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# Observers for Lipschitz Nonlinear Systems

Rajesh Rajamani

***Abstract*—This paper presents some fundamental insights into observer design for the class of Lipschitz nonlinear systems. The stability of the nonlinear observer for such systems is not determined purely by the eigenvalues of the linear stability matrix. The correct necessary and sufficient conditions on the stability matrix that ensure asymptotic stability of the observer are presented. These conditions are then reformulated to obtain a sufficient condition for stability in terms of the eigenvalues and the eigenvectors of the linear stability matrix. The eigenvalues have to be located sufficiently far out into the left half-plane, and the eigenvectors also have to be sufficiently well-conditioned for ensuring asymptotic stability. Based on these results, a systematic computational algorithm is then presented for obtaining the observer gain matrix so as to achieve the objective of asymptotic stability.**

***Index Terms*— Conditioning of eigenvectors, Lipschitz nonlinearity, nonlinear systems, observer design.**

## I. Introduction

In this paper, we present some new results on designing observers for the class of nonlinear systems described by

$$\dot{x} = Ax + \Phi(x, u) \tag{1a}$$

$$y = Cx \tag{1b}$$

where $\Phi(x, u)$ is a Lipschitz nonlinearity with a Lipschitz constant $\gamma$, i.e.,

$$\|\Phi(x, u) - \Phi(\hat{x}, u)\|_2 \leq \gamma \|x - \hat{x}\|. \tag{1c}$$

To begin with, note that any nonlinear system of the form $\dot{x} = f(x, u)$ can be expressed in the form of (1a), as long as $f(x, u)$ is differentiable with respect to $x$. To ensure that (1c) is satisfied, many nonlinearities can be regarded as Lipschitz, at least locally. For instance, the sinusoidal terms usually encountered in many problems in robotics are all globally Lipschitz. Even terms like $x^2$ can be regarded as Lipschitz, provided we know that the operating range of $x$ is bounded. Thus, the class of systems covered by this paper is fairly general, with the linearity assumption being made only on the output vector $y$.

The observer will be assumed to be of the form

$$\dot{\hat{x}} = A\hat{x} + \Phi(\hat{x}, u) + L[y - C\hat{x}]. \tag{2}$$

The estimation error dynamics are then seen to be given by

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + [\Phi(x, u) - \Phi(\hat{x}, u)] \tag{3}$$

where $\tilde{x} = x - \hat{x}$.

## II. Background Results

This section presents results available in literature for the above class of nonlinear systems. Comments on the practical use of these results and on their accuracy are included. The following sections then develop new results and a systematic procedure for observer design for the same class of systems.

We first present Thau's well-known result in this area [14].

*Theorem 1 [14]:* If a gain matrix $L$ can be chosen such that

$$\gamma < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \tag{4a}$$

where $(A - LC)^T P + P(A - LC) = -Q$, then (2) yields asymptotically stable estimates for (1).

The ratio in (4a) is maximized when $Q = I$ [10]. The problem is then reduced to that of choosing $L$ to satisfy

$$\gamma < \frac{1}{2\lambda_{\max}(P)} \tag{4b}$$

where $(A - LC)^T P + P(A - LC) = -I$.

However, choosing the observer gains so as to satisfy (4b) is not straightforward. For instance, merely placing the eigenvalues of $(A - LC)$ far into the left half-plane will not ensure that (4b) is satisfied. There is no clear relation between the eigenvalues of $(A - LC)$ and $\lambda_{\max}(P)$.

Thau's result, therefore, merely serves as a method for checking stability *once the gain matrix $L$ has already been chosen*. It is not useful from the point of view of the observer design task.

The observer design problem has been considered by several other researchers in literature. Raghavan [11] has proposed the following design algorithm for choosing $L$ to ensure stability.

For some small $\varepsilon$ check if the Riccati equation

$$AP + PA^T + P\left(\gamma^2 I - \frac{1}{\varepsilon} C^T C\right)P + I + \varepsilon I = 0 \tag{5a}$$

has a positive definite solution $P$. If it does, then a choice of

$$L = \frac{1}{2\varepsilon} PC^T \tag{5b}$$

is shown to stabilize the error dynamics (3).

However, the algorithm does not succeed for all observable $(A, C)$ and unfortunately does not provide insights on what conditions the matrix $(A - LC)$ must satisfy for ensuring stability. We have seen that placing the eigenvalues of $(A - LC)$ far into the left half-plane certainly does not do the job.

