KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2023/2024 ACADEMIC YEAR

THIRD YEAR **SECOND** SPECIAL/SUPPLEMENTARY EXAMINATIONS

FOR THE DEGREE OF:

BACHELOR OF SCIENCE WITH EDUCATION AND BACHELOR OF EDUCATION SCIENCE

COURSE CODE: MAT 311

COURSE TITLE: REAL ANALYSIS II

DATE: 16th **July, 2024** **TIME: 12.00PM - 2.00PM**

Instructions: See Inside

Answer question **ONE** in section A and any other **TWO** questions from section B.

SECTION A

Answer **all** questions from this section

QUESTION ONE (30 marks)

a. Prove that $\lim_{x\to 2}(2x+3)=7$. **[3 marks]**

b. Give the formal definition of limit of a function. **[3 marks]**

c. Show that if f and g are functions of bounded variation on $I=[a,b]$, then $f+g$ is a function of bounded variation on I and $V(f+g,I)\leq V(f)+V(g)$. **[5 marks]**

d. Verify Clairaut's Theorem for the following function. **[4 marks]**

$$f(x,y)=x^3y^2-\frac{4y^6}{x^3}$$

e. State the Mean Value Theorem. **[3 marks]**

f. Determine all the number(s) c which satisfy the conclusion of Rolle's Theorem for $f(x)=x^2-2x-8$ on $[-1,3]$ **[3 marks]**.

g. Given $f(x)=2x+1$ on $I=[0,1]$ and $P=\{0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1\}$ determine $U(f,P)$ and $L(f,P)$. **[5 marks]**

h. If Q is a refinement of P, show that **[4 marks]**

$$L(f,P)\leq L(f,Q)\leq U(f,Q)\leq U(f,P).$$

SECTION B

Answer **any TWO** questions from this section

QUESTION TWO (20 marks)

a. Given that $h(x,y) = (2x^2y+1)^3$, determine $h_x(x,y)$. **[4 marks]**

b. Determine all the numbers c which satisfy the conclusions of the Mean Value Theorem for the function $f(x) = x^3 + 2x^2 - x$ on the interval $[-1, 2]$. **[4 marks]**

c. Prove that if f is a monotone function, then f is Riemann integrable. **[4 marks]**

d. Evaluate the point-wise limit of the functions $f_n(x) = \frac{n^2x}{1+n^2x}$ for each $n \in \mathbb{N}$ and $\forall x \in \mathbb{R}$. **[3 marks]**

e. Consider the functional sequence $\left(x + \frac{1}{n}\right)_{n\in\mathbb{N}}$ defined on $\mathbb{R}$. Write the first five terms of the sequence. **[5 marks]**

QUESTION THREE (20 marks)

a. Given that $f(x,y) = x^2 + y^2$. Compute the partial derivative $\frac{\partial f}{\partial x}$ using the definition of derivatives. **[3 marks]**

b. If $f(x) = \cos x$ is a bounded function on $[0, \frac{\pi}{2}]$ and $\alpha(x) = \frac{1}{2}x^2$ a monotonically increasing function on $[0, \frac{\pi}{2}]$. Find $\int_0^{\frac{\pi}{2}} f(x)\, d\alpha(x)$. **[4 marks]**

c. Let $f(x) = x^2$ and $\alpha(x) = \frac{1}{3}x^3$. Let $P = \{0, 1, 2, 3, 4, 5\}$ of $I = [0, 5]$. Find $U(P, f, \alpha)$ and $L(P, f, \alpha)$. Hence obtain $\int_0^5 f\, d\alpha$. **[7 marks]**

d. Define uniform convergence of a sequence of functions. **[2 marks]**

e. Prove that if f is continuous on $[a, b]$, then f is Riemann integrable. **[4 marks]**

QUESTION FOUR (20 marks)

a. State the weierstrass M-Test criterion. **[2 marks]**

b. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \frac{x^2 \cos x - y^2 \cos y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Show that $\lim_{\substack{x\to 0 \\ y\to 0}} f(x,y)$ does not exist. **[4 marks]**

c. Determine the Taylor series for the function $f(x) = \ln(1+x)$ at $x = 0$. **[4 marks]**

d. Given that $g(x,y,z) = \frac{xy}{z^2}$, compute the partial derivatives.

i) $g_y(x,y,z)$. **[1 mark]**

ii) $g_{yx}(x,y,z)$. **[2 mark]**

e. State and proof the Cauchy Criterion for Uniform Convergence. **[7 marks]**

QUESTION FIVE (20 marks)

a. Show that $f(x) = 4x^5 + x^3 + 7x - 2$ has exactly one real root between 0 and 1. **[3 marks]**

b. Define the following terms in reference to function of several variables.

i) Open set. **[1 mark]**

ii) Domain. **[1 mark]**

c. Let $f(x,y)$ and $g(x,y)$ be defined in a domain D, and suppose that $\lim_{\substack{x\to x_0 \\ y\to y_0}} f(x,y) = u$ and $\lim_{\substack{x\to x_0 \\ y\to y_0}} g(x,y) = v$. Show that $\lim_{\substack{x\to x_0 \\ y\to y_0}} [f(x,y) + g(x,y)] = u + v$. **[4 marks]**

d. Test for Maxima and minima in the function $z = x^3 - 3xy^2$. **[7 marks]**

e. Prove that a monotone function on an interval $I = [a,b]$ is of bounded variation on I and $V(f) = \Big|f(b) - f(a)\Big|$. **[4 marks]**