Polynomials

Some facts to know about polynomials:

- We can factor out $x - a$ from a polynomial $p(x)$ if and only if $p(a) = 0$.
- A polynomial of degree n has exactly n complex roots (counting multiplicities). Thus if a polynomial has more distinct real roots than its degree, it must be the zero polynomial. Hence if two polynomials agree at more real numbers than the larger of their degrees, then the two polynomials are the same.
- Given distinct real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, there is a unique polynomial $p(x)$ of degree at most n that satisfies $p(a_i) = b_i$ for all $i = 1, \ldots, n$, called the Lagrange interpolation polynomial. It is given by

$$p(x) = \sum_{i=1}^{n} b_i \frac{\prod\limits_{1 \le j \le n, j \ne i} (x - a_j)}{\prod\limits_{1 \le j \le n, j \ne i} (a_i - a_j)} = \sum_{i=1}^{n} b_i \frac{(x - a_1) \cdots (x - a_{i-1})(x - a_{i+1}) \cdots (x - a_n)}{(a_i - a_1) \cdots (a_i - a_{i-1})(a_i - a_{i+1}) \cdots (a_i - a_n)}$$

1. (1967, A-3)* Consider polynomial forms $ax^2 - bx + c$ with integer coefficients which have two distinct zeros in the open interval $0 < x < 1$. Exhibit with a proof the least positive integer value of a for which such a polynomial exists.

2. (1969, A-1)** Let $f(x, y)$ be a polynomial with real coefficients in the real variables x and y defined over the entire x-y plane. What are the possibilities for the range of $f(x, y)$?

3. (1970, A-2)* Consider the locus given by the real polynomial equation

$$Ax^2 + Bxy + Cy^2 + Dx^3 + Ex^2y + Fxy^2 + Gy^3 = 0,$$

where $B^2 - 4AC < 0$. Prove that there is a positive number δ such that there are no points of the locus in the punctured disk $0 < x^2 + y^2 < \delta^2$.

4. (1971, A-2)* Determine all polynomials $P(x)$ such that $P(x^2 + 1) = (P(x))^2 + 1$ and $P(0) = 0$.

5. (1971, A-4)*** Show that for $0 < \epsilon < 1$ the expression $(x + y)^n(x^2 - (2 - \epsilon)xy + y^2)$ is a polynomial with positive coefficients for n sufficiently large and integral. For $\epsilon = .002$ find the smallest admissible value of n.

6. (1972, B-4)** Let n be an integer greater than 1. Show that there exists a polynomial $P(x, y, z)$ with integral coefficients such that $x \equiv P(x^n, x^{n+1}, x + x^{n+2})$.

7. (1976, A-2)* Let $P(x, y) = x^2y + xy^2$ and $Q(x, y) = x^2 + xy + y^2$. For $n = 1, 2, 3, \ldots,$ let $F_n(x, y) = (x + y)^n - x^n - y^n$ and $G_n(x, y) = (x + y)^n + x^n + y^n$. One observes that $G_2 = 2Q$, $F_3 = 3P$, $G_4 = 2Q^2$, $F_5 = 5PQ$, $G_6 = 2Q^3 + 3P^2$. Prove that, in fact, for each n either F_n or G_n is expressible as a polynomial in P and Q with integer coefficients.

8. (1978, B-3)** The sequence $\{Q_n(x)\}$ of polynomials is defined by

$$Q_1(x) = 1 + x, \quad Q_2(x) = 1 + 2x,$$

and, for $m \geq 1$, by

$$\begin{aligned} Q_{2m+1}(x) &= Q_{2m}(x) + (m+1)xQ_{2m-1}(x), \\ Q_{2m+2}(x) &= Q_{2m+1}(x) + (m+1)xQ_{2m}(x). \end{aligned}$$

Let x_n be the largest real solution of $Q_n(x) = 0$. Prove that $\{x_n\}$ is an increasing sequence and that $\lim_{n\to\infty} x_n = 0$.

Hints:

1. Write down the conditions that follow from the roots being different and between 0 and 1, and check small values of a for solutions until you find one.

2. Consider first the question in one variable using continuity to limit the possibilities. Show that if the range is a finite interval, then it must be one point. Give examples for all possibilities.

3. Show that the two-variable polynomial on the left side of the equation has either a local minimum or maximum at $(0, 0)$.

4. Find the values of $P(x)$ at 1, $2 = 1^1 + 1$, $5 = 2^1 + 1$, $26 = 5^2 + 1$, etc., then consider how many zeros $P(x) - x$ has.

5. Find the coefficient of $x^{k+1}y^{n+1-k}$ of the expansion, factor out $\binom{n}{k}$, and find the minimum of the resulting expression with respect to k.

6. First consider a few small values of n and try to write x as a combination of x^n, x^{n+1}, and $x + x^{n+2}$.

7. Try to write F_5 and G_6 in terms of F_3, G_4 and G_2. Based on these, guess a recursion for G_{2n} and P_{2n-1} of the same form and show it using induction.

8. Use induction to show that $x_1 < x_2 < \cdots < x_n < 0$. To show that the limit is 0, consider the value of $Q_{2m+2}(x)$ at $x = -1/(m + 1)$.