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To cite this article: Chun Kiat Tan, Jianliang Wang, Yew Chai Paw & Fang Liao (2018): Robust Linear Output Feedback Controller for Autonomous Landing of a Quadrotor on a Ship Deck, International Journal of Control, DOI: [10.1080/00207179.2018.1459859](https://doi.org/10.1080/00207179.2018.1459859)

To link to this article: <https://doi.org/10.1080/00207179.2018.1459859>



Accepted author version posted online: 02 Apr 2018.



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Publisher: Taylor & Francis



Journal: *International Journal of Control*

DOI: <https://doi.org/10.1080/00207179.2018.1459859>

Robust Linear Output Feedback Controller for Autonomous Landing of a Quadrotor on a Ship Deck

Chun Kiat Tan^a, Jianliang Wang^a, Yew Chai Paw^b, Fang Liao^c

^aNanyang Technological University, Singapore; ^bDSO National Laboratories, Singapore;

^cTemasek Laboratories, National University of Singapore, Singapore

ARTICLE HISTORY

Compiled March 29, 2018

ABSTRACT

In the ship deck landing problem of a quadrotor, the heave motion of the ship deck increases the difficulty to perform a smooth landing. Landing controllers therefore requires certain robustness with respect to ship heave motion. Typical quadrotor systems only provides relative height feedback, therefore an observer is required to estimate the relative heave rate information used by typical full state feedback controller. In this paper, a linear output feedback control consisting of a full state feedback controller and a Luenberger observer is used in a ship deck landing problem. Invariant ellipsoid method is used to formulate an estimation of a bound on the response of a generic linear output feedback controlled system subjected to external disturbances and measurement noise for guaranteed robustness. The controller and observer gains that result in a minimum bounds or invariant ellipsoid are optimized using a gradient descent iterative approach proposed in this paper where the invariant ellipsoid condition is linearized into a tractable LMI condition. This approach is applied in a numerical simulation of a quadrotor landing on a ship deck in the presence of wind disturbance and relative height measurement noise. Results are also compared with other gains. The gains selected using the proposed approach exhibits improved robustness to external disturbances and measurement noise.

KEYWORDS

Quadrotor, ship deck landing, invariant ellipsoid, linear output feedback.

1. Introduction

The use of unmanned aerial vehicle (UAV) in open seas has provided military and civilian operators with the capability to conduct reconnaissance and surveillance missions on the vast ocean efficiently. These missions are conducted far away from any land or air bases thus requiring the UAV to operate from a ship or vessel, sometimes in rough sea condition. Development and advancement in vertical takeoff and landing

(VTOL) UAV, especially quadrotors, allow for easier and safer UAV deployment, thus leading to increasing interests in using UAV onboard ships.

However, ship-wave interactions result in a pseudorandom ship motion that increases difficulty and risk during landing. During landing, the UAV needs to take into account both the relative velocity and distance between the UAV and the ship in order to achieve safe and smooth landing. This is because erratic ship motion can lead to high impact velocity, damaging the UAV or the ship.

Autonomous landing requires relative height information so that ship motion information can be included in the computation. Measurement and estimation of the relative height above the landing platform on the ship deck can be obtained by different sensors. Range sensors such as radar and lidar altimeter were used by (Arora, 2013; Garratt, 2009; Hervas, 2014) to provide reliable relative height information. In recent years, vision-based control is increasingly useful in ship deck landing applications (Herisse, 2012; Sanchez-Lopez, 2013; Xu, 2009). In addition to relative height measurement, vision sensors can also provide the flight controller with an estimate of the UAV pose relative to the ship thus localizing the UAV's lateral position with respect to the landing platform. Fusion technique (Yang, 2011) and redundancy with other sensors (Arora, 2013) are also employed. Infrared sensor is sometimes used (Yakimenko, 2002) to complement vision sensor.

Linear controllers such as linear quadratic integral (LQI), model predictive control (MPC) and loop shaping design (LSD) were analyzed and compared by (Sandino, 2011). In particular, linear quadratic method was used by (Esmailifar & Saghaei, 2009; Saghaei & Esmailifar, 2007) to design a controller for a helicopter landing on a 6DOF moving platform similar to a ship. Nonlinear controllers such as sliding mode control were also used (Hervas, 2014).

Classical linear controllers such as PID control (Garratt, 2009; Huh & Shim, 2010) were more commonly used in quadrotors and its ship deck landing operation. However, this requires meticulous tuning of the controller gains to achieve robustness with respect to ship deck motion. Nevertheless due to the simplicity of the structure of a PID or linear controller, there is a huge motivation to develop a gain tuning method that enable the selection of a suitable gain for UAV ship deck landing operation.

The invariant ellipsoid method was used on the ship deck landing problem by (Tan & Wang, 2014b,a) to obtain optimal gains for their ship landing controller. Tan further developed an optimal full state feedback linear controller based on invariant ellipsoid method for this problem. However, full state feedback controllers require relative heave rate feedback which was difficult to be obtained using sensors.

Therefore, this paper proposed an output feedback controller which uses only relative height feedback that is commonly available in quadrotor using commercial off-the-shelf sensors. The proposed controller uses a linear controller structure in the form of a PID or PD controller coupled with a Luenberger observer using only the information of relative height above the ship as the measurement output. The output is affected by measurement noise. The ship deck motion and wind disturbance are treated as external disturbances acting on the system. This approach uses the bounded-input bounded-output (BIBO) nature of a stable linear system and reduces the worst case bound on the response. The local invariant set theory is used to compute the controller and observer gains that result in a minimal bounded response of the closed-loop system in this optimal controller synthesis approach.

Invariant ellipsoid method was applied to an output feedback control problem and was analyzed and solved by (Gonzalez-Garcia, 2008). However, the solution is conservatively estimated and fails to produce solution in some cases. In the proposed

method, a path following iterative method is used to produce the optimized controller and observer gains. This method iteratively solves a linear matrix inequality (LMI) that is obtained by linearizing the full bilinear matrix inequality (BMI) with small step approximation.

The contributions of this paper are as follows:

- Formulation of a quadrotor ship deck landing problem using output feedback controller due to limitation of feedback information.
- A novel approach to obtain an optimal controller and observer gains of the output feedback problem based on linearized solution and gradient descent.

This paper is organized in the following way. The main results of the output feedback controller analyzed by the invariant ellipsoid method are given in Section 2. This is followed by Section 3 which outlines the controller and observer synthesis approach. The proposed output feedback control is applied to the ship deck landing problem in Section 4. Numerical simulation and discussions are given in Section 5 and some concluding remarks are given in Section 6.

2. Output Feedback Control using Invariant Ellipsoid Method

The invariant ellipsoid method is based on the general theory developed by (Chellaboina, 1999; Khlebnikov, 2011; Nazin, 2007). It is applied in situations such as spacecraft stabilization (Gonzalez-Garcia, 2009) and sliding mode control (Polyakov & Poznyak, 2011). Extensions to this method are developed in (Azhmyakov, 2011; Azhmyakov, 2013).

This section aims to extend the invariant ellipsoid method to an output feedback controller for a linear time-invariant (LTI) system subjected to external disturbances and measurement noises.

Consider a system

$$\dot{x} = Ax + Bu + \sum_{i=1}^N D_i w_i, \quad (1a)$$

$$y = Cx + n, \quad (1b)$$

where $x \in \mathbb{R}^n$ is the state of the system, $u \in \mathbb{R}^m$ is the system input, $w_i \in \mathbb{R}^{p_i}$ are the external disturbances, $y \in \mathbb{R}^k$ is the measured output, $n \in \mathbb{R}^k$ is the measurement noise, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{k \times n}$, $D_i \in \mathbb{R}^{n \times p_i}$ are the system matrices for $i = 1, 2, \dots, N$.

Remark 1. Multiple state disturbances that arise from different sources are represented in this system. Measurement noises for different channels are also effectively represented.

The external disturbances are bounded by

$$\|w_i\|_{F_i} < 1, \quad (2a)$$

$$\|n\|_G < 1 \quad (2b)$$

where $0 < F_i \in \mathbb{R}^{p_i \times p_i}$ and $0 < G \in \mathbb{R}^k$ for all $i = 1, 2, \dots, N$.

Remark 2. The Q -vector norm is defined as $\|x\|_Q \triangleq x^T Q x$, for any $Q > 0$.

In output feedback control, only the output y is available for feedback. Conventional approach to synthesize a dynamic controller decouples the problem into an observer and controller design problem.

A Luenberger state observer is chosen as follows,

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}), \quad (3)$$

where $\hat{x} \in \mathbb{R}^n$ is the observed state and $L \in \mathbb{R}^{n \times k}$ is the observer gain. Substituting with (1b), (3) becomes

$$\dot{\hat{x}} = A\hat{x} + Bu + LC(x - \hat{x}) + Ln. \quad (4)$$

On the other hand, the controller is formulated with the estimated state as

$$u = K\hat{x}, \quad (5)$$

where $K \in \mathbb{R}^{m \times n}$ is the controller gain.

