KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2024/2025 ACADEMIC YEAR

SECOND YEAR **FIRST** SEMESTER REGULAR EXAMINATIONS **FOR THE DEGREE OF**:

BACHELOR OF SCIENCE (SCIENCE, WITH EDUCATION**,** IN APPLIED STATISTICS AND COMPUTING **AND** IN ACTURIAL SCIENCE

BACHELOR OF EDUCATION (IN SCIENCE AND IN ARTS)

COURSE CODE: MAT 213

COURSE TITLE: LINEAR ALGEBRA II

DATE: 13TH DECEMBER 2024 **TIME:** 3:00PM – 5:00PM

INSTRUCTION TO CANDIDATES

- SEE INSIDE

INSTRUCTION: Answer all questions in section A and any other TWO in section B

SECTION A (30 Marks)

Question One (30 Marks)

a) Verify the Cauchy-Schwarz inequality for the functions $f(x) = 1$ and $g(x) = x$ over the closed interval [0,1]. (4 marks)

b) Suppose that $u, v \in V$,then prove that $\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$. (3 marks)

c) Given that T (x, y, z) = (2x+y, 3y−z). Write down the standard matrix of T and use it to find T (0, 1, −1). (3 marks)

d) (i) Find a basis for the eigenspace associated with the matrix $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$. (2 marks)

(ii) Determine whether A is diagonalizable. (3 marks)

e) Given that $p(x) = 1 - 2x^2$ and $q(x) = 4 - 2x + x^2$ are polynomials in $P_n(x)$, for an inner product in the polynomial space, Determine:

(i) $\|q\|$. (2 marks)

(ii) $d(p, q)$ (3 marks)

f) Consider the following two bases of $\Re^2$: $S = \{u_1, u_2\} = \{(1,2), (3,5)\}$ and $S' = \{v_1, v_2\} = \{(1,-1), (1,-2)\}$. Find the change-of-basis matrix P from S to S′. (5marks)

g) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ using the Cayley-Hamilton theorem. (5 marks)

SECTION B

Question Two (20 Marks)

a) Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two vectors on $\Re^2$. Define on this space, an inner product by $\langle u, v \rangle = 3u_1v_1 + 2u_2v_2$. Show that this defines an inner product. (5 marks)

b) Determine the minimal polynomial of $A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ (4 marks)

c) A map is defined with a transformation $T : \Re^2 \to \Re^3$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \\ x_2 \end{bmatrix}$,

(i) Show that T is a linear transformation. (4 marks)

(ii) Find the matrix A such that $T(x) + Ax$ for each $x \in \mathfrak{R}^2$. (3 marks)

(iii) Describe the null space (kernel) and the range of T and give the rank and nullity of T. (4 marks)

Question Three (20 Marks)

a) Let $T : \mathfrak{R}^3 \rightarrow \mathfrak{R}^3$: be a linear transformation such that $T(1,0,0) = (2,4,-1), T(0,1,0) = (1,3,-2), T(0,0,1) = (0,-2,2)$. Compute $T(-2,4,-1)$. (4 marks)

b) Let V =C [0,1] with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$ for $f, g, \in V$. Let $f(x) = 2x$ and $g(x) = x^2 + x + 1$. Compute the orthogonal projection of g onto f. (5 marks)

c) Find the real symmetric matrix which corresponds to the polynomial

$Q(x_1, x_2, x_3) = 5x_1^2 + 11x_2^2 + 4x_3^2 - 16x_1x_2 + 22x_2x_3 - 8x_1x_3$ (3 marks)

d) Consider the basis $B = \{(1,2,-1), (1,0,1), (1,2,1\}$ and $C = \{(1,1,0), (0,1,1), (1,1,1)\}$

i) Find the transition matrix from basis C to B (4 marks)

ii) Find the transition matrix from basis B to C (4 marks)

Question four (20 Marks)

a) Let W be a subspace of the inner product space in $\mathfrak{R}^4$ with the dot product. W has a basis of $\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}4\\-1\\2\\0\end{bmatrix}, \begin{bmatrix}5\\1\\4\\0\end{bmatrix}\right\}$. Use Gram-Schmidt process to find the orthonormal basis for W. (8 marks)

b) Transform the quadratic form $Q(x) = x_1^2 - 8x_1x_2 - 5x_2$ into quadratic form with no cross-product term. (5 marks)

c) Consider the following two bases of $\mathfrak{R}^2$: $S = \{u_1, u_2\} = \{(1,2), (3,5)\}$ and $S' = \{v_1, v_2\} = \{(1,-1), (1,-2)\}$. Find the change-of-basis matrix P from S to S′. (4 marks)

d) The linear transformation $T : \mathfrak{R}^n \rightarrow \mathfrak{R}^m$ is given by $T(x) = Ax$. Given that

$A = \begin{bmatrix}1 & 2 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$ Find the nullity and the rank of T and determine whether T is one to one, onto or neither. (3 marks)

Question Five (20 Marks)

a) Show that $\langle u+v, u-v\rangle = \|u\|^2 - \|v\|^2$. For every $u, v \in V$. (2 marks)

b) Diagonalize the matrix $C = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ (13 marks)

c) Consider a linear operator $T : \Re^2 \to \Re^2$, such that $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Find the matrix of T with respect to the basis $V_1 = (3,1)$, $V_2 = (2,1)$. (5 marks)