Inspiring Innovation and Leadership

KARATINA UNIVERSITY

UNIVERSITY EXAMINATIONS

2024/2025 ACADEMIC YEAR

FIRST YEAR SECOND SEMESTER REGULAR EXAMINATIONS

FOR THE DEGREE OF MSC IN PURE MATHEMATICS

COURSE CODE: MAT 819

COURSE TITLE: ALGEBRAIC TOPOLOGY

DATE: th ., 2025 **TIME:**

Instructions: See Inside

Answer question **ONE** in section A and any other **Two** from section B.

SECTION A

Question ONE is Compulsory

QUESTION ONE (20 marks)

(a) Define the following terms

i) A loop. **[2 marks]**

ii) The Fundamental group $\pi_1(X, x_0)$. **[2 marks]**

(b) Compute the fundamental group of a convex subset in $\mathbb{R}^n$. **[3 marks]**

(c) Let $X = \mathbb{R}$ with the standard topology. Suppose $f : [0, 1] \to \mathbb{R}$ is a path defined by $f(t) = 2t + 1$ for $t \in [0, 1]$.

Compute the reversal of the path f. **[3 marks]**

(d) If the maps $f, g : X \to Y$ are both null-homotopic and Y is path-connected, then they are homotopic to each other. Prove. **[5 marks]**

(e) Discuss the concept of retractions and deformation retractions.

Provide an example. **[5 marks]**

SECTION B

Answer **any Two** questions from this section

QUESTION TWO (20 marks)

(a) Let X and Y be topological spaces. Prove that the homotopy relation $\sim$ is an equivalence relation on the set of all continuous maps from X to Y. **[6 marks]**

(b) At any time, there is always a pair of antipodal points (opposite points) on the Earth with equal temperature and equal atmospheric pressure. **[5 marks]**

(c) Construct a simplicial complex that represents a cube. How many 0-, 1-, 2-, and 3-simplices does it contain? **[9 marks]**

QUESTION THREE (20 marks)

(a) Let $p : X \to Y$ be a continuous surjective map. When do we say that p *evenly covers* Y ? **[3 marks]**

(b) Explain the path lifting property and the path homotopy lifting property of covering maps. Illustrate each with an example. **[5 marks]**

(c) Show that the set $\pi_1(X, x_0)$ forms a group under the operation of concatenation of loops. **[12 marks]**

QUESTION FOUR (20 marks)

(a) Let X and Y be two topological spaces. When are they said to be homotopy equivalent? **[3 marks]**

(b) Show that the point, the disk, and the Euclidean plane are all homotopy equivalent. **[5 marks]**

(c) Let α be a path in X from x_0 to x_1. Then we can define a map

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

by the equation

$$\hat{\alpha}([f]) = [\alpha] * [f] * [\alpha^{-1}].$$

Show that this is a well-defined group isomorphism. **[6 marks]**

(d) Determine the number of possible orientations for a 4-simplex. **[6 marks]**