Zak's [15] suggested design procedure is directly related to the matrix $(A - LC)$.

A choice of $L$ such that

$$\sigma_{\min}(A - LC) > \gamma \tag{6}$$

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will ensure $\gamma < 1/2\lambda_{\max}(P)$. The singular values of $(A-LC)$ do indeed play a role in the stability of the observer. The above result in general is, however, *incorrect* as demonstrated by the following counterexample.

*Counterexample 1:* Let

$$(A-LC)=\begin{bmatrix}0 & 1\\ -\varepsilon^2-1 & -2\varepsilon\end{bmatrix}, \qquad E=\begin{bmatrix}2\varepsilon & 0\\ 0 & 2\varepsilon\end{bmatrix}.$$

Let $\Phi(x, u) = Ex$. Then $\gamma = 2\varepsilon$

$$\sigma_{\min}(A-LC) = \sqrt{1+3\varepsilon^2+\frac{\varepsilon^4}{2}-\frac{1}{2}\sqrt{\varepsilon^8+36\varepsilon^4+12\varepsilon^6+16\varepsilon^2}}.$$

Choosing $\varepsilon$ small enough, we can easily ensure that (6) is satisfied. But the matrix $(A-LC+E)$

$$(A-LC+E)=\begin{bmatrix}2\varepsilon & 1\\ -\varepsilon^2-1 & 0\end{bmatrix}$$

is clearly unstable, since the sum of eigenvalues is equal to the trace $2\varepsilon$!

The inaccuracy of the above result is intuitively understandable. The singular values of a matrix determine how close to singular the matrix is. A matrix could be far from being singular but still have its eigenvalues close to the imaginary axis. This distinction becomes clear in light of the new result presented in Theorem 2.

## III. Necessary and Sufficient Conditions on the Matrix $(A-LC)$

A sufficient condition specifically related to the matrix $(A-LC)$ for asymptotic stability of the observer is presented and proved in Theorem 2 below.

*Theorem 2:* For (1), with $(A, C)$ observable and $\Phi(x, u)$ Lipschitz in $x$ with a Lipschitz constant $\gamma$, the observer given by (2) is asymptotically stable if $L$ can be chosen so as to ensure that $(A-LC)$ is stable and

$$\min_{\omega \in R^+} \sigma_{\min}(A-LC-j\omega I) > \gamma. \tag{7}$$

Here $\omega \in R^+$ implies $\omega \geq 0$.

*Proof:* The proof of this theorem is done in three parts.

*Part 1:* If $\min_{\omega\in R^+} \sigma_{\min}(A-LC-j\omega I) > \gamma$, then there exists $\varepsilon > 0$ such that the matrix

$$H=\begin{bmatrix}(A-LC) & \gamma^2 I\\ -I-\varepsilon I & -(A-LC)^T\end{bmatrix} \tag{8}$$

has no imaginary axis eigenvalues.

*Part 2:* If the matrix $H$ has no imaginary axis eigenvalues and if $(A-LC)$ is stable, then there exists a symmetric positive definite solution $P = P^T$ to the Riccati equation

$$(A-LC)^T P + P(A-LC) + \gamma^2 PP + I + \varepsilon I = 0. \tag{9}$$

*Part 3:* The existence of a positive definite matrix $P$ satisfying (9) ensures that the observer (2) for the system (1) is asymptotically stable.

*Proof of Part 1:* Since the singular values of a matrix are continuous functions of the elements of the matrix and since $\sigma_{\min}(A-LC-j\omega I) \to \infty$ as $\omega \to \infty$, there exists a finite $\omega_1 \in R^+$ such that

$$\min_{\omega\in R^+} \sigma_{\min}(A-LC-j\omega I) = \sigma_{\min}(A-LC-j\omega_1 I) = \gamma_{\min}.$$

Hence, for all $\omega$, we have $(A-LC-j\omega I)^*(A-LC-j\omega I) \geq \gamma_{\min}^2 I$, where $^*$ indicates the Hermitian matrix.

From (7), $\gamma < \gamma_{\min}$. Choose $\varepsilon$ such that

$$\gamma^2(1+\varepsilon) < \gamma_{\min}^2 \tag{10}$$

to obtain

$$(A-LC-j\omega I)^*(A-LC-j\omega I) > \gamma^2 I + \gamma^2\varepsilon I. \tag{11}$$

The eigenvalues $\lambda$ of the Hamiltonian matrix $H$ are given by (12), as shown at the bottom of the page.

