KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2024/2025 ACADEMIC YEAR

SECOND SEMESTER REGULAR EXAMINATIONS

FOR THE DEGREE OF:

MASTER OF SCIENCE IN APPLIED MATHEMATICS

COURSE CODE: MAT 806

COURSE TITLE: ORDINARY DIFFERENTIAL EQUATIONS II

DATE: 12th MAY, 2025 **TIME: 2:00PM - 5:00 PM**

Instructions: See Inside

Answer **all** questions in section $\boldsymbol{A}$ and any other **two** from section $\boldsymbol{B}$.

SECTION A

Answer **all** questions from this section

QUESTION ONE (20 MARKS)

(a) A competitive interaction is described by the Lotka-Voltera model

$$\begin{aligned} x' &= 0.004x(50 - x - 0.75y) \\ y' &= 0.001y(100 - y - 3.0x) \end{aligned}$$

classify all the critical points of the system **[5 Marks]**

(b) Solve the system

$$\boldsymbol{X'} = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

on the interval $(-\infty, \infty)$ **[7 Marks]**

(c) When expressed in polar coordinates, a plane autonomous system takes the form

$$\begin{aligned} \frac{dr}{dt} &= 0.05r(3 - r) \\ \frac{d\theta}{dt} &= -1 \end{aligned}$$

Show that for $(x, y) = (0, 0)$ is unstable critical point **[8 Marks]**

SECTION B

Answer **any TWO** questions from this section

QUESTION TWO (20 MARKS)

(a) Show that the critical points $(x, y) = (0, 0)$ of the autonomous system

$$\begin{aligned} \frac{dx}{dt} &= -x - xy^2 \\ \frac{dy}{dt} &= -y - x^2 y \end{aligned}$$

is asymptotically stable **[6 Marks]**

(b) Using the equation

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Find the linear system corresponding to the pendulum equation **[6 Marks]**

(c) Solve the initial value problem (IVP)

$$X' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} x, \qquad X(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

[8 Marks]

QUESTION THREE (20 MARKS)

(a) Qualitative Analysis the model with constant yield harvesting expressed as

$$\frac{dP}{dt} = rP\Big(1 - \frac{P}{\kappa}\Big) - H$$

[10 Marks]

(b) Find the general solution to the non-homogeneous system

$$\boldsymbol{X'} = \begin{pmatrix} \boldsymbol{-3} & \boldsymbol{1} \\ \boldsymbol{2} & \boldsymbol{-4} \end{pmatrix} \boldsymbol{X} + \begin{pmatrix} \boldsymbol{3t} \\ \boldsymbol{e^t} \end{pmatrix} \qquad (-\infty, \infty)$$

[10 marks]

QUESTION FOUR (20 MARKS)

Determine the stability of the fixed points of the SIR model with vital dynamics

$$\frac{\boldsymbol{ds}}{\boldsymbol{dt}} = \boldsymbol{\mu - \beta si - \mu s} \tag{1}$$

$$\frac{\boldsymbol{di}}{\boldsymbol{dt}} = \boldsymbol{\beta si - \gamma i - \mu i} \tag{2}$$

$$\frac{\boldsymbol{dr}}{\boldsymbol{dt}} = \boldsymbol{\gamma i - \mu r} \tag{3}$$

[20 Marks]