Consider the case where $\zeta = x - \hat{x}$ is the observation error, its derivative is

$$\dot{\zeta} = (A - LC)\zeta - Ln + \sum_{i=1}^N D_i w_i. \quad (6)$$

Furthermore (1a) is re-expressed as

$$\dot{x} = (A + BK)x - BK\zeta + \sum_{i=1}^N D_i w_i. \quad (7)$$

By separation principle, the combined system of (6) and (7) is stable if and only if $A - LC$ and $A + BK$ are stable. It is also known that a stable LTI system is also bounded input bounded output stable, thus the combined system is also bounded given that w_i and n_j are bounded for all $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, k$. In the following part, the invariant ellipsoid condition is established that determine whether a given ellipsoid bounds the state trajectory when a LTI system is subjected to bounded disturbances.

Theorem 2.1 (Invariant ellipsoid condition). *For an output feedback controlled linear time-invariant system given by (1), (3) and (5), an ellipsoid defined by $\xi^T P \xi < 1$, where $\xi = (x \quad \zeta^T)^T$ and $P = \begin{pmatrix} P_1^{-1} & 0 \\ 0 & P_2 \end{pmatrix}$ is attractive and invariant if it satisfies*

$$\begin{pmatrix} \Pi_1 & -BK & 0 & D \\ * & \Pi_2 & -P_2 L & P_2 D \\ * & * & -\tau_n G & 0 \\ * & * & * & -T_d F \end{pmatrix} \leq 0,$$

where

$$\begin{aligned}\Pi_1 &= (A + BK)P_1 + (*) + \alpha P_1, \\ \Pi_2 &= P_2(A - LC) + (*) + \alpha P_2, \\ \alpha &= \sum_{i=1}^N \tau_{d,i} + \sum_{j=1}^k \tau_{n,j}, \\ D &= (D_1 \ D_2 \ \cdots \ D_N), \\ T_d &= \text{diag}(\tau_{d,1}I, \tau_{d,2}I, \cdots, \tau_{d,N}I), \\ F &= \text{diag}(F_1, F_2, \cdots, F_N).\end{aligned}$$

Remark 3. Note that $X + (*) \triangleq X + X^T$ and $\begin{pmatrix} X + (*) & Y \\ * & Z \end{pmatrix} \triangleq \begin{pmatrix} X + X^T & Y \\ Y^T & Z \end{pmatrix}$.

Proof. Consider (6), (7) and a control Lyapunov function

$$V = \begin{pmatrix} x^T & \zeta^T \end{pmatrix} \begin{pmatrix} P_1^{-1} & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} x \\ \zeta \end{pmatrix} \triangleq \xi^T P \xi, \quad (8)$$

where $0 < P_1 \in \mathbb{R}^{n \times n}$ and $0 < P_2 \in \mathbb{R}^{n \times n}$ define the minimum invariant ellipsoid, $\xi = \begin{pmatrix} x^T & \zeta^T \end{pmatrix}^T$ and $P = \begin{pmatrix} P_1^{-1} & 0 \\ 0 & P_2 \end{pmatrix}$. The time derivative of V is

$$\begin{aligned}\dot{V} &= x^T P_1^{-1} \left[(A + BK)\hat{x} - BK\zeta + \sum_{i=1}^N D_i w_i \right] \\ &\quad + \zeta^T P_2 \left[(A - LC)\zeta - Ln + \sum_{i=1}^N D_i w_i \right] + (*). \quad (9)\end{aligned}$$

For an ellipsoid given by $V = \xi^T P \xi < 1$ to be attractive and invariant, its time derivative must be $\dot{V} < 0$ for all $V = \xi^T P \xi \geq 1$. This condition is expressed as

$$z^T \begin{pmatrix} P_1^{-1}(A + BK) + (*) & -P_1^{-1}BK & 0 & P_1^{-1}D \\ * & P_2(A - LC) + (*) & -P_2L & P_2D \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} z \leq 0, \quad (10)$$

where $z = \begin{pmatrix} x^T & \zeta^T & n^T & w_1 & \cdots & w_N \end{pmatrix}^T$.

Inequality (10) is to be satisfied for

$$z^T \begin{pmatrix} P_1^{-1} & 0 & 0 & 0 \\ * & P_2 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} z \geq 1, \quad (11)$$

$$z^T \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & G & 0 \\ * & * & * & 0 \end{pmatrix} z < 1, \quad (12)$$

and

$$z^T \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & F^{(i)} \end{pmatrix} z < 1, \quad (13)$$

where

$$F^{(i)} = \text{diag}(0^{(1)}, \dots, 0^{(i-1)}, F_i, 0^{(i+1)}, \dots, 0^{(N)}) \quad (14)$$

for all $i = 1, 2, \dots, N$. (11) is derived from $\xi^T P \xi \geq 1$, (12) and (13) are derived from (2).

S-procedure states that a sufficient convex condition for $x^T A_0 x < \alpha_0$ for all x that satisfy $x^T A_i x \leq \alpha_i$ for $i = 1, 2, \dots, m$ is $x^T \left(A_0 - \sum_{i=1}^m \tau_i A_i \right) x \leq 0$ where $\alpha_0 - \sum_{i=1}^m \tau_i \alpha_i \geq 0$ and $\tau_i \geq 0$ for $i = 1, 2, \dots, m$.

Applying S-procedures to the aforementioned inequalities, we get

$$\begin{pmatrix} \Sigma_1 & -P_1^{-1}BK & 0 & P_1^{-1}D \\ * & \Sigma_2 & -P_2L & P_2D \\ * & * & -\tau_p G & 0 \\ * & * & * & -T_d F \end{pmatrix} \leq 0, \quad (15)$$

where

$$\Sigma_1 = P_1^{-1}(A + BK) + (*) + \tau P_1^{-1}, \quad (16)$$

$$\Sigma_2 = P_2(A - LC) + (*) + \tau P_2, \quad (17)$$

for some $\tau \geq \sum_{i=1}^N \tau_{d,i} + \tau_n \geq 0$.

Observe that (15) can be expressed as

$$\begin{pmatrix} \Pi_1' & -P_1^{-1}BK & 0 & P_1^{-1}D \\ * & \Pi_2 & -P_2L & P_2D \\ * & * & -\tau_n G & 0 \\ * & * & * & -T_d F \end{pmatrix} + \tau_1 \begin{pmatrix} P_1^{-1} & 0 & 0 & 0 \\ * & P_2 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} \leq 0, \quad (18)$$

where

$$\Pi'_1 = P_1^{-1}(A + BK) + (*) + \alpha P_1^{-1}, \quad (19)$$

$$\Pi_2 = P_2(A - LC) + (*) + \alpha P_2, \quad (20)$$

$$\alpha = \sum_{i=1}^N \tau_{d,i} + \tau_n, \quad (21)$$

for some $\tau_1 \geq 0$. Since $\begin{pmatrix} P_1^{-1} & 0 & 0 & 0 \\ * & P_2 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} \geq 0$, there always exists $\alpha = \sum_{i=1}^N \tau_{d,i} + \sum_{j=1}^M \tau_{n,j}$

that satisfies inequality (15) for any solution to the inequality

$$\begin{pmatrix} \Pi'_1 & P_1^{-1}BK & 0 & P_1^{-1}D \\ * & \Pi_2 & -P_2L & P_2D \\ * & * & -\tau_n G & 0 \\ * & * & * & -T_d F \end{pmatrix} \leq 0. \quad (22)$$

Hence (22) is a sufficient condition for (15) and for the ellipsoid to be attractive and invariant.

Let $T_1 = \begin{pmatrix} P_1 & 0 & 0 & 0 \\ * & I & 0 & 0 \\ * & * & I & 0 \\ * & * & * & I \end{pmatrix}$ and perform a congruent transformation¹ such that (22) becomes

$$\begin{pmatrix} \Pi_1 & -BK & 0 & D \\ * & \Pi_2 & -P_2L & P_2D \\ * & * & -\tau_n G & 0 \\ * & * & * & -T_d F \end{pmatrix} \leq 0, \quad (23)$$

where

$$\Pi_1 = P_1 \Pi'_1 P_1 = (A + BK)P_1 + (*) + \alpha P_1. \quad (24)$$

Inequality (23) is the invariant ellipsoid condition that determines whether an ellipsoid defined by P_1 and P_2 is attractive and invariant for a system given by (1), (3) and (5). \square

2.1. Existence of Solution

Theorem 2.1 states the inequality used in the determination of an invariant ellipsoid. The following theorem provides the existence condition for an invariant ellipsoid. This is required in the synthesis of the controller and observer gains.

Definition 2.2. A matrix A is considered to be α -stable if and only if the matrix $A + \alpha I_n$ is asymptotically stable.

¹A congruent transformation of a matrix A by a transformation matrix T is defined as $T^T A T$.

The definition of an α -stable matrix A is equivalent to the existence of a $P = P^T > 0$ such that $AP + (*) + 2\alpha P \leq 0$ or $PA + (*) + 2\alpha P \leq 0$. In addition, for any $Q = Q^T \geq 0$, there exists $P = P^T > 0$ such that $AP + (*) + 2\alpha P \leq -Q$ or $PA + (*) + 2\alpha P \leq -Q$. Furthermore, a matrix that is α -stable has the real part of all the eigenvalues of the state matrix A no larger than $-\alpha$. ie. $\text{Re}(\lambda_i) \leq -\alpha$ where $\lambda_i \in \text{eig}(A)$. This notion is stronger than Lyapunov stability.