We shall now prove Part 1 by the method of contradiction. Let us assume that the matrix $H$ does have imaginary axis eigenvalues. Since $H$ is a real matrix, its eigenvalues occur in complex conjugate pairs. Without loss of generality, we can assume that an imaginary axis eigenvalue is represented by $j\omega$ where $\omega \in R^+$. Substituting $\lambda = j\omega$ in (12), we have

$$\det\{\gamma^2+\gamma^2\varepsilon I - [(A-LC)-j\omega I]^* \cdot [(A-LC)-j\omega I]\} = 0.$$

This contradicts (11). Therefore, the matrix $H$ cannot have any purely imaginary eigenvalues.

*Proof of Part 2:* We will use the following well-known result from $H_\infty$ theory.

If the matrix

$$G =: \begin{bmatrix}A & R\\ Q & -A^T\end{bmatrix}$$

has no imaginary axis eigenvalues, $R$ is either positive semidefinite or negative semidefinite and $(A, R)$ is stabilizable, then there exists a symmetric solution to the algebraic Riccati equation

$$A^T X + XA + XRX - Q = 0.$$

(for proof, see [4]). Furthermore, if $A$ is stable, then $X$ is positive definite.

In our case, $R = \gamma^2 I$ and so the fact that $H$ has no imaginary axis eigenvalues is sufficient to ensure that there exists a symmetric solution to our Riccati equation (9). Since $(A-LC)$ is stable, this solution is also positive definite.

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$$\det\begin{bmatrix}\lambda I-(A-LC) & -\gamma^2 I\\ I+\varepsilon I & \lambda I+(A-LC)^T\end{bmatrix}=0$$

i.e.,

$$\det\left\{\begin{bmatrix}\lambda I-(A-LC) & -\gamma^2 I\\ I+\varepsilon I & \lambda I+(A-LC)^T\end{bmatrix}\begin{bmatrix}I & 0\\ \frac{1}{\gamma^2}(\lambda I-(A-LC)) & I\end{bmatrix}\right\}=0$$

i.e.,

$$\det\begin{bmatrix}0 & -\gamma^2 I\\ I+\varepsilon I+\frac{1}{\gamma^2}(\lambda I+(A-LC)^T)(\lambda I-(A-LC)) & \lambda I+(A-LC)^T\end{bmatrix}=0$$

or

$$\det\{\gamma^2+\gamma^2\varepsilon I+[\lambda I+(A-LC)^T][\lambda I-(A-LC)]\}=0 \tag{12}$$

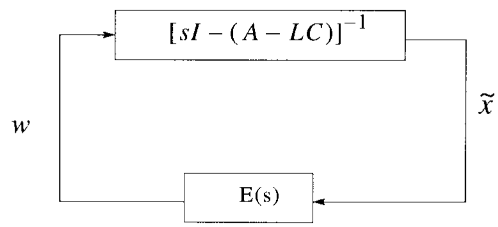


Fig. 1. Standard feedback loop for linear time-invariant systems.

*Proof of Part 3:* Suppose there exist matrices $L$ and $P$ which satisfy (9). Let this choice of $L$ be used in the observer (2) for state estimation of the system given by (1). Consider the Lyapunov function candidate

$$V = \tilde{x}^T P \tilde{x}. \tag{13}$$

Its derivative is

$$\begin{aligned}\dot{V} &= \tilde{x}^T[(A-LC)^T P + P(A-LC)]\tilde{x} \\ &\quad + 2\tilde{x}^T P[\Phi(x, u) - \Phi(\hat{x}, u)]. \end{aligned}\tag{14}$$

We have $2\tilde{x}^T P[\Phi(x, u) - \Phi(\hat{x}, u)] \le 2\|P\tilde{x}\|\, \|\Phi(x, u) - \Phi(\hat{x}, u)\|$.

Using the Lipschitz property of $\Phi(x, u)$, $2\tilde{x}^T P[\Phi(x, u) - \Phi(\hat{x}, u)] \le 2\gamma\|P\tilde{x}\|\, \|\tilde{x}\|$.

