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A new methodology for an adaptive state observer design for a class of nonlinear systems with unknown parameters in unmeasured state dynamics

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Abstract

An adaptive state observer is an adaptive observer that does not require the persistent excitation condition to estimate the state. The usual structural requirement for designing this kind of observers is that the unknown parameters explicitly appear in the measured state dynamics. This paper deals with the problem of adaptive state observer synthesis for a class of nonlinear systems with unknown parameters in unmeasured state dynamics. The novelty of the proposed approach is that it requires neither a canonical form nor the approximation of some of the output's time derivatives. Firstly, we establish a new matrix equality that characterizes the structure of almost all systems found in the very small literature dealing with this problem. Then, this equality is exploited in the construction of the adaptation law. This simplifies the design procedure and makes it very similar to the conventional adaptive state observer design procedure. The problem of finding the observer gains is expressed as a linear matrix inequalities optimization problem. Two examples are given to demonstrate the validity of the proposed scheme.

Keywords

Nonlinear adaptive observer, adaptive state observer, persistent excitation, observer matching condition

Introduction

In the formulation of practical state feedback control problems, it is often the case where only a part of the overall state of the plant is measurable. Furthermore, it is generally difficult to obtain a precise mathematical model of the plant because of parasitic dynamics, unknown system parameters, external disturbances, etc. These facts have motivated the development of a wide variety of robust observer design techniques for both linear and nonlinear systems (Misawa and Hedrick, 1989; Radke and Gao, 2006; Trinh and Fernando, 2011). Adaptive observers have been developed to cope with the particular case where the model uncertainties are due to unknown parameters. To keep the convergence of the estimated state to the real state despite the effect of the unknown parameters, an on-line adaptation law is appended to the state observer. A particular feature of adaptive observers is that, based on some *persistent excitation* (PE) condition, they enable the estimation of the state and any number of the unknown parameters regardless of the number of the measured outputs.

Adaptive observers for linear systems were first proposed in Carroll and Lindorff (1973) and Luders and Narendra (1973) for *single-input single-output* (SISO) systems. Afterwards, this area has attracted much attention in the control community. Some notable results can be found in Ioannou and Sun (1996). For the nonlinear case, adaptive observers have been developed for specific classes of nonlinear systems (Bastin and Gevers, 1988; Besançon, 2000; Cho and Rajamani, 1997; Marino, 1990; Marino and Tomei, 1992). In Bastin and Gevers (1988), the authors have considered a class of SISO nonlinear systems expressed in a special form called the adaptive observer form, in which the nonlinearities are functions only of measurable signals. It has been shown that the state estimate converges to the real state under PE. The geometric conditions characterizing nonlinear systems that can be turned into the adaptive observer form have been investigated in Marino (1990). Also, a simple adaptive observer that achieves convergence of the state estimate without requiring PE has been proposed. However, the stability of this observer needs a *strictly positive real* (SPR) requirement with respect to the unknown parameters. To enlarge the class of nonlinear systems that can be turned into the adaptive observer form, a filtered nonlinear transformation, namely a transformation that depends upon the unknown parameters, has been given in Marino and Tomei (1992). This has lead to a design without PE of an adaptive observer for an equivalent state related to the original state through the

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unknown parameters. Nevertheless, PE is needed to achieve the original state estimation. In Cho and Rajamani (1997), the authors have proposed a systematic algorithm based on linear matrix inequalities (LMI) for an adaptive observer synthesis for *multi-input multi-output* (MIMO) Lipschitz nonlinear systems, where the regressor can depend on the unmeasured states. This design can achieve asymptotic stability of the state estimation error even without PE. Moreover, under PE, the parameters estimation error goes to zero. Nevertheless, the design procedure requires that the unknown parameters enter, with respect to some outputs, in such a way that an SPR-like condition is satisfied. In Besançon (2000), it has been shown that most of the systems for which adaptive observers are designable can be turned (possibly after a filtered transformation) into a canonical form in which the unknown parameters do not appear in the unmeasured state dynamics.

In relatively recent years, within the framework of high-gain nonlinear observer theory, the work in Zhang (2002) on an adaptive observer for linear time-varying systems has been generalised to a class of nonlinear systems with triangular form, namely uniformly observable nonlinear systems. A high gain adaptive observer with a relaxed PE condition has been given in Xu and Zhang (2004) for the class of systems which have a single output. However, the proposed algorithm is complicated as it is based on building up a composite system corresponding to several delayed versions of the original system. A procedure to construct a high gain adaptive observer for MIMO uniformly observable nonlinear systems which are both linearly and nonlinearly parametrised has been provided in Farza et al. (2009). High gain adaptive observers are a powerful method for the joint estimation of the state and the unknown parameters. However, they guarantee correct state estimation only when under PE.

The existing adaptive observers for both linear and nonlinear systems can be broadly classified into two main groups: *joint state-parameter observers* and *adaptive state observers* (Besançon, 2007; Ekramian et al., 2013). In joint state-parameter observers, the state estimate is somehow coupled to the parameter estimates. The main drawback of this approach is that the state estimation is guaranteed only when under PE. Adaptive observers with additional auxiliary filters (Bastin and Gevers, 1988; Carroll and Lindorff, 1973), filtered transformation based adaptive observers (Marino and Tomei, 1992), and high-gain adaptive observers (Farza et al., 2009; Xu and Zhang, 2004; Zhang, 2002) fall into this category. In adaptive state observers, the state estimation does not need PE. Furthermore, when PE holds, then the parameter estimates go to the actual parameters. However, the application of the adaptive state observers is restricted to those systems in which the unknown parameters explicitly appear in the measured state dynamics. Unfortunately, the fulfilment of this structural requirement is an exception in engineering systems.

Very few solutions have been proposed in the literature to construct adaptive state observers for systems with parametric uncertainities in the unmeasured state dynamics (Stamnes et al., 2011, 2009; Vijayaraghavan, 2013; Zhu and Khayati, 2011). Using a special canonical form, a procedure for the construction of a reduced order adaptive state observer for a class of MIMO nonlinear systems has been provided in Stamnes et al. (2011). However, the proposed adaptation law relies on finding a solution to a certain system of partial differential equations. In Zhu and Khayati (2011) an adaptive state observer for a class of SISO second order nonlinear systems has been proposed. This approach can be easily extended to a class of MIMO second order nonlinear systems. A different design that does not require any canonical form has been investigated in Vijayaraghavan (2013). This method is based on the approximation of the time derivatives of some system outputs using a high gain linear filter. Nevertheless, this scheme is only applicable for a class of nonlinear systems in which the regressor is linear in the state.

Motivated by the results in Stamnes et al. (2011) and Zhu and Khayati (2011), in this paper we present a new methodology for designing an adaptive state observer for a class of nonlinear systems in which the unknown parameters lie in the unmeasured state dynamics. The proposed method overcomes some difficulties faced in the above related works and has the potential to benefit from the useful implementation features of the conventional adaptive state observer design methodology, namely its simple and systematic design and the efficient LMI computations of the observer gains. Compared to the existing works on adaptive state observers, the main contributions of this paper are as follows.

1. A simple and new adaptive state observer design methodology is proposed for a class of nonlinear systems whose structure does not satisfy the structural requirement needed for the conventional adaptive state observer design.
2. The proposed method can deal with the structures of the nonlinear systems considered in Stamnes et al. (2011) and Zhu and Khayati (2011). However, unlike in these works, we do not require to change the system model into a special form or the resolution of a system of partial differential equations.
3. To simplify the design procedure, a new matrix equality that characterizes the structure of the class of systems under consideration is established. This equality, which involves some observer gains, can be solved together with the Lyapunov inequality using LMI-based computations. This makes the proposed scheme similar to the conventional adaptive observer design methodology.

