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Observing the state of non-linear dynamic systems†

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In this paper observers which approximately reconstruct state variables of classes of non-linear systems are presented. The question of convergence of the reconstruction error is examined for special cases. Simulation results of the performance of some non-linear observers are also included.

1. Introduction and general formulation

The problem of approximately reconstructing state variables of dynamic systems when only incomplete output information is available has been the concern of a number of investigators during the past several years. An introduction to the subject with a rather complete set of references is contained in Luenberger (1971). For linear systems the state reconstruction problem bears a close relationship to the concepts of controllability, observability, and the design of linear regulators.

In this paper the state reconstruction problem for certain non-linear systems is considered and the question of convergence of the reconstruction error is examined for special case. Consider the non-linear nth-order system

$$\dot{x} = f(t, x), \tag{1}$$

where it is assumed that $f(\cdot, \cdot)$ is such that (1) yields a unique solution starting from any initial state $x(0)$. Measurements of the form

$$y = h(t, x), \tag{2}$$

where $h(\cdot, \cdot)$ is an $m \times 1$ vector, are to be used as the input to an observer dynamic system,

$$\dot{z} = g(t, y, z), \tag{3}$$

where z is an $r \times 1$ vector. It is of interest to find conditions under which the state of the observer system (3) can be made to approach some transformation (possibly non-linear) of the system state

$$\Phi(x) \tag{4}$$

where $\Phi(x)$ is an r vector with components

$$\phi_i(x), \quad i = 1, 2, \ldots, r.$$

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Primarily, of course, one is interested in cases where the transformation $\Phi(\cdot)$ is invertible so that if error

$$e = z - \Phi(x) \tag{5}$$

approaches zero, then one could 'reconstruct' the entire state vector by a transformation of the observer state

$$\hat{x} \approx \Phi^{-1}(z).$$

Here it is shown first that if

$$\left(\frac{\partial \Phi}{\partial x}\right) f(t, x) = g(t, h(t, x), \Phi(x)) \tag{6}$$

for all x, t, then the error e satisfies

$$\dot{e} = g(t, y, \Phi(x) + e) - g(t, y, \Phi(x)). \tag{7}$$

To verify (7), note that

$$\dot{e} = g(t, y, z) - \frac{\partial \Phi}{\partial x} f(t, x). \tag{8}$$

Using (5) and (6) in (8) yields (7).

Next, conditions under which the error will approach zero are examined. Write (7) as

$$\dot{e} = g_1(t, e) - g_2(t), \tag{9}$$

where

$$\left.\begin{aligned} g_1(t, e) &= g(t, h(t, x(t)), \Phi(x(t)) + e), \\ g_2(t) &= g(t, h(t, x(t)), \Phi(x(t))). \end{aligned}\right\} \tag{10}$$

Assume that $g_1(t, e)$ can be expanded in a Taylor series about $e = 0$,

$$g_1(t, e) = g_2(t) + G_1(t)e + G_2(t, e)e, \tag{11}$$

where $G_1(t)$ is the Jacobian matrix of g_1 evaluated at $e = 0$, t and $G_2(t, e)$ is a matrix with the property

$$\|G_2(t, e\| / \|e\| \to 0 \quad \text{uniformly in } t, \text{ as } \|e\| \to 0. \tag{12}$$

Now, if

$$\|G_1(t)\| \leqslant c \quad \text{for all } t, \text{ where } c > 0, \tag{13}$$

and if the linear system

$$\dot{e} = G_1(t)e \tag{14}$$

is uniformily asymptotically stable, then Example 9 of Kalman and Bertram (1960) reveals that the equilibrium state $e = 0$ of (9) is locally uniformily asymptotically stable.

Although the foregoing conditions insure only local asymptotic stability, the results of the following sections indicate that global asymptotic stability can be established in certain cases and that estimates of the extent of asymptotic stability can also be obtained.

