Determinants, Matrices, Linear Algebra

Properties of determinants: Its value does not change if a multiple of a row is added to another row (same with columns).

The characteristic polynomial of a square matrix M is $\det(M - \lambda I)$. Its roots are the eigenvalues of M, whose sum is the trace of M.

1. (1968, B-5)** Let p be a prime number. Let J be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ whose entries are chosen from $\{0, 1, 2, \ldots, p-1\}$ and satisfy the conditions $a + d \equiv 1 \pmod{p}$, $ad - bc \equiv 0 \pmod{p}$. Determine how many members J has.

2. (1969, A-2)* Let D_n be the determinant of order n of which the element in the ith row and jth column is the absolute value of the difference of i and j. Show that D_n is equal to
$$(-1)^{n-1}(n-1)2^{n-2}.$$

3. (1969, B-6)** Let A and B be matrices of size 3×2 and 2×3 respectively. Suppose that their product in the order AB is given by
$$AB = \begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{bmatrix}.$$
Show that the product BA is given by
$$BA = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}.$$

4. (1977, A-2)* Determine all solutions in real numbers x, y, z, w of the system
$$x + y + z = w,$$
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{w}.$$

5. (1978, A-2)** Let $a, b, p_1, p_2, \ldots, p_n$ be real numbers with $a \neq b$. Define $f(x) = (p_1 - x)(p_2 - x)(p_3 - x) \cdots (p_n - x)$. Show that
$$\det \begin{pmatrix} p_1 & a & a & a & \ldots & a & a \\ b & p_2 & a & a & \ldots & a & a \\ b & b & p_3 & a & \ldots & a & a \\ b & b & b & p_4 & \ldots & a & a \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ b & b & b & b & \ldots & p_{n-1} & a \\ b & b & b & b & \ldots & b & p_n \end{pmatrix} = \frac{bf(a) - af(b)}{b - a}.$$

6. (1984, A-3)*** Let n be a positive integer. Let a, b, x be real numbers, with $a \neq b$, and let M_n denote the $2n \times 2n$ matrix whose (i, j) entry m_{ij} is given by

$$m_{ij} = \begin{cases} x & \text{if } i = j, \\ a & \text{if } i \neq j \text{ and } i + j \text{ is even,} \\ b & \text{if } i \neq j \text{ and } i + j \text{ is odd.} \end{cases}$$

Thus, for example, $M_2 = \begin{pmatrix} x & b & a & b \\ b & x & b & a \\ a & b & x & b \\ b & a & b & x \end{pmatrix}$. Express $\lim_{x \to a} \det M_n / (x - a)^{2n-2}$ as a polynomial in a, b, and n, where $\det M_n$ denotes the determinant of M_n.

7. (1985, B-1)* Let k be the smallest positive integer for which there exist distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

8. (1999, B-2)** Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots.

Hints:

1. Consider cases depending on the value of a (0, 1, or something different).

2. Use row operations to get a determinant containing a lot of 0s.

3. Show that the first two rows of A are linearly independent. Solve for A in terms of B, then compute BA using that.

4. Introduce new variables for $x + y$ and xy and rewrite the equations.

5. Use induction on n. You can check first the coefficients of p_i on both sides, then assume $p_i = 0$ for all i to compare coefficients of a and b. For the induction step you can subtract a row from another one to get a lot of zeros in a row before expending the determinant.

6. Find the eigenvalues of the determinant when $x = a$, then based on that find the characteristic polynomial of M_n.

7. Show that you cannot have exactly 1 or 2 nonzero coefficients, then find an example with 3.

8. Use proof by contradiction and assume that $P(x)$ has a root with multiplicity m, where $2 \leq m < n$, so $P(x) = a(x - c)^m R(x)$, where $R(x)$ is a monic polynomial such that $R(c) \neq 0$. Show that $Q(x)$ must have c as a double root, so it is a multiple of $(x - c)^2$. Compare coefficients of x^n in the equation to get a contradiction.