The remainder of this paper is organized as follows: The next section specifies the class of nonlinear systems under consideration and provides the problem formulation. Section 'Proposed approach to adaptive state observer design' gives the proposed adaptive observer design methodology. Two illustrative examples with simulation results are provided in the section 'Simulation examples' to confirm the validity of the proposed scheme. The final section gives some conclusions.

Notation. Throughout the paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ the set of real $n-$ vectors and $\mathbb{R}^{n\times m}$ the set of real $n\times m$ matrices. The symbols I_n and $0_{n\times m}$ represent, respectively, the $n\times n$ identity matrix and $n\times m$ zero matrix and 0 denotes the scalar zero or the zero vector. $||.||$ denotes the Euclidean norm or its induced norm. Finally, for a square matrix A, $A>0$ denotes a positive definite matrix.

System description and problem formulation

System description

In this paper, we consider a class of nonlinear systems described by the following set of equations

$$\dot{x} = Ax + \Omega(y, u) + B\Phi(x, u) + E\Psi(x)\theta \tag{1a}$$

$$y = Cx \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input vector, $y \in \mathbb{R}^p$ is the output vector and $\theta \in \mathbb{R}^r$ is an unknown parameter vector. $\Omega(y, u) : \mathbb{R}^p \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $\Phi(x, u) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^s$ are known nonlinear function vectors and the regressor $\Psi(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{q \times r}$ is a known function matrix which may depend nonlinearly on x. $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times s}$, $E \in \mathbb{R}^{n \times q}$ and $C \in \mathbb{R}^{p \times n}$ are known constant matrices with (C, A) being an observable pair. Without loss of generality we assume that $rank(C) = p$ and $rank(E) = q$. Also, it is natural to consider that u is bounded and ensures the boundedness of x, and that the nonlinear terms do not alter the observability of the system.

Throughout this paper, the following assumptions are made on system (1).

Assumption 1. The vector function $\Omega(y, u)$ is continuous in its variables and $\Phi(x, u)$ and $\Psi(x)$ satisfy the Lipschitz propriety with respect to x, that is, there exist positive constants γ_1 and γ_2 such that

$$\| \Phi(x, u) - \Phi(\hat{x}, u) \| \leq \gamma_1 \| x - \hat{x} \| \tag{2}$$

$$\| \Psi(x) - \Psi(\hat{x}) \| \leq \gamma_2 \| x - \hat{x} \| \tag{3}$$

where γ_1 and γ_2 are known.

Assumption 2. The unknown parameter vector θ is constant and bounded in the following sense

$$\|\theta\| \leq \gamma_3 \tag{4}$$

where γ_3 is a known positive constant.

Remark 1. The results of this paper can be applied locally or globally depending on whether the Lipschitz conditions (2) to (3) are satisfied locally or globally. Systems with Lipschitz nonlinearities include a large class of nonlinear systems. It is stated in Pertew et al. (2006) that any system of the form $\dot{x} = f(x, u, t)$ can be expressed in a form with a linear part plus a nonlinear part satisfying the Lipschitz propriety at least locally, if $f(x, u, t)$ is continuously differentiable with respect to x. Assumption 2 is a standard assumption in the adaptive observer design literature when the regressor depends on the unmeasured state variables (Pourgholi and Majd, 2011; That and Ding, 2014). In practice, the bound γ_3 can be set by exploiting some knowledge on the physical nature of θ.

Definition 1. (Besançon, 2007) A signal $\omega(\tau) : \mathbb{R}^+ \rightarrow \mathbb{R}^{\rho \times \mu}$ satisfies the property of persistent excitation if there exist positive constants T, k_1, and k_2 such that for all $t \geq 0$

$$I_\rho k_1 \geq \int_t^{t+T} \omega(\tau)\omega(\tau)^T d\tau \geq I_\rho k_2 \tag{6}$$

Remark 2. The PE of the regressor plays a key role in ensuring the correct parameter identification in adaptive systems. In adaptive observers, a common objective is for a term of the form $\omega(t)\tilde{\theta}$ to converge to the origin, where $\tilde{\theta}$ is the parameter estimation error. However, this does not guarantee for $\tilde{\theta}$ to also converge to the origin. Indeed, $\tilde{\theta}$ converges to a vector in the null space of $\omega(t)$ which is not necessarily the zero vector. The PE of $\omega(t)$ requires that the integral of $\omega(t)\omega(t)^T$ is full rank over any time interval $[t, t + T]$. This guarantees that the null space of $\omega(t)$ keeps changing with time such that the only value of $\tilde{\theta}$ that always ensures $\omega(t)\tilde{\theta} = 0$ is $\tilde{\theta} = 0$. The constant k_2 in (5) is known as the *level of excitation* and it has an important effect on the speed of convergence of $\tilde{\theta}$ (Ioannou and Sun, 1996). In practice, PE is often difficult to check since it may depend on the system inputs and external disturbances (Mahyuddin et al., 2014).

Problem formulation

The typical form of the adaptive state observer dealing with the class of nonlinear systems (1) is given by the following equations (Cho and Rajamani, 1997; That and Ding, 2014)

$$\dot{\hat{x}} = A\hat{x} + \Omega(y, u) + B\Phi(\hat{x}, u) + E\Psi(\hat{x})\hat{\theta} + L(y - C\hat{x}) \tag{6}$$

$$\dot{\hat{\theta}} = \rho^{-1}\Psi(\hat{x})^T FC(x - \hat{x}) \tag{7}$$

where $\hat{x}$ is the state estimate, $\hat{\theta}$ is the unknown parameter vector estimate, $L \in \mathbb{R}^{n \times p}$ is the observer gain, ρ is a positive constant and $F \in \mathbb{R}^{q \times p}$ is a matrix to be designed.

Theorem 1. *Consider the nonlinear system (1) along with the adaptive state observer (6) to (7). Under Assumptions 1 and 2, the state estimate $\hat{x}$ converges to the actual state x and $E\Psi(\hat{x})\hat{\theta}$ converges to $E\Psi(x)\theta$, if there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a matrix $F \in \mathbb{R}^{q \times p}$ such that*

$$\begin{aligned} &(A - LC)^T P + P(A - LC) \\ &+ (\gamma_1 \| B\|^2 + \gamma_2\gamma_3 \| E\|^2)PP + (\gamma_1 + \gamma_2\gamma_3)I_n < 0 \end{aligned} \tag{8}$$

$$E^T P = FC \tag{9}$$

Moreover, if $E\Psi(x)$ satisfy the PE condition, then $\hat{\theta}$ converges to θ.

For the proof of the above theorem, the reader is referred to Cho and Rajamani (1997).

The structural constraint (9), relying E to C, is a common requirement in adaptive state observers design (Ekramian et al., 2013; Mahyuddin et al., 2014; Pourgholi and Majd, 2011; Tyukin et al., 2013). Clearly, it severely restricts the class of systems admitting adaptive state observers. It is stated in Cho and Rajamani (1997) that the existence of $P > 0$ to satisfy (9) is guaranteed when at least some of the measured outputs are such that the transfer functions between these outputs and the unknown parameters are dissipative or SPR. Notice that in the literature on adaptive state observer design, equality (9) is presented in different forms, namely $E^T PC^{\perp} = 0_{q \times (n-p)}$, $E^T P = C_r$ and

$E^T P = C$ (Besançon, 2000; Cho and Rajamani, 1997; Ekramian et al., 2013), where $C^{\perp} \in \mathbb{R}^{n \times (n-p)}$ is any full rank matrix whose columns span the null space of C and $C_r \in \mathbb{R}^{q \times n}$ spans the rows of C. The following lemma gives more insight into equality (9).

