KARATINA UNIVERSITY

UNIVERSITY SPECIAL/ SUPPLEMENTARY EXAMINATIONS

2024/2025 ACADEMIC YEAR

THIRD YEAR **FIRST** SEMESTER EXAMINATION

FOR THE DEGREE OF:

BACHELOR OF EDUCATION (E101)

COURSE CODE: MAT 313

COURSE TITLE: GROUP THEORY

DATE: **TIME:**

INSTRUCTIONS TO CANDIDATES

- SEE INSIDE

INSTRUCTIONS: Answer question ONE in SECTION A and any other TWO in SECTION B

SECTION A

QUESTION ONE: (3O MARKS)

a) Let $S = \{f_1, f_2, f_3, f_4\}$ where $f_i: \mathbb{R}^* \to \mathbb{R}^*$ define by $f_1(x) = x$, $f_2(x) = \frac{1}{x}$, $f_3(x) = -x$, $f_4(x) = -\frac{1}{x}$. Let o denote composition of mappings.

(i) Construct the Cayley table for S under o. (3 Marks)

(ii) Is o a commutative operation? Explain (3 Marks)

b) Find the order of $(123)(45)$ in S_5. (2 Marks)

c) Let G be S_3 and $H = \{1, (1,2)\}$. Is H a normal subgroup of G. (2 Marks)

d) Let S be the set $S = \{(a, b): a, b \in \mathbb{R}\}$. Define $*$ on S by $(a, b) * (c, d) = (ac, ad + b)$, for all $a, b, c, d \in \mathbb{R}$. Determine whether $*$ a commutative operation is. (2 Marks)

e) i) Define the center of a group. (2 Marks)

ii) Hence find the center of the quaternion group. (1 Mark)

f) Define Homomorphism (2 Marks)

g) Let F be the additive group of all functions mapping $\mathbb{R}$ into $\mathbb{R}$, let $\mathbb{R}$ be the additive group of real numbers and let c be any real number. Let $\emptyset_c: F \to \mathbb{R}$ be defined by $\emptyset_c(f) = f(c)$ for $f \in F$.

(i) Find $\emptyset_3(2x^3 - 2x^2 + 1)$ (1 Mark)

(ii) Show that $\emptyset_c$ is a homomorphism. (2 Marks)

(h) Let S be a non-empty set and $*$ be an operation on the set S. State two conditions that $*$ should satisfy to be a binary operation on the set S. (2 Marks)

(i) Find the cyclic subgroup of the quaternion group generated by . (2 Marks)

(j) Determine if the permutation $(123)(45678)$ is even or odd (2 Marks)

(k) Determine whether subtraction is a binary operation on $\mathbb{Z}^+$. (2 Marks)

(l) Find gcd of 168 and 198 by the Euclidean algorithm. (3 Marks)

SECTION B

QUESTION TWO (20 MARKS)

a) Define $*$ be on $\mathbb{R}$ by $a * b = 2a + 2b + ab + 2$. Determine the identity element for $*$ (2 Marks)

b) Find the gcd of 1547 and 3059 and hence express the gcd in the form

$1547m + 3059n = gcd$ (8 Marks)

c) Proof all the group axioms for the Abelian group $\langle \mathbb{Z}, + \rangle$ (5 Marks)

d) List the elements of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hence show that $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ has seven subgroups of order 2. (5 Marks)

QUESTION THREE (20 MARKS)

a) Find the order of the factor group $(\mathbb{Z}_{12} \times \mathbb{Z}_{18})/\langle(4,3)\rangle$. (2 Marks)

b) Let $G = \mathbb{R}^2$ (operation vector addition), let $H = \mathbb{R}$ (operation addition) and define $\psi: \mathbb{R}^2 \to \mathbb{R}$ by $\psi(x, y) = x$. Thus ψ is projection onto the x-axis

(i) Show that ψ is a homomorphism (3 Marks)

(ii) Find $\ker \psi$ (2 Marks)

c) Let $\delta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{bmatrix}$ and $\tau = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \end{bmatrix}$

(i) Express δ and τ in disjoint cycle form (2 Marks)

(ii) Find . (1 Mark)

(iii) Find δ^2. (1 Mark)

(iv) Find δ^4. (1 Mark)

(v) Find δ^{11}. (1 Mark)

(vi) Find τ^{-1} (1 Mark)

d) Define $*$ on $\mathbb{Z}$ by $a * b = a + b - ab$. $*$ is a binary operation $\mathbb{Z}$ with identity element 0. If $a \in \mathbb{Z}$, what is the inverse of a ? (3 Marks)

e) The collection of all the even permutations in S_n is a subgroup called the alternating group denoted by A_n of order $\frac{n!}{2}$. List the elements of A_3 (3 Marks)

QUESTION FOUR (20 MARKS)

a) List and describe all the subgroups of $\langle \mathbb{Z}_{30}, + \rangle$ (16 Marks)

b) Let G be a group. Show that the left cancellation laws hold, i.e. $a * b = a * c$ implies $b = c$ for all $a, b, c \in G$. (2 Marks)

c) The ring $R = \{0, 2, 4, 6, 8\}$ under addition and multiplication modulo 10 has a unity. Find it. (2 Marks)

QUESTION FIVE (20 MARKS)

a) Consider the various symmetry transformations of a rectangle that form the symmetry group of a rectangle.

(i) With the help of a diagram, compute and list all the elements in the symmetry group of rectangle. (4 Marks)

(ii) Construct the Cayley table for the symmetry group. (3 Marks)

(iii) List all non-trivial subgroup of this symmetry group. (3 Marks)

(iv) Determine if this is an abelian group. (1 Mark)

(v) What is the order of the non-identity elements in this group? (1 Mark)

(vi) The symmetry group of a rectangle is also referred to as? (1 Mark)

b) Let S be the set $S = \{(a, b): a, b \in \mathbb{R}\}$. Define $*$ on S by $(a, b) * (c, d) = (ac, a + bd)$, for all $a, b, c, d \in \mathbb{R}$. Determine whether $*$ is a binary operation. (3 Marks)

c) Let $= \left\{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} : \lambda_1, \lambda_2 \in \mathbb{R}\right\}$, the set of all 2×2 diagonal matrices. Let $*$ be matrix multiplication. Is $*$ a commutative operation on S? (4 Marks)