Lemma 2.3. *The invariant ellipsoid condition in Theorem 2.1 is satisfied only if the matrices $A + BK$ and $A - LC$ are $\frac{\alpha}{2}$ -stable.*

Proof. The invariant ellipsoid condition in Theorem 2.1 is satisfied only if the diagonal terms are negative semi-definite. By definition, $-\tau_n G \leq 0$ and $-T_d F \leq 0$. Furthermore $\Pi_1 \leq 0$ implies that

$$(A + BK)P_1 + (*) + \alpha P_1 \leq 0. \quad (25)$$

Therefore the matrix $A + BK$ must be $\frac{\alpha}{2}$ -stable by definition.

Likewise can be proven for $\Pi_2 \leq 0$, and the matrix $A - LC$ must be $\frac{\alpha}{2}$ -stable. \square

To prove that the converse is true, the following lemma is stated.

Lemma 2.4. *For any $\frac{\alpha}{2}$ -stable matrices $A + BK$ and $A - LC$, and any $Q = Q^T \geq 0$, there exists $0 < P_1 = P_1^T \in \mathbb{R}^{n \times n}$ and $0 < P_2 = P_2^T \in \mathbb{R}^{n \times n}$ such that*

$$\begin{pmatrix} (A + BK)P_1 + (*) + \alpha P_1 & -BK \\ * & P_2(A - LC) + (*) + \alpha P_2 \end{pmatrix} \leq -Q.$$

Proof. Consider $0 \leq Q = Q^T = \begin{pmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{pmatrix}$, where $0 \leq Q_{11} = Q_{11}^T \in \mathbb{R}^{n \times n}$, $0 \leq Q_{22} = Q_{22}^T \in \mathbb{R}^{n \times n}$ and $Q_{12} \in \mathbb{R}^{n \times n}$, the inequality is expressed as

$$\begin{pmatrix} (A + BK)P_1 + (*) + \alpha P_1 + Q_{11} & -BK + Q_{12} \\ * & P_2(A - LC) + (*) + \alpha P_2 + Q_{22} \end{pmatrix} \leq 0. \quad (26)$$

Since matrix $A - LC$ is $\frac{\alpha}{2}$ -stable, there exist $P_2 = P_2^T > 0$ such that $P_2(A - LC) + (*) + \alpha P_2 + Q_{22} \leq 0$ by definition.

Therefore using the method of Shur complement, the inequality in Lemma 2.4 becomes

$$(A + BK)P_1 + (*) + \alpha P_1 + Q_{11} - R_1 \Xi_1^{-1} R_1^T \leq 0, \quad (27)$$

where $R_1 = -BK + Q_{12}$ and $\Xi_1 = P_2(A - LC) + (*) + \alpha P_2 + Q_{22} \leq 0$. By definition of a $\frac{\alpha}{2}$ -stable matrix $A + BK$, there exists $P_1 = P_1^T > 0$ such that (27) is satisfied since $Q_{11} - R_1 \Xi_1^{-1} R_1^T \geq 0$. Hence the lemma is proven. \square

With Lemma 2.4, the converse of Lemma 2.3 is stated.

Lemma 2.5. *For any $\frac{\alpha}{2}$ -stable matrices $A + BK$ and $A - LC$, there exist $0 < P_1 = P_1^T \in \mathbb{R}^{n \times n}$ and $0 < P_2 = P_2^T \in \mathbb{R}^{n \times n}$ such that the invariant ellipsoid condition in Theorem 2.1 is satisfied.*

Proof. Let $\Xi_2 = \begin{pmatrix} -\tau_n G & 0 \\ 0 & -T_d F \end{pmatrix} \leq 0$, the method of Schur complement transforms the inequality in Theorem 2.1 into

$$\begin{pmatrix} (A+BK)P_1 + (*) + \alpha P_1 & -BK \\ * & P_2(A-LC) + (*) + \alpha P_2 \end{pmatrix} - \Gamma R_2 \Xi_2^{-1} R_2^T \Gamma \leq 0, \quad (28)$$

where $R_2 = \begin{pmatrix} 0 & D \\ -L & D \end{pmatrix}$ and $\Gamma = \begin{pmatrix} I & 0 \\ 0 & P_2 \end{pmatrix}$.

Using the identity $XY^T + YX^T \leq X\Lambda X^T + Y\Lambda^{-1}Y^T$ with $X = \Gamma > 0$, $Y = I$ and $\Lambda = -R_2 \Xi_2^{-1} R_2^T$, we get $\Gamma R_2 \Xi_2^{-1} R_2^T \Gamma \leq (R_2 \Xi_2^{-1} R_2^T)^{-1} - 2\Gamma \leq (R_2 \Xi_2^{-1} R_2^T)^{-1}$. Therefore satisfying

$$\begin{pmatrix} (A+BK)P_1 + (*) + \alpha P_1 & -BK \\ * & P_2(A-LC) + (*) + \alpha P_2 \end{pmatrix} - (R_2 \Xi_2^{-1} R_2^T)^{-1} \leq 0 \quad (29)$$

implies (28).

According to Lemma 2.4, there exists $P_1 = P_1^T > 0$ and $P_2 = P_2^T > 0$ such that

$$\begin{pmatrix} (A+BK)P_1 & -BK \\ 0 & P_2(A-LC) \end{pmatrix} + (*) + \alpha \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \leq -Q \quad (30)$$

for $Q = -(R_2 \Xi_2^{-1} R_2^T)^{-1}$, hence the lemma is proven. \square

From Lemma 2.3 and 2.5, the following theorem is stated.

Theorem 2.6 (Existence condition). *There exist $0 < P_1 = P_1^T \in \mathbb{R}^{n \times n}$ and $0 < P_2 = P_2^T \in \mathbb{R}^{n \times n}$ that satisfy the invariant ellipsoid condition in Theorem 2.1 if and only if the matrices $A+BK$ and $A-LC$ are $\frac{\alpha}{2}$ -stable.*

Proof. See Lemma 2.3 and 2.5. \square

3. Controller and Observer Gain Synthesis

In this section, the controller and observer gains synthesis approach is outlined. The controller and observer gains K and L are determined such that the minimum invariant ellipsoid given by $\xi^T P \xi < 1$ is the smallest.

Remark 4. Note that the size of an ellipsoid given by $x^T P x < 1$ is determined by the size of matrix $P > 0$. A smaller matrix P^{-1} gives rise to a smaller ellipsoid. In this context, the size of P^{-1} is measured by its largest eigenvalue. Other methods of measurements such as trace and determinant are also possible.

To determine the controller and observer gains K and L that minimizes the invariant ellipsoid in Theorem 2.1 effectively, it is desirable to transform invariant ellipsoid condition from a BMI into a LMI.

An approximate method proposed by (Gonzalez-Garcia , 2008) uses a congruent transformation with $T_2 = \begin{pmatrix} P_2 & 0 & 0 & 0 \\ * & I & 0 & 0 \\ * & * & I & 0 \\ * & * & * & I \end{pmatrix}$ on the invariant ellipsoid condition and

using the identity $P_2 \Pi_1 P_2 \leq -2P_2 - \Pi_1^{-1} \leq R$ where $0 \geq R \in \Re^{n \times n}$. This results in an over-conservative estimate, and sometimes fails to produce solution over certain regions in the domain despite satisfying Theorem 2.6.

Based on Theorem 2.6, this paper proposed a novel approach to synthesize an optimal controller and observer gains K and L for the output feedback controller given in (3) and (5). In this method, the controller and observer gains K and L are improved iteratively from an initial estimate by solving n LMIs obtained from linearizing the full BMI and using a small step approximation. The initial estimates for the controller and observer gains K and L are obtained from a controller and observer sub-problem respectively. The initial estimates obtained from the proposed method yields sufficiently good result for the optimized gains with fast convergence speed.

3.1. Sub-problem: Controller Gain Synthesis

To obtain an initial controller gain K , the following system is considered.

$$\dot{x} = (A + BK)x + \sum_{i=1}^N D_i w_i. \quad (31)$$

The controller gain K that results in a minimum invariant ellipsoid for the above system is synthesized. The following theorems and lemmas are required to establish this optimization problem.

Remark 5. Note that the system considered in (31) is a full state feedback of the system in (1a).

