Some Inequality Problems

The Arithmetic–Geometric Mean (AGM) Inequality is an essential inequality to solve many problems. It asserts that if $a_1, a_2, \ldots, a_n > 0$, then

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \ldots + a_n}{n},$$

and equality holds if and only if $a_1 = a_2 = \ldots = a_n$. Here is a generalization that has a simpler proof as it fits induction easily. If $\mu_1, \mu_2, \ldots, \mu_n > 0$, $\mu_1 + \mu_2 + \cdots + \mu_n = 1$, and $a_1, a_2, \ldots, a_n > 0$, then $a_1^{\mu_1} a_2^{\mu_2} \cdots a_n^{\mu_n} \leq \mu_1 a_1 + \mu_2 a_2 + \cdots + \mu_n a_n$, and equality holds only if $a_1 = a_2 = \cdots = a_n$.

$\frac{1}{2}$. Prove that $x + \dfrac{1}{x} \geq 2$ for every $x > 0$.

1. Prove that $(a+b)(b+c)(c+a) \geq 8abc$ for all $a, b, c > 0$.

2. Prove that $\dfrac{x^2+2}{\sqrt{x^2+1}} \geq 2$ for all real x.

3. Show that $1 + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n}} \geq 2\sqrt{n+1} - 2$ for every positive integer n.

4. Show that if $a_i > 0$ for $i = 1, 2, \ldots, n$ and $a_1 a_2 \cdots a_n = 1$, then $(1+a_1)(1+a_2)\cdots(1+a_n) \geq 2^n$.

5. Show that $\dfrac{a}{b+c} + \dfrac{b}{a+c} + \dfrac{c}{a+b} \geq \dfrac{3}{2}$ for all $a, b, c > 0$.

6. Let $a \geq 1$, and let n be a positive integer. Prove that $a^n - 1 \geq n\big(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}}\big)$.

Hints:

- $\frac{1}{2}$ Use AGM or move every term to one side and complete the square.

1. Expand and use the fact that $x^2 + y^2 \geq 2xy$.
2. Use $\frac{1}{2}$.
3. Use induction on n.
4. Use $\frac{1}{2}$, expand and group terms that are "complementary" with respect to the a_is.
5. Assume without loss of generality that a is the smallest of the three numbers. Divide numerator and denominator by a and reduce the inequality to an inequality in two variables $s, t \geq 1$. Show that the partial derivative of the expression with respect to t is positive when $t > s > 1$. Then let $t = s$ and examine the derivative when $s > 1$.
6. Divide by $a^{\frac{n}{2}}$ and apply the mean value theorem.