Non-linear observers may be constructed for forced non-linear systems by a technique that has been used for forced linear systems. Suppose the system to be observed is

$$\dot{x} = f(t, x) + f_2(u), \tag{15}$$

where u is an sth-order control vector, and suppose the observer system is given by

$$\dot{z} = g(t, y, z) + g_2(u), \tag{16}$$

where the measurements y are of the form (2). Suppose T is an $r \times n$ matrix such that

$$Tf(t, x) = g(t, h(t, x), Tx) \tag{17}$$

for all x, t, and

$$Tf_2(u) = g_2(u) \quad \text{for all } u. \tag{18}$$

Then it is easily shown that the error, $e = z - Tx$, satisfies (7) with $\Phi(x) = Tx$, and that $e = 0$ is locally asymptotically stable if $G_1(t)$ and $G_2(t, e)$ possess the properties indicated above. Hence, in the remainder of this paper the discussion is confined to unforced systems.

2. Special cases

In this section the discussion is restricted to unforced systems of the form

$$\dot{x} = Ax + \mu f(x), \tag{19}$$

where A is an $n \times n$ constant matrix and μ is a scalar†. Measurements

$$y = Hx, \tag{20}$$

where H is an $m \times n$ matrix, are assumed to be available to an observer whose dynamics are

$$\dot{z} = Bz + g(z) + Ky. \tag{21}$$

It is easily shown that if there is a matrix T such that

$$TA - BT = KH, \tag{22 a}$$

$$\mu Tf(x) = g(Tx) \quad \text{for all } x, \tag{22 b}$$

then the error $e = z - Tx$ satisfies

$$\dot{e} = Be + g(Tx + e) - g(Tx). \tag{23}$$

To obtain (22), form $\dot{z} - T\dot{x}$ using (19) and (20), and then apply conditions (22 *a*) and (22 *b*).

The convergence of the observation error (23) will be examined for the case $T = I$. It is known (Luenberger 1971) that if the pair (H, A) is completely observable, and if $T = I$, then the eigenvalues of B,

$$B = A - KH, \tag{24}$$

can be made to correspond to the eigenvalues of any $n \times n$ real matrix by suitable choice of the matrix K.

† $f(x)$ is assumed to contain only terms of second-order or higher.

Assume that K has been chosen so that eigenvalues of B all have negative real part. Then for any positive definite matrix Q, there is a unique positive definite P such that

$$PB+B'P=-Q. \tag{25}$$

Consider the positive definite function

$$V(e)=e'Pe \tag{26}$$

and evaluate the derivative of V along the solution of (23) (with $T=I$),

$$\dot{V}(e)=-e'Qe+2e'P[g(x+e)-g(x)]. \tag{27}$$

Using (22 *b*), with $T=I$, in (27) yields

$$\dot{V}(e)=-e'Qe+2\mu e'P[f(x+e)-f(x)]. \tag{28}$$

Now, if $f(x)$ satisfies a Lipschitz condition,

$$\|f(x)-f(y)\|\leqslant k\|x-y\| \tag{29}$$

for all x and y, in a certain region R containing the origin, where k is a constant independent of x and y, and if there is a constant C_0 such that

$$C_0 I\leqslant Q,$$

then

$$\dot{V}(e)\leqslant -C_0\|e\|^2+2\mu k\|Pe\|\,\|e\| \tag{30}$$

for all e in R, where $\|e\|$ can be taken as the Euclidean norm, $\left(\sum_{i=1}^{n} e_i^2\right)^{1/2}$. If $\|P\|=K$, then

$$\dot{V}(e)\leqslant(-C_0+2\mu kK)\|e\|^2. \tag{31}$$

Hence, if

$$C_0>2\mu kK, \tag{32}$$

then $e=0$ is asymptotically stable and every solution of (23) in R tends to the origin as $t\to\infty$.