Lemma 1. *(Corless and Tu, 1998; Raoufi et al., 2010) There exist matrices $P = P^T > 0$ and F verifying the equality $E^T P = FC$ if and only if*

$$\operatorname{rank}(CE) = \operatorname{rank}(E) \tag{10}$$

Equality (10) is known as the observer matching condition (Floquet et al., 2007). It is also a common requirement in sliding mode observers and unknown input observers (Mondal, 2015; Raoufi et al., 2010). An immediate implication of the observer matching condition is that $\operatorname{rank}(E) \leq \operatorname{rank}(C)$, that is, $q \leq p$. Satisfying the observer matching condition by system (1) means that all of the entries of the vector $\Psi(x)\theta$, and hence also the unknown parameters, explicitly appear in at least q measured state dynamics. This fact can be understood by taking the first time derivative of the output y in (1), which gives

$$\dot{y} = C(Ax + \Omega(y,u) + B\Phi(x,u)) + CE\Psi(x)\theta \tag{11}$$

It is clear from (11) that, if the observer matching condition is satisfied, that is, the distribution matrices of $\Psi(x)\theta$ in (1) and (11) have the same rank q, then all of the entries of $\Psi(x)\theta$ that appear in (1) also appear in (11).

Unfortunately, for many practical systems the observer matching condition is not satisfied, and as a consequence, the conventional adaptive state observer is not implementable for these systems.

Proposed approach to adaptive state observer design

The purpose of this section is to design an adaptive state observer for system (1) when the unknown parameters lie in the unmeasured state dynamics, that is, when condition (9) is not satisfied. Like in conventional adaptive state observer synthesis, the proposed scheme is based on the reformulation of the location of the unknown parameters with respect to the measured outputs into a matrix equality. In order to achieve this reformulation, first we introduce a structural assumption.

A structural assumption

To characterise the structure of the systems considered in this paper, let us formulate the following assumption.

Assumption 3 Matrices A, B, E and C satisfy

$$C[B \quad E] = [0_{p\times s} \quad 0_{p\times q}] \tag{12}$$

$$\operatorname{rank}(CAE) = \operatorname{rank}(E) \tag{13}$$

Basically, the conditions in Assumption 3 play the same role as the observer matching condition in the conventional adaptive state observer design. To give an explanation of the structural conditions (12) and (13), we use the concept of relative degree. We recall that the relative degree from an output y_i $(i = 1, \ldots, p)$ to a parameter θ_j $(j = 1, \ldots, r)$ is defined as the number of times y_i needs to be differentiated so that θ_j appears explicitly.

Clearly, in (12), $CB = 0_{p\times s}$ means that the nonlinear term $\Phi(x,u)$ does not enter the measured state dynamics, that is, the relative degrees from the outputs to $\Phi(x,u)$ are greater than one. Equality $CE = 0_{p\times q}$ in (12) along with (13) means that the relative degrees from at least q outputs to $\Psi(x)\theta$ are two. This fact can be understood by taking the second time derivative of y in (1). A second implication of (13) is that $\operatorname{rank}(E) \leq \operatorname{rank}(C)$, that is, $q \leq p$.

Although Assumption 3 may appear very restrictive, it is actually satisfied by many systems which do not satisfy the observer matching condition. Some examples of real-life systems that satisfy Assumption 3 are given in Table 1.

Remark 3. In Stamnes et al. (2011, 2009), the authors have considered a class of nonlinear systems in a special canonical form that shows that the unknown parameters and the unmeasured nonlinear term lie in the unmeasured state dynamics. However, this canonical form does not show the precise location of the unknown parameters with respect to the outputs. Nevertheless, one can easily check that the proposed adaptation law in Stamnes et al. (2011, 2009) is feasible only for those systems whose structure satisfies the conditions specified in Assumption 3. Furthermore, the class second order of nonlinear systems considered in Zhu and Khayati (2011) satisfies Assumption 3.

Remark 4. The class of nonlinear systems considered in Vijayaraghavan (2013) does not necessarily satisfy Assumption 3. Indeed, the proposed approach can handle systems with relative degrees from the measured outputs to the unknown parameters higher than two. The design is based on using a high gain linear filter to approximate some measured outputs time derivatives. These signals are used as auxiliary outputs to estimate the unknown parameters. However, this approach can be applied only to systems in which the regressor is linear in the state. Moreover, the proposed adaptation law requires the inversion of a matrix that depends on some measured outputs and their approximate time derivatives. This leads to the assumption that the system evolves such that the aforementioned matrix remains invertible at any time, which cannot be a priori checked.

A preliminary result

The lemma established in this subsection is the counterpart of Lemma 1 when Assumption 3 is satisfied.

Lemma 2. *Given system (1) with $CE = 0_{p\times q}$, then $\operatorname{rank}(CAE) = \operatorname{rank}(E)$ if and only if there exist matrices $P = P^T > 0$, H and G such that*

$$HCA - GC = E^T P \tag{14}$$

Proof. (*Necessity*) Right-multiplying (14) by E and using $CE = 0_{p\times q}$, it follows that

$$HCAE = E^T PE \tag{15}$$

Since E is full column rank and P is positive definite, then $E^T PE$ is also positive definite (this is because for any vector

Table 1. Examples of real-life systems satisfying Assumption 3.

System model	Parameters for which Assumption 3 is satisfied
Mechanical systems (Dixon and Nagarkatti, 2003) $\begin{cases} \dot{q}_1 = q_2 \\ \dot{q}_2 = M(q_1)^{-1}(C(q_1, q_2)q_2 + G(q_1) - u). \\ y = q_1 \qquad (q_i = [x_{i1} \cdots x_{ip}]^T \quad i = 1, 2) \end{cases}$	All parameters for which the system is affine.
Contois model of bio-reactor (Zhang and Besançon, 2005) $\begin{cases} \dot{x}_1 = x_2 + x_1 u \\ \dot{x}_2 = -\dfrac{a_2 x_2(x_2^2 - a_1 u x_1^2) + (a_1 x_1 - x_2)^2(a_4 u - a_3 x_2)}{a_1 a_2 x_1^2} \\ y = x_1 \end{cases}$	a_3 and a_4.
Drilling system (Stamnes et al., 2009) $\begin{cases} \dot{x}_1 = \dfrac{\beta_d}{V_d}(u_p - x_2) \\ \dot{x}_2 = \dfrac{1}{M}(x_1 - x_3) - \dfrac{F_d}{M}x_2^2 - \dfrac{F_a}{M}(x_2 + q_{\text{res}})^2 + g\dfrac{\rho_d - \rho_a}{M}h_{\text{bit}} \\ \dot{x}_3 = -\dfrac{\beta_a}{V_a}(x_2 - q_{\text{res}} + u) \\ y = x_1 \end{cases}$	$\frac{1}{M}$, F_d, F_a, ρ_a and ρ_d.
Multi-axial electromagnetically actuated punch (Riva et al., 2014) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{c_z}{m_z}x_1 - \dfrac{d_z}{m_z}x_2 - \dfrac{d_{z2}}{m_z}\tanh(d_{z3}x_2) + u \\ y = x_1 \end{cases}$	c_z, d_z, d_{z2} and $\frac{1}{m_z}$.
Electro-hydraulic manipulator (Feki, 2001) $\begin{cases} \dot{x}_1 = \dfrac{4B}{V_t}\left(ku\sqrt{d\dfrac{P_s - P_r - \text{sign}(u)x_1}{2}} - \dfrac{\alpha}{1+\gamma\lvert u\rvert}x_1 - Sx_2\right) \\ \dot{x}_2 = -\dfrac{1}{m_t}(Sx_1 - bx_2 - k_l x_3) \\ x_3 = x_2 \\ y = x_1 \end{cases}$	$\frac{1}{m_t}$, b and k_l.

