KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2024/2025 ACADEMIC YEAR

FOURTH YEAR SPECIAL/SUPPLEMENTARY EXAMINATIONS

FOR THE DEGREE OF BACHELOR OF EDUCATION (E100, E101, E111, E113)

COURSE CODE: MAT 413

COURSE TITLE: TOPOLOGY I

DATE: ...$^{\text{th}}$ DEC, 2024 **TIME:**

Instructions: See Inside

Answer question **ONE** in section A and any other **Two** from section B.

SECTION A

Question ONE is compulsory

QUESTION ONE (30 marks)

(a) Let X be a non-empty set. Define a topology τ on X. **[4 marks]**

(b) List of all topologies on a 2-point set $\{0,1\}$ **[4 marks]**

(c) Let (X,τ) be a topological space. When is X said to be an Hausdorff space or T_2 space. **[2 marks]**

(d) Define the discrete topology on a set X . **[2 marks]**

(e) Show that any set endowed with the discrete topology is a Hausdorff space. **[3marks]**

(f) Consider the set $\mathbb{Q}$ of rationals. Find the closure of $\mathbb{Q}$, i.e. $\bar{\mathbb{Q}}$. **[2 marks]**

(g) Let X and Y be topological spaces. Suppose $f : X \to Y$. When is f called ;

i) an open function. **[2 marks]**

ii) a closed function. **[1 mark]**

(h) Let (X,τ) and $(X,\tau)^*$ be two topological spaces. State what is meant by:

i) $f : X \to Y$ is a homeomorphism. **[2 marks]**

ii) $f : X \to Y$ is a continuous map. **[2 marks]**

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = c$ for all $x \in \mathbb{R}$, where c is a constant. Show that f is continuous relative to any topology τ. **[2 marks]**

(j) When is a property P of sets called a topological invariant? **[2 marks]**

(k) In $\mathbb{R}$, is length a topological property? Explain **[2 marks]**

SECTION B

Answer **any TWO** questions from this section

QUESTION TWO (20 marks)

(a) Let τ_1 and τ_2 be two topologies on X.

Show that $\tau_1 \cap \tau_2$ is also a topology on X. **[6 marks]**

(b) Show that $\tau_1 \cup \tau_2$ may not be a topology on X even though τ_1 and τ_2 are topologies on X. **[4 marks]**

(c) Let $X = \{a, b, c, d\}$ and $A = \{\{a, b\}, \{b, c\}, \{d\}\}$. Find the topology on X generated by A. **[4 marks]**

(d) Consider the following topologies on $X = \{a, b, c, d\}$ and $Y = \{x, y, z, w\}$ respectively:

$\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$ and $\tau^* = \{Y, \emptyset, \{x\}, \{y\}, \{x, y\}, \{y, z, w\}\}$

Also, consider the functions f and g from X to Y defined by:

$f = \{(a, y), (b, z), (c, w), (d, z)\}$ and $g = \{(a, x), (b, x), (c, z), (d, w)\}$

Determine whether each of the given function is continuous. **[6 marks]**

QUESTION THREE (20 marks)

(a) Let τ be the class of subsets of $\mathbb{N}$ consisting of $\emptyset$ and all subsets of $\mathbb{N}$ of the form $E_n = \{n, n+1, n+2, ...\}$ with $n \in \mathbb{N}$. List the open sets containing the positive integer 4. **[2 marks]**

(b) Consider the following topology on $X = \{a, b, c, d, e\}$:

$\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}\}$

(i) Define the neighbourhood system of a point p in a topological space X . **[2 marks]**

(ii) List the neighbourhoods of e in X. **[2 marks]**

(iii) List the members of the relative topology τ_A on $A = \{b, c, e\}$. **[3 marks]**

(iv) List the closed subsets of X. **[2 marks]**

(v) Determine the closure of the set $\{a\}$. **[2 marks]**

(vi) Determine the set $\{a\}$ is dense in X. **[3 marks]**

(vii) Find the interior points of the subset $A = \{a, b, c\}$ of X . **[2 marks]**

(viii) Find the exterior points of the subset $A = \{a, b, c\}$ of X . **[2 marks]**

QUESTION FOUR (20 marks)

(a) Define the following concepts on the set $\mathbb{R}$

i) an interior point. **[2 marks]**

ii) an isolated point. **[2 marks]**

(b) Let $E = (-1, 5] \cup \{7\} \cup (10, \infty)$.

i) Determine whether 5 is an interior point. **[2 marks]**

ii) Determine whether $\{7\}$ is an accumulation point **[2 marks]**

iii) Determine the boundary points. **[2 marks]**

iv) Determine whether $\{7\}$ is an isolated point **[2 marks]**

(c) When is a property P of a topological space X said to be hereditary? **[2marks]**

(d) Prove that the property of being Hausdorff is hereditary. **[3 marks]**

(e) Show that the real line $\mathbb{R}$ and $X = (-1, 1)$ are homeomorphic. **[3 marks]**

QUESTION FIVE (20 marks)

(a) Distinguish between first countable and second countable spaces. **[4 marks]**

(b) Determine whether the real numbers with the standard Euclidean topology is a first countable space. **[2 marks]**

(c) When is a topological space said to be separable ? **[2 marks]**

(d) Show that $\mathbb{R}$, the set of real numbers with respect to the Euclidean topology is separable. **[2marks]**

(e) Let $X = \{a, b, c\}$. List down all the topologies on X that consists of exactly four members. **[3 marks]**

(f) Consider the discrete topology $\mathcal{D}$ on $X = \{a, b, c, d, e\}$.

i) Find a base for the discrete topology $\mathcal{D}$ **[2 marks]**

ii) Find a subbase δ for $\mathcal{D}$ which does not contain any singleton sets. **[2 marks]**

(g) Consider the topology $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$ on the set $X = \{a, b, c\}$. Determine whether or not (X, τ) is a Regular space. Explain. **[3 marks]**