Inspiring Innovation and Leadership

KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2023/2024 ACADEMIC YEAR

FOURTH YEAR SPECIAL/ SUPPLIMENTARY EXAMINATION

FOR THE DEGREE OF

BACHELOR OF SCIENCE WITH EDUCATION, BACHELOR OF EDUCATION SCIENCE & BACHELOR OF SCIENCE

COURSE CODE: MAT 421

COURSE TITLE: COMPLEX ANALYSIS II

DATE: 17th JULY 2024 TIME: 12.00noon -2.00pm

INSTRUCTION TO CANDIDATES

SEE INSIDE

ANSWER **QUESTION ONE (COMPULSORY**) AND ANY **OTHER TWO QUESTIONS**

QUESTION ONE (30 marks)

a) Show that the real part of the complex logarithm $\log z$ is given by $\frac{1}{2}\log(x^2+y^2)$ (3marks)

b) Given the polynomial $u(x,y) = x^3 - 3xy^2 - 3x^2y^2 + y^3$, determine whether it satisfies the Laplace equation. (3marks)

c) Show that $ph\,\sqrt{z} = \frac{1}{2}\theta$ where z is a complex number and θ is the phase of z (3marks)

d) Determine whether the function $f(z) = \frac{1}{z^2}$ is harmonic. (3marks)

e) Given the harmonic complex binomial $(z+1)^n$ find an expression for the real and imaginary terms. (4marks)

f) Show that the Joukowski map $\varsigma = \frac{1}{2}(z + \frac{1}{z})$ is conformal except at points $z = \pm 1$ (6 marks)

g) Describe the analytic continuation of a function f. (3marks)

h) Show that the function $f(x) = e^{z^2}$ is analytic and that it converges for all values of z. (5marks)

QUESTION TWO (**20marks**)

a) Show that if $\prod_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 1$ (4marks)

b) Show that the complex trigonometric function, $cos(z) = cos(x)\,cosh(y) - i\,sin(x)\,sinh(y)$. (6marks)

c) Using the definition of a derivative of a complex function derive the Cauchy-Riemann equations. (10marks)

QUESTION THREE (20 marks)

a) Define any two types of singularities of a complex function $f(z)$ (2marks)

b) Determine whether the function $u(x, y) = lnz^2e^z$ has a complex derivative.

(7marks)

c) Consider the complex potential function $W(z) = \frac{1}{2}z^2$, determine the streamlines and equipotential lines. (5marks)

d) Determine the harmonic conjugate to the harmonic polynomial

$u(x, y) = x^3 - 3x^2y - 3xy^2 + y^3)$ (6marks)

QUESTION FOUR (**20marks**)

a) Consider the Real part of the function $\varphi\,(x, y) = y - \frac{1}{z^2}$, show that the first term of the 2-D Laplace equation $\nabla\varphi = 0$is $2y(x^2 + y^2)^2 - 8x^2y(x^2 + y^2)$

(5marks)

b) Determine the sum of the residues of the function $f(z) = \frac{e^z}{z^3 + 4z^2 - 11z - 30}$

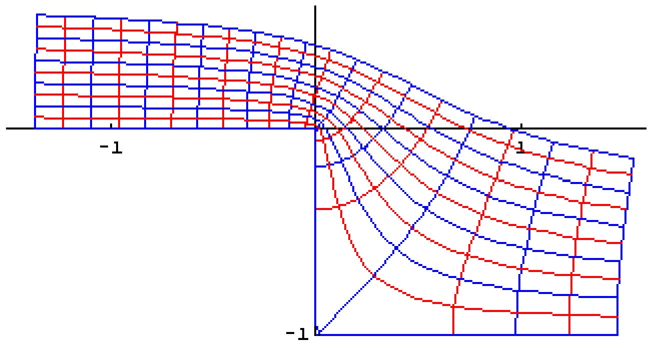
(8marks

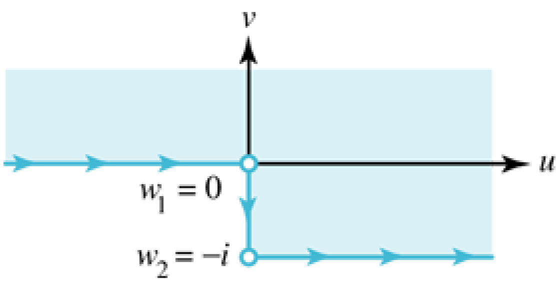
c) Use the residue theorem to integrate $\int_0^{2\pi} \frac{d\theta}{2+cos\theta}$ (7marks)

QUESTION FIVE (**20marks**)

a) Show that $w = f(z) = \frac{1}{\pi}(z^2 - 1)^{\frac{1}{2}} + \frac{1}{\pi}log(z + (z^2 - 1)^{\frac{1}{2}}) - i$ maps the upper half-plane onto the domain indicated in the Figure below. (8marks)

Hint: Set $x_1 = -1, x_1 = 1\ and\ w_1 = 0, w_1 = -i$.





b) Determine and classify the singular points of the function $f(z) = \frac{e^z}{z^3 - z^2 - 5z - 3}$

(5marks)

c) State the Cauchy's Integral Formula, hence compute the integral $\oint_C \frac{e^z\,dz}{z^3 - z^2 - 5z - 3}$ around the circle of radius 2 centred at the origin. (7marks)