Theorem 3.1 (Invariant ellipsoid condition for controller sub-problem). *For a system given by (31), an ellipsoid defined by $x^T P_1^{-1} x < 1$ is attractive and invariant if it satisfies*

$$\begin{pmatrix} \Psi_1 & D \\ * & -T_d F \end{pmatrix} \leq 0, \quad (32)$$

where

$$\begin{aligned} \Psi_1 &= AP_1 + BY_1 + (*) + \alpha_1 P_1, \\ \alpha_1 &= \sum_{i=1}^N \tau_{d,i}, \\ Y_1 &= K_1 P_1. \end{aligned}$$

Proof. Consider the control Lyapunov function, $V_1 = x^T P_1^{-1} x$ where $0 < P_1 \in \Re^{n \times n}$,

its time derivative is

$$\dot{V}_1 = x^T P_1^{-1} \left((A + BK)x + \sum_{i=1}^N D_i w_i \right) + (*). \quad (33)$$

For the ellipsoid given by $x^T P_1^{-1} x < 1$ to be attractive and invariant, $\dot{V}_1 < 0$ for all $x^T P_1^{-1} x \geq 1$ and $\|w_i\|_{F_i} < 1$ where $i = 1, 2, \dots, N$. By S-procedure,

$$\begin{pmatrix} \Psi_1 & D \\ * & -T_d F \end{pmatrix} \leq 0, \quad (34)$$

where

$$\Psi_1 = (A + BK)P_1 + (*) + \alpha_1 P_1, \quad (35)$$

$$\alpha_1 = \sum_{i=1}^N \tau_{d,i}. \quad (36)$$

Let $Y_1 = KP_1 \in \mathbb{R}^{m \times n}$, (34) is transformed into a LMI as in (32). Hence the theorem is proven. \square

Remark 6. Note that the derivation of Theorem 3.1 is similar to that of Theorem 2.1.

Similarly, the existence of an invariant ellipsoid is stated in the following theorem.

Theorem 3.2. *There exists $0 < P_1 = P_1^T \in \mathbb{R}^{n \times n}$ that satisfy the invariant ellipsoid condition in Theorem 3.1 if and only if the matrix $A + BK$ is $\frac{\alpha_1}{2}$ -stable.*

Proof. For necessity, observe that the invariant ellipsoid condition in Theorem 3.1 implies that $\Psi_1 \leq 0$ thus the matrix $A + BK$ is $\frac{\alpha_1}{2}$ -stable by definition.

For sufficiency, observe that $-T_d K_d \leq 0$ and using the method of Schur complement,

$$\Psi_1 + D(T_d K_d)^{-1} D^T \leq 0, \quad (37)$$

where $D(T_d K_d)^{-1} D^T \geq 0$. If the matrix $A + BK$ is $\frac{\alpha_1}{2}$ -stable, there always exists a $P_1 = P_1^T > 0$ that satisfies (37).

Hence the theorem is proven. Note that this proof is similar to that of Theorem 2.6. \square

Since the controller gain K is related to the control action, it is necessary to limit K such that the control action within the invariant ellipsoid is does not cause controller saturation. The following theorem provides the control action condition.

Theorem 3.3 (Control action condition). *For a system given by (31), the control action is $\|u\|_{Q_u} < \nu$ for all x in the invariant ellipsoid given by $x^T P_1^{-1} x \leq 1$ if and only if*

$$\begin{pmatrix} P_1 & Y_1^T \\ * & \nu Q_u^{-1} \end{pmatrix} \geq 0.$$

Proof. Define an upper bound to the control action ν as

$$\|u\|_{Q_u}^2 = \|Kx\|_{Q_u}^2 < \nu, \quad (38)$$

where $0 < Q_u = Q_u^T \in \mathbb{R}^{m \times m}$. This is rewritten as

$$x^T K^T Q_u K x < \nu. \quad (39)$$

This condition is to be satisfied for all $x^T P_1^{-1} x \leq 1$.

Using S-procedure, the following inequality is obtained.

$$K^T Q_u K \leq \tau_u P_1^{-1}, \quad (40)$$

where $0 \leq \tau_u \leq \nu$. It is noted that τ_u can take any non-negative value up to ν therefore $\tau_u P_1^{-1} \leq \nu P_1^{-1}$. Hence a sufficient condition for (40) is

$$K^T Q_u K \leq \nu P_1^{-1}, \quad (41)$$

or

$$\nu P_1 - P_1^T K^T Q_u K P_1 \geq 0 \quad (42)$$

by pre- and post-multiplying by P_1 . Furthermore the method of Schur complement gives,

$$\begin{pmatrix} P_1 & (K P_1)^T \\ * & \nu Q_u^{-1} \end{pmatrix} \geq 0, \quad (43)$$

or

$$\begin{pmatrix} P_1 & Y_1^T \\ * & \nu Q_u^{-1} \end{pmatrix} \geq 0, \quad (44)$$

where $Y_1 = K P_1$. □

Remark 7. In addition to control constraint, state and output constraints can also be applied by following through the same derivation with $K = I$ and C respectively.

The controller gain K that gives the minimum invariant ellipsoid with limited control action to this sub-problem is obtained by the following theorem.

Theorem 3.4. *The controller gain K that gives a minimum invariant ellipsoid with a compensation to the control action $\|u\|_{Q_u} < \nu$ within the ellipsoid for a system given by (31) is the solution to the following optimization:*

$$\min_{P_1 > 0, Y_1, t_1 > 0, \nu > 0} \lambda t_1 + \mu \nu$$

subjected to

$$\begin{aligned} \begin{pmatrix} \Psi_1 & D \\ * & -T_d F \end{pmatrix} &\leq 0, \\ \begin{pmatrix} P_1 & Y_1^T \\ * & \nu Q_u^{-1} \end{pmatrix} &\geq 0, \\ P_1 - t_1 I &\leq 0, \end{aligned}$$

where

$$\begin{aligned} \Psi_1 &= AP_1 + BY_1 + (*) + \alpha_1 P_1, \\ \alpha_1 &= \sum_{i=1}^N \tau_i. \end{aligned}$$

Proof. The inequalities

$$\begin{pmatrix} \Psi_1 & D \\ * & -T_d F \end{pmatrix} \leq 0, \quad (45)$$

$$\begin{pmatrix} P_1 & Y_1^T \\ * & \nu Q_u^{-1} \end{pmatrix} \geq 0, \quad (46)$$

are the invariant ellipsoid and control action conditions respectively and are a consequence of Theorem 3.1 and 3.3. Any solution must satisfy these conditions.

Furthermore, the minimum ellipsoid is determined by the minimization of λ_1 , the largest eigenvalue of P_1 . Let $t_1 \geq \lambda_1 > 0$ where $P_1 - t_1 I \leq 0$, therefore minimizing t_1 also minimizes λ_1 .

The objective function $\lambda t_1 + \mu \nu$ is weighted with the invariant ellipsoid and the control action performance objectives by $\lambda > 0$ and $\mu > 0$.

Hence the theorem is proven. \square

3.2. Sub-problem: Observer Gain Synthesis

On the other hand, an observer gain estimate L is obtained through the following system.

$$\dot{\zeta} = (A - LC)\zeta - Ln + \sum_{i=1}^N D_i w_i, \quad (47)$$

Remark 8. (47) is identical to (6), hence the optimal L necessarily satisfy the invariant ellipsoid condition.

The invariant ellipsoid condition is stated in the following theorem.

Theorem 3.5 (Invariant ellipsoid condition for observer sub-problem). *For a system*

given by (47), an ellipsoid defined by $\zeta^T P_2 \zeta < 1$ is attractive and invariant if it satisfies

$$\begin{pmatrix} \Psi_2 & -Y_2 & P_2 D \\ * & -\tau_n G & 0 \\ * & * & -T_d F \end{pmatrix} \leq 0,$$

where

$$\Psi_2 = P_2 A + Y_2 C + (*) + \alpha_2 P_2,$$

$$\alpha_2 = \sum_{i=1}^N \tau_{d,i} + \tau_n,$$

$$Y_2 = P_2 L.$$

Proof. Consider the control Lyapunov function, $V_2 = \zeta^T P_2 \zeta$ where $0 < P_2 \in \mathbb{R}^{n \times n}$, its time derivative is

$$\dot{V}_2 = \zeta^T P_2 \left((A - LC)\zeta - Ln + \sum_{i=1}^N D_i w_i \right) + (*). \quad (48)$$

For the ellipsoid given by $\zeta^T P_2 \zeta < 1$ to be attractive and invariant ellipsoid, $\dot{V}_2 < 0$ for all $\zeta^T P_2 \zeta \geq 1$, $\|w_i\|_{F_i} < 1$ and $\|n\|_G < 1$ where $i = 1, 2, \dots, N$. By S-procedure,

$$\begin{pmatrix} \Psi'_2 & -P_2 L & P_2 D \\ * & -\tau_n G & 0 \\ * & * & -T_d F \end{pmatrix} \leq 0, \quad (49)$$

where

$$\Psi'_2 = P_2 (A - LC) + (*) + \alpha_2 P_2, \quad (50)$$

$$\alpha_2 = \sum_{i=1}^N \tau_{d,i} + \tau_n. \quad (51)$$

Let $Y_2 = P_2 L \in \mathbb{R}^{n \times k}$, (49) is transformed into a LMI

$$\begin{pmatrix} \Psi_2 & -Y_2 & P_2 D \\ * & -\tau_n G & 0 \\ * & * & -T_d F \end{pmatrix} \leq 0, \quad (52)$$

where

$$\Psi_2 = P_2 A + Y_2 C + (*) + \alpha_2 P_2. \quad (53)$$

Hence the theorem is proven. \square

Similarly, the existence of an invariant ellipsoid is stated in the following theorem.

Theorem 3.6. *There exists $0 < P_2 = P_2^T \in \mathbb{R}^{n \times n}$ that satisfies the invariant ellipsoid condition in Theorem 3.5 if and only if the matrix $A - LC$ is $\frac{\alpha_2}{2}$ -stable.*

Proof. For necessity, observe that the invariant ellipsoid condition in Theorem 3.5 implies that $\Psi_2 \leq 0$ thus the matrix $A - LC$ is $\frac{\alpha_2}{2}$ -stable by definition.

