Inspiring Innovation and Leadership

KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2023/2024 ACADEMIC YEAR

FOURTH YEAR SPECIAL/ SUPPLIMENTARY

EXAMINATION

FOR THE DEGREE OF

BACHELOR OF SCIENCE (P106, P103), BACHELOR OF EDUCATION (E100, E101, E111)

COURSE CODE: MAT 418

COURSE TITLE: PARTIAL DIFFERENTIAL EQUATIONS I

DATE: 16th JULY 2024 **TIME: 3.00pm-5.00pm**

INSTRUCTION TO CANDIDATES

SEE INSIDE

ANSWER **QUESTION ONE (COMPULSORY)** AND ANY OTHER **TWO QUESTIONS**

QUESTION ONE (30 marks)

a) Compute the following partial derivative

$$\frac{\partial}{\partial y}(y^2 \sin xy^3) \qquad \text{(2marks)}$$

b) Given the partial differential equation below

$uu_{xx} + u_{yy} - u = 0$

and defining the linear differential operator as L, express the above PDE using L. (2marks)

c) State the order of each of the following PDEs and classify them as linear or non linear

i. $x^2 u_{xxy} + y^2 u_{yy} - \log(1+y^2)u = 0$ (2marks)

ii. $uu_{xx} + u_{yy} - u = 0$ (2marks)

d) Consider the transport equation $au_t(x,t) + bu_x(x,t) = 0$ where a and b are constants. Show that $u(x,t) = f(bt - ax)$ is a solution where f is an arbitrary differentiable function in one variable. (3marks)

e) Compute the following partial derivatives

$$\frac{\partial^2}{\partial x^2}(x^3 e^{x+y})^2 \qquad \text{(2marks)}$$

f) Using the change of variables technique with $v = y - 2x$ and $w = x$. Solve the PDE $u_x + 2u_y = 1$ (5marks)

g) Use the definition of the partial derivative to find $\frac{\partial f}{\partial x}$ as a function of x and y, for $f(x,y) = 3x^2 + xy$ (4marks)

h) Determine whether the function $u(x,y) = e^{-\lambda^2 \alpha^2 t}(\cos \lambda x - 2\sin \lambda x)$ is a solution to the PDE $u_t - \alpha^2 u_{xx} = -u_x$ (4marks)

i) Find the solution to $-3u_x + u_y = 0$, $u(x,0) = e^{-x^2}$ (4marks)

QUESTION TWO (20 marks)

a) Show that $u(x,y) = e^{-2y}\sin(x-y)$ is the solution to the initial value problem

$$u_x + u_y + 2u = 0 \text{ for x,y>1}$$
$$\text{u(x,0)=sinx}$$
(4marks)

b) Find the general solution of the PDE $yu_x + xu_y = x^2 + y^2$ (7marks)

c) Solve the problem below using linear change of variables technique (9marks)

$$5\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = x$$
$$u(x,0) = \sin 2\pi x \text{ for } -\infty < x < \infty$$

QUESTION THREE (20 marks)

a) Solve the initial value problem below using the method of Cauchy. (7marks)

$$u_x + u_y = 1$$
$$u(x,0) = f(x)$$

b) By use of the transformations $s = ax - \frac{a^2}{b}y$ and $t = aby$ show that the solution to the PDE $u_x + bu_y = 2$ is $u(x,y) = \frac{1}{ab^2}[2aby + f(ax - \frac{a^2}{b}y)$

(6marks)

c) Solve the following partial differential equation

$z^2 = pqxy$

using Jacobi's method (7marks)

QUESTION FOUR (20 marks)

a) Solve the partial differential equation $u_x + u_y = 1$ by introducing the change of variables $s = x + y$ and $t = x - y$. (6marks)

b) Let $G(x,y) = \ln(x^2 + y) + xy^2$ Find

i. The level surface of G (1mark)

ii. The normal to the level surface above (3marks)

iii. The directional derivative of G at point (1,2) in the direction of $\vec{v} = 2i + 3j$ (3marks)

c) Solve the following non-linear PDE using the Charpit method

$$px + qy = pq$$ (7marks)

QUESTION FIVE (20 marks)

a) For $x \in R$ and $t > 0$ we consider the initial value problem

$$u_{tt} + u_{xx} = 0$$
$$u(x,0) = u_t(x,0) = 0$$

Clearly $u(x,t) = 0$ is a solution to this problem

Let $0 < \varepsilon < 1$ be a very small number. Show that the function

$u_\varepsilon(x,t) = \varepsilon^2 \sin(\frac{x}{\varepsilon}) \sinh(\frac{t}{\varepsilon})$ where sinh x= $\frac{e^x - e^{-x}}{2}$ is a solution to the problem

$$u_{tt} + u_{xx} = 0$$
$$u(x,0) = 0$$ (6marks)
$$u_t(x,0) = \varepsilon \sin(\frac{x}{\varepsilon})$$

b) Find the general solution of the equation $xu_x + yu_y = xe^{-u}$, $x > 0$ (5marks)

c) Determine every function u(x,t) that solves the function using the linear change of variable technique (9marks)

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$$

Where v is a fixed constant.