Note that the above result implies that if the non-linear term in the system to be observed satisfies a Lipschitz condition in a region R containing the origin, and if there is an initial observation error e that is confined to R, then the reconstruction error $e=z-x$ that results from the use of the reconstructor,

$$\dot{z}=Az+K[y-Hz]+\mu f(z) \tag{33}$$

will eventually approach zero if (32) is satisfied.

If $f(x)$ satisfies a global Lipschitz condition and if the parameter μ is sufficiently small so that (32) is satisfied, then, despite *any* initial observation error, the state of the reconstructor (33) will approach that of the system (19).

The following special case is of some interest: consider the problem of designing a state reconstructor of the form

$$\dot{z}=Bz+Ky+k(y) \tag{34}$$

for the system (19) with measurements (20). It is easily seen that if (22 *a*) is satisfied and if

$$\mu T f(x) = k(Hx) \quad \text{for all } x, \tag{35}$$

then the error, $e = z - Tx$, satisfies the homogeneous equation

$$\dot{e} = Be. \tag{36}$$

The restriction (35) implies that if all state variables that appear non-linearly in (19) can be measured, then, as one would intuitively expect, a reconstructor can always be built that will yield an asymptotically stable observation error.

In the next section some examples are given to illustrate the type of performance that can be obtained with observers of the form (21). The question of reducing the order of the observer dynamic system is examined in § 4.

3. Examples

Case I

Consider the system

$$\left.\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -a x_2 |x_2| \end{aligned}\right\} \tag{37}$$

with measurement

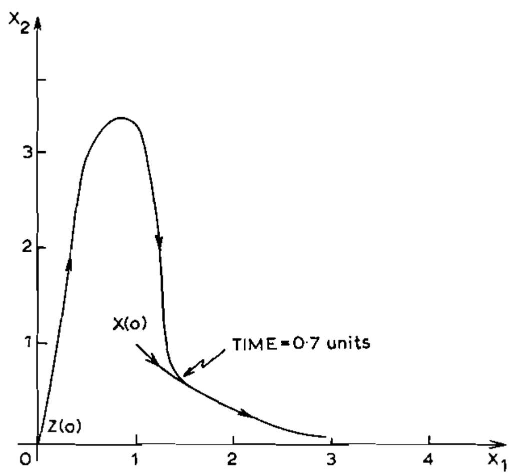
$$y = x_1. \tag{38}$$

A second-order observer of the form (21), with $T = I$, for this system is

$$\left.\begin{aligned} \dot{z}_1 &= -20 z_1 + z_2 + 20 y \\ \dot{z}_2 &= -100 z_1 - a z_2 |z_2| + 100 y. \end{aligned}\right\} \tag{39}$$

Figure 1 indicates the result of a digital computer simulation of the motion of

Fig. 1



Performance of a second-order non-linear observer. System to be observed: $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_2|x_2|$, $y = x_1$. Observer dynamics: $\dot{z}_1 = -20z_1 + z_2 + 20y$, $\dot{z}_2 = -100z_1 - z_2|z_2| + 100y$.

the system (37) and the observer (39) for $a=1$. The process to be observed (37) started at the initial point $x_1(0)=1$, $x_2(0)=1$. The observer (39) started at the initial point $z_1(0)=0$, $z_2(0)=0$. The trajectories of (37) and (39) indicate the convergence of the solution of (39) to that of (37). Since the measurement (38) provides exact knowledge of the state x_1, one would expect that a first-order observer could be constructed to provide an estimate of a linear combination of x_1 and x_2, and that the measurement (38) and the output of the first-order observer could be combined to yield an estimate of x_2. Section 4 contains a technique for accomplishing the reduction of observer dynamic dimension.