Using

$$2\gamma\|P\tilde{x}\|\, \|\tilde{x}\| \le \gamma^2 \tilde{x}^T PP\tilde{x} + \tilde{x}^T \tilde{x} \tag{15}$$

(14) then becomes

$$\dot{V} \le \tilde{x}^t[(A-LC)^T P + P(A-LC) + \gamma^2 PP + I]\tilde{x} \tag{16}$$

i.e., $\dot{V} < 0$ and so the system is asymptotically stable.

*Necessity:* We showed that (7) is *sufficient* to ensure stability of the error dynamics (3). Here we state and prove two theorems which show that (7) is in fact a tight bound and is also *necessary* in the following sense.

*Theorem 3:* If the observer gain matrix $L$ is chosen such that

$$\min_{\omega \in R+} \sigma_{\min}(A - LC - j\omega I) \le \gamma \tag{17}$$

then there exists at least one matrix $E \in C^{n\times n}$ such that the function $\Phi(x, u) = Ex$ has a Lipschitz constant $\gamma$ and the error dynamics (3) are unstable.

*Proof:* From (17), since the singular values of a matrix are continuous functions of its elements, there exists $\omega_1 \ge 0$ such that

$$\sigma_{\min}(A - LC - j\omega_1 I) < \gamma_{\min}$$

with $\gamma_{\min} < \gamma$.

Choose $\varepsilon > 0$ such that $\gamma_{\min} + \varepsilon < \gamma$.

Since $\sigma_{\min}(A - LC - j\omega_1 I) < \gamma_{\min}$, by definition of the singular value, there exists a matrix $F \in C^{n\times n}$ with $\|F\|_2 < \gamma_{\min}$ such that $(A - LC - j\omega_1 I + F)$ is singular.

Let $E = F + \varepsilon I$. Then $\|E\| < \gamma$ and the matrix $(A - LC + E - \varepsilon I - j\omega_1 I)$ is singular.

This implies that the matrix $(A - LC + E)$ has an eigenvalue of $\varepsilon + j\omega_1$. Hence the matrix $(A - LC + E)$ is unstable.

*Theorem 4:* If the observer gain matrix $L$ is chosen such that (17) is true, then for every initial condition $\tilde{x}(0)$, there exists at least one time-varying matrix $E \in R^{n\times n}$ such that the nonlinearity $\Phi(x, t, u) = E(t)x$ has a Lipschitz constant $\gamma$ and the error dynamics (3) are unstable.

*Proof:* Consider the feedback system shown below in Fig. 1. The transfer function of the upper block between $\tilde{x}$ and $w$ is

$$\frac{\tilde{x}}{w} = [sI - (A - LC)]^{-1}. \tag{18}$$

Condition (17) implies that the gain of this block is

$$\|[sI - (A - LC)]^{-1}\| \ge \frac{1}{\gamma}.$$

By the small gain theorem which is necessary and sufficient for linear time-invariant systems, there exists a dynamic matrix $E(s)$ with a gain $\|E(s)\| \le \gamma$ such that the closed-loop system is unstable. Note that the closed-loop system under such feedback is given by

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + w(t)$$

with

$$W(s) = E(s)\tilde{X}(s). \tag{19}$$

For every given initial condition $\tilde{x}(0)$, this means that there exists a nonlinearity $\Phi(x, t, u) = E(t)\tilde{x}$ of Lipschitz constant $\gamma$ such that the observer error dynamics (3) are unstable.

*Relation to Results from $H_\infty$ Theory:* The condition $\min_{\omega\in R+} \sigma_{\min}\ (A - LC - j\omega I) > \gamma$ can be rewritten as

$$\|[sI - (A - LC)]^{-1}\|_\infty < \frac{1}{\gamma}. \tag{20}$$

For the system

$$\begin{aligned} \dot{x} &= Ax + Iu + Iw \\ y &= Cx \\ z &= Ix \end{aligned} \tag{21}$$

(20) is equivalent to the minimization of the $H_\infty$ norm of the transfer function between $z$ and $w$, with $y$ being the measurement and $u$ the control input. However, standard solutions from $H_\infty$ theory, for instance from [3], are not directly applicable to the above system. For instance, (21) does not satisfy the assumptions $D_{12}^T D_{12} = I$ and $D_{12}^T C = 0$ upon which the results of [3] are based. Results in [7], [12], and [13] have contributed to the relaxation of such assumptions. Based on the type of approach (linear matrix inequalities or algebraic Riccati equations), there is a substantial difference in the type of existence conditions obtained in these results. The present work takes an approach involving eigenvector selection which is completely different from $H_\infty$ results.