$a \neq 0$ of dimension q, $a^T(E^TPE)a = (aE)^TP(Ea) = b^TPb > 0$ where the vector $b = Ea \neq 0$ as E is full rank). Therefore E^TPE is full rank, and thus

$$\text{rank}(E^TPE) = \text{rank}(E) \tag{16}$$

From (16) and (17) it follows that

$$\text{rank}(HCAE) = \text{rank}(E) \tag{17}$$

and hence

$$\text{rank}(CAE) = \text{rank}(E) \tag{18}$$

(*Sufficiency*) For convenience, the proof of sufficiency is divided into two parts.
Part 1: Partitioning C and E as

$$C = [\,C_1 \quad C_2\,], E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{19}$$

where $C_1 \in \mathbb{R}^{p\times p}$ and $E_1 \in \mathbb{R}^{p\times q}$. Without loss of generality, we assume that C_1 is full rank. We introduce a non-singular state transformation as follows

$$T = \begin{bmatrix} C_1 & C_2 \\ 0_{(n-p)\times p} & I_{(n-p)} \end{bmatrix} \tag{20}$$

Therefore, using equality $CE = 0_{p\times q}$, in the new coordinates, system (1) is transformed into the following form

$$\dot{z} = TAT^{-1}z + T\Omega(y, u) + TB\Phi(T^{-1}z, u) + TE\Psi(T^{-1}z)\theta \tag{21a}$$

$$y = CT^{-1}z \tag{21b}$$

where

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, z_1 = y, T^{-1} = \begin{bmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0_{(n-p)\times p} & I_{(n-p)} \end{bmatrix},$$
$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{22}$$
$$TE = \begin{bmatrix} 0_{p\times q} \\ E_2 \end{bmatrix}, CT^{-1} = [\,I_{p\times p} \quad 0_{p\times(n-p)}\,]$$

Application of (13) to the transformed system (21) yields

$$\text{rank}(A_{12}E_2) = \text{rank}(E_2) \tag{23}$$

Equality (23) has the same form as (10). Therefore, by applying Lemma 1 we conclude that (23) holds if and only if there exist matrices $P_3 = P_3^T > 0$ and H such that

$$E_2^T P_3 = HA_{12} \tag{24}$$

Part 2: Application of (14) to the transformed system (21) yields

$$HCT^{-1}TAT^{-1} - GCT^{-1} = (TE)^T(T^T)^{-1}PT^{-1} \tag{25}$$

Let's define the symmetric positive definite matrix $(T^T)^{-1}PT^{-1}$ as

$$(T^T)^{-1}PT^{-1} = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \tag{26}$$

where P_3 satisfy (24). Substituting the structures of TAT^{-1}, CT^{-1}, and TE given in (22) into (25) yields

$$\begin{bmatrix} HA_{11} - G & HA_{12} \end{bmatrix} = \begin{bmatrix} E_2^T P_2^T & E_2^T P_3 \end{bmatrix} \tag{27}$$

Thus, considering (24), we deduce that if we choose

$$G = HA_{11} - E_2^T P_2^T \tag{28}$$

then the matrix equality (25) holds. This completes the proof of sufficiency.

Adaptive state observer design

For system (1) satisfying Assumptions 1 to 3, a Luenbeger-like observer of the following form can be designed to estimate the state

$$\dot{\hat{x}} = A\hat{x} + \Omega(y,u) + B\Phi(\hat{x},u) + E\Psi(\hat{x})\hat{\theta} + L(y - C\hat{x}) \tag{29}$$

where the parameter estimate vector $\hat{\theta}$ is the solution of the following adaptation law, which is partially inspired from Zhu and Khayati (2011)

$$\hat{\theta} = W + \Gamma\Psi^T(\hat{x})Hy \tag{30}$$

$$\dot{W} = -\Gamma\frac{d\Psi^T(\hat{x})}{dt}Hy - \Gamma\Psi^T(\hat{x})(HC(A\hat{x} + \Omega(y,u)) + G(y - C\hat{x})) \tag{31}$$

where H and G are constant matrices to be designed and $\Gamma = \Gamma^T > 0$ is the learning rate matrix.

Remark 5. As $\dot{\hat{x}}$ is available from (29), the term $\frac{d\Psi^T(\hat{x})}{dt}$ in (31) is implemented using the chain rule $\frac{\partial\Psi^T(\hat{x})}{\partial\hat{x}}\dot{\hat{x}}$. Also, since the regressor $\Psi(x)$ is assumed to be Lipschitz, $\frac{\partial\Psi^T(\hat{x})}{\partial\hat{x}}$ exists and is bounded. However, if the regressor explicitly depends upon u, then the first time derivative of u will be involved in the adaptation law (31). In this case, the proposed approach could be used only when u is time differentiable and $\frac{\partial\Psi^T(\hat{x},u)}{\partial u}\dot{u}$ is known and bounded. Notice that the same limitation is encountered in related works (Stamnes et al., 2011, 2009; Zhu and Khayati, 2011).

Remark 6. It should be pointed out that all the nonlinear adaptive state observers for systems with the unknown parameters in unmeasured state dynamics (Stamnes et al., 2011, 2009; Vijayaraghavan, 2013; Zhu and Khayati, 2011) are sensitive to measurement noise. Our approach is no exception to this. Indeed, it is clear from (30) that in the case of noisy output measurements, $\hat{\theta}$ will be directly affected by the measurement noise, unlike in conventional adaptive state observers where the integration between y and $\hat{\theta}$ attenuates the high frequency noise components (see (7)).

A sufficient condition for the asymptotic stability of the adaptive state observer (29) to (31) is presented in the following theorem.

Theorem 2. *Consider system* (1) *along with the adaptive state observer* (29) *to* (31). *Under Assumptions 1 to 3 the state estimate $\hat{x}$ converges to the actual state x asymptotically while $E\Psi(\hat{x})\hat{\theta}$ converges to $E\Psi(x)\theta$ if there exist positive real numbers ε_1 and ε_2 and matrices $P = P^T > 0$, H and G such that*

$$\begin{aligned}(A - LC)^T P + P(A - LC) + \varepsilon_1 PBB^T P \\ + \varepsilon_2 PEE^T P + \varepsilon_1^{-1}\gamma_1^2 I_n + \varepsilon_2^{-1}\gamma_2^2\rho^2 I_n < 0\end{aligned} \tag{32}$$

$$HCA - GC = E^T P \tag{33}$$

Moreover, if $E\Psi(x)$ satisfies the PE condition, then $\hat{\theta}$ converges to θ.

Proof. Let $\tilde{x} = x - \hat{x}$ and $\tilde{\theta} = \theta - \hat{\theta}$ and notice that $\dot{\theta} = 0$. Then, from (1) and (29) to (31), it follows that

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + B\tilde{\Phi} + E(\tilde{\Psi}\theta + \Psi(\hat{x})\tilde{\theta}) \tag{34}$$

$$\dot{\tilde{\theta}} = -\Gamma\Psi^T(\hat{x})(HCA - GC)\tilde{x} \tag{35}$$

where $\tilde{\Phi} = \Phi(x,u) - \Phi(\hat{x},u)$ and $\tilde{\Psi} = \Psi(x,u) - \Psi(\hat{x},u)$.

