimorphic. Show that F is irreducible iff there is an epimorphism $\mathcal{C}(A, -) \twoheadrightarrow F$ for some $A \in \mathrm{ob}\,\mathcal{C}$.

(ii) Deduce that F is irreducible and projective iff there is a split epimorphism $\mathcal{C}(A, -) \twoheadrightarrow F$ for some A.

(iii) Hence show that if all idempotents in $\mathcal{C}$ split, then the irreducible projectives in $[\mathcal{C}, \mathbf{Set}]$ are exactly the representable functors.

(iv) Deduce that if $\mathcal{C}$ and $\mathcal{D}$ are small categories then $[\mathcal{C}, \mathbf{Set}] \simeq [\mathcal{D}, \mathbf{Set}]$ iff the categories $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ defined as in Exercise 1.17(iii) are equivalent.

Exercise 2.18 Let $\mathcal{D}$ be the full subcategory of the category $\mathcal{C}$ in Exercise 2.15 with objects A, B and D, and let $\mathbf{2}$ be the category with two objects 0 and 1 and one non-identity morphism $0 \to 1$. Find an example of a morphism in the functor category $[\mathbf{2}, \mathcal{D}]$ which is epic but not pointwise epic.