Case II

Consider the norm-invariant system

$$\left.\begin{aligned}\dot{x}_1&=x_2x_1^2+x_2,\\ \dot{x}_2&=-x_1^3-x_1,\end{aligned}\right\} \tag{40}$$

with measurement

$$y=x_1. \tag{41}$$

A second-order observer of the form (21), with $T=I$, for this system is

$$\left.\begin{aligned}\dot{z}_1&=-5z_1+z_2+5y+z_1^2z_2,\\ \dot{z}_2&=-11z_1-z_1^3+10y.\end{aligned}\right\} \tag{42}$$

Figure 2 shows the phase-plane motion of system (40) and the observer (42). The state of the norm-invariant system (40) is seen to orbit the origin. The observer (42) was started in the zero state and converged to the state of the norm-invariant system at a rate indicated in fig. 3. After approximately 1.3 time units the second component z_2 of the observer state is very close to the corresponding component of the norm-invariant system.

4. Reduction of observer dynamic order

In this section unforced systems of the form (19) with measurements (20) are again considered. It is shown below that if the following six conditions are satisfied :

(1) H has rank m†,
(2) the pair (H, A) is completely observable,
(3) there is a unique $(n-m)\times n$ matrix T such that (22 a) is satisfied,
(4) the $n\times n$ matrix

$$\begin{array}{r}n-m\,\updownarrow\\ m\,\updownarrow\end{array}\begin{bmatrix}T\\ \hdashline H\end{bmatrix}$$

has an inverse

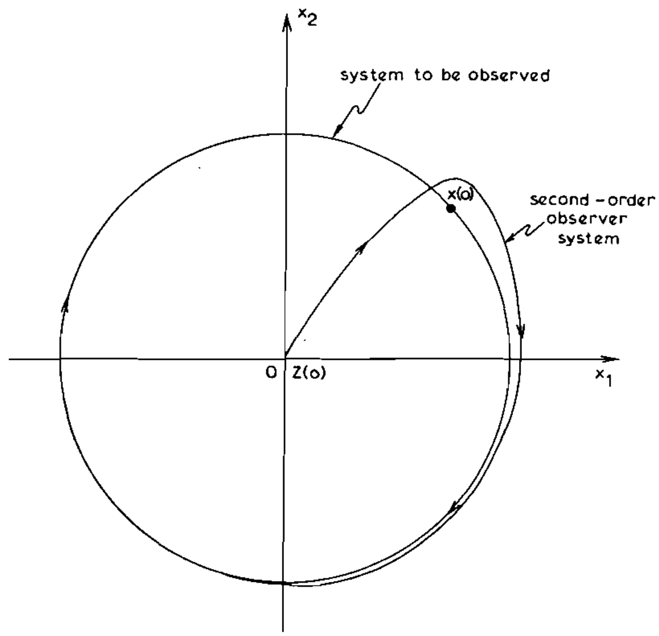
$$[L_1 \,\vdots\, L_2]=\begin{bmatrix}T\\ \hdashline H\end{bmatrix}^{-1},$$

(5) there is a function $g(z, y)$ such that

$$\mu Tf(x)=g(Tx, Hx)\quad \text{for all } x,$$

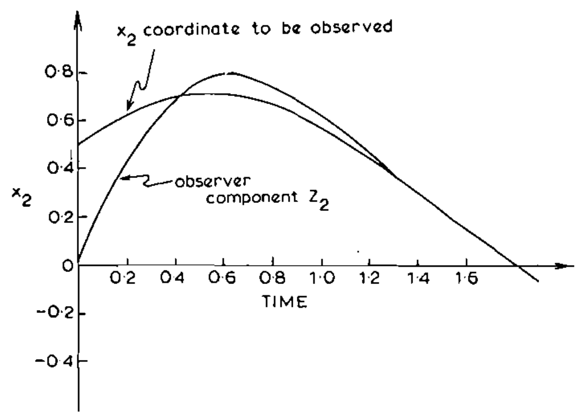
† This can be assumed without loss of generality.

Fig. 2



Observing the state of a norm-invariant system.

Fig. 3



Convergence of second-order observer of a norm-invariant system.