Now, consider the following Lyapunov function

$$V = \tilde{x}^T P\tilde{x} + \tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \tag{36}$$

The time derivative of V along with the trajectories of the error system (34) to (35) is

$$\begin{aligned}\dot{V} = \tilde{x}^T[(A - LC)^T P + P(A - LC)]\tilde{x} + 2\tilde{x}^T PB\tilde{\Phi} \\ + 2\tilde{x}^T PE\tilde{\Psi}\theta + 2\tilde{x}^T PE\Psi(\hat{x},u)\tilde{\theta} \\ -2\tilde{\theta}^T\Psi^T(\hat{x},u)(HCA - GC)\tilde{x}\end{aligned} \tag{37}$$

Using (33), we get

$$\dot{V} = \tilde{x}^T[(A - LC)^T P + P(A - LC)]\tilde{x} + 2\tilde{x}^T PB\tilde{\Phi} + 2\tilde{x}^T PE\tilde{\Psi}\theta \tag{38}$$

By using Young's inequality $2v^T w \le \varepsilon v^T v + \varepsilon^{-1}w^T w$ ($\varepsilon > 0$), the Lipschitz conditions (2) to (3), and inequality (4), we obtain the following inequalities

$$\begin{aligned}2\tilde{x}^T PB\tilde{\Phi} &\le \varepsilon_1\tilde{x}^T PBB^T P\tilde{x} + \varepsilon_1^{-1}\tilde{\Phi}^T\tilde{\Phi} \\ &\le \varepsilon_1\tilde{x}^T PBB^T P\tilde{x} + \varepsilon_1^{-1}\tilde{x}^T\gamma_1^2\tilde{x}\end{aligned} \tag{39}$$

$$
\begin{aligned}
2\tilde{x}^T PE\tilde{\Psi}\theta &\leq \varepsilon_2 \tilde{x}^T PEE^T P\tilde{x} + \varepsilon_2^{-1}\theta^T\tilde{\Psi}^T\tilde{\Psi}\theta \\
&\leq \varepsilon_2 \tilde{x}^T PEE^T P\tilde{x} + \varepsilon_2^{-1}\tilde{x}^T\gamma_2^2\gamma_3^2\tilde{x}
\end{aligned}
\tag{40}
$$

where ε_1 and ε_2 are positive constants. Substituting (39) and (40) into (38), we obtain

$$\dot{V} \leq -\tilde{x}^T Q\tilde{x} \tag{41}$$

where

$$
\begin{aligned}
Q = &-\big((A-LC)^T P + P(A-LC) + \varepsilon_1 PBB^T P \\
&+ \varepsilon_2 PEE^T P + \varepsilon_1^{-1}\gamma_1^2 I_n + \varepsilon_2^{-1}\gamma_2^2\rho^2 I_n\big)
\end{aligned}
\tag{42}
$$

Therefore, if inequality (32) is satisfied (i.e. Q is positive definite), then $\tilde{x} \in L_\infty$ and $\tilde{\theta} \in L_\infty$. Hence, it can also be concluded from (34) that $\dot{\tilde{x}} \in L_\infty$ since $\Phi(x,u)$ and $\Psi(x)$ are Lipschitz. To prove the asymptotic stability of $\tilde{x}$, we integrate both sides of the inequality (41) from $t = 0$ to $t = t_f$. It follows that

$$V(t_f) \leq V(0) - \int_0^{t_f} \tilde{x}^T Q\tilde{x}dt \tag{43}$$

Since $V(t) \geq 0$ and finite and Q is positive definite, it follows from (43) that $\tilde{x} \in L_2$. Therefore, $\tilde{x} \in L_\infty \cap L_2$ and $\dot{\tilde{x}} \in L_\infty$. Application of Barbalat's lemma (Slotine and Li, 1991) yields $\lim_{t\to\infty} \tilde{x} = 0$.

We prove now that the parameter estimate error converges to the origin under PE. To do this, we show first that $\dot{\tilde{x}}$ is uniformly continuous by checking the uniform continuity of the right-hand side of (34). Since $\dot{\tilde{x}} \in L_\infty$ and $\Phi(x,u)$ and $\Psi(x)$ are Lipschitz, it can be confirmed straightforwardly that the first three terms on the right-hand side of (34) are uniformly continuous. Also, from (29) and (35), it can be checked that $\dot{\hat{x}} \in L_\infty$ and $\dot{\tilde{\theta}} \in L_\infty$, respectively, which leads to the uniform continuity of the fourth term in the right-hand side of (34). We conclude from the above that $\dot{\tilde{x}}$ is uniformly continuous. On the other hand, we have

$$\int_0^\infty \dot{\tilde{x}}dt = \lim_{t\to\infty}\tilde{x}(t) - \tilde{x}(0) = -\tilde{x}(0) \tag{44}$$

which is finite. Hence, again an application of Barbalat's Lemma yields $\lim_{t\to\infty}\dot{\tilde{x}} = 0$. So, considering (34) when $t \to \infty$, we conclude that

$$\lim_{t\to\infty} E\hat{\Psi}\tilde{\theta} = 0 \tag{45}$$

Consequently, if the regressor satisfies the PE condition, then $\lim_{t\to\infty}\tilde{\theta} = 0$. □

LMI formulation

Finding the observer gains by solving (32) under equality constraint (33) is not a trivial task. To make this problem easily tractable, we intend in this subsection to transform it into an LMI optimization problem. By introducing the following change of variable

$$L = P^{-1}M \tag{46}$$

where $M \in \mathbb{R}^{n\times p}$ and using the Schur complement (Boyd et al., 1997), inequality (32) can be turned into the following LMI in the variable P

$$
\begin{bmatrix}
\Lambda & PB & PE \\
B^T P & -\varepsilon_1^{-1} I_s & 0_{s\times q} \\
E^T P & 0_{q\times s} & -\varepsilon_2^{-1} I_q
\end{bmatrix} < 0
\tag{47}
$$

where $\Lambda = A^T P + PA - C^T M^T - MC + \varepsilon_1^{-1}\gamma_1^2 I_n + \varepsilon_2^{-1}\gamma_2^2\gamma_3^2 I_n$. Also, by using the same idea used in Corless and Tu (1998) to turn (9) into an LMI problem, we can turn the problem of the resolution of equality (33) into the following LMI optimization problem

$$
\begin{gathered}
\text{Minimize } \eta \text{ subject to} \\
\begin{bmatrix}
\eta I_q & HCA - GC - E^T P \\
(HCA - GC - E^T P)^T & \eta I_n
\end{bmatrix} \geq 0
\end{gathered}
\tag{48}
$$

where η is a positive scalar. Therefore, computing P,H, and G involves solving LMIs (47) and (48), simultaneously, for $P = P^T > 0$, $M = M^T$, G, H, and η where L is given in (46).

Simulation examples

In order to evaluate the proposed scheme, we present in this section two numerical examples that were chosen to show some difficulties and limits of the related works (Stamnes et al., 2011; Vijayaraghavan, 2013; Zhu and Khayati, 2011) from the point of view of the feasibility and the design complexity. The LMIs were solved using CVX, a Matlab-based modelling system for convex optimization (Grant et al., 2016), and the numerical simulations were conducted in Simulink using fixed-step fourth order Runge–Kutta solver with integration time step of 0.001 s.

Example 1

Consider the following Rössler system

$$
\begin{cases}
\dot{x}_1 = -(x_2 + x_3) \\
\dot{x}_2 = x_1 + ax_2 \\
\dot{x}_3 = b + x_3(x_1 - c) \\
y = x_1 + x_2
\end{cases}
\tag{49}
$$

We suppose that $a = 0.2$ and b and c are unknown parameters that belong to the ranges $[0 \quad 7]$ and $[0 \quad 4]$, respectively. Accordingly, the bound in Assumption 2 can be chosen as $\gamma_3 = 8.1$. Note that it is well known that this system exhibits chaotic behaviour when $a = 0.2$, $b = 0.2$, and $c = 5.7$. The above system can be described in the form of system (1) with

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \theta = \begin{bmatrix} b \\ c \end{bmatrix}, A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$

$$
E = B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = [1 \quad 1 \quad 0],
$$

$$\Omega(x,u) = 0, \Phi(x) = x_1 x_3, \Psi(x) = [1 \quad -x_3].$$

The functions $\Phi(x)$ and $\Psi(x)$ are Lipschitz as required in Assumption 1. Obviously, $\Psi(x)$ is globally Lipschitz with the Lipschitz constant $\gamma_2 = 1$. However, $\Phi(x)$ is locally Lipschitz, which means that its Lipschitz constant γ_1 depends on the region where the system operates. Suppose that the trajectories of the system evolve in the set $\Omega_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 / |x_1| \leq 20, |x_2| \leq 20, |x_3| \leq 20\}$, then $\Phi(x)$ can be extended into $\Phi(\mathrm{sat}(x))$ which coincides with $\Phi(x)$ in Ω_1, and thus the system becomes globally Lipschitz (Farza et al., 2009). γ_1 can be computed using the relation $\gamma_1 = \max\left\|\frac{\partial\Phi(\mathrm{sat}(x))}{\partial x}\right\|$ (Khalil, 2002), which yields $\gamma_1 = 28.3$. Finally, it is easy to check that the structural requirements in Assumption 3 are satisfied. Notice that the application of the method in Stamnes et al. (2011) to system (49) requires first to turn it into a special form. The adaptive state observer in Zhu and Khayati (2011) cannot be applied to this system as it is not a second order system. Moreover, as the regressor $\Psi(x)$ is affine in the state but it is not linear, this system does not match the system structure considered in Vijayaraghavan (2013).

