[14] L. Rosier, "Homogeneous Lyapunov function for homogeneous continuous vector field," *Syst. Contr. Lett.*, vol. 19, pp. 467–473, 1992.

[15] L. Rosier, "Stabilization of a system with integrator, "Thése de l'Ecole Normale Supérieure de Cachan," 1993.

[16] C. Samson, "Velocity and torque feedback control of a nonholonomic cart," in *Proc. Int. Wkshp. Adaptative Nonlinear Control: Issues Robotics*, Grenoble, France, 1990; also in *Proc. Advanced Robot Control*, vol. 162. New York: Springer-Verlag, 1991.

[17] E. D. Sontag and H. J. Sussmann, "Remarks on continuous feedback," in *Proc. 19th IEEE Conf. Decision Contr.*, Albuquerque, NM, 1980, pp. 916–921.

[18] G. C. Walsh, R. Montgomery, and S. S. Sastry, "Orientation control of the dynamic satellite," in *Proc. Amer. Contr. Conf.*, Baltimore, MD, 1994, pp. 138–142.

A Systematic Approach to Adaptive Observer Synthesis for Nonlinear Systems

Young Man Cho and Rajesh Rajamani

***Abstract*— Geometric techniques of controller design for nonlinear systems have enjoyed great success. A serious shortcoming, however, has been the need for access to full-state feedback. This paper addresses the issue of state estimation from limited sensor measurements in the presence of parameter uncertainty. An adaptive nonlinear observer is suggested for Lipschitz nonlinear systems, and the stability of this observer is shown to be related to finding solutions to a quadratic inequality involving two variables. A coordinate transformation is used to reformulate this inequality as a linear matrix inequality. A systematic algorithm is presented, which checks for feasibility of a solution to the quadratic inequality and yields an observer whenever the solution is feasible. The state estimation errors then are guaranteed to converge to zero asymptotically. The convergence of the parameters, however, is determined by a persistence-of-excitation-type constraint.**

***Index Terms*—Adaptive observer, interior point method, linear matrix inequality, nonlinear systems.**

I. Introduction

Observer design and adaptive control for nonlinear systems have both been very active fields of research during the last decade. The introduction of geometric techniques has led to great success in the development of controllers for nonlinear systems. Many attempts have been made to achieve results of equally wide applicability for state estimation and adaptation. The observer problem has, however, turned out to be much more difficult than the controller problem [1], [2].

An adaptive observer performs the twin tasks of state estimation and parameter identification. The two tasks are performed simultaneously and cannot be separated. The identification algorithm has to be defined using access to only the measured outputs and the estimated states. The state estimation algorithm has to work in the presence of uncertain parameters. This makes the problem very challenging.

The design of an adaptive observer for a linear time invariant system has been well analyzed [3]. In this case the order of the plant "n" is assumed to be known, nothing else about the plant need be known. The output of the plant is described as the output of a first-order differential equation whose input is a linear combination of "$2n$" signals. The coefficients of these signals represent the unknown parameters of the plant. The adaptive observer is also described by a similar first-order equation using the input and output of the plant, with its parameters being adjustable. An adaptation/estimation law is derived, and its uniform stability about the origin for the state estimation error can be shown without any further assumptions on the plant. If the plant is stable, the convergence of the state estimation error to zero can also be concluded. The parameters of the observer are adjusted using stable adaptation laws so that the error between the plant and observer outputs converges to zero. The convergence of the parameters to the desired values, however, depends on the persistent excitation of the input signals.

In the case of nonlinear systems, Sastry and Isidori presented results on the use of parameter adaptive control for obtaining asymptotically exact cancellation for the class of nonlinear systems which can be feedback linearized [4]. The full-state was assumed to be available, however, for the controller. Papers by Marino *et al.* on adaptive observers attempted to find a coordinate transformation so that the estimation error dynamics would be linearized in the new coordinates [5], [6]. They provide necessary and sufficient conditions for the existence of such a coordinate transformation. Even if these conditions are satisfied, the construction of the observer still remains a difficult task due to the need to solve a set of simultaneous partial differential equations to obtain the actual transformation function. An intuitively appealing and systematic treatment of the output feedback and adaptive observer problem for nonlinear systems has been developed by Kokotovic *et al.* [7]–[9]. Here the authors develop a set of tools which the user can attempt to customize for his specific problem. There has also been work by authors to propose adaptive observers for very special classes of nonlinear systems [10], [11].

