KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2024/2025 ACADEMIC YEAR

FOURTH YEAR FIRST SEMESTER REGULAR EXAMINATIONS

FOR THE DEGREE OF:

BACHELOR OF SCIENCE(P106), BACHELOR OF EDUCATION (E100, E101, E103, E111, E112, E113)

COURSE CODE: MAT 413

COURSE TITLE: TOPOLOGY I

DATE: th DEC., 2024 **TIME:**

Instructions: See Inside

Answer question **ONE** in section A and any other **Two** from section B.

SECTION A

Question ONE is Compulsory

QUESTION ONE (30 marks)

(a) Let X be a non-empty set. Define a topology τ on X. **[4 marks]**

(b) Let $X = \{a, b, c\}$ be a set. Determine whether the following collections of subsets of X are topologies on X.

i) $\mathcal{T}_1 = \{\emptyset, \{a, b\}, \{a, c\}, \{a\}, \{a, b, c\}\}$ **[2 marks]**

ii) $\mathcal{T}_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ **[2 marks]**

(c) Let τ be the class of subsets of $\mathbb{N}$ consisting of $\emptyset$ and all subsets of $\mathbb{N}$ of the form $E_n = \{n, n+1, n+2, ...\}$ with $n \in \mathbb{N}$.

i) List the open sets containing the positive integer 4 **[2 marks]**

ii) Find the accumulation points of the set $A = \{4, 13, 28, 37\}$ **[2 marks]**

iii) Determine those subsets E of $\mathbb{N}$ for which $E' = \mathbb{N}$ **[2 marks]**

(d) Consider the following topologies on $X = \{a, b, c, d\}$ and $Y = \{x, y, z, w\}$ respectively:

$\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$ and $\tau^* = \{Y, \emptyset, \{x\}, \{y\}, \{x, y\}, \{y, z, w\}\}$

Also, consider the functions f and g from X to Y defined by:

$f = \{(a, y), (b, z), (c, w), (d, z)\}$ and $g = \{(a, x), (b, x), (c, z), (d, w)\}$

Determine whether each of the given function is continuous. **[6 marks]**

(e) Define a Hausdorff space. **[2 marks]**

(f) Determine whether the following spaces are Hausdorff:

i) The Sierpinki space **[1 mark]**

ii) The discrete space **[1 mark]**

(g) Distinguish between first countable and second countable spaces. **[4 marks]**

(h) Determine whether the real numbers with the standard Euclidean topology is a first countable space. **[2 marks]**

SECTION B

Answer **any Two** questions from this section

QUESTION TWO (20 marks)

(a) Let $X = \{a, b, c, d\}$ and $A = \{\{a, b\}, \{b, c\}, \{d\}\}$. Find the topology on X generated by A. **[3marks]**

(b) Let (X, τ) and $(X, \tau)^*$ be two topological spaces. State what is meant by:

i) $f : X \to Y$ is a homeomorphism. **[2 marks]**

ii) $f : X \to Y$ is a continuous map. **[2 marks]**

(c) Show that the real line $\mathbb{R}$ and $X = (-1, 1)$ are homeomorphic. **[2 marks]**

(d) When is a topological space said to be separable ? **[2 marks]**

(e) Show that $\mathbb{R}$, the set of real numbers with respect to the Euclidean topology is separable. **[2marks]**

(f) When is a property P of a topological space X said to be hereditary? **[2 marks]**

(g) In $\mathbb{R}$, is length a topological property? **[2 marks]**

(h) Prove that the property of being Hausdorff is hereditary. **[3 marks]**

QUESTION THREE (20 marks)

Consider the following topology on $X = \{a, b, c, d, e\}$:

$\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}\}$

(i) List the neighbourhoods of e in X. **[3 marks]**

(ii) List the members of the relative topology τ_A on $A = \{b, c, e\}$. **[3 marks]**

(iii) List the closed subsets of X. **[2 marks]**

(iv) Determine the closures of the sets $\{a\}, \{b\}$ and $\{c, e\}$. **[6 marks]**

(v) Which sets in ii)are dense in X? **[2 marks]**

(vi) Find the interior points of the subset $A = \{a, b, c\}$ of X . **[2 marks]**

(vii) Find the exterior points of the subset $A = \{a, b, c\}$ of X . **[2 marks]**

QUESTION FOUR (20 marks)

(a) Define the discrete topology on a set X . **[2 marks]**

(b) Determine the coarsest and the finest topologies on the set $A = \{x, y, z\}$ **[4 marks]**

(c) Consider the topology $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$ on the set $X = \{a, b, c\}$. Determine whether or not (X, τ) is a:

i) T_1- Space . Explain. **[3 marks]**

ii) Regular space. Explain. **[3 marks]**

(d) Consider the discrete topology $\mathcal{D}$ on $X = \{a, b, c, d, e\}$.

i) Find a base for the discrete topology $\mathcal{D}$ **[2 marks]**

ii) Find a subbase δ for $\mathcal{D}$ which does not contain any singleton sets. **[2 marks]**

(e) Show that in a T_2 -space, each singleton set is a closed set. **[4 marks]**

QUESTION FIVE (20 marks)

(a) Let τ_1 and τ_2 be two topologies on X.

Show that $\tau_1 \cap \tau_2$ is also a topology on X. **[6 marks]**

(b) Show that $\tau_1 \cup \tau_2$ may not be a topology on X even though τ_1 and τ_2 are topologies on X. **[3 marks]**

(c) Show that any set endowed with the discrete topology is a Hausdorf space. **[3 marks]**

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = c$ for all $x \in \mathbb{R}$, where c is a constant. Show that f is continuous relative to any topology τ. **[3 marks]**

(e) Show that the set of real numbers $\mathbb{R}$ with the usual topology is a Hausdorff space. **[5 marks]**