We start the observer design by computing the observer gains. Let $\epsilon_1 = 150$ and $\epsilon_2 = 15$. After simultaneously solving (47) and (48), we obtain the following results

$$P = \begin{bmatrix} 253.6936 & 245.5466 & 5.7071 \\ 245.5466 & 415.4404 & 6.2103 \\ 5.7071 & 6.2103 & 0.2795 \end{bmatrix}, M = \begin{bmatrix} 551.7183 \\ 192.0864 \\ -223.6543 \end{bmatrix},$$
$$H = -0.2795, G = -5.9867, \eta = 0.1, L = 10^3 \times \begin{bmatrix} 0.0337 \\ 0.0042 \\ -1.5809 \end{bmatrix}$$

where L is computed from the relation (46). We may now proceed to the design of adaptation law (30) to (31). The learning rate matrix is chosen as $\Gamma = 100 \times I_2$ and by using the chain rule, we get $\frac{d\Psi(\hat{x})^T}{dt} = [0 \; -\dot{\hat{x}}_3]^T$.

The simulation was performed using the initial conditions $x = [-5 \quad 5 \quad -5]$, $\hat{x} = [0 \quad 0 \quad 0]$ and in order to get $\hat{\theta}(0) = [0 \quad 0]^T$, considering (30), $W(0)$ was selected as $W(0) = -\Gamma H \Psi(\hat{x}(0)) y(0) = [0 \quad 0]$. To show how capable the observer is in estimating the unknown parameters, θ was initially selected as $[0.2 \quad 5.7]$, then after 40 s it begins drifting slowly according to $[0.2 + 0.05(t-40) \quad 5.7 - 0.05(t-40)]$. The state estimation results are depicted in Figures 1 and 2 and the parameter estimation results are given in Figures 3 and 4. As it can be seen, the state estimate converges quickly to the real state, regardless of the rate of convergence of the parameter estimates. Also, it is clear that as the PE condition is satisfied the parameter estimates converge to their constant real parameters and track the slow time-varying actual parameters with satisfying accuracy.

Example 2

Consider the following nonlinear system (Stamnes et al., 2011)

$$\begin{cases} \dot{x}_1 = & x_2 + u \\ \dot{x}_2 = & -x_1 - 2\sin(x_2) + \arctan(x_2)\theta_1 + \cos(x_1 x_2)\theta_2 + \Delta(x_1, x_2, t) \\ y = & x_1 \end{cases} \quad (50)$$

where θ_1 and θ_2 are unknown constant parameters and $\Delta(x_1, x_2, t)$ is a bounded unknown term which consists of nonparametric model uncertainties and disturbances. We assume that θ_1 and θ_2 are known a priori to belong to the range $[0 \quad 2.5]$, thus Assumption 2 is satisfied with $\gamma_3 = \sqrt{12.5}$. For $\Delta(x_1, x_2, t) = 0$, the above system can be described in the form of system (1) with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$
$$E = B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \quad 0],$$
$$\Omega(y, u) = [u \quad 0]^T, \Phi(x) = -2\sin(x_2),$$
$$\Psi(x) = [\arctan(x_2) \quad \cos(x_1 x_2)]$$

It is not hard to confirm that Assumption 1 is met. The function $\Phi(x)$ is globally Lipschitz with the Lipschitz constant $\gamma_1 = 2$. For the nonlinear vector function $\Psi(x)$, since the term $\cos(x_1 x_2)$ is locally Lipschitz, its Lipschitz constant γ_2 is region-based. Suppose that the control input is chosen such that it keeps the state bounded in the set $\Omega_2 = \{(x_1, x_2) \in \mathbb{R}^2 / |x_1| \leq 5, |x_2| \leq 3\}$, then by extending $\Psi(x)$ into

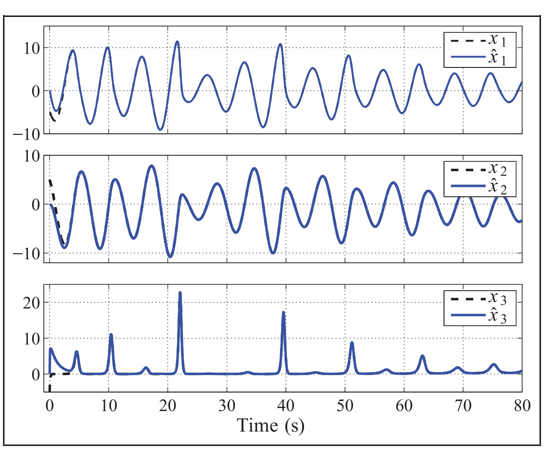

Figure 1. Actual state variables and their estimates (Example 1).

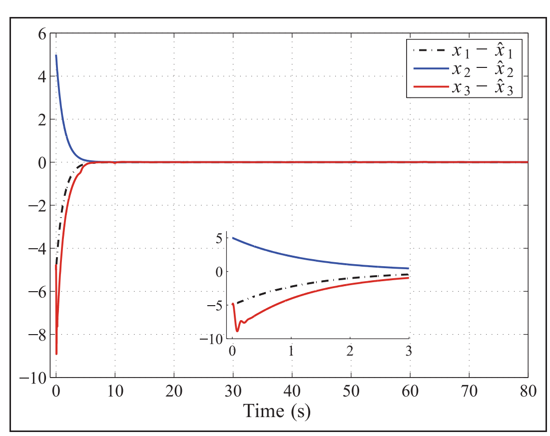

Figure 2. State estimation error (Example 1).

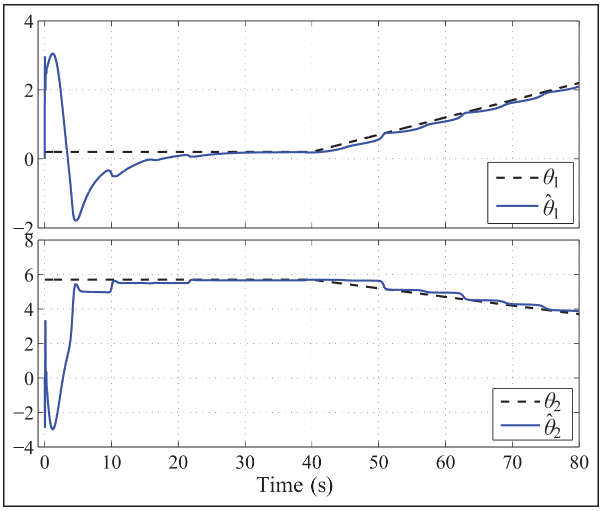

Figure 3. Actual parameters and their estimates (Example 1).

