Discrete Mathematics

Some important tools:

- The pigeon-hole principle: if we put more than kn objects into k boxes, then at least one of the boxes will have at least $n+1$ objects.

- The inclusion–exclusion principle: If $A_1, A_2, \ldots, A_n$ are sets, then the cardinality of their union is given by

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

- Some expressions can be factored. The most important ones:

$$\begin{aligned} a^2 - b^2 &= (a-b)(a+b) \\ a^3 - b^3 &= (a-b)(a^2+ab+b^2) \\ a^3 + b^3 &= (a+b)(a^2-ab+b^2) \end{aligned}$$

  In general, one can factor out $x - c$ from a polynomial $p(x)$ for a constant c if and only if c is a zero of $p(x)$, i.e., $p(c) = 0$. Using this one can see that we can factor out $a - b$ from $a^n - b^n$ for every positive integer n, and also $a + b$ from $a^n + b^n$ for every *odd* n.

1. (1966, B-4)** Let $0 < a_1 < a_2 < \cdots < a_{mn+1}$ be $mn + 1$ integers. Prove that you can select either $m + 1$ of them no one of which divides any other, or $n + 1$ of them each dividing the following one.

2. (1971, A-1)* Let there be given nine lattice points (points with integral coordinates) in three-dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.

3. (1968, A-3)* Prove that a list can be made of all subsets of a finite set in such a way that (i) the empty set is first in the list, (ii) each subset occurs exactly once, and (iii) each subset in the list is obtained either by adding one element to the preceding subset or by deleting one element of the preceding subset.

4. (1974, A-1)* Call a set of positive integers "conspiratorial" if no three of them are pairwise relatively prime. (A set of integers is "pairwise relatively prime" if no pair of them has a common divisor greater than 1.) What is the largest number of elements in any "conspiratorial" subset of the integers 1 through 16?

5. (1979, A-1)* Find positive integers n and a_1, a_2,..., a_n, such that

$$a_1 + a_2 + \cdots + a_n = 1979$$

   and the product $a_1 a_2 \cdots a_n$ is as large as possible.

6. (1980, A-2)* Let r and s be positive integers. Derive a formula for the number of ordered quadruples (a, b, c, d) of positive integers such that

$$3^r \cdot 7^s = \text{lcm}[a, b, c] = \text{lcm}[a, b, d] = \text{lcm}[a, c, d] = \text{lcm}[b, c, d].$$

The answer should be a function of r and s. (Note that $\text{lcm}[x, y, z]$ denotes the least common multiple of x, y, z.)

7. (1983, A-1)* How many positive integers n are there such that n is an exact divisor of at least one of the numbers 10^{40}, 20^{30}?

8. (1989, A-1)* How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

Hints:

1. For each i, let b_i be the length of the longest subsequence starting with a_i such that each divides the next. Use the pigeon-hole principle on the b_i's.

2. Consider the parity of the coordinates, and show that the midpoint of one of the points has integral coordinates.

3. Use induction on the number of elements in the set.

4. How many of the primes below 16 can we have in a conspiratorial set?

5. Consider small values of n first, then use that result to restrict what values a_1, a_2,... can take to maximize their product.

6. Find what form a, b, c, and d must take, then count how many ways that can happen.

7. Use the inclusion–exclusion principle.

8. Write the 1's representing powers of 10, use the geometric series formula to find a closed form, then try to factor its numerator.