KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2024/2025 ACADEMIC YEAR

THIRD YEAR FIRST SEMESTER REGULAR EXAMINATIONS

FOR THE DEGREE OF:

BACHELOR OF SCIENCE(P102 P103, P106, P107), BACHELOR OF EDUCATION (E100, E101, E103, E111, E112)

COURSE CODE: MAT 310

COURSE TITLE: REAL ANALYSIS I

DATE: 10^{th} **DEC, 2024** **TIME: 12 PM-2 PM**

Instructions: See Inside

Answer question **ONE** in section A and any other **Two** from section B.

SECTION A

Question ONE is compulsory

QUESTION ONE (30 marks)

(a) State the Nested Interval Property of the real numbers $\mathbb{R}$. **[2 marks]**

(b) Solve $|3x-1| = |5x+9|$. **[4 marks]**

(c) For the set $[-1,2]$, find the:

i) Boundary **[1 mark]**

ii) Interior **[1 mark]**

iii) Accumulation Points **[1 mark]**

iv) Closure of the set A. **[1 mark]**

(d) Show that the function $f(x) = 2x$ is Uniformly Continuous on $\mathbb{R}$. **[4 marks]**

(e) Let $\{B_\alpha\}$ be a collection of open sets. Show that the union $\bigcup_\alpha B_\alpha$ is open. **[3 marks]**

(f) Show that $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. **[4 marks]**

(g) Determine if the following sequence $\left\{\dfrac{3n^2-1}{10n+5n^2}\right\}_{n=2}^{\infty}$ converges or diverges. If the sequence converges determine its limit. **[4 marks]**

(h) State the Cauchy-Schwarz inequality and verify it in $\mathbb{R}^2$ using $x = (2,3)$ and $y = (4,1)$. **[5 marks]**

SECTION B

Answer **any Two** questions from this section

QUESTION TWO (20 marks)

(a) Write down the first six terms the sequence $\left\{\dfrac{n+1}{n^2}\right\}_{n=1}^{\infty}$. **[3 marks]**

(b) Given that $\{\alpha_n\}$ is the sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$, give the even subsequence of $\{\alpha_n\}$. **[2 marks]**

(c) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. Does the series converge or diverge? Explain. **[3 marks]**

(d) Show that the $\lim\limits_{x \to 0} \frac{x}{|x|}$ does not exist. **[3 marks]**

(e) By use of a counter-example, show that set of rational numbers does not satisfy the completeness axiom. **[3 marks]**

(f) Let $A \subseteq \mathbb{R}$ be non-empty and bounded above. Let $c \in \mathbb{R}$. Define a new set $c + A = \{c + x : x \in A\}$. Show that Sup $(c + A) = c+$Sup (A). **[6 marks]**

QUESTION THREE (20 marks)

(a) Determine the 2-tail of the sequence $\left\{\dfrac{1}{n^2+1}\right\}_{n=1}^{\infty}$. **[2 marks]**

(b) Explain one significance of the Archimedean property of real numbers. **[2 marks]**

(c) Given that $x = -3$ and $\epsilon = 2$, determine the deleted neighborhood of x. **[3 marks]**

(d) Consider the sequence defined by

$$\alpha_n = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

that is, $\{\alpha_n\} = \{0, 1, 0, 1, 0, \ldots\}$. Show that this sequence diverges. **[3 marks]**

(e) Define a cauchy sequence. **[2 marks]**

(f) Every convergent sequence is Cauchy. Prove. **[4 marks]**

(g) Show that the infinite series $\sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)}\right)$ converges to 1. **[4 marks]**

QUESTION FOUR (20 marks)

(a) Define the continuity of a function. **[2 marks]**

(b) Show that the function defined below is continuous at $x = 1$. **[3 marks]**

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & \text{if } x \neq 1, \\ 2, & \text{if } x = 1. \end{cases}$$

(c) Write $\sum_{n=1}^{\infty} \frac{n^2}{1-3^{n+1}}$ as a series that starts at $n = 3$. **[3 marks]**

(d) Define ratio test for convergence. **[4 marks]**

(e) Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$. **[4 marks]**

(f) Prove that a sequence $\{a_n\}$ converges if and only if it is a Cauchy sequence. **[4 marks]**

QUESTION FIVE (20 marks)

(a) Compute $|-7|$, $|0.2|$, and $|0|$. **[3 marks]**

(b) Let $f(x) = x^2 + 1$ for all $x \in \mathbb{R}$. Prove that $\lim_{x\to 2} f(x) = 5$. **[5 marks]**

(c) Compute the distances $d_1(f,g)$ and $d_\infty(f,g)$ where $d_1(f,g) = \int_0^1 |f-g|\, dx$, $d_\infty(f,g) = \max_{0\leq x\leq 1} |f-g|$ and $f, g \in C[0,1]$ are the functions defined by $f(x) = x$ and $g(x) = x^3$. **[5 marks]**

(d) Show that the set of real numbers $\mathbb{R}$ is a metric space with the metric $d(x,y) = |x-y|$. **[7 marks]**