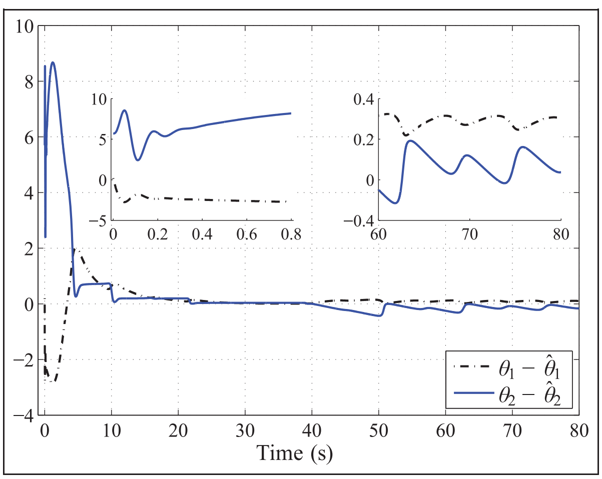

Figure 4. Parameter estimation error (Example 1).

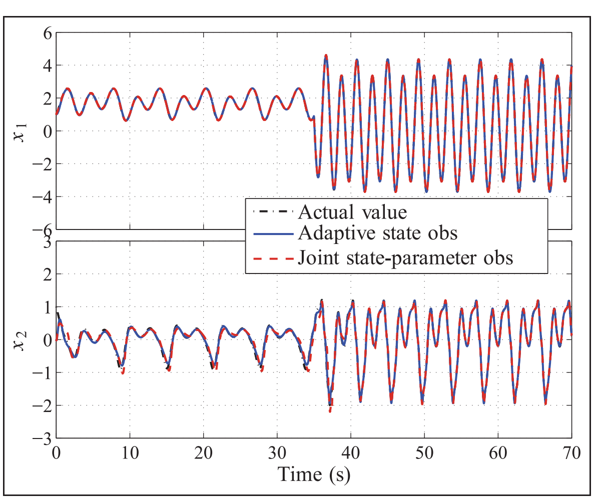

Figure 5. Actual state variables and their estimates (Example 2: Case 1).

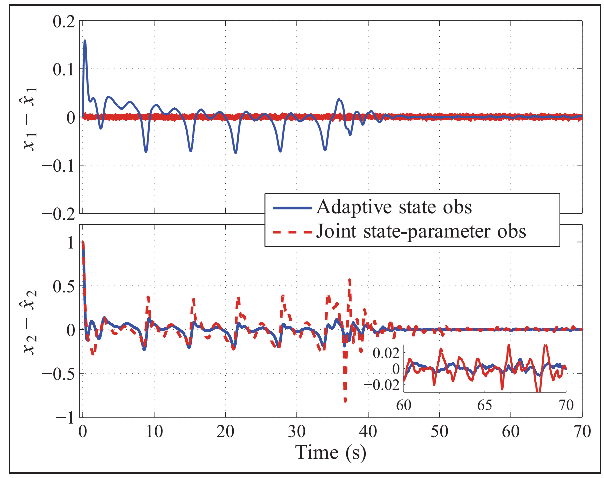

Figure 6. State estimation error (Example 2: Case 1).

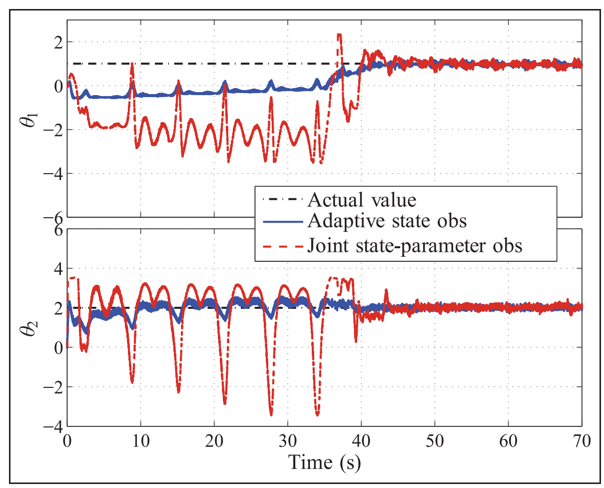

Figure 7. Actual parameters and their estimates (Example 2: Case 1).

the bounded function $\Psi(\text{sat}(x))$ (such that the restriction of $\Psi(x)$ to Ω_2 coincide with $\Psi(\text{sat}(x))$), γ_2 can be computed as $\gamma_2 = \max\left\|\frac{\partial\Psi(\text{sat}(x))}{\partial x}\right\|$. We found that we can take $\gamma_2 = 5.83$. Moving to Assumption 3, it is straightforward to confirm that the structural requirements (12) and (13) needed in the proposed approach are satisfied.

To compute the observer gains and design the adaptation law, we proceed as in the previous example. We choose $\epsilon_1 = 10^3$, $\epsilon_2 = 3 \times 10^3$, and $\Gamma = 100 \times I_2$. The resolution of (47) and (48) simultaneously, yields

$$P = \begin{bmatrix} 1.9379 & -0.2155 \\ -0.2155 & 0.1320 \end{bmatrix}, M = \begin{bmatrix} 0.9860 \\ 1.2479 \end{bmatrix},$$
$$H = 0.1320, G = 0.2155, \eta = 0.1,$$
$$L = \begin{bmatrix} 1.9066 \\ 12.5699 \end{bmatrix}$$

where L is computed from the relation (46).

It should be pointed out that, for system (50), the construction of the adaptation law in Stamnes et al. (2011) has required some complicated analytical computations to solve a system of partial differential equations. Moreover, the design in Vijayaraghavan (2013) cannot be applied to this system as the regressor is nonlinear. However, the approach proposed in Zhu and Khayati (2011) can be readily applied to system (50).

For comparison purposes, the following joint state-parameter observer, presented in Besançon (2007) for a class of SISO nonlinear systems, is also implemented

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + \bar{\Phi}(\hat{\bar{x}}, u) + \bar{\Psi}(\hat{\bar{x}})\hat{\bar{\theta}} + \lambda\Lambda^{-1}(\lambda)\left(K + \Upsilon\Upsilon^T C^T\right)(y - C\hat{x}) \\ \dot{\hat{\theta}} = \lambda^n \Upsilon^T C^T (y - X\hat{x}) \\ \dot{\Upsilon} = \lambda(A - KC) + \lambda\bar{\Psi}(\hat{\bar{x}}) \\ \hat{\bar{x}} = \hat{x} \quad if \quad ||\hat{x}|| \le X, \quad \dfrac{\hat{x}}{||\hat{x}||}X \quad \text{otherwise} \\ \hat{\bar{\theta}} = \hat{\theta} \quad if \quad ||\hat{\theta}|| \le \gamma_3, \quad \dfrac{\hat{\theta}}{||\hat{\theta}||}\gamma_3 \quad \text{otherwise} \end{cases} \tag{51}$$

The design parameters are selected as: $\lambda = 100$, $K = [1.8 \quad -0.2]^T$, $X = \sqrt{34}$ and $\gamma_3 = \sqrt{12.5}$. Notice that, for both observers, the design parameters are chosen as a good compromise between fast convergence and satisfactory behaviour under measurement noise.

To investigate the capabilities of the proposed approach when compared to the joint state-parameter observer (51), the simulations were conducted in two cases. In all of the simulations, the output y has been corrupted by an additive white noise with a zero mean value and a standard deviation of 10^{-2}, and as in Stamnes et al. (2011), we selected: $\theta = [1 \quad 2]^T$, $x(0) = [1 \quad 1]^T$, $\hat{x}(0) = [1 \quad 0]^T$ and $\hat{\theta}(0) = [0 \quad 0]^T$.