For sufficiency, observe that $\begin{pmatrix} -\tau_n G & 0 \\ * & -T_d F \end{pmatrix} \leq 0$ and using the method of Schur complement,

$$\Psi_2 + P_2 R_3 \Xi_3^{-1} R_3^T P_2 \leq 0, \quad (54)$$

where $R_3 = \begin{pmatrix} -L & D \end{pmatrix}$, $\Xi_3 = \begin{pmatrix} \tau_n G & 0 \\ * & T_d F \end{pmatrix} \geq 0$. If the matrix $A - LC$ is $\frac{\alpha_2}{2}$ -stable, there always exists a $P_2 = P_2^T > 0$ that satisfies (54) as $R_3 \Xi_3^{-1} R_3^T \geq 0$.

Hence the theorem is proven. Note that this proof is similar to that of Theorem 2.6. \square

The observer gain L that gives a minimum invariant ellipsoid to this sub-problem is obtained using the following theorem.

Theorem 3.7. *The observer gain L that gives a minimum invariant ellipsoid for a system given by (47) is the solution to the following optimization:*

$$\min_{P_2 > 0, H > 0, Y_2, t_2 > 0} t_2$$

subjected to

$$\begin{pmatrix} \Psi_2 & -Y_2 & P_2 D \\ * & -\tau_n G & 0 \\ * & * & -T_d F \end{pmatrix} \leq 0, \\ \begin{pmatrix} H & I \\ I & P_2 \end{pmatrix} \geq 0, \\ H - t_2 I \leq 0.$$

where

$$\Psi_2 = P_2 A + Y_2 C + (*) + \alpha_2 P_2,$$

$$\alpha_2 = \sum_{i=1}^N \tau_{d,i} + \tau_n.$$

Proof. The inequality

$$\begin{pmatrix} \Psi_2 & -Y_2 & P_2 D \\ * & -\tau_n G & 0 \\ * & * & -T_d F \end{pmatrix} \leq 0, \quad (55)$$

is the invariant ellipsoid condition and a consequence of Theorem 3.5. Any solution must satisfy this condition.

Furthermore, the minimum invariant ellipsoid is determined by the minimization of λ_2 , the largest eigenvalue of P_2^{-1} . Define $0 < H = H^T \in \mathbb{R}^{n \times n}$ such that

$$P_2^{-1} \leq H. \quad (56)$$

By the method of Schur's complement, (56) is equivalent to

$$\begin{pmatrix} H & I \\ I & P_2 \end{pmatrix} \geq 0. \quad (57)$$

Furthermore minimizing H implies minimizing P_2^{-1} , t_2 is minimized where

$$H - t_2 I \leq 0, \quad (58)$$

and t_2 is the upper-bound for the maximum eigenvalue of H .

Hence the above optimization problem is thus justified. \square

3.3. Optimal Gain Synthesis

The optimal controller and observer gains are obtained by iteratively improving the solution. The full BMI given by the invariant ellipsoid condition in Theorem 2.1 is linearized to form a LMI by considering changes to previous solution and neglecting higher order terms. This is possible by restricting the changes in the solution to be small at each iteration step.

The initial solution is given by the controller and observer sub-problems (Theorem 3.4 and 3.7) as $K^{(1)}$ and $L^{(1)}$ respectively. These gains guaranteed that $A + BK^{(1)}$ is $\frac{\alpha_1}{2}$ -stable and $A - L^{(1)}C$ is $\frac{\alpha_2}{2}$ -stable by Theorem 3.2 and 3.6 respectively. Theorem 2.6 in turn guarantees the existence of $P_1^{(1)}$ and $P_2^{(1)}$ for the closed-loop system with controller and observer gains $K^{(1)}$ and $L^{(1)}$ if $\alpha_1, \alpha_2 \geq \alpha$. $P_1^{(1)}$ and $P_2^{(1)}$ are obtained using the following optimization.

$$\min_{P_1^{(1)} > 0, P_2^{(1)} > 0, H > 0, t_1 > 0, t_2 > 0} t_1 + t_2$$

subjected to

$$\begin{pmatrix} \Pi_1 & -BK^{(1)} & 0 & D \\ * & \Pi_2 & -P_2^{(1)}L^{(1)} & P_2^{(1)}D \\ * & * & -\tau_n G & 0 \\ * & * & * & -T_d F \end{pmatrix} \leq 0,$$

$$\begin{pmatrix} H & I \\ I & P_2^{(1)} \end{pmatrix} \geq 0,$$

$$P_1^{(1)} - t_1 I \leq 0,$$

$$H - t_2 I \leq 0,$$

where

$$\begin{aligned}\Pi_1 &= (A + BK^{(1)})P_1^{(1)} + (*) + \alpha P_1^{(1)}, \\ \Pi_2 &= P_2^{(1)}(A - L^{(1)}C) + (*) + \alpha P_2^{(1)}, \\ \alpha &= \sum_{i=1}^N \tau_{d,i} + \tau_n.\end{aligned}$$

The set of parameters $\{P_1^{(1)}, P_2^{(1)}, K^{(1)}, Y_2^{(1)} = P_2^{(1)}L^{(1)}\}$ satisfies the invariant ellipsoid condition and forms an initial estimate to the optimal controller and observer gains.

Let the set of parameters at the i^{th} iteration step be $\{P_1^{(i)}, P_2^{(i)}, K^{(i)}, Y_2^{(i)}\}$ and the parameters at the $i + 1^{th}$ iteration step be $P_1^{(i+1)} = P_1^{(i)} + \delta P_1$, $P_2^{(i+1)} = P_2^{(i)} + \delta P_2$, $K^{(i+1)} = K^{(i)} + \delta K$ and $Y_2^{(i+1)} = Y_2^{(i)} + \delta Y_2$. Assuming that δP_1 , δP_2 , δK and δY_2 are small, thus neglecting $\delta P_1 \delta K$, the invariant ellipsoid condition is rewritten as

$$\begin{pmatrix} \Pi_1^{(i+1)} & -BK^{(i+1)} & 0 & D \\ * & \Pi_2^{(i+1)} & -Y_2^{(i+1)} & P_2^{(i+1)}D \\ * & * & -\tau_n G & 0 \\ * & * & * & -T_\alpha F \end{pmatrix} \leq 0, \quad (59)$$

where

$$\begin{aligned}\Pi_1^{(i+1)} &= AP_1^{(i+1)} + BK^{(i)}P_1^{(i)} + BK^{(i)}\delta P_1 + B\delta KP_1^{(i)} + (*) \\ &\quad + \alpha P_1^{(i+1)}, \\ \Pi_2^{(i+1)} &= P_2^{(i+1)}A - Y_2^{(i+1)}C + (*) + \alpha P_2^{(i+1)}.\end{aligned}$$

The linearized invariant ellipsoid condition (59) is a LMI and can be efficiently solved by semi-definite programming.

The following optimization (reduced LMI optimization) solves for the solution at the $i + 1^{th}$ iteration step given the solution at the i^{th} iteration step.

$$\min_{\delta P_1 > 0, \delta P_2 > 0, \delta K, \delta Y_2, H > 0, t_1 > 0, t_2 > 0} t_1 + t_2$$

subjected to (59) and

$$\begin{aligned}\begin{pmatrix} H & I \\ I & P_2^{(i+1)} \end{pmatrix} &\geq 0, \\ P_1^{(i+1)} - t_1 I &\leq 0, \\ H - t_2 I &\leq 0, \\ \|\delta P_1\|_{Fr} &\leq \epsilon \|P_1^{(i)}\|_{Fr}, \\ \|\delta P_2\|_{Fr} &\leq \epsilon \|P_2^{(i)}\|_{Fr}, \\ \|\delta K\|_{Fr} &\leq \epsilon \|K^{(i)}\|_{Fr}, \\ \|\delta Y_2\|_{Fr} &\leq \epsilon \|Y_2^{(i)}\|_{Fr}.\end{aligned}$$

Remark 9. $\|A\|_{Fr} = \sqrt{\text{tr}(AA^T)}$ is the Frobenius norm of a real valued matrix A and is used to denote the change in size of the matrices $P_1^{(i)}, P_2^{(i)}, K^{(i)}, Y_2^{(i)}$ for successive iteration steps. The step size is determined by ϵ .

Furthermore, there exists a feasible domain for the reduced LMI optimization at each iteration step if the solution at each step satisfies the invariant ellipsoid condition. This is stated in the following theorem.