The present work deals with a fairly general class of nonlinear systems, in which the nonlinearities are assumed to be Lipschitz. A systematic algorithm is provided which checks for the feasibility of an asymptotically stable adaptive observer. If the feasibility condition is satisfied, the algorithm provides the observer gains.

II. Background

This section presents results which will be used in the construction of our proposed observer.

A. Adaptive Observers for a Class of Nonlinear Systems

We begin with the adaptive observer proposed for a class of nonlinear systems in [15]. The class of systems we consider are linear in the unknown parameters and nonlinear in the states, with the nonlinearities assumed to be Lipschitz as described in (1) below. This is a fairly general class, since most nonlinearities can be bounded in a Lipschitz manner if the states can be assumed to be bounded. Further, many nonlinearities, like the sinusoidal terms encountered in robotics, are globally Lipschitz. The success of the adaptive observer method as outlined below, however, depends on being able to find a positive definite matrix $\boldsymbol{P}$ and an observer gain matrix $\boldsymbol{L}$ to satisfy (2) and (6). For proof of the Theorem, refer to [15].

Theorem II.1: Consider the class of nonlinear dynamical systems described by

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{u}) + \boldsymbol{b}\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{\theta} \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} \end{aligned} \tag{1}$$

Manuscript received April 4, 1995; revised January 29, 1996 and June 24, 1996.

The authors are with the United Technologies Research Center, East Hartford, CT 06108 USA.

Publisher Item Identifier S 0018-9286(97)02810-9.

where

$$\boldsymbol{x} \in \Re^n, \quad \boldsymbol{y} \in \Re^m, \boldsymbol{\theta} \in \Re^p, \quad \boldsymbol{b} \in \Re^{n\times s}, \quad \boldsymbol{C} \in \Re^{m\times n}$$
$$\boldsymbol{f} : \Re^n \to \Re^{s\times p}, \quad \boldsymbol{\Phi} : \Re^n \to \Re^n.$$

If:

1) there exists a positive definite symmetric matrix $\boldsymbol{P}$ such that

$$\boldsymbol{b}^T\boldsymbol{P} = \boldsymbol{C}_1 \tag{2}$$

where each row of $\boldsymbol{C}_1$ lies in span (rows of $\boldsymbol{C}$);

2) $\boldsymbol{\Phi}$ and $\boldsymbol{f}$ are Lipschitz in x with Lipschitz constants γ_1 and γ_2, respectively, i.e.,

$$\|\boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{\Phi}(\hat{\boldsymbol{x}},\boldsymbol{u})\| < \gamma_1\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| \tag{3}$$

and

$$\|\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{f}(\hat{\boldsymbol{x}},\boldsymbol{u})\| < \gamma_2\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|; \tag{4}$$

3) the vector of unknown parameters $\boldsymbol{\theta}$ is bounded in the following sense:

$$\|\boldsymbol{\theta}\|_2 \leq \gamma_3; \tag{5}$$

4) a gain matrix $\boldsymbol{L}$ can be chosen such that

$$\gamma_1 + \gamma_2\gamma_3\|\boldsymbol{b}\| < \frac{\lambda_{\min}(\boldsymbol{Q})}{2\lambda_{\max}(\boldsymbol{P})} \tag{6}$$

where $\boldsymbol{Q}$ is a positive definite symmetric matrix satisfying the Lyapunov equation

$$(\boldsymbol{A} - \boldsymbol{LC})^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A} - \boldsymbol{LC}) = -\boldsymbol{Q};$$

then the adaptive observer

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{\Phi}(\hat{\boldsymbol{x}},\boldsymbol{u}) + \boldsymbol{b}\boldsymbol{f}(\hat{\boldsymbol{x}},\boldsymbol{u})\hat{\boldsymbol{\theta}} + \boldsymbol{L}[\boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}}] \tag{7}$$
$$\dot{\hat{\boldsymbol{\theta}}} = \frac{\boldsymbol{f}(\hat{\boldsymbol{x}},\boldsymbol{u})^T\boldsymbol{C}_1\tilde{\boldsymbol{x}}}{\rho}, \quad \text{for} \quad \rho > 0 \tag{8}$$

is convergent, $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\boldsymbol{x}} \to 0$ as $t \to \infty$ and $[\boldsymbol{bf}(\boldsymbol{x},\boldsymbol{u})\boldsymbol{\theta} - \boldsymbol{bf}(\hat{x},\boldsymbol{u})\hat{\boldsymbol{\theta}}] \to 0$ as $t \to \infty$.