## IV. Eigenvalues and Eigenvectors of $(A - LC)$

In the last section we derived a sufficient condition that the matrix $(A - LC)$ should satisfy for asymptotic stability and also showed that this sufficient condition is in fact a tight bound. That result, however, does not tell us *how* one would choose $L$ to satisfy that condition. The following theorem, inspired by the well-known Bauer–Fike's theorem in Linear Algebra [6] relates the results of the previous section to the eigenvalues and eigenvectors of $(A - LC)$. This theorem also gives a very good insight into why the stability of the error dynamics do not depend in a straightforward way on just the eigenvalues. The theorem also explains why the fact that $(A, C)$ is observable is not sufficient to guarantee the existence of a stable observer.

*Theorem 5:* If all the eigenvalues $\lambda$ of $(A - LC)$ are chosen such that

$$\mathrm{Re}(-\lambda) > K_2(T)\gamma \tag{22}$$

where $(A - LC) = T\Lambda T^{-1}$ and $K_2(T)$ is the condition number of the matrix $T$, then

$$\min_{\omega\in R+} \sigma_{\min}(A - LC - j\omega I) > \gamma. \tag{23}$$

*Proof:* We are given that (22) is satisfied. Suppose (23) is not satisfied.

Then

$$\begin{aligned}
&\min_{\omega \in R+} \sigma_{\min}(A - LC - j\omega I) \\
&\quad = \min_{\omega \in R+} \sigma_{\min}(T\Lambda T^{-1} - j\omega I) \\
&\quad \geq \min_{\omega \in R+} \sigma_{\min}(T)\sigma_{\min}(T^{-1})\sigma_{\min}(j\omega I - \Lambda) \\
&\quad \geq \frac{1}{K_2}(T) \min_{l} \operatorname{Re}(-\lambda_l)
\end{aligned}$$

i.e., $\gamma K_2(T) \geq \min_l \operatorname{Re}(-\lambda_l)$ which is a contradiction.

*Interpretation:* It is not enough to place the eigenvalues of $(A - LC)$ far into the left half-plane. One also has to ensure that the matrix $T$, i.e., the matrix of the eigenvectors of $(A - LC)$, is well-conditioned. Observability is, therefore, not sufficient to ensure that a stable observer exists. The condition number of the matrix $T$ depends on several factors including the location of the eigenvalues [the maximum condition number of any $(A - \lambda_i I)$], the distance to unobservability of the pair $(A, C)$, and the condition number of $C$. For the analogous state feedback case involving $(A - BK)$, results in [2] show that a lower bound on the condition number of $T$ is given essentially by the reciprocal of the product of the above three terms. For a given set of eigenvalues, one can in general say that the larger the distance to unobservability, the smaller the condition number. Thus, we not only need the pair $(A, C)$ to be observable but also need its "distance to unobservability" to be large.

## V. An Algorithm for Stable Observer Design

In the case of a single output system, the location of the eigenvalues of $(A - LC)$ uniquely determine the matrix $L$. In the case of multi-output systems, however, one can achieve the desired closed-loop pole configuration with many different $L$. The extra degrees of freedom available can be utilized to choose that value of $L$ which gives the lowest condition number for $T$. Further, in many applications, one may not be particular about where exactly the eigenvalues of $(A - LC)$ should lie. It may be sufficient if they lie in some designated area of the complex plane. In that case one can develop an algorithm to choose both the eigenvalues as well as the eigenvectors so as to achieve the best possible conditioning.

Several researchers have worked on the problem of achieving good conditioning while using state feedback [1], [2], [5], [8], [9], though for an entirely different objective. It is well known that minimizing the condition number of the closed-loop matrix leads to low control inputs, good robustness, and good transient response in the case of a linear time-invariant system.

Most of the methods for improving conditioning assume the location of the closed-loop poles to be fixed and then work on finding the optimal $L$ to achieve this pole configuration. Recently, [5] and [8] have worked on problems where the location of the closed-loop poles is itself flexible. Reference [5] develops a method of choosing a set of eigenvalues from a larger allowable set so as to achieve the best possible conditioning. In the case where the allowable set can be an entire region of the complex plane, [5] develops a gradient-based method to continuously change the locations of the eigenvalues so as to keep reducing the condition number. This method is designed to converge to a local minimum of the condition number.

