KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2024/2025 ACADEMIC YEAR

SECOND SEMESTER REGULAR EXAMINATIONS

FOR THE DEGREE OF:

MASTER OF SCIENCE IN APPLIED MATHEMATICS

COURSE CODE: MAT 810

COURSE TITLE: NUMERICAL ANALYSIS II

DATE: 2st MAY, 2025 **TIME: 9:00AM -12:00NOON**

Instructions: See Inside

Answer **all** questions in section $\boldsymbol{A}$ and any other **two** from section $\boldsymbol{B}$.

SECTION A

Answer **all** questions from this section

QUESTION ONE (20 MARKS)

(a) Show that the given Partial Differential Equation (PDE)

$$U_t = kU_{xx}$$

with the following initial and boundary conditions

$$\text{ICs}: \quad U(0,x) = 2\sin\left(\frac{\pi x}{\ell}\right)$$

$$\text{BCs}: \quad U(t,0) = 0, \quad U(t,\ell) = 0$$

has the solution

$$U(t,x) = 2\sin\left(\frac{\pi x}{\ell}\right)\exp\left(\frac{-k\pi^2 t}{\ell^2}\right)$$

[7 Marks]

(b) Use first and second order accurate finite differences to approximate the first and second derivative of

$$f(x) = \cos(x) \quad \text{at} \quad x = \frac{\pi}{4}$$

Using a step length of $h = 0.1$ compute the absolute error between your approximation and the correct value **[7 Marks]**

(c) Use Von-Neumann Stability Analysis to show that First Order Upwind (FOU) is stable for

$$\Delta t \leq \frac{\Delta x}{|\mu|}$$

[6 Marks]

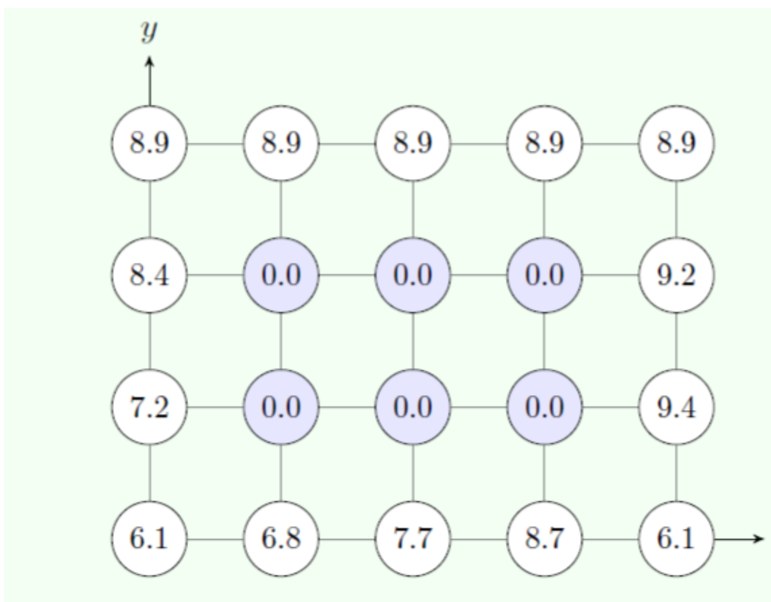
SECTION B

Answer **any TWO** questions from this section

QUESTION TWO (20 MARKS)

(a) Use the method of undetermined coefficients to derive one-sided second order finite difference approximation of $\boldsymbol{f_x(x)}$ **[10 Marks]**

(b) Perform two iterations of the Jacobi Method to solve Laplace equation for the finite difference grid shown below with zero starting values for the computational nodes **[10 Marks]**



QUESTION THREE (20 MARKS)

(a) The concentration of a pollutant in a river is to be modelled using advection equation. The stratch of a river under study is of length $\mathbf{10}$m with an average flow velocity of $\boldsymbol{v = 5ms^{-1}}$. At time $\boldsymbol{t = 0}$ a poluutant is released into the river such that the concentration of the pollutant in the river $\boldsymbol{u(t, x)}$ can be described by

$$\boldsymbol{u(0, x)} = \begin{cases} \mathbf{1}, & \boldsymbol{2 \leq x \leq 4}; \\ \\ \mathbf{0}, & \text{elsewhere.} \end{cases}$$

The domain is discretized using a finite difference grid of six nodes and a first order zero gradient boundary condition are used to compute ghost nodes such that

$$\boldsymbol{u_{-1} = u_0} \qquad\qquad \boldsymbol{u_N = u_{N-1}}$$

Use the First Order Upwind (FOU) scheme with $\boldsymbol{\Delta t = 0.25}$ to solve the advection equation to model the concentration of the pollutant after $\boldsymbol{t = 1s}$ **[15 Marks]**