Remarks:

1) The above theorem provides sufficient conditions for checking the stability of the observer, once it has been designed. Designing the observer, however, is no trivial task. Finding matrices $\boldsymbol{P}$ and $\boldsymbol{L}$ to satisfy (2) and (6) is not intuitive. Indeed, there is no straightforward way of choosing $\boldsymbol{L}$ to satisfy (6). For instance, placing the eigenvalues of $\boldsymbol{A} - \boldsymbol{LC}$ far into the left half-plane will not ensure that (6) is satisfied [11], [15].
2) The existence of a positive definite matrix $\boldsymbol{P}$ to satisfy (2) is guaranteed when at least some of the measured outputs are such that the transfer functions between these outputs and the unknown parameters are dissipative or strictly positive real [3]. However, the same matrix $\boldsymbol{P}$ has to be used in (6) for ensuring that $\boldsymbol{L}$ satisfies the required conditions.
3) The Lipschitz constant can be well approximated by the bound on the 2-norm of the Jacobian of the nonlinear system. The norm of the Jacobian provides a tight bound on the Lipschitz constant for a smooth and differentiable nonlinear system.
4) The region of stability for the above adaptive observer depends on the region over which the Lipschitz constant is defined. If the system is globally Lipschitz, the adaptive observer is guaranteed to be globally convergent. When the system is locally Lipschitz, the region of stability can be computed, and its computation is shown in Section III.

B. Linear Matrix Inequality

A linear matrix inequality (LMI) has the form

$$\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{F}_0 + \sum_{i=1}^{m} x_i\boldsymbol{F}_i > 0 \tag{9}$$

where $\boldsymbol{x} \in \Re^m$ is the variable and the symmetric matrices $\boldsymbol{F}_i = \boldsymbol{F}_i^T \in \Re^{n\times n}, i = 0,\cdots,m$ are given. The inequality symbol in (9) means that $\boldsymbol{F}(\boldsymbol{x})$ is positive-definite, i.e., $\boldsymbol{u}^T\boldsymbol{F}(x)\boldsymbol{u} > 0$.

Now consider inequalities in which the variables to be solved for are matrices, e.g., the Lyapunov inequality

$$\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{PA} < 0$$

where $\boldsymbol{A} \in \Re^{n\times n}$ is given and $\boldsymbol{P} = \boldsymbol{P}^T$ is the variable. This problem can be reformulated into the form (9) as follows. Let $\boldsymbol{P}_1,\cdots,\boldsymbol{P}_m$ be a basis for symmetric $n \times n$ matrices $(m = n(n+1)/2)$. Then, take $\boldsymbol{F}_0 = 0$ and $\boldsymbol{F}_i = -\boldsymbol{A}^T\boldsymbol{P}_i - \boldsymbol{P}_i\boldsymbol{A}$. Thus the problem of finding $\boldsymbol{P}$ now becomes one of solving (9) for $\boldsymbol{x}$.

An LMI of the form (9) can be solved in a computationally efficient manner using techniques like the interior point method [12], [13]. For a detailed discussion of LMI's and their solutions, refer to [13] and the references therein. Special purpose software packages like those designed by Boyd *et al.* from Stanford University can be used for the solution of such LMI's [14].

Nonlinear (convex) inequalities can be converted into LMI's using Schur complements. The basic idea is as follows: the LMI

$$\begin{bmatrix} \boldsymbol{Q}(\boldsymbol{x}) & \boldsymbol{S}(\boldsymbol{x}) \\ \boldsymbol{S}(\boldsymbol{x})^T & \boldsymbol{R}(\boldsymbol{x}) \end{bmatrix} > 0 \tag{10}$$

where $\boldsymbol{Q}(\boldsymbol{x}) = \boldsymbol{Q}(\boldsymbol{x})^T, \boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{R}(\boldsymbol{x})^T$, and $\boldsymbol{S}(\boldsymbol{x})$ depend affinely on $\boldsymbol{x}$, is equivalent to

$$\boldsymbol{R}(\boldsymbol{x}) > 0, \boldsymbol{Q}(\boldsymbol{x}) - \boldsymbol{S}(\boldsymbol{x})\boldsymbol{R}(\boldsymbol{x})^{-1}\boldsymbol{S}(\boldsymbol{x})^T > 0. \tag{11}$$

In other words, the set of nonlinear inequalities (11) can be represented as the LMI (10).

