KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2023/2024 ACADEMIC YEAR

SPECIAL/SUPPLEMENTARY EXAMINATIONS FOR THE DEGREE OF:

Bsc. W/EDUCATION (P106)

Bed. SCIENCE (E101)

COURSE CODE: MAT 304

COURSE TITLE: FUNCTIONAL ANALYSIS I

DATE: 17th JULY, 2024 **TIME:** 12PM - 2PM

INSTRUCTIONS : ***See inside***

INSTRUCTIONS

Answer **ALL** questions in *Section A* and **ANY** other **TWO** from section B.

SECTION A

Answer **ALL** questions from this section

QUESTION ONE (30 marks)

a) Give an example of incomplete normed linear space. **(2 marks)**

b) Let $X = C[0,1]$. Consider three metrics d_1 and d_∞ defined as:
$d_1(f,g) = \int_a^b |f(x) - g(x)|dx$ and $d_\infty(f,g) = \max\{|f(x) - g(x)|: a \leq x \leq b\}$.
On $X = C[0,1]$ take $f(x) = x$ and $g(x) = x^3$. Calculate:

i) $d_1(f,g)$. **(3 marks)**

ii) $d_\infty(f,g)$. **(3 marks)**

c) Let $X = (0,1]$ and a metric be defined by $d(x,y) = |x - y|$. Let $x_n = \frac{1}{n}$ be a sequence in X . Determine if $(X,\ d)$ is complete. **(4 marks)**

d) Let be a vector space of all polynomials defined on the interval $[a,b]$. $x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$, $n \geq 1$ and $t \in [a,b]$. Define an operator by $Tx = \frac{d}{dx} x(t)$. Show that T is not bounded. **(5 marks)**

e) Consider the vector $\vec{u} = (11, 8, 7, -4, 6, -19)$. Determine the:

i) One norm. **(2 marks)**

ii) Two norm. **(3 marks)**

iii) Infinity norm. **(3 marks)**

f) Show that every convergent sequence in a metric space *(X, d)* is Cauchy. **(5 marks)**

SECTION B

Answer **ANY TWO** questions from this section

QUESTION TWO (20 marks)

a) Show that the differential operator is linear. **(5 marks)**

b) Show that the differential operator is linear. **(6 marks)**

c) Let $X = \mathbb{R}^n$ be the Euclidean vector space. Define on X the one norm on by $\|\vec{u}\|_1 = |u_1| + |u_2| + |u_3| + \cdots + |u_n|$. Show that $\|\vec{u}\|_1$ defines a norm. **(9 marks)**

QUESTION THREE (20 marks)

a) i) Define the differential operator. **(3 marks)**

ii) Show that the differential operator is unbounded. **(4 marks)**

b) i) Let *(X, d)* be a metric space. Define an open ball in relation to *(X, d)* . **(3 marks)**

ii) Let $\mathbb{R}^2, d$ be the Euclidean metric in $\mathbb{R}^2$. Find and describe $B_d((0,0),1)$. **(4 marks)**

c) Let *(X, d)* be a metric space. Define $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$ for all $x, y \in X$. Show that $d_1(x,y)$ satisfies the symmetry and triangular inequality properties of a metric space. **(6 marks)**

QUESTION FOUR (20 marks)

a) Define the term "norm of an operator". **(3 marks)**

b) i) Define the term continuous function" for a function $f : V \to \mathbb{R}$ where V is a metric space. **(3 marks)**

ii) Give an example of a continuous function in a metric space $V = \mathbb{R}$. Show that this function is continuous. **(4 marks)**

c) Let $V = \mathbb{R}^2$. Let $(x, y) \in V$ with x and y written as $x = (\xi_1, \eta_1)$ and $y = (\xi_2, \eta_2)$. Let a metric on this set be defined by $d_{tc}(x, y) = \sqrt{|\xi_1 - \xi_2| + |\eta_1 - \eta_2|}$. Show that $d_{tc}(x, y)$ defines a metric on V. **(10 marks)**

QUESTION FIVE (20 marks)

a) Show that the space of rational numbers is incomplete. **(4 marks)**

b) Give an example of a complete normed space. **(2 marks)**

c) Consider the operator $T : C[0, 5] \to C[0, 5]$ defined by $S(f(t)) = x^3 f(x)$. Show that T is:

i) Linear. **(3 marks)**

ii) Bounded. **(3 marks)**

d) Define a Cauchy sequence in a normed space X **(4 marks)**

e) Define a norm on a real vector space X. **(4 marks)**