Theorem 3.8. *For the reduced LMI optimization given above, there exists a connected feasible domain which also satisfies the invariant ellipsoid condition for the optimization if the solution to the i^{th} iteration step satisfies*

$$\begin{pmatrix} \Pi_1 & -BK^{(i)} & 0 & D \\ * & \Pi_2 & -Y_2^{(i)} & P_2^{(i)}D \\ * & * & -\tau_n G & 0 \\ * & * & * & -T_d F \end{pmatrix} \leq 0,$$

where

$$\begin{aligned} \Pi_1 &= AP_1^{(i)} + BK^{(i)}P_1^{(i)} + (*) + \alpha P_1^{(i)}, \\ \Pi_2 &= P_2^{(i)}A - Y_2^{(i)}C + (*) + \alpha P_2^{(i)}, \\ \alpha &= \sum_{i=1}^N \tau_{d,i} + \tau_n. \end{aligned}$$

Proof. This theorem is easily proved by considering (59) as

$$\begin{pmatrix} \Pi_1 & -BK^{(i)} & 0 & D \\ * & \Pi_2 & -Y_2^{(i)} & P_2^{(i)}D \\ * & * & -\tau_n G & 0 \\ * & * & * & -T_d F \end{pmatrix} + \begin{pmatrix} \pi_1 & -B\delta K & 0 & 0 \\ * & \pi_2 & -\delta Y_2 & \delta P_2 D \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} \leq 0, \quad (60)$$

where

$$\begin{aligned} \Pi_1 &= AP_1^{(i)} + BK^{(i)}P_1^{(i)} + (*) + \alpha P_1^{(i)}, \\ \Pi_2 &= P_2^{(i)}A - Y_2^{(i)}C + (*) + \alpha P_2^{(i)}, \\ \pi_1 &= A\delta P_1 + BK^{(i)}\delta P_1 + B\delta K P_1^{(i)} + (*) + \alpha \delta P_1, \\ \pi_2 &= \delta P_2 A - \delta Y_2 C + (*) + \alpha \delta P_2, \\ \alpha &= \sum_{i=1}^N \tau_{d,i} + \tau_n. \end{aligned}$$

It is obvious that a feasible solution is obtained when $\delta P_1 = 0, \delta P_2 = 0, \delta K = 0$ and $\delta Y_2 = 0$. Furthermore by continuity of matrix positiveness, there also exist a connected neighborhood around $\delta P_1 = 0, \delta P_2 = 0, \delta K = 0$ and $\delta Y_2 = 0$ such that the solution satisfy (59) and the invariant ellipsoid condition. \square

Theorem 3.8 proves the existence of a connected feasible domain which satisfies the invariant ellipsoid condition at each iteration step for sufficiently small

$\{\delta P_1, \delta P_2, \delta K, \delta Y_2\}$. Therefore by iteratively solving the linearized BMI (which is a LMI) while limiting the variables $\{\delta P_1, \delta P_2, \delta K, \delta Y_2\}$ to be small, a locally optimized solution of the full BMI is obtained. Figure 1 highlights this equivalent gradient descent approach at which a small step is allowable for each iteration step.

The iteration procedure is stated as follows:

- (1) Obtain a set of initial controller and observer gains $\{K^{(1)}, L^{(1)}\}$ using the controller and observer sub-problems. Then solves for the initial invariant ellipsoid that bounds the closed-loop system obtaining the initial solution set $\{P_1^{(1)}, P_2^{(1)}, K^{(1)}, Y_2^{(1)} = P_2^{(1)} L^{(1)}\}$.
- (2) Iterate by solving the solution $\{P_1^{(i+1)}, P_2^{(i+1)}, K^{(i+1)}, Y_2^{(i+1)}\}$ for the i^{th} iteration step using the reduced LMI optimization.
- (3) Check for the feasibility of the $i + 1^{th}$ solution $\{P_1^{(i+1)}, P_2^{(i+1)}, K^{(i+1)}, Y_2^{(i+1)}\}$ using Theorem 3.8. If the $i + 1^{th}$ solution $\{P_1^{(i+1)}, P_2^{(i+1)}, K^{(i+1)}, Y_2^{(i+1)}\}$ is found to be infeasible, re-iterate with a reduced step size ϵ . Otherwise, repeat step 2 for the next iteration.
- (4) Stop iteration if there is no feasible solution for a sufficiently small step size or if the changes in solution for successive iteration steps are small.

Remark 10. The initial estimates of controller and observer gains are solution of LMI problems, hence are optimal with respect to each problem. Therefore the locally optimal solution obtained by the proposed gradient descent approach is guaranteed to have reasonable performance.

4. Quadrotor Ship Deck Landing

In this section, the output feedback controller discussed in Section 2 with gains synthesized in Section 3 is implemented in an quadrotor in a ship deck landing application. This section is divided into quadrotor modelling and controller synthesis.

4.1. Modelling

The full six degree of freedom (6dof) dynamics of a quadrotor is decoupled into planar and heave motion. Interested reader can refer to Tan for the formulation. The planar motion is controlled to track the ship (Tan, 2016) while the heave motion is controlled to achieve a soft landing. It is also noted that a quadrotor is sufficiently small in size, hence the rolling and pitching motion of the ship has limited impact on landing operation.

The heave dynamics of a quadrotor is the primary concern of landing operation and is written as

$$m\ddot{h} = T \cos \phi \cos \theta - mg + mw_d, \quad (61)$$

where h is the height of the quadrotor above the mean sea level, m is the mass of the quadrotor, T is the thrust acting on the quadrotor, ϕ and θ are the roll and pitch angles respectively, g is the gravitational acceleration and w_d is the air drag per unit mass due to wind.

This solution seeks to formulate a controller for a quadrotor to land on a heaving

ship, therefore the error

$$e_1 = h - H_s - H, \quad (62)$$

where H_s is the height of the ship above the mean sea level and H is the reference height above the mean sea level, is considered. Its second derivative is

$$\ddot{e}_1 = \frac{T}{m} \cos \phi \cos \theta - g + w_d - \ddot{H}_s - \ddot{H}. \quad (63)$$

An equivalent control input

$$u = \frac{T}{m} \cos \phi \cos \theta - g - \ddot{H}, \quad (64)$$

is chosen such that (62) is simplified as

$$\ddot{e}_1 = u + w_d + w_s, \quad (65)$$

where $w_s = \ddot{H}_s$.

The external disturbances due to ship motion w_s and wind drag w_d are bounded as

$$\|w_s\|_{K_s} < 1, \quad (66a)$$

$$\|w_d\|_{K_d} < 1, \quad (66b)$$

where $0 < K_s \in \mathfrak{R}$ and $0 < K_d \in \mathfrak{R}$ are parameters indicating the size of disturbances. These bounds on the external disturbances w_s and w_d are estimated using historical data.

Expressed in the state space form, (65) becomes

$$\dot{e} = Ae + Bu + B(w_d + w_s), \quad (67)$$

where $e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ is the state, $e_2 = \dot{e}_1$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the state matrix and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the control matrix.

The measured output of the quadrotor landing system is the height above the ship deck, y which is

$$y = Ce + w_n, \quad (68)$$

where $C = (1 \ 0)$ is the output matrix and w_n is the measurement noise with

$$\|w_n\|_{K_n} < 1, \quad (69)$$

where $0 < K_n \in \mathfrak{R}$ is a parameter indicating the size of noise. Similarly K_n is estimated using the sensor precision.

Remark 11. This quadrotor model allows the formulation of the lateral controller to be independent of the heave controller.

4.2. Controller Synthesis

Using the proposed output feedback control structure, the observer is formulated as

$$\dot{\hat{e}} = A\hat{e} + Bu + L(y - C\hat{e}), \quad (70)$$

where $L = (L_1 \ L_2)^T$ is the observer gain and \hat{e} is the estimated state.

Similarly, the controller is formulated as

$$u = K\hat{e}, \quad (71)$$

where $K = (K_1 \ K_2)$ is the controller gain.

The controller and observer gains K and L are synthesized using the method proposed in Section 2 with the following parameters,

$$D = (B \ B), \quad (72)$$

$$T_d = \begin{pmatrix} \tau_s I & 0 \\ 0 & \tau_d I \end{pmatrix}, \quad (73)$$

$$G = K_n, \quad (74)$$

$$F = \begin{pmatrix} K_s & 0 \\ 0 & K_d \end{pmatrix}. \quad (75)$$

5. Numerical Simulation

The ship deck landing problem is numerically simulated using Matlab. This section presents the simulation results and discussion. The ship deck heave model is first described, followed by the parameters used in the simulation. Then the landing operation of a quarotor on a ship deck is simulated. The results are then discussed followed by comparison with other gain values.

5.1. Ship Deck Heave Model

The ship deck is assumed to exhibit heave motion from the interaction with the sea wave motion. A ship with a unity Response Amplitude Operator (RAO) is assumed without any loss of generality.

A sea wave model is derived using the energy spectrum and the ship deck is assumed to perform only heave motion in sync with the wave. The sea wave elevation is approximated into a Fourier series consisting of sinusoidal waves at different energy content (amplitude), frequency and phase (Brekken, 2012; Tristan & Blanke, 2002).

The energy content at a particular sea state depends on the wave height spectrum. The modified Pierson-Moskowitz (PM) spectrum is used in this simulation. The parameters used in the modified PM spectrum depends on the significant wave height and period which can be obtained from the sea state according to

$$S(\omega) = H_{1/3}^2 T_1 \frac{0.11}{2\pi} \left(\frac{\omega T_1}{2\pi} \right)^{-5} \exp \left[-0.44 \left(\frac{\omega T_1}{2\pi} \right)^{-4} \right], \quad (76)$$

where $S(\omega)$ is the wave height spectrum, $H_{1/3}$ is the significant wave height, T_1 is the mean wave period and ω is the frequency. $H_{1/3}$ and T_1 is dependent on the steady wind speed and sea state.

