Integrals and Power Series

In many cases an appropriate substitution will transform the integral into an integral very similar to the original. If the integral is on an interval $[a, b]$, it is worth trying the substitution $y = b - x$. In addition to the usual techniques to compute an integral, one can also try to find the Taylor series representation of the function and integrate term-by-term or use a couple of those terms to estimate the integral.

1. (1965, B-1)* Evaluate

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \cos^2\left\{\frac{\pi}{2n}(x_1 + x_2 + \cdots + x_n)\right\} dx_1\, dx_2 \cdots dx_n.$$

2. (1968, A-1)* Prove

$$\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx.$$

3. (1968, B-4)**** Show that if f is real-valued and continuous on $(-\infty, \infty)$ and $\int_{-\infty}^{\infty} f(x)\, dx$ exists, then

$$\int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx = \int_{-\infty}^{\infty} f(x)\, dx.$$

4. (1969, A-4)*** Show that

$$\int_0^1 x^x dx = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-n}.$$

   (The integrand is taken to be 1 at $x = 0$.)

5. (1970, B-1)* Evaluate

$$\lim_{n\to\infty} \frac{1}{n^4} \prod_{i=1}^{2n} (n^2 + i^2)^{1/n}.$$

6. (1976, B-1)** Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\left[\frac{2n}{k}\right] - 2\left[\frac{n}{k}\right]\right)$$

   and express your answer in the form $\log a - b$, with a and b positive integers.

   Here $[x]$ is defined to be the integer such that $[x] \le x < [x] + 1$ and $\log x$ is the logarithm of x to base e.

7. (1979, B-2)** Let $0 < a < b$. Evaluate

$$\lim_{t\to 0} \left\{\int_0^1 [bx + a(1-x)]^t dx\right\}^{1/t}.$$

   [The final answer should not involve any operations other than addition, subtraction, multiplication, division, and exponentiation.]

8. (1981, A-3)** Find

$$\lim_{t\to\infty}\left[e^{-t}\int_0^t\int_0^t \frac{e^x-e^y}{x-y}\,dx\,dy\right]$$

or show that the limit does not exist.

9. (1982, B-2)** Let $A(x,y)$ denote the number of points (m,n) in the plane with integer coordinates m and n satisfying $m^2+n^2\le x^2+y^2$. Let $g=\sum_{k=0}^{\infty} e^{-k^2}$. Express

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(x,y)e^{-x^2-y^2}\,dx\,dy$$

as a polynomial in g.

10. (1989, A-2)* Evaluate $\displaystyle\int_0^a\int_0^b e^{\max\{b^2x^2,a^2y^2\}}\,dy\,dx$ where a and b are positive.

11. (1997, A-3)*** Evaluate

$$\int_0^{\infty}\left(x-\frac{x^3}{2}+\frac{x^5}{2\cdot 4}-\frac{x^7}{2\cdot 4\cdot 6}+\cdots\right)$$
$$\left(1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}+\frac{x^6}{2^2\cdot 4^2\cdot 6^2}+\cdots\right)dx.$$

Hints:

1. Try the substitution $y_i = 1 - x_i$.

2. This is an easy integral that can be evaluated using regular Calculus techniques by expanding the numerator, performing long division, then integrating the resulting terms.

3. Rewrite the left-hand side as a sum of four improper integrals. In each we want to use a substitution $x = f(u)$ (like in a trigonometric substitution) such that $u = x - 1/x$, then combine the resulting integrals.

4. Rewrite x^x using the Taylor series of e^x, then integrate each term using integration by parts repeatedly.

5. Take logarithm of the product to make it a sum, then recognize it as a Riemann sum of a definite integral.

6. Recognize the limit as the Riemann sum of an integral of a function $f(x)$ on $[0, 1]$.

7. Use substitution to compute the integral, then L'Hospital's rule to find the limit.

8. Use the Taylor series of e^x to estimate the integrand, then L'Hospital's rule to find the limit.

9. Rewrite the integral as a doubly infinite sum over all integer pairs (x, y), then use polar coordinates to compute the integral.

10. Split the region into parts depending on which of b^2x^2 and a^2y^2 is larger before integrating.

11. Recognize the first factor in the integrand as a power series of a function, then multiply it out and integrate term-by-term using integration by parts, then recognize the resulting power series.