Case 1. In order to show the effect of the level of excitation generated by the input signal on the rate of convergence of the estimated variables, the simulation was tested using the input

$$u = \begin{cases} \sin(2t) & \text{for} \quad t < 35 \quad \text{s} \\ 10\sin(3t) & \text{for} \quad t \ge 35 \quad \text{s} \end{cases} \tag{52}$$

The nonparametric model uncertainties and perturbations are not considered in this case. The state estimation results are shown in Figures 5 and 6. It is seen that the excitation signal $u = 10\sin(3t)$ improves significantly the state estimation performances of both observers compared to the excitation signal $u = \sin(2t)$. However, it is clear that, for both excitations, the proposed approach provides a better estimation of the unmeasured state than the joint state-parameter observer. Remarkably, the measured state estimate provided by the joint state-parameter observer maintains its performance no matter if the system is under an excitation of sufficient level or not, but it is clear that it is noise sensitive. The parameter estimation results are depicted in Figures 7 and 8. These results show that parameter estimates of both observers are noise sensitive. Moreover, it is clear that even the excitation level generated by the signal $u = \sin(2t)$ is not enough to provoke the convergence of the parameter estimates of the joint state-parameter observer, it seems that it leads to a very slow convergence of the adaptive state observer. However, when the input signal $u = 10\sin(10t)$ is applied all of the parameter estimates go to within small vicinities of their actual values. What is interesting about this simulation test is that it shows that even the state estimate of the adaptive state observer do not need PE for their convergence, the level of excitation has an important impact on its rate of convergence.

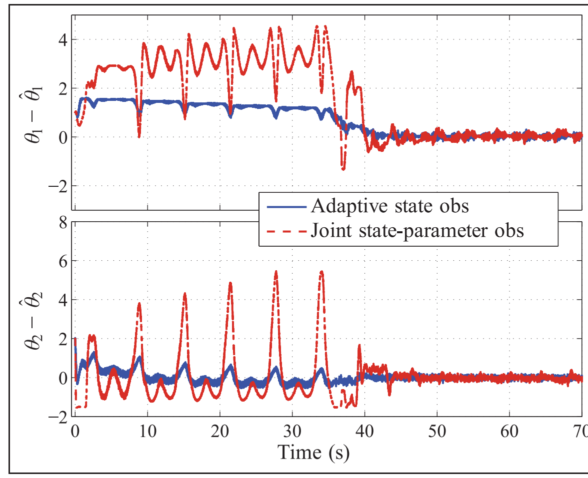

Figure 8. Parameter estimation error (Example 2: Case 1).

Case 2. This case is associated with the presence of nonparametric uncertainties and disturbances. More specifically, $\Delta(x_1, x_2, t) = 0.5(1 + \cos(x_2)\arctan(0.25x_1x_2) + 2\sin(t))$ was introduced at $t = 30$ s. The simulation was conducted using $u = 10\sin(3t) - x_1 - x_2$ which provides a sufficient level of excitation (the state feedback term is added to prevent instability due to the nonparametric model uncertainties and disturbances). The results are shown in Figures 9 to 12. It is seen that, in the absence of nonparametric model uncertainties and disturbances, both observers provide good estimations of the state and parameters. The introduction of the nonparametric model uncertainties and disturbances at $t = 30$ s severely affects the performances of the joint state-parameter observer, except for the measured state estimate. Although there is a significant degradation in the parameter estimate of adaptive state observer due to the nonparametric model uncertainties and disturbances, the state estimate is only slightly affected.

Conclusion

A new approach for adaptive state observer design is presented for a class of nonlinear systems. The proposed method does not require the system structure to satisfy the usual observer matching condition needed in the conventional adaptive state observer design. The scheme is based on a matrix equality that specifies the location of the unknown parameters with respect to the measured outputs. Through Lyapunov analysis and using the aforementioned matrix equality, the asymptotic stability of the state estimate error is proven and the observer gain calculation problem is formulated into an LMI optimization problem. Two simulation

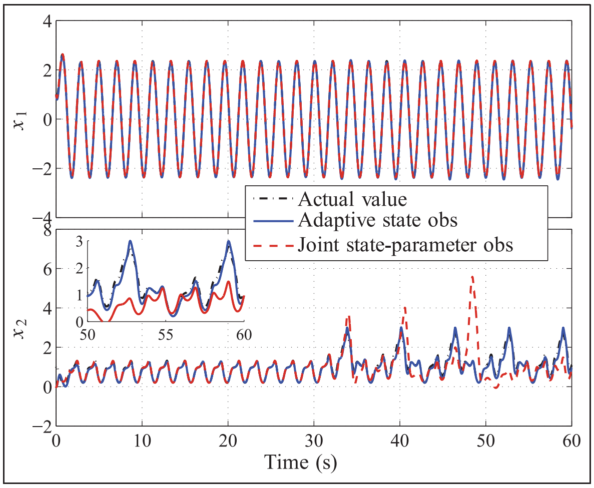

Figure 9. Actual state variables and their estimates (Example 2: Case 2).

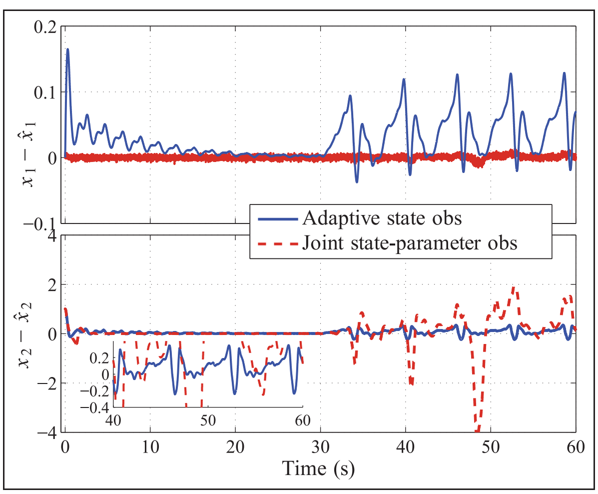

Figure 10. State estimation error (Example 2: Case 2).

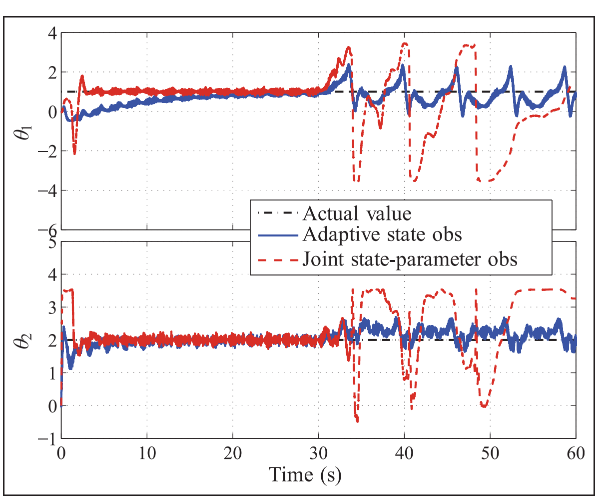

Figure 11. Actual parameters and their estimates (Example 2: Case 2).

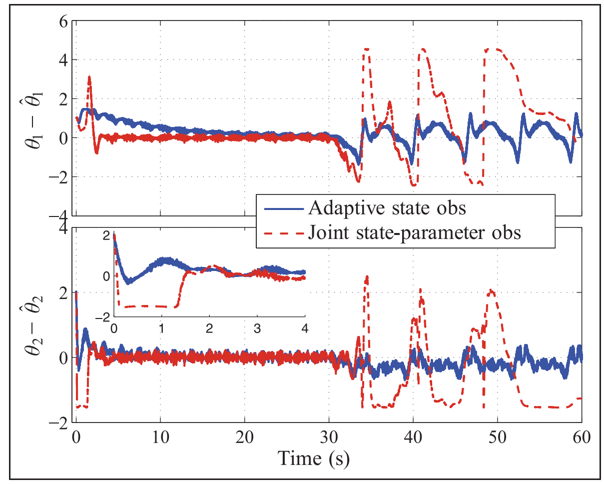

Figure 12. Parameter estimation error (Example 2: Case 2).

examples are presented to emphasize the theoretical results of this paper.

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