



KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2024/2025 ACADEMIC YEAR

FOURTH YEAR FIRST SEMESTER REGULAR
EXAMINATIONS

FOR THE DEGREE OF:

BACHELOR OF SCIENCE(P102 P103, P106),
BACHELOR OF EDUCATION (E101)

COURSE CODE: MAT 412

COURSE TITLE: MEASURE THEORY

DATE: th DEC., 2024

TIME:

Instructions: See Inside

Answer question **ONE** in section A and any other **Two** from section B.

SECTION A

Question ONE is Compulsory

QUESTION ONE (30 marks)

- (a) Explain three advantages of the Lebesgue integral in comparison to the Riemann integrals. [3 marks]
- (b) What is a sigma algebra? [4 marks]
- (c) List all sigma algebras on $X = \{1, 2, 3\}$. [5 marks]
- (d) Let (X, \mathcal{A}) be a measurable space. When is a function $f : X \rightarrow \mathbb{R}$ said to be \mathcal{A} -measurable? [2 marks]
- (e) Show that the function $f(x) = x$ is Borel measurable. [2 marks]
- (f) Define a measure. [4 marks]
- (g) Let A be a measurable set. When do we say that a measurable function $\psi : A \rightarrow \mathbb{R}$ is simple? [2 marks]
- (h) When is a set $E \subseteq X$ said to be m^* -measurable? [2 marks]
- (i) Show that the sets \emptyset and \mathbb{R} are Lebesgue measurable. [3 marks]
- (j) Find $\int_E \chi_Q dm^*$ where $E = [0, 1]$, Q the set of rational numbers and m^* the Lebesgue measure. [3 marks]

SECTION B

Answer **any Two** questions from this section

QUESTION TWO (20 marks)

(a) Show that the intersection of any two sigma algebras is a sigma algebra. [6 marks]

(b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3x + 2 & \text{if } x \in [0, 4] \\ 2x - 4 & \text{if } x \in (5, 10) \\ 0 & \text{if otherwise} \end{cases}$$

show that $38 \leq \int f \, dm \leq 136$ where m is the Lebesgue measure. [8 marks]

(c) Let $(\mathbf{X}, \mathcal{A}, \mu)$ be a measure space, and $\{E_n\}$ be a monotone sequence in \mathcal{A} .

If $\{E_n\}$ is increasing, show that $\lim_{n \rightarrow \infty} \mu \{E_n\} = \mu \left(\lim_{n \rightarrow \infty} E_n \right)$ [6 marks]

QUESTION THREE (20 marks)

(a) Show that a measurable function f is integrable if and only if $|f|$

is integrable. [4 marks]

(b) Consider the function $f(x) = x^2 + 1$ defined on the interval $[0, 4]$.

i) Find the upper and lower Riemann sums when the interval is partitioned into 4 equal intervals. [4 marks]

ii) Find the exact Riemann integral. [5 marks]

iii) Compute $\int_0^4 (x^2 + 1) \, dx$. [3 marks]

(c) Let X be a set, \mathcal{A} a sigma-algebra on X , and $A \in \mathcal{A}$. The characteristic (indicator) function of A , denoted by χ_A , is defined as:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Show that χ_A is a measurable function with respect to the sigma-algebra \mathcal{A} .

[4 marks]

QUESTION FOUR (20 marks)

- (a) Let $M^+(X, \mathcal{A})$ denote the collection of all positive measurable functions on X .

Let $f, g \in M^+(X, \mathcal{A})$. If $f(x) \leq g(x)$ for all $x \in X$, show that

$$\int_X f \, d\mu \leq \int_X g \, d\mu. \quad [4 \text{ marks}]$$

- (b) Let $M^+(X, \mathcal{A})$ denote the collection of all positive measurable functions on X .

Let $f \in M^+(X, \mathcal{A})$, and let $B, C \in \mathcal{A}$ with $B \subset C$.

Show that: $\int_B f \, d\mu \leq \int_C f \, d\mu$. [4 marks]

- (c) Let $X = \{1, 2, 3, 4\}$, $\mathcal{A} = \{\emptyset, X, \{1\}, \{2, 3, 4\}\}$, $Y = \{a, b, c\}$ and

$$\mathcal{B} = \{\emptyset, Y, \{b\}, \{a, c\}\}.$$

Define $f : X \rightarrow Y$ by $1 \rightarrow a, 2 \rightarrow a, 3 \rightarrow b, 4 \rightarrow c$ and

$g : X \rightarrow Y$ by $1 \rightarrow b, 2 \rightarrow a, 3 \rightarrow c, 4 \rightarrow c$.

Determine whether each of these functions is measurable or not. [6 marks]

- (d) Show that the Lebesgue outer measure is translation invariant. That is:

$$m^*(A + b) = m^*(A) \quad [6 \text{ marks}]$$

QUESTION FIVE (20 marks)

(a) The function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is defined as:

$$\psi(x) = \begin{cases} 5 & \text{if } x \in [0, 6] \\ 2 & \text{if } x \in \{7, 8, 9\} \\ 4 & \text{if } x \in (9, 12) \\ 0 & \text{if otherwise} \end{cases}$$

Let m be the Lebesgue outer measure. Evaluate

i) $\int \psi \, dm$ **[5 marks]**

ii) $\int_E \psi \, dm$ where $E = (4, 11)$. **[5 marks]**

(b) Let ϕ and ψ be simple functions on (X, \mathcal{A}, μ) , show that

$$\int (\phi + \psi) \, d\mu = \int \phi \, d\mu + \int \psi \, d\mu$$
 [10 marks]