Inspiring Innovation and Leadership

# KARATINA UNIVERSITY

## UNIVERSITY EXAMINATIONS

## 2023/2024 ACADEMIC YEAR

**FOURTH** YEAR **FIRST** SEMESTER SUPPLEMENTARY/SPECIAL EXAMINATION

**FOR THE DEGREE OF:**

**BACHELOR OF EDUCATION (E101)**

**BACHELOR OF SCIENCE (P102 & P106)**

**COURSE CODE: MAT 412**

**COURSE TITLE: MEASURE THEORY**

**DATE: 16TH JULY, 2024** **TIME: 3:00PM-5:00PM**

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**INSTRUCTIONS TO CANDIDATES**

- SEE INSIDE

**<u>INSTRUCTIONS:</u> Answer <u>all</u> questions in SECTION A and any other <u>TWO</u> in SECTION B**

## <u>SECTION A (30 MARKS)</u>

### QUESTION ONE (30 MARKS)

a) Define the Lebesque measure $m$ and the Borel measure $m_\beta$ on the real line. What are the relations between $m^*, m$ and $m_\beta$. (4 marks)

b) Show that the Borel measure $m_\beta$ is not complete (4 marks)

c) Let $A \subseteq B \subseteq \mathbb{R}$ , show that $m^*(A) \leq m^*(B)$ (3 marks)

d) When do we say a function is measurable? Give an example of a measurable function. (3 marks)

e) Explain the "almost everywhere" concept on measure space $(X, S, \mu)$. (2 marks)

f) If $E \in m$ and $F \subseteq \mathbb{R}'$, show that $m^*(E \cap F) + m^*(E \cup F) = m^*(E) + m^*(F)$ (5 marks)

**g)** Let $(X, S)$ be a measurable space and $u_1, u_2, \dots \dots . ., u_k$ be a measure on $(X, S)$, where $a_1, a_2, \dots \dots . ., a_k$ are non-negative reals. Show that the set function $\lambda: S \to \mathbb{R}'_e$, defined by $\lambda = a_1 u_1 + a_2 u_2 \dots . . + a_k u_k$ is a measure on $(X, S)$. (4 marks)

h) Show that if $(X, S, \mu)$ is a measure space and $f_n$ a sequence of elements in $m^+(X, S)$ then $\int (\underline{\lim}_{n\to\infty} f_n) d\mu \leq \underline{\lim}_{n\to\infty} \int f_n d\mu$. (5 marks)

## <u>SECTION B (40 MARKS)</u>

### QUESTION TWO (20 MARKS)

a) Show that outer lebesque measure is countably sub additive (9 marks)

b) Let $(X, S)$ be a measurable space and $f: X \to \mathbb{R}'$ be a function, show that the following statements are equivalent.

(i) $f^{-1}(B) \in S$ for each $B \in B(\mathbb{R}')$ (2 marks)

(ii) $f^{-1}((a, \infty)) \in S$ for each $a \in \mathbb{R}'$ (3 marks)

(iii) $f^{-1}([a, \infty)) \in S$ for each $a \in \mathbb{R}'$ (2 marks)

(iv) $f^{-1}((\infty, a)) \in S$ for each $a \in \mathbb{R}'$ (2 marks)

(v) $f^{-1}((\infty, a]) \in S$ for each $a \in \mathbb{R}'$ (2 marks)

**QUESTION THREE (20 MARKS)**

a) Let $(X, S)$ be a measurable space and $f: X \to \mathbb{R}'$ be S measurable, show that the functions $c + f$, $c.f$, $|f|$, and $f^2$ are all S measurable. Here c is a real number.

(8 marks)

b) Suppose A and B are subsets $\mathbb{R}$ with $m^*(A) = 0$, show that $m^*(A \cup B) = m^*(B)$

(6 marks)

c) Let $(X, S, \mu)$ be a measure space, show that $\mu$ is countably sub additive (6 marks)

**QUESTION FOUR (20 MARKS)**

a) Define a simple function. Write down the canonical representation of the function $S = 2\chi_{[0,2]} + 3\chi_{[1,3]}$. (5 marks)

b) Let $(X, S, \mu)$ be a measure space and A a fixed member of S. Define a set function $\lambda: S \to \mathbb{R}'_e$ by $\lambda(E) = \mu(E \cap A)$ for each $E \in S$.

(i) Show that $\lambda$ is a measure. (8 marks)

(ii) If $\mu$ is $\delta$ finite what can we say about $\lambda$. (5 marks)

c) Show that $m^*(\emptyset) = 0$ (2 marks)

**QUESTION FIVE (20 MARKS)**

a) Show that if $f \in m(X, S)$ then $f$ is integrable iff $|f|$ is integrable (5 marks)

b) If A and B are subsets of $\mathbb{R}$ with finite outer measure show that

$|m^*(A) - m^*(B)| \leq m^*(A \,\Delta\, B)$. (7 marks)

c) Show that $f(x) = \begin{cases} 1 \; if \; x \; irrational \\ 0 \; if \; x \; rational \end{cases}$ is lebesque integrable but not Riemann integrable. (5 marks)

d) Consider the measure space $(\mathbb{R}, B(\mathbb{R}), m)$ and a function $f$ defined by

$f(x) = \begin{cases} 1 \; if \; x \in \mathbb{R} - \mathbb{Q} \\ 0 \; if \; x \in \mathbb{Q} \end{cases}$. Does $f(x)$ hold $m - almost \; everywhere$ on $\mathbb{R}$ . (3 marks)