For purposes of the observer design problem discussed in this paper, the algorithm in [5] was modified so as to maximize the ratio $\min_l [\operatorname{Re}(-\lambda_l)/K_2(T)]$. This gradient-based algorithm for choosing $L$ has been coded into MATLAB in the form of a user friendly package. The user supplies the matrices $A$ and $C$ and an initial guess for the eigenvalues to the package. The package automatically continues iterations until a local maximum of $\gamma_1 = \min_{\lambda_l} \operatorname{Re}(-\lambda_l)/K_2(T)$ is reached. The package then returns this maximum value and the corresponding value of $L$ to the user. The observer designed using this $L$ is guaranteed to be stable for all nonlinearities with Lipschitz constant of magnitude less than $\gamma_1$. Note that the $L$ obtained by this algorithm is locally optimum. If the user-supplied initial eigenvalues are changed significantly, the $L$ matrix calculated by the algorithm could vary.

The MATLAB code is available upon request from the author. The efficacy of the algorithm is demonstrated by the following numerical example. The initial guess for $\Lambda_0$ seems reasonable but leads to very bad conditioning. The algorithm converges in a few steps to a $\Lambda$ that yields stability for much larger Lipschitz constants.

*Example 1:*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let us start with the initial choice of closed-loop eigenvalues as

$$\Lambda = [-10 + 0.1i, \ -10 - 0.1i].$$

The condition number of the matrix $T$ is found to be 1010.1! This means that the maximum Lipschitz constant for which the observer could be guaranteed to be stable was $\gamma = 10/1010.1 = 0.01$. After just seven iterations, the algorithm converged to a condition number for $T$ of 12.823 with the location of closed-loop poles being

$$\Lambda = [-6.2839 + 5.3911i, \ -6.2839 - 5.3911i].$$

This means that the observer can now be guaranteed to be stable for a Lipschitz constant of $\gamma = 6.2839/12.823 = 0.49$. The corresponding $L$ is found to be $[69.5523 \ 11.5679]^T$.

## VI. Concluding Remarks

In this paper we considered observer design for the class of systems with Lipschitz nonlinearities. We made an extensive study of stability and obtained some good insights into the conditions needed for stability. A systematic computational procedure was developed to enable asymptotically stable observer design for systems in this class.

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# Equivalence of $n$-Dimensional Roesser and Fornasini–Marchesini Models

S. A. Miri and J. D. Aplevich

***Abstract*— An equivalent Roesser model is obtained for an arbitrary Fornasini–Marchesini model in $n$-dimensions and vice-versa. Both the regular and singular cases are covered.**

***Index Terms*—Linear systems, $n$-D systems, realizations.**

## I. Introduction

Two important two-dimensional (2-D) linear system models are the discrete model proposed by Roesser [1] and the model of Fornasini and Marchesini [2]. In this paper, an $n$-D Roesser model equivalent to the Fornasini–Marchesini model is derived and vice-versa. The extension to $n$-D requires careful attention to notation and has not appeared in the literature. The development is explicit and is illustrated by examples. The Roesser model allows the generalization of certain analysis, structure, and design results for one-dimensional (1-D) state-space systems by allowing the local state of its model to be divided into a *horizontal* and a *vertical* state which are propagated, respectively, horizontally and vertically by first-order difference equations. The Fornasini–Marchesini model includes the notion of a *global* state as well as of a local state. Extension of these models from 2-D to $n$-D has been suggested by Klamka [3] and Kaczorek [4], [5]. The relationship between the general $n$-D Roesser and Fornasini–Marchesini models has not been made explicitly clear, although for 2-D systems, this relationship has been extensively studied [2], [6], [7]. Such a relationship is useful since often in the early stages of the modeling of a given physical system a Fornasini–Marchesini model is obtained, whereas an equivalent first-order Roesser model would simplify analysis. The contribution of this paper is to give a systematic way to translate between local–state models via a simplified notation.

