KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2023/2024 ACADEMIC YEAR

THIRD YEAR SPECIAL/SUPPLEMENTARY EXAMINATIONS

FOR THE DEGREE OF:

BED (E100, E101) BSC (P106

COURSE CODE: MAT 313

COURSE TITLE: ABSTRACT ALGEBRA/GROUP THEORY

DATE: 19th **JULY, 2024** **TIME: 3.00 PM - 5.00 PM**

Instructions: Answer **all** questions in section A and any other **TWO** from section B.

SECTION A

Answer **all** questions from this section

QUESTION ONE (30 marks)

(a) Find the cyclic subgroup of the quarternion group generated by j. **[2 marks]**

(b) Determine if the permutation (123)(45678) is even or odd. **[2 marks]**

(c) Let S be a non-empty set and $\star$ be an operation on the set S.State two conditions that $\star$ should satisfy to be a binary operation on the set S. **[2 marks]**

(d) Determine whether subtraction is a binary operation on $\mathbb{Z}^+$. **[2 marks]**

(e) Find the gcd of 168 and 198 by the Euclidean Algorithm. **[3 marks]**

(f) Let $S = \{f_1, f_2, f_3, f_4\}$ where $f_i : \mathbb{R}^\star \to \mathbb{R}^\star$ defined by $f_1(x) = x$, $f_2(x) = \frac{1}{x}$, $f_3(x) = -x$, $f_4(x) = -\frac{1}{x}$. Let $\circ$ denote composition of mappings.

i) Construct the Cayley table for S under $\circ$. **[2 marks]**

ii) Is $\circ$ a commutative operation? Explain. **[2 marks]**

(g) Find the order of (123)(45) in S_5. **[2 marks]**

(h) Let G be S_3 and $H = \{1, (1, 2)\}$.

Determine whether H is a normal subgroup of G? **[2 marks]**

(i) Let S be the set $S = \{(a, b) : a, b \in \mathbb{R}\}$.

Define $\star$ on S by $(a, b) \star (c, d) = (ac, ad + b)$, for all $a, b, c, d \in \mathbb{R}$.

Determine whether $\star$ is a commutative operation. **[2 marks]**

(j) i) Define the center of a group . **[2 marks]**

ii) Hence find the center of the quarternion group. **[1 mark]**

(k) Define an Homomorphism. **[2 marks]**

(l) Let F be the additive group of all functions mapping $\mathbb{R}$ into $\mathbb{R}$, let $\mathbb{R}$ be the additive group of real numbers, and let c be any real number. Let $\emptyset_c : F \to \mathbb{R}$ be defined by $\emptyset_c(f) = f(c)$ for $f \in F$.

i) Find $\emptyset_3(2x^3 - 2x^2 + 1)$ **[1 mark]**

ii) Show that $\emptyset_c$ is a homomorphism . **[2 marks]**

iii) What is the name of this homomorphism? **[1 mark]**

SECTION B

Answer **any Two** questions from this section

QUESTION TWO (20 marks)

(a) Define $\star$ on $\mathbb{R}$ by $a \star b = 2a + 2b + ab + 2$.

Determine the identity element for $\star$. **[4 marks]**

(b) Find the gcd of 1547 and 3059 and hence express the gcd in the form $1547m + 3059n = gcd$. **[8 marks]**

(c) Proof all the group axioms for the Abelian group $\langle \mathbb{Z}, + \rangle$. **[8 marks]**

QUESTION THREE (20 marks)

(a) Find the order of the factor group $(\mathbb{Z}_{12} \times \mathbb{Z}_{18})/\langle(4, 3)\rangle$. **[3 marks]**

(b) Let $G = \mathbb{R}^2$ (operation vector addition), let $H = \mathbb{R}$ (operation addition) and define $\psi : \mathbb{R}^2 \to \mathbb{R}$ by $\psi(x, y) = x$. Thus ψ is projection onto the $x-$axis.

i) Show that ψ is a homomorphism **[3 marks]**

ii) Find ker ψ **[2 marks]**

(c) Let $\delta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{bmatrix}$ and $\tau = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \end{bmatrix}$.

(i) Express δ and τ in disjoint cycle form. **[2 marks]**

(ii) Find $\delta\tau$. **[2 marks]**

(iii) Find δ^2. **[2 marks]**

(iv) Find $\delta^2\tau$. **[2 marks]**

(v) Find δ^{11}. **[2 marks]**

(vi) Find τ^{-1}. **[2 marks]**

QUESTION FOUR (20 marks)

(i) List and describe all the subgroups of $\langle \mathbb{Z}_{30}, + \rangle$. **[13 marks]**

(ii) Consider the various symmetry transformations of a rectangle that form the symmetry group of a rectangle.

i) With the help of a diagram, compute and list all the elements in the symmetry group of a rectangle **[4 marks]**

ii) Construct the Cayley table for the symmetry group. **[3 marks]**

QUESTION FIVE (20 marks)

(a) Let G be a group. Show that the left cancellation laws hold, i.e. $a \star b = a \star c$ implies $b = c$ for all $a, b, c \in G$ **[4 marks]**

(b) The ring $R = \{0, 2, 4, 6, 8\}$ under addition and multiplication modulo 10 has a unity. Find it. **[4 marks]**

(c) Define $\star$ on $\mathbb{Z}$ by $a \star b = a + b - ab$. $\star$ is a binary operation on $\mathbb{Z}$ with identity element 0. If $a \in \mathbb{Z}$, what is the inverse of a ? **[4 marks]**

(d) The collection of all the even permutations in S_n is a subgroup called the alternating group denoted by A_n of order $\frac{n!}{2}$. List the elements of A_3 **[4 marks]**

(e) List the elements of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hence show that $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ has seven subgroups of order 2. **[4 marks]**