Figure 2 shows the PM spectrum generated for sea state 6 at a constant wind speed of 30kts.

Based on the wave height spectrum, a discretization can be performed to obtain a time series for the wave elevation.

$$d = \sum_{j=1}^N \sqrt{2S(\omega_j)\Delta\omega_j} \sin(\omega_j t + \epsilon_j), \quad (77)$$

where d is the wave elevation, N is the number of waves to be modelled, $\Delta\omega_j$ is the discretization interval, ω_j and ϵ_j are the frequency and phase of the j^{th} wave respectively.

The time series of the wave elevation is simulated for two different initial conditions at sea state 6 and a constant wind speed of 30kts and is shown in Figure 3 to highlight the pseudorandomness of the wave elevation at a discretization of $N = 500$.

5.2. Simulation Results

Numerical simulation is performed on a generic quadrotor using parameters obtained from (Gong, 2012). The parameters are listed in Table 1.

The quadrotor is initially placed at a 10m height above the mean sea level ($h = 10$). The landing pad is located 1m above the sea level. The controller is commanded to bring and maintain the quadrotor at a spot 10m above the landing location for 50s. In this phase, the quadrotor attempts to follow the ship deck heave motion while limiting the relative motion error to within the invariant ellipsoid. Thereafter the height of the quadrotor is reduced gradually in a landing phase. A smooth sigmoid function is chosen as the reference path,

$$H = H_0 + \frac{(H_f - H_0)}{1 + \exp\left(\frac{-6(2(t-t_0)-\Delta t)}{\Delta t}\right)}, \quad (78)$$

where H_0 , H_f are the initial and final height, t_0 is the time when the height is commanded to decrease and Δt is the time duration of the descend. In this simulation, the following parameters are used. $H_0 = 10$, $H_f = 1$, $t_0 = 50$ and $\Delta t = 30$.

In addition to the ship deck motion, external disturbance due to wind is also included. The constants K_s and K_d indicating the bounds are $K_s = \frac{1}{9}$ and $K_d = 1$ respectively. Similarly, measurement noise is added to the relative height measurement (e_1) with bound is affected by $K_n = 100$.

The optimal controller and observer gains K and L are obtained using $\tau_n = \tau_d = \tau_s = 5$. The initial estimate of the controller gain $K^{(1)} = (-442 \quad -42)$ is obtained by solving the controller sub-problem, while the initial estimate of the observer gain $L^{(1)} = (31 \quad 237)^T$ is obtained by solving the observer sub-problem. The optimal controller and observer gains $K = (-338 \quad -34)$ and $L = (96 \quad 914)^T$ are obtained after 127 iterations with an objective function $t_1 + t_2 = 72.3$ from an initial value of $t_1 + t_2 = 151.6$.

The effect of the gradient descent approach in decreasing the objective function

value each iteration is shown in Figure4(a). The same approach is also performed with another initial estimate obtained from pole placement with $K^{(1)} = (-72 \ -17)$ and $L^{(1)} = (17 \ 72)^T$. Its objective function value changes with iteration is shown in Figure4(b).

The initial estimate using the controller and observer sub-problems exhibits a higher convergence rate requiring only 127 iterations as compared to 1363 iterations. Furthermore, both initial estimates converged to a same K and L as a result of choosing the initial estimates belonging to a connected feasible region. This shows the local optimality of the proposed iterative process.

The simulation results of the ship deck landing performed with the optimal controller are shown in Figure5 and 6.

From Figure5(b), it is shown that the quadrotor is able to perform smooth and soft landing on the ship deck with the proposed control structure and optimized gains. Figure5(c) further support this claim by showing that the tracking error e_1 is maintained sufficiently small at about within 0.1m after initial stabilization. This is achieved with a thrust input within the actuation limits as shown in Figure5(d). In addition, ship heave, wind disturbance and measurement noise are also shown in Figure5(a), 5(f) and 5(e). The quadrotor response as a function of $V = \xi^T P \xi$ is also shown in Figure6. The ellipsoid given by P is shown to be attractive and invariant ($V < 1$).

5.3. Comparison using other Gains

The performance of the quadrotor landing operation using the optimal controller and observer gains obtained from the proposed approach in section 3 is compared with gains obtained using pole placement method. This serves to highlight the superiority of the optimal controller and observer gains and its effectiveness when used in disturbed and noisy systems. The gains selected are $K = (-12 \ -7)$ and $L = (7 \ 12)^T$.

The simulation results using the pole placed controller and observer gains are shown in Figure7.

The performance parameters, relative height of the quadrotor above the ship deck and the tracking error, are shown in Figure7(b) and 7(c) respectively. It is obvious that the performance using non-optimal controller and observer gains is not acceptable. Large bounds on the tracking error e_1 (about 1.5m) leads to high impact velocity and potential damage to the quadrotor upon landing. This highlights the importance of using the invariant ellipsoid method to select a controller and observer gains in relation to the landing performance. The controller and observer gains that produce a minimum invariant ellipsoid results in a small height variation with respect to the ship deck hence ensuring safety during landing.

6. Concluding Remarks

In this paper, the problem of a quadrotor landing on a ship deck is solved using a linear output feedback controller consisting of a full state feedback controller and a Luenberger observer. In addition to the ship deck heave motion, the wind disturbance and measurement noise are also considered in this formulation. The invariant set theory based invariant ellipsoid method is used to determine an optimal controller and observer gains such that the closed-loop system has a minimum bounded response in the presence of external disturbances and noises. A general formulation of the problem and

gain synthesis approach is given. The gain synthesis approach linearizes the invariant ellipsoid condition to form a LMI that is easily solved in an optimization iteratively from an initial estimation obtained from a controller and an observer sub-problems. Finally, numerical simulation of the quadrotor landing on a ship deck is performed using the gains obtained from the proposed approach and compared with gains obtained from pole placement method. The optimal gains obtained from the proposed method exhibits significantly better performance. This highlights the importance of using the invariant ellipsoid method in the selection of gains.

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Figure 1. The gradient descent approach in the iterative solution of the full BMI invariant ellipsoid condition. The objective function is reduced at each iteration step with a limit set on the improvements in the solution such that a LMI condition is solved. Iteratively, the local optimal solution is obtained within a connected feasible domain containing the initial condition.

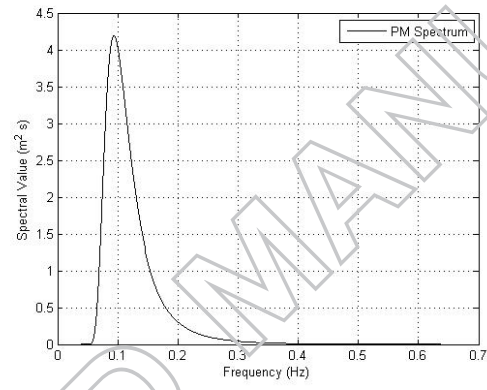
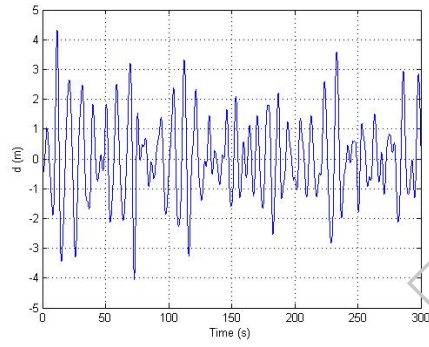
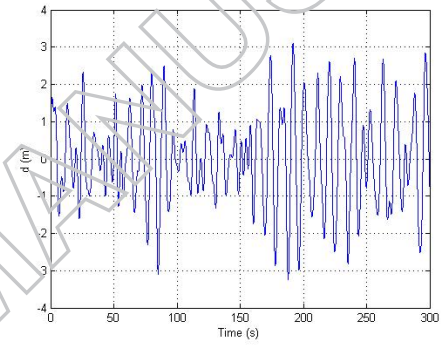


Figure 2. The modified Pierson-Moskowitz spectrum at sea state 6 and a constant wind speed of 30kts.

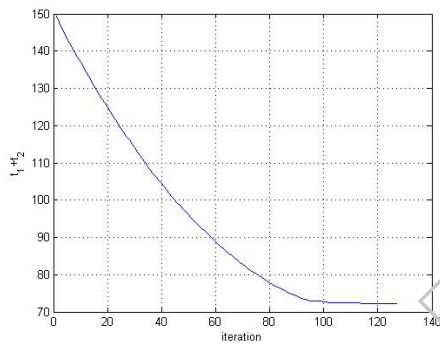


(a) Case I

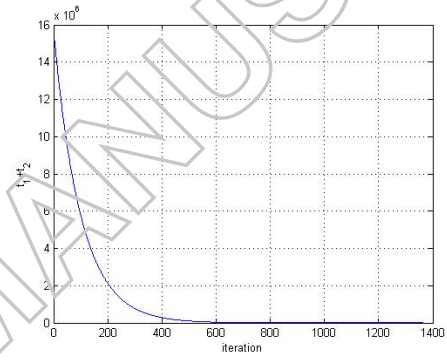


(b) Case II

Figure 3. Simulation of the ship deck motion at sea state 6 at a constant wind speed of 30kts for two different initial conditions. The simulation shows that different waveforms are obtained by differing the initial conditions.