(6) $g(z, y)$ satisfies a uniform Lipschitz condition,

$$\|g(z_1, y)-g(z_2, y)\| \leqslant k\|z_1-z_2\| \tag{43}$$

for all y and for z_1 and z_2 in a region R containing the origin, where k is independent of z and y, then the $(n-m)$th-order observer

$$\dot{z}=Bz+Ky+g(z, y) \tag{44}$$

can be constructed so that the error $e=z-Tx$, satisfying

$$\dot{e}=Be+g(Tx+e,\ Hx)-g(Tx,\ Hx) \tag{45}$$

is asymptotically stable in R.

This result is established as follows : conditions (1) and (2) above guarantee that if B has all of its eigenvalues different from those of A, then (3) can be satisfied. If (4) and (5) hold, then the function $g(z,\ y)$ can be constructed from

$$g(z,\ y)=\mu Tf(L_1z+L_2y). \tag{46}$$

Condition (6) guarantees the asymptotic stability of (45) by an argument similar to that of § 3.

By combining the $(n-m)$th-order observer state z and the m measurements y as follows :

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{array}{c} n-m\updownarrow \\ m\updownarrow \end{array} \begin{bmatrix} T \\ \hline H \end{bmatrix} x + \begin{bmatrix} e \\ \hline 0 \end{bmatrix}, \tag{47}$$

an estimate of the state x can be obtained as

$$\hat{x}\approx L_1z+L_2y. \tag{48}$$

This is illustrated in the following example : consider Case I of § 4, where now

$$T=[t_{11},\ t_{12}]. \tag{49}$$

Using the above conditions, a first-order observer of the form

$$\dot{z}=bz+ky+g(z,\ y) \tag{50}$$

can be constructed where

$$\left.\begin{aligned} -bt_{11}&=k, \\ t_{11}&=bt_{12}. \end{aligned}\right\} \tag{51}$$

Hence, if $k=1$ and if we choose $b=-1$, then

$$\left.\begin{aligned} t_{11}=1,\quad t_{12}&=-1, \\ g(z,\ y)&=-a(z-y)|z-y|, \end{aligned}\right\} \tag{52}$$

and the first-order observer is

$$\dot{z}=-z+y-a(z-y)|z-y|. \tag{53}$$

From (48), an estimate of the state x_2 is then

$$\hat{x}_2\approx -z+y. \tag{54}$$

5. Conclusion

In this paper it was shown that observer dynamic systems can be constructed for classes of non-linear systems that have some state variables not available for direct measurement. In § 2 it was shown that for non-linear time-varying systems the error between a non-linear transformation of the system state (4) and the state of the observer (3) can be made to approach zero if the error at time $t=0$ is sufficiently small.

For the special case of autonomous systems of the form (19) it was shown that if linear and non-linear algebraic eqns. (22) are satisfied, and if the system non-linearities satisfy Lipschitz conditions, then the reconstruction error approaches zero in the same finite region. Global asymptotic stability of the reconstruction error can be established in special cases.

It was also shown that if the dynamics of the observer includes a non-linear transformation of the available measurements, as in (44), then reduction of the observer dynamic order is possible.

When an observer of the types described above is used to approximate a control law $u=F(x)$ as $\hat{u}=F(\hat{x})$, where

$$\hat{x}=L_1z+L_2y$$

in the following closed-loop system :

$$\begin{aligned}\dot{x}&=Ax+\mu f(x)+DF(\hat{x}),\\ \dot{z}&=Bz+g(z)+KHx+TDF(\hat{x}),\\ \hat{x}&=L_1z+L_2Hx,\end{aligned}$$

then the stability of the closed-loop system using the non-linear observer can be established as in Luenberger (1966). However, in this case, because the reconstruction error may not be globally asymptotically stable, the asymptotic stability of the $\{x, z\}$ system can be established only in a region R containing the origin.

A number of other problems related to the performance of sub-optimum feedback control systems employing non-linear observers are the subject of current research.

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