III. Adaptive Observer Synthesis

We now present our main result on designing an adaptive observer for system (1). The following theorem provides sufficient conditions for convergence of the state estimates. Following the proof, these conditions are converted into a form in which an efficient algorithm (interior point method) can be used for the actual observer design process.

Theorem III.1: Consider the same class of nonlinear dynamical systems described by (1) and (3)–(5). If there exists a positive definite symmetric matrix $\boldsymbol{P}$ and a matrix $\boldsymbol{L}$ such that

$$(\boldsymbol{A} - \boldsymbol{LC})^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A} - \boldsymbol{LC}) + (\gamma_1 + \gamma_2\gamma_3\|\boldsymbol{b}\|)PP + (\gamma_1 + \gamma_2\gamma_3)\boldsymbol{I} < \boldsymbol{O} \tag{12}$$

$$\boldsymbol{b}^T PC^{\perp} = O \tag{13}$$

then the adaptive observer (7), (8) is convergent, $\tilde{\boldsymbol{x}} \to 0$ as $t \to \infty$ and $[\boldsymbol{bf}(\boldsymbol{x},\boldsymbol{u})\boldsymbol{\theta} - \boldsymbol{bf}(\hat{x},\boldsymbol{u})\hat{\boldsymbol{\theta}}] \to 0$ as $t \to \infty$. $\boldsymbol{C}^{\perp}$ is the orthogonal projection on to $\text{null}(\boldsymbol{C})$ [16].

Proof—Part A—Lyapunov Stability): Let $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ be the estimation error. The error dynamics are then given by

$$\dot{\tilde{\boldsymbol{x}}} = (\boldsymbol{A} - \boldsymbol{LC})\tilde{\boldsymbol{x}} + \Phi(\boldsymbol{x},\boldsymbol{u}) - \Phi(\hat{\boldsymbol{x}},\boldsymbol{u}) + \boldsymbol{bf}(\boldsymbol{x},\boldsymbol{u})\boldsymbol{\theta} - \boldsymbol{bf}(\hat{\boldsymbol{x}},\boldsymbol{u})\hat{\boldsymbol{\theta}}. \tag{14}$$

Consider the Lyapunov function candidate $V = \tilde{\boldsymbol{x}}^T \boldsymbol{P}\tilde{\boldsymbol{x}} + \rho\tilde{\boldsymbol{\theta}}^T\tilde{\boldsymbol{\theta}}$. Then