(a) Proposed initial estimate



(b) Initial estimate from pole placement

Figure 4. The decrease in objective function value with the number of iterations using a gradient descent approach using the initial estimate from controller and observer sub-problems (Figure4(a)) and pole placement (Figure4(b)).

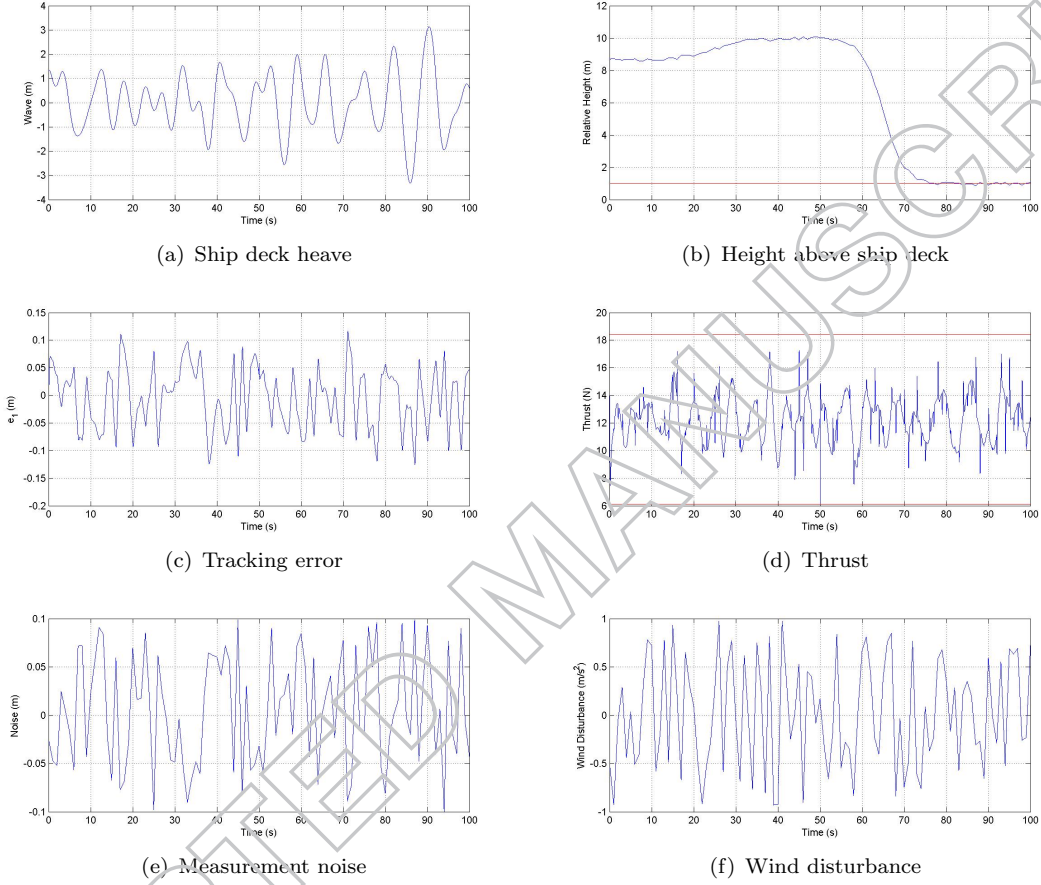


Figure 5. Simulation results of a quadrotor landing on a ship deck. Figure5(a) shows the heave motion of the ship deck with the mean sea level at $d = 0$. Figure5(b) shows the height of the quadrotor relative to the ship deck. The landing pad is located at 1m above the ship deck. Figure5(c) shows the tracking error e_1 of the quadrotor, a small tracking error indicates that the quadrotor is able to track the ship deck motion. Figure5(d) shows the thrust input of the quadrotor and its limits. Figure5(e) shows the measurement noise acting on the relative position. Figure5(f) shows the wind disturbance acting on the quadrotor

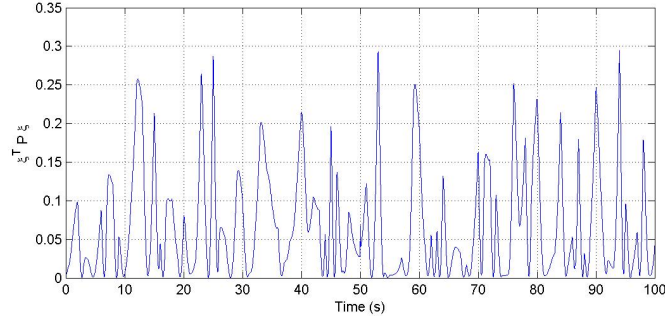


Figure 6. The value of $V = \xi^T P \xi$. $V < 1$ indicates that the state of the system remains inside the ellipsoid. Furthermore, this ellipsoid is shown to be attractive and invariant.

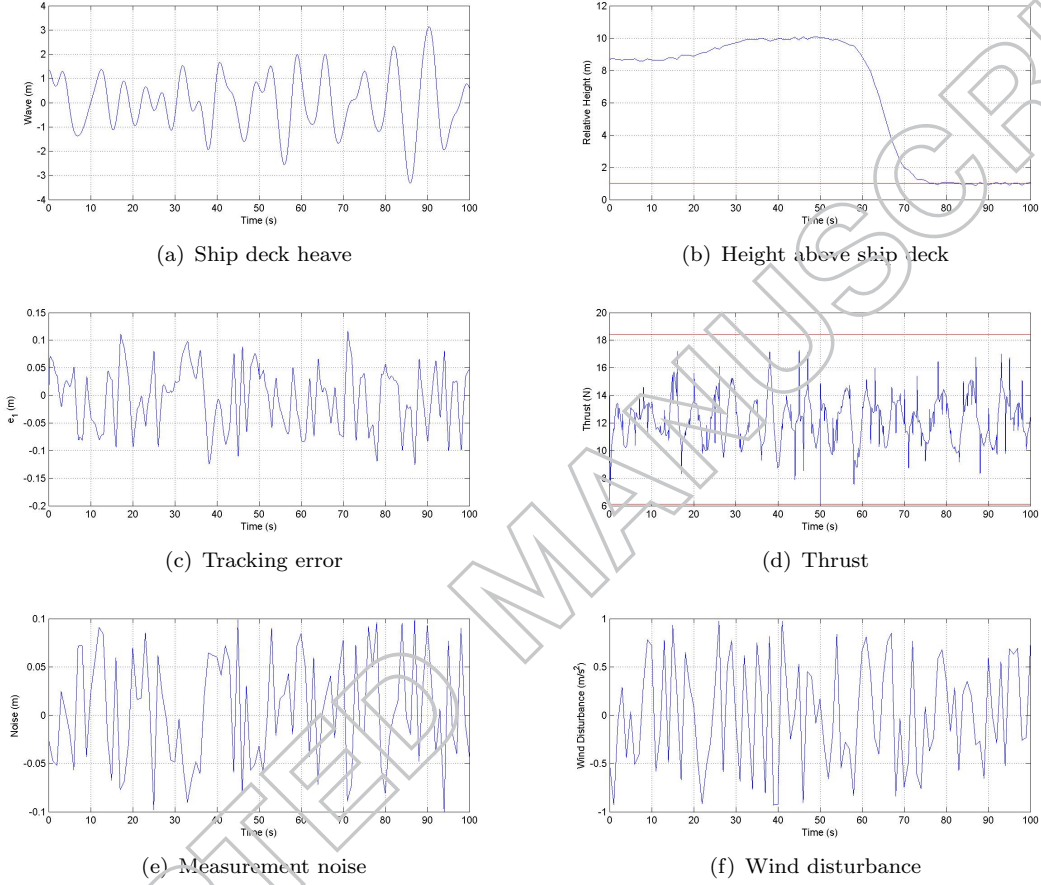


Figure 7. Simulation results of a quadrotor landing on a ship deck. Figure7(a) shows the heave motion of the ship deck with the mean sea level at $d = 0$. Figure7(b) shows the height of the quadrotor relative to the ship deck. The landing pad is located at 1m above the ship deck. Figure7(c) shows the tracking error e_1 of the quadrotor, a small tracking error indicates that the quadrotor is able to track the ship deck motion. Figure7(d) shows the thrust input of the quadrotor and its limits. Figure7(e) shows the measurement noise acting on the relative position. Figure7(f) shows the wind disturbance acting on the quadrotor

Table 1. A list of quadrotor parameters used in the simulation (Gong, 2012).

Parameters	Values
Mass (m)	1.25kg
Distance between the rotor and center (l)	0.5m
Moment of inertia to x -axis (J_{xx})	$8.1 \times 10^{-3} \text{Nms}^{-2}$
Moment of inertia to y -axis (J_{yy})	$8.1 \times 10^{-3} \text{Nms}^{-2}$
Moment of inertia to z -axis (J_{zz})	$14.2 \times 10^{-3} \text{Nms}^{-2}$
Lift factor k_l	$54.2 \times 10^{-6} \text{Ns}^2$
Drag factor k_d	$1.1 \times 10^{-6} \text{Ns}^2$