KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2023/2024 ACADEMIC YEAR

FOURTH YEAR SPECIAL/SUPPLIMENTARY EXAMINATION

FOR THE DEGREE OF

BACHELOR OF SCIENCE WITH EDUCATION, BACHELOR OF EDUCATION SCIENCE AND BACHELOR OF SCIENCE

COURSE CODE: MAT 419

COURSE TITLE: PARTIAL DIFFERENTIAL EQUATIONS II

DATE: 16th JULY 2024 TIME: 12.00noon-2.00pm

INSTRUCTION TO CANDIDATES

SEE INSIDE

ANSWER **QUESTION ONE (COMPULSORY) AND ANY OTHER TWO QUESTIONS**

SECTION A

QUESTION ONE (30 marks)

a) Show that the equation below is satisfied by $u = f(x+ct)+F(x-ct)$. (3marks)

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

b) Solve the equation $\frac{\partial^2 u}{\partial x^2} = 12x^2(t+1)$ given that at $x=0, u=\cos 2t$ and $\frac{\partial u}{\partial x} = \sin t$ (4marks)

c) Find the general Solution of $2\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial y \partial x} + 5\frac{\partial^2 u}{\partial y^2} = 0$ (4marks)

d) Transform the following differential equation into canonical form

$5\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + 3u = 0$ using the transformations $\xi = 2x - y$ and $\eta = x + 2y$. (4marks)

e) Find the general solution of $\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} = 0$ by setting $\frac{\partial u}{\partial x} = v$. (4marks)

f) By using the method of separation of variables form a set of two ODEs

$x\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} = 0$ (4marks)

g) Show that the transformation $\xi = y - \frac{x^2}{2}$, $\eta = x$ reduces the equation

$\frac{\partial^2 u}{\partial x^2} + 2x\frac{\partial^2 u}{\partial y \partial x} + x^2\frac{\partial^2 u}{\partial y^2} = 0$, to $\frac{\partial^2 u}{\partial \eta^2} = \frac{\partial u}{\partial \xi}$ (7marks)

QUESTION TWO (20 marks)

a) Consider a stretched string of length L fixed at the end points where $u(x,t)$ is the position of the string at time t after an initial disturbance is given
   i. State five assumptions that must be made for in derivation of one-dimensional wave equation. (5marks)
   ii. Show that the one-dimensional wave equation is given by $\frac{\rho}{T}u_{tt} = u_{xx}$ where ρ and T are density and tension of the string. (8marks)

b) The solution of the wave equation below can be expressed in the form Y = X(x)T(t), in the process of constructing Y show why the value of k cannot be positive. (7marks)

$$\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0$$

$$y(0,t) = 0, \quad 0 \le t < \infty$$

$$y(\pi,t) = 0, \quad 0 \le t < \infty$$

$$y(x,0) = f(x), \quad 0 \le x < \pi$$

$$\frac{\partial y(x,0)}{\partial t} = g(x), \quad 0 \le x < \pi$$

<u>QUESTION THREE (20 marks</u>)

a) Determine the real characteristics of the function $8\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial y \partial x} - 3\frac{\partial^2 u}{\partial y^2} = 0$ and sketch these families of curves (8marks)

b) Classify the following partial differential equation

$$2\frac{\partial^2 u}{\partial X^2} - x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} - xy\frac{\partial u}{\partial x} = 0$$ (3marks)

c) Using the definition of the derivative of a function in two independent variables x and y determine the value of $x^2\frac{\partial^2 u}{\partial x^2} - 2y\frac{\partial u}{\partial y} + u$ given that $u(x,y) = 4x^2y$ (9marks)

<u>QUESTION FOUR</u> (<u>20 marks</u>)

a) Consider the partial differential equation $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$

And the five real functions defined for all real t by

$x(t) = t, \; y(t) = 0, \quad u(t) = t^4 + t^2$

$p(t) = 4t^3 + 2t, \qquad q(t) = 4t$

i. Show that the five functions satisfy the strip condition. (2marks)

ii. Show that the function $\emptyset$ defined for all (x,y) by $\emptyset(x,y) = x^4 + 6x^2y^2 + y^4 + x^2 + y^2 + 4xy$ is a solution for the initial value problem in (ii) above. (3marks)

b) Transform the differential equation $\frac{d^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = 0$ into canonical form using the transformations $\xi = 2x$ and $\eta = y$ (6marks)

c) The ends of an insulated rod AB, 10 units long are maintained at 0^0c at t=0 the temperature within the rod rises uniformly from each end reaching 2^0 c at the midpoint of AB.

i. Write down an expression for the heat equation together with its boundary conditions. (2marks)

ii. Determine an expression for the temperature at u(x,t) at any point in the rod, distant x from the left hand end at any subsequent time t. (7marks)

<u>QUESTION FIVE</u> (**<u>20 marks</u>**)

a) Determine the general solution of $2\frac{\partial^2 u}{\partial y \partial x} + 3\frac{\partial^2 u}{\partial y^2} = 0$ (3marks)

b) Use the method of separation of variables to find a formal solution $u(x, y)$of the Laplace equations (9marks)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

And the boundary conditions

$$u(0, y) = 0, \quad 0 \leq y \leq \pi$$

$$u(\pi, y) = 0, \quad 0 \leq y \leq \pi$$

$$u(x, \pi) = 0, \quad 0 \leq x \leq \pi$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq \pi$$

Where f is a specified function of x, $0 \leq x \leq \pi$

c) Given the parabolic equation $\frac{\partial^2 u}{\partial x^2} = a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} + cu + f$ where the coefficients are constants. By the substitution $u = ve^{\frac{1}{2}bx}$ for the case $c = -(b^2/4)$, show that the given equation is reduced to the heat equation $v_{xx} = av_t + g$ where $g = fe^{\frac{-bx}{2}}$.
(8marks)