$$\begin{aligned}\dot{V} =& \tilde{\boldsymbol{x}}^T[(\boldsymbol{A}-\boldsymbol{LC})^T\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{A}-\boldsymbol{LC})]\tilde{\boldsymbol{x}}\\ &+2\tilde{\boldsymbol{x}}^T\boldsymbol{P}[\boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{u})-\boldsymbol{\Phi}(\hat{\boldsymbol{x}},\boldsymbol{u})]\\ &+2[\boldsymbol{bf}(\boldsymbol{x},\boldsymbol{u})\boldsymbol{\theta}-\boldsymbol{bf}(\hat{\boldsymbol{x}},\boldsymbol{u})\hat{\boldsymbol{\theta}}]^T\boldsymbol{P}\tilde{\boldsymbol{x}}+2\rho\tilde{\boldsymbol{\theta}}^T\dot{\tilde{\boldsymbol{\theta}}}\\ \le& \tilde{\boldsymbol{x}}^T[(\boldsymbol{A}-\boldsymbol{LC})^T\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{A}-\boldsymbol{LC})]\tilde{\boldsymbol{x}}+2\gamma_1\|\tilde{\boldsymbol{x}}^T\boldsymbol{P}\|\|\tilde{\boldsymbol{x}}\|\\ &+2[\boldsymbol{bf}(\boldsymbol{x},\boldsymbol{u})\boldsymbol{\theta}-\boldsymbol{bf}(\hat{\boldsymbol{x}},\boldsymbol{u})\boldsymbol{\theta}]^T\boldsymbol{P}\tilde{\boldsymbol{x}}\\ &+2[\boldsymbol{bf}(\hat{\boldsymbol{x}},\boldsymbol{u})\tilde{\boldsymbol{\theta}}]^T\boldsymbol{P}\tilde{\boldsymbol{x}}+2\rho\tilde{\boldsymbol{\theta}}^T\dot{\tilde{\boldsymbol{\theta}}}\\ \le& \tilde{\boldsymbol{x}}^T[(\boldsymbol{A}-\boldsymbol{LC})^T\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{A}-\boldsymbol{LC})]\tilde{\boldsymbol{x}}\\ &+\gamma_1(\tilde{\boldsymbol{x}}^T\boldsymbol{PP}\tilde{\boldsymbol{x}}+\tilde{\boldsymbol{x}}^T\tilde{\boldsymbol{x}})+2\gamma_2\gamma_3\|\boldsymbol{b}\|\|\tilde{\boldsymbol{x}}\|\|\boldsymbol{P}\tilde{\boldsymbol{x}}\|\\ &+2[\boldsymbol{bf}(\hat{\boldsymbol{x}},\boldsymbol{u})\tilde{\boldsymbol{\theta}}]^T\boldsymbol{P}\tilde{\boldsymbol{x}}+2\rho\tilde{\boldsymbol{\theta}}^T\dot{\tilde{\boldsymbol{\theta}}}\\ \le& \tilde{\boldsymbol{x}}^T[(\boldsymbol{A}-\boldsymbol{LC})^T\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{A}-\boldsymbol{LC})+\gamma_1\boldsymbol{PP}+\gamma_1\boldsymbol{I}\\ &+\gamma_2\gamma_3\|\boldsymbol{b}\|\boldsymbol{PP}+\gamma_2\gamma_3\boldsymbol{I}]\tilde{\boldsymbol{x}}\\ &+2[\boldsymbol{bf}(\hat{\boldsymbol{x}},\boldsymbol{u})\tilde{\boldsymbol{\theta}}]^T\boldsymbol{P}\tilde{\boldsymbol{x}}+2\rho\tilde{\boldsymbol{\theta}}^T\dot{\tilde{\boldsymbol{\theta}}}\end{aligned}\tag{15}$$

where the following inequality was used:

$$2\|\boldsymbol{P}\tilde{\boldsymbol{x}}\|\|\tilde{\boldsymbol{x}}\| \le \tilde{\boldsymbol{x}}^T\boldsymbol{PP}\tilde{\boldsymbol{x}}+\tilde{\boldsymbol{x}}^T\tilde{\boldsymbol{x}}.$$

Now we determine the adaptation law from the last inequality by setting

$$2[\boldsymbol{bf}(\hat{\boldsymbol{x}},\boldsymbol{u})\tilde{\boldsymbol{\theta}}]^T\boldsymbol{P}\tilde{\boldsymbol{x}}+2\rho\tilde{\boldsymbol{\theta}}^T\dot{\tilde{\boldsymbol{\theta}}}=0.$$

In order to satisfy this equation without knowing $\tilde{\boldsymbol{\theta}}$, we have

$$\begin{aligned}\dot{\tilde{\boldsymbol{\theta}}} &= -\frac{\boldsymbol{f}(\hat{\boldsymbol{x}},\boldsymbol{u})^T\boldsymbol{b}^T\boldsymbol{P}\tilde{\boldsymbol{x}}}{\rho}\\ &= -\frac{\boldsymbol{f}(\hat{\boldsymbol{x}},\boldsymbol{u})^T\boldsymbol{C}_1\tilde{\boldsymbol{x}}}{\rho}\end{aligned}\tag{16}$$

where $\boldsymbol{C}_1=\boldsymbol{b}^T\boldsymbol{P}$. Here the term $\boldsymbol{C}_1\tilde{\boldsymbol{x}}$ can be computed if and only if each row of $\boldsymbol{C}_1$ lies in span (rows of $\boldsymbol{C}$). This condition can be compactly written as $\boldsymbol{b}^T\boldsymbol{PC}^\perp=\boldsymbol{O}$, where $\boldsymbol{CC}^\perp=\boldsymbol{O}$ or $\boldsymbol{C}^\perp$ is the orthogonal projection on to null $(\boldsymbol{C})$ [16]. Later in this section, we show how $\boldsymbol{C}^\perp$ and $\boldsymbol{C}_1\tilde{\boldsymbol{x}}$ can be computed using the singular value decomposition (SVD) of $\boldsymbol{C}$.

