KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2023/2024 ACADEMIC YEAR

SPECIAL/SUPPLEMENTARY EXAMINATIONS
FOR THE DEGREE OF:

Bsc. W/EDUCATION (P106)

Bed. SCIENCE (E101)

COURSE CODE: MAT 326 / MAT 410

COURSE TITLE: RINGS AND MODULES

DATE: 17th JULY, 2024 **TIME:** 9AM - 11AM

INSTRUCTIONS : ***See inside***

INSTRUCTIONS

Answer **ALL** questions in *Section A* and **ANY** other **TWO** from section B.

SECTION A

Answer **ALL** questions from this section

QUESTION ONE (30 marks)

a) List all the zero divisors in the ring $\mathbb{Z}_{12}$. **(3 marks)**

b) Give an example of a commutative ring without unit. **(2 marks)**

c) Let $R = \mathbb{C}$ and $S = \mathbf{M}\ (2, \mathbb{R})$. (ring of real valued matrices of order 2). Define a map $\phi : R \to S$ by $\phi(a + bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

Show that ϕ is a ring homomorphism. **(6 marks)**

d) Compute $(4x^4 + 3x^2) \cdot (2x^4 + 5x^3 + 3x)$ in $\mathbb{Z}_6[x]$. **(4 marks)**

e) Consider the ring $\mathbf{M(2,}\ \mathbb{R}\mathbf{)}$ consisting of all real valued matrices of order 2. That is, the set $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$.

Next consider the set $S \subset R$ given as $S = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$. Show that the set S is a subring of R. **(5 marks)**

f) Construct the addition and multiplication table of the quotient ring $\dfrac{\mathbb{Z}}{4\mathbb{Z}}$. **(4 marks)**

g) Let S be a subring of $R = \mathbb{Z}_8$ where $R = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$.
Determine the characteristic of S . **(3 marks)**

h) Let S be a ring. Show that if $s \in S$ is a unit, then s is not a zero divisor. **(3 marks)**

SECTION B

Answer **ANY TWO** questions from this section

QUESTION TWO (20 marks)

a) Determine if the following are irreducible.

   i) $2x^2 + x + 1$ in $\mathbb{Z}_3[x]$. **(4 marks)**

   ii) $x^4 + 2x^3 + 3x + 1$ in $\mathbb{Z}_5[x]$. **(4 marks)**

b) Show that the intersection of two ideals of a ring R is an ideal of R. **(5 marks)**

c) Mark each of the following as True Or False.

   (i) Every ring with unity has at most two units. **(1 mark)**

   (ii) Every field is an integral domain. **(1 mark)**

   (iii) Every element in division ring is invertible. **(1 mark)**

   (iv) Addition in every ring is commutative. **(1 mark)**

d) State the rational roots test for irreducibility of a polynomial $p(x)$. **(3 marks)**

QUESTION THREE (20 marks)

a) Show that if D is an integral domain and if $na = 0$ for some $a \neq 0$ in D and some integer $n \neq 0$, then D is of finite characteristic. **(4 marks)**

b) Show that the intersection of two subrings H_1 and H_2 of a ring R is a subring of R. **(5 marks)**

c) i) State the Einsteins criterion on irreducibility of a polynomial in $\mathbb{Q}[x]$. **(3 marks)**

   ii) Use (i) above to show that the polynomial $4x^4 - 6x^3 + 9x - 21$ is irreducible. **(3 marks)**

d) Let U, V be ideals of a ring R. Define $U + V = \{u + v : u \in U, v \in V\}$. Show that $U + V$ is also an ideal of R. **(5 marks)**

QUESTION FOUR (20 marks)

a) In the ring M of 2×2 matrices over integers consider the set $U = \left\{ \begin{bmatrix} 0 & r \\ 0 & s \end{bmatrix} : r, s \in \mathbb{Z} \right\}$. Show that U is a right ideal of M. **(6 marks)**

b) Let $R = \left\{ \begin{bmatrix} p & q & r \\ 0 & s & t \\ 0 & u & v \end{bmatrix} : p, q, r, st \in \mathbb{Z} \right\}$. Also let $S = \mathbf{M}\ (2, \mathbb{R})$ (ring of real valued matrices of order 2). Define a map $\pi : R \to S$ by $\pi \begin{bmatrix} p & q & r \\ 0 & s & t \\ 0 & u & v \end{bmatrix} = \begin{bmatrix} s & t \\ u & v \end{bmatrix}$. Determine if π is a ring homomorphism. **(8 marks)**

c) Let R be a commutative ring and let $a \in R$. Show that $< a >$, the principal ideal generated by a is a two sided ideal in R. **(6 marks)**

QUESTION FIVE (20 marks)

a) Give an example of:

i) An integral domain. **(2 marks)**

ii) A ring with only two units. **(2 marks)**

b) i) Show that structure $(\mathbb{Z}_8, +_8, \times_8)$ is a ring. **(7 marks)**

ii) State whether its a Commutative ring. **(2 marks)**

iii) State whether its a ring with unit. **(2 marks)**

c) Consider the set $S = \left\{ x = p + q\sqrt{2} \mid p, q \in \mathbb{Z} \right\}$. Show that the associativity (additive) axiom of a ring hold for the structure $(S, +, \times)$, where $+$ and $\times$ are the usual addition and multiplication in the rationals set. **(5 marks)**