(b) For each of the PDEs given below, classify them as either elliptic, parabolic or hyperbolic PDE

$$\boldsymbol{(i)\ U_{xx} + U_{yy} = 0, \quad (ii)\ U_t = kU_{xx}, \quad (iii)\ U_{tt} = c^2 U_{xx}}$$

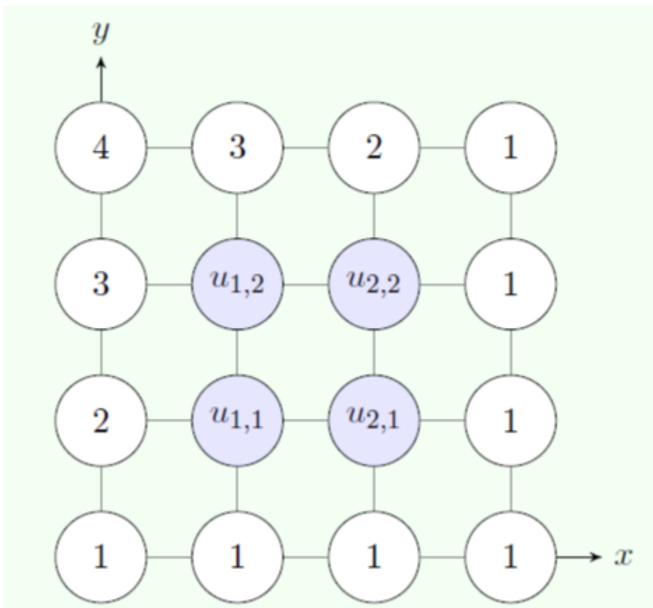
[5 marks]

QUESTION FOUR (20 MARKS)

(a) An electrical field over two dimensional domain is to be modelled using Poisson's equation

$$U_{xx} + U_{yy} = f(x, y)$$

where $f(x, y) = x + y$, the domain $0 \leq x, y \leq 1$ is described using 4 nodes in x and y directions with Dirichlet boundary conditions providing the value of the nodes as shown in the diagram below



Compute the first two iterations of Jacobi **[10 Marks]**

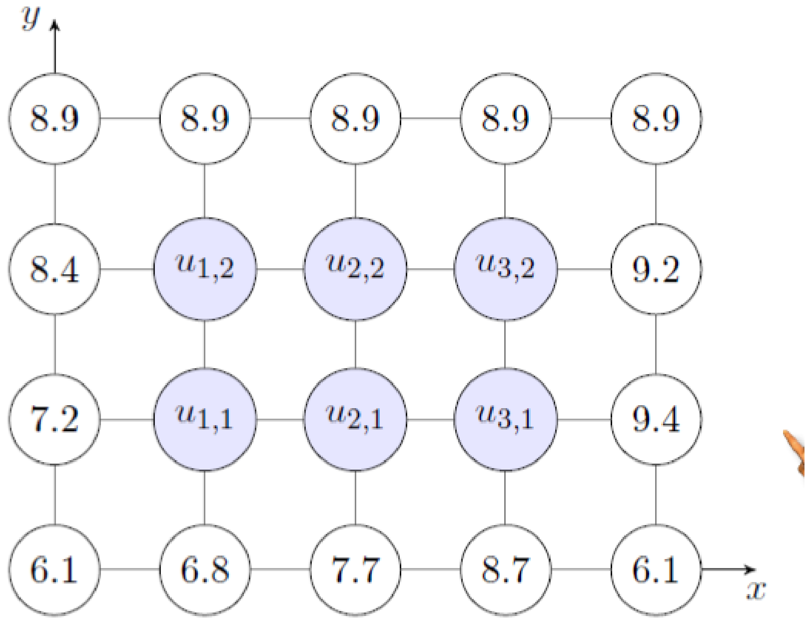
(b) Using second order finite difference scheme

$$u_{i,j} = \frac{\Delta y^2(u_{i-1,j+u_{i+1,j}}) + \Delta x^2(u_{i,j-1} + u_{i,j+1})}{2(\Delta x^2 + \Delta y^2)}$$

Solve the Laplace equation given the domain

$$0 \leq x \leq 4 \qquad \text{and} \quad 0 \leq y \leq 3$$

discretized using **5** nodes in the x direction and **3** in the y direction such that $\Delta x = \Delta y = 1$.



Dirichlet boundary conditions are implemented at all the **4** boundaries are constant **[10 marks]**