KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2024/2025 ACADEMIC YEAR

FOURTH YEAR SPECIAL/SUPPLEMENTARY EXAMINATIONS

FOR THE DEGREE OF:

BACHELOR OF SCIENCE(P102 P103, P106),

BACHELOR OF EDUCATION (E101)

COURSE CODE: MAT 412

COURSE TITLE: MEASURE THEORY

DATE: th DEC., 2024 **TIME:**

Instructions: See Inside

Answer question **ONE** in section A and any other **Two** from section B.

SECTION A

Question ONE is Compulsory

QUESTION ONE (30 marks)

(a) List three limitations associated with Riemann integration. **[3 marks]**

(b) What is a sigma algebra? **[4 marks]**

(c) List all sigma algebras on $X = \{a, b, c\}$. **[5 marks]**

(d) Let $(X, \mathcal{A})$ be a measurable space. When is a function $f : X \to \mathbb{R}$ said to be $\mathcal{A}-$ measurable? **[2 marks]**

(e) Show that the function $f(x) = c$, where c is a scalar, is Borel measurable. **[5 marks]**

(f) Define a measure. **[4 marks]**

(g) Let A be a measurable set.When do we say that a measurable function $\psi : A \to \mathbb{R}$ is simple? **[2 marks]**

(h) When is a set $E \subseteq X$ said to be m^*- measurable? **[2 marks]**

(i) Show that the sets $\emptyset$ and $\mathbb{R}$ are Lebesgue measurable. **[3 marks]**

m^* the Lebesgue measure. **[3 marks]**

SECTION B

Answer **any Two** questions from this section

QUESTION TWO (20 marks)

(a) Show that the intersection of any two sigma algebras is a sigma algebra. **[6 marks]**

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3x+2 & \text{if } x \in [0,4] \\ 2x-4 & \text{if } x \in (5,10) \\ 0 & \text{if otherwise} \end{cases}$$

show that $38 \leq \int f\ dm \leq 136$ where m is the Lebesque measure. **[8 marks]**

(c) Let $(\mathbf{X}, \mathcal{A}, \mu)$ be a measure space, and $\{E_n\}$ be a monotone sequence in $\mathcal{A}$.

If $\{E_n\}$ is increasing, show that $\lim\limits_{n\to\infty} \mu\{E_n\} = \mu\left(\lim\limits_{n\to\infty} E_n\right)$ **[6 marks]**

QUESTION THREE (20 marks)

(a) Show that a measurable function f is integrable if and only if $|f|$ is integrable. **[4 marks]**

(b) Consider the function $f(x) = x^2 + 1$ defined on the interval $[0, 4]$.

i) Find the upper and lower Riemann sums when the interval is partitioned into 4 equal intervals. **[4 marks]**

ii) Find the exact Riemann integral. **[5 marks]**

iii) Compute $\int_0^4 (x^2+1)\,dx$. **[3 marks]**

(c) Let X be a set, $\mathcal{A}$ a sigma-algebra on X, and $A \in \mathcal{A}$. The characteristic (indicator) function of A, denoted by χ_A, is defined as:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Show that χ_A is a measurable function with respect to the sigma-algebra $\mathcal{A}$.

[4 marks]

QUESTION FOUR (20 marks)

(a) Let $M^+(X,\mathcal{A})$ denote the collection of all positive measurable functions on X.

Let $f, g \in M^+(X,\mathcal{A})$. If $f(x) \leq g(x)$ for all $x \in X$, show that

$\int_X f\,d\mu \leq \int_X g\,d\mu.$ **[4 marks]**

(b) Let $M^+(X,\mathcal{A})$ denote the collection of all positive measurable functions on X.

Let $f \in M^+(X,\mathcal{A})$, and let $B, C \in \mathcal{A}$ with $B \subset C$.

Show that: $\int_B f\,d\mu \leq \int_C f\,d\mu$. **[4 marks]**

(c) Let $X = \{1,2,3,4\}$, $\mathcal{A} = \{\emptyset, X, \{1\}, \{2,3,4\}\}$, $Y = \{a,b,c\}$ and

$\mathcal{B} = \{\emptyset, Y, \{b\}, \{a,c\}\}$.

Define $f : X \to Y$ by $1 \to a$, $2 \to a$, $3 \to b$, $4 \to c$ and

$g : X \to Y$ by $1 \to b$, $2 \to a$, $3 \to c$, $4 \to c$.

Determine whether each of these functions is measurable or not. **[6 marks]**

(d) Show that the Lebesque outer measure is translation invariant.That is:

$m^*(A+b) = m^*(A)$ **[6 marks]**

QUESTION FIVE (20 marks)

(a) The function $\psi : \mathbb{R} \to \mathbb{R}$ is defined as:

$$\psi(x) = \begin{cases} 5 & \text{if } x \in [0, 6] \\ 2 & \text{if } x \in \{7, 8, 9\} \\ 4 & \text{if } x \in (9, 12) \\ 0 & \text{if otherwise} \end{cases}$$

Let m be the Lebesque outer measure. Evaluate

i) $\int \psi \, \delta m$ **[5 marks]**

ii) $\int_E \psi \, \delta m$ where $E = (4, 11)$. **[5 marks]**

(b) Let ϕ and ψ be simple functions on $(X, \mathcal{A}, \mu)$, show that

$\int (\phi + \psi) \, d\mu = \int \phi \, d\mu + \int \psi \, d\mu$ **[10 marks]**