## II. Equivalence of Representations

In the models considered, the notation $x(i_1, \cdots, i_n)$ will represent a variable dependent on $n$ independent indexes, written as a vector $\mathcal{I} = (i_1, \cdots, i_n)$. Then the general $n$-D Roesser model can be written as

$$\hat{E} \overbrace{\begin{bmatrix} \hat{x}_1(\mathcal{I}+e_1) \\ \hat{x}_2(\mathcal{I}+e_2) \\ \vdots \\ \hat{x}_n(\mathcal{I}+e_n) \end{bmatrix}}^{\hat{x}'} = \hat{A} \overbrace{\begin{bmatrix} \hat{x}_1(\mathcal{I}) \\ \hat{x}_2(\mathcal{I}) \\ \vdots \\ \hat{x}_n(\mathcal{I}) \end{bmatrix}}^{\hat{x}} + \hat{B}u(\mathcal{I}) \tag{1}$$

$$y(\mathcal{I}) = \hat{C}\hat{x}(\mathcal{I}) + \hat{D}u(\mathcal{I}) \tag{2}$$

where $e_j$ is a vector which is zero, except in the $j$th entry where it is one.

Using a similar notation, with $\mathcal{V} = (1, 1, \cdots, 1)$, the general $n$-D Fornasini–Marchesini model can be written as

$$\begin{aligned} Ex(\mathcal{I}+\mathcal{V}) = {} & A_0 x(\mathcal{I}) + \sum_{j=1}^{n} A_j x(\mathcal{I}+e_j) + \cdots \\ & + \sum_{j=1}^{n} A_{1,\cdots,j-1,j+1,\cdots,n} x(\mathcal{I}+\mathcal{V}-e_j) + B_0 u(\mathcal{I}) \\ & + \sum_{j=1}^{n} B_j u(\mathcal{I}+e_j) + \cdots \\ & + \sum_{j=1}^{n} B_{1,\cdots,j-1,j+1,\cdots,n} u(\mathcal{I}+\mathcal{V}-e_j) \end{aligned} \tag{3}$$

$$y(\mathcal{I}) = Cx(\mathcal{I}) + Du(\mathcal{I}), \qquad \forall i_1, i_2, \cdots, i_n \geq 0. \tag{4}$$

The above equations are a direct $n$-D generalization of the model given by Kurek [8]. Generalizations of the two proposed Fornasini–Marchesini models [2], [9] can be obtained by setting appropriate $A$'s and $B$'s in (3) to zero. The notation used above is similar to those developed in [10] and [11]. As a notational example, a three-dimensional (3-D) Fornasini–Marchesini model has the form

$$\begin{aligned} Ex(\mathcal{I}+\mathcal{V}) = {} & A_0 x(\mathcal{I}) + A_1 x(\mathcal{I}+e_1) + A_2 x(\mathcal{I}+e_2) \\ & + A_3 x(\mathcal{I}+e_3) + A_{12} x(\mathcal{I}+e_1+e_2) \\ & + A_{13} x(\mathcal{I}+e_1+e_3) + A_{23} x(\mathcal{I}+e_2+e_3) \\ & + B_0 u(\mathcal{I}) + B_1 u(\mathcal{I}+e_1) + B_2 u(\mathcal{I}+e_2) \\ & \times B_3 u(\mathcal{I}+e_3) + B_{12} u(\mathcal{I}+e_1+e_2) \\ & + B_{13} u(\mathcal{I}+e_1+e_3) + B_{23} u(\mathcal{I}+e_2+e_3) \end{aligned} \tag{5}$$

$$y(\mathcal{I}) = Cx(\mathcal{I}) + Du(\mathcal{I}). \tag{6}$$

To recast the Roesser model (1) and (2) into the Fornasini–Marchesini form (3) and (4), consider one of the following.

*Case 1:* If the matrix $\hat{E}$ is nonsingular, both sides of (1) can be premultiplied by $\hat{E}^{-1}$ to obtain a regular Roesser representation, and thus without loss of generality, assume that $\hat{E} = I$, the identity matrix. An equivalent Fornasini–Marchesini model can be obtained by the substitutions $x(\mathcal{I}) = \hat{x}(\mathcal{I})$, $C = \hat{C}$, $D = \hat{D}$, $E = \hat{E} = I$, $A_{1,2,\cdots,i-1,i+1,\cdots,n} = [a_{hk}]_{m\times m}$, where

$$a_{hk} = \begin{cases} \hat{A}_{hk}, & \text{for } h = i \\ 0, & \text{otherwise} \end{cases}$$

and $m = \sum_{j=1}^{n} n_j$, where $n_j$ is the order of $x_j(\mathcal{I})$. Similarly, $B_{1,2,\cdots,i-1,i+1,\cdots,n} = [b_{hk}]_{m\times p}$, where

$$b_{hk} = \begin{cases} \hat{B}_h, & \text{for } h = i \\ 0, & \text{otherwise} \end{cases}$$

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