KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2024/2025 ACADEMIC YEAR

FOURTH YEAR **FIRST** SEMESTER **REGULAR**

EXAMINATIONS FOR THE DEGREE OF:

BACHELOR OF SCIENCE W/EDUCATION (P106);

BACHELOR OF EDUCATION (E101).

COURSE CODE: MAT 416

COURSE TITLE: FUNCTIONAL ANALYSIS II

DATE: 13th DEC., 2024 **TIME:** 3:00PM - 5:00PM

INSTRUCTIONS : ***See Inside***

Answer question **ONE** in *Section* **A** and any other **Two** from *Section* **B**.

SECTION A

QUESTION ONE (30 marks)

a) Let $\mathcal{H} = L^2[0,\ 1]$ be the space of square - integrable functions over $[0,\ 1]$. Find the norm of the function $f(x) = x^2 + i$. **(3 marks)**

b) Define

i) A Pre - Hilbert Space. **(4 marks)**

ii) Hilbert adjoint operator. **(3 marks)**

c) Consider "the differentiation operator" defined on the Hilbert space $L^2(\mathbb{R})$, that is: $T : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad T(f) = \dfrac{d}{dx} f(x)$, where $L^2(\mathbb{R})$ is the space of square-integrable functions.

Show that this operator is:

i) Linear. **(3 marks)**

ii) Not bounded. **(4 marks)**

d) State and prove the Cauchy Schwartz inequality for a Hilbert space $\mathcal{H}$. **(5 marks)**

e) Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in a Hilbert space $\mathcal{H}$ such that

$\langle \mathbf{u},\ \mathbf{v} \rangle = 2,\ \langle \mathbf{v},\ \mathbf{w} \rangle = -3,\ \langle \mathbf{u},\ \mathbf{w} \rangle = 4$, and

$\|\mathbf{u}\| = 1,\ \|\mathbf{v}\| = 3,\ \|\mathbf{w}\| = 5$.

Evaluate $\langle \mathbf{2v} + \mathbf{w},\ \mathbf{3u} - \mathbf{2w} \rangle$. **(4 marks)**

f) i) Define "*a linear Functional*" in a Hilbert Space $\mathcal{H}$. **(2 marks)**

ii) Give two examples of linear functionals on some Hilbert space H. **(2 marks)**

QUESTION TWO (20 marks)

a) Prove that $\mathbb{R}^n$ is a Hilbert Space. **(8 marks)**

b) Let $(X, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space, and let $U, V \subseteq X$ be a subspace of X.

i) Define the term "orthogonal complement" of U in X. **(2 marks)**

ii) Show that $U \subseteq V \Longrightarrow U^{\perp} \supseteq V^{\perp}$. **(4 marks)**

c) Let $\mathcal{H}$ be a Hilbert Space.

i) Define an orthogonal complement $M^{\perp}$ of a $M \subseteq \mathcal{H}$. **(2 marks)**

ii) Show that $M^{\perp}$ is a closed subspace of $\mathcal{H}$. **(4 marks)**

QUESTION THREE (20 marks)

a) Give an example of an incomplete Pre - Hilbert space. Illustrate why the space is incomplete. **(5 marks)**

b) Let $\mathcal{H}$ be a Hilbert space.

i) Define the term "norm" on $\mathcal{H}$. **(2 marks)**

ii) Prove that the norm defined in part (i) is indeed a norm. **(4 marks)**

c) i) State the Pythagorean Theorem for orthogonal vectors in a Hilbert space $\mathcal{H}$. **(2 marks)**

ii) Prove the Pythagorean Theorem for orthogonal vectors in a Hilbert space $\mathcal{H}$. **(3 marks)**

d) State and prove the parallelogram law with reference to a Hilbert space $\mathcal{H}$. **(4 marks)**

QUESTION FOUR (20 marks)

a) Give any two properties of the Hilbert adjoint operator. **(2 marks)**

b) Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and let $T^* : \mathcal{H}_2 \to \mathcal{H}_1$ be the Hilbert adjoint operator of T. Also let α be a scalar. Show that

i) $\langle T^* y, x\rangle = \langle y, Tx\rangle$ for all $x, y \in \mathcal{H}_1$.. **(2 marks)**

ii) $(S+T)^* = S^* + T^*$, where $S \text{and} T$ are bounded linear operators from Hilbert spaces while $S^* \text{and} T*$ are the Hilbert adjoint operators of $S \text{and} T$ respectively. **(3 marks)**

iii) $(\alpha T)^* = \overline{\alpha} T^*$ **(2 marks)**

c) i) Define the term "*a bounded linear functional*" on a Hilbert space H. **(2 marks)**

ii) Give two examples of bounded linear functionals on some Hilbert space H. **(4 marks)**

d) Let $f(x) = e^x$ and $g(x) = 1$ be two functions in the Hilbert space $L^2([0, \ln 2])$.

i) Determine if f and g are orthogonal. **(2 marks)**

ii) Find $d(f, g)$. **(3 marks)**

QUESTION FIVE (20 marks)

a) Give any:

i) Two similarities of the Banach and Hilbert spaces. **(2 marks)**

ii) Two differences between Banach spaces and Hilbert spaces. **(3 marks)**

b) State (without proof) any two fundamental theorems related to Hilbert spaces: **(4 marks)**

c) Give two examples of linear functionals on different Hilbert spaces H. **(3 marks)**

d) Define an orthonomal basis of a Hilbert space $\mathcal{H}$ **(3 marks)**

e) Let$\{v_k\}_{k=1}^{\infty}$ be a sequence of unit vectors in a Hilbert space $\mathcal{H}$. Define a mapping $\Phi : \mathcal{H} \longrightarrow \mathbb{C}$ by : $\Phi(v) = \sum_{k=1}^{\infty} \frac{1}{k^2} \langle \vec{v}, \vec{v_k} \rangle$.. Show that Φ is:

i) Well defined. **(3 marks)**

ii) Bounded. **(2 marks)**