With (16), (15) reduces to

$$\begin{aligned}\dot{V} \le& \tilde{\boldsymbol{x}}^T[(\boldsymbol{A}-\boldsymbol{LC})^T\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{A}-\boldsymbol{LC})\\ &+(\gamma_1+\gamma_2\gamma_3\|\boldsymbol{b}\|)\boldsymbol{PP}+(\gamma_1+\gamma_2\gamma_3)\boldsymbol{I}]\tilde{\boldsymbol{x}}.\end{aligned}$$

To make $\dot{V}$ negative semidefinite, we end up with the following inequality:

$$\begin{aligned}&(\boldsymbol{A}-\boldsymbol{LC})^T\boldsymbol{P}+P(\boldsymbol{A}-\boldsymbol{LC})+(\gamma_1+\gamma_2\gamma_3\|\boldsymbol{b}\|)\boldsymbol{PP}\\ &\quad+(\gamma_1+\gamma_2\gamma_3)\boldsymbol{I}<\boldsymbol{O}\end{aligned}$$

with the equality constraint $\boldsymbol{b}^T\boldsymbol{PC}^\perp=\boldsymbol{O}$.

Part B—Convergence of $\tilde{\boldsymbol{x}}$: Let β be such that

$$\begin{aligned}&(\boldsymbol{A}-\boldsymbol{LC})^T\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{A}-\boldsymbol{LC})+(\gamma_1+\gamma_2\gamma_3\|\boldsymbol{b}\|)\boldsymbol{PP}\\ &\quad+(\gamma_1+\gamma_2\gamma_3)\boldsymbol{I}\le-\beta\boldsymbol{I}.\end{aligned}$$

Then

$$\dot{V}\le-\beta\tilde{\boldsymbol{x}}^T\tilde{\boldsymbol{x}}.$$

Integrating

$$V(t)\le V(0)-\beta\int_0^t\tilde{\boldsymbol{x}}^T\tilde{\boldsymbol{x}}\,dt.$$

Since $V(t)\in L_\infty$ and $V(0)$ is finite, this implies that $\tilde{\boldsymbol{x}}\in L_2$. Also, from (14) for $\dot{\tilde{\boldsymbol{x}}}$, we see that $\dot{\tilde{\boldsymbol{x}}}\in L_\infty$ (this is because $\tilde{\boldsymbol{x}}\in L_\infty$ and both $\Phi(\boldsymbol{x},\boldsymbol{u})$ and $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})$ are Lipschitz). Thus, $\tilde{\boldsymbol{x}}\in L_\infty, \tilde{\boldsymbol{x}}\in L_2$ and $\dot{\tilde{\boldsymbol{x}}}\in L_\infty$. Therefore, by Barbalat's lemma [3], $\tilde{\boldsymbol{x}}\to 0$.

Part C—Convergence of $\tilde{\boldsymbol{\theta}}$:

$$\int_0^\infty\dot{\tilde{\boldsymbol{x}}}\,dt=\lim_{t\to\infty}\tilde{\boldsymbol{x}}(t)-\tilde{\boldsymbol{x}}(0)=-\tilde{\boldsymbol{x}}(0)$$

which is finite. Also, from (14), using the Lipschitz continuity of $\boldsymbol{\Phi}, \dot{\tilde{\boldsymbol{x}}}$ is uniformly continuous. Hence, again by Barbalat's lemma [3], $\dot{\tilde{\boldsymbol{x}}}\to 0$. From (14), this implies that $[\boldsymbol{bf}(\boldsymbol{x},\boldsymbol{u})\boldsymbol{\theta}-\boldsymbol{bf}(\hat{\boldsymbol{x}},\boldsymbol{u})\hat{\theta}]\to 0[$. ■

Remark: In terms of parameter convergence, all we have been able to prove is that $[\boldsymbol{bf}(\boldsymbol{x},\boldsymbol{u})\boldsymbol{\theta}-\boldsymbol{bf}(\hat{\boldsymbol{x}},\boldsymbol{u})\hat{\theta}]\to 0$. Since $\hat{\boldsymbol{x}}\to\boldsymbol{x}$ and $\|\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})-\boldsymbol{f}(\hat{\boldsymbol{x}},\boldsymbol{u})\|<\gamma_2\|\boldsymbol{x}-\hat{\boldsymbol{x}}\|$, it is easy to prove that $\boldsymbol{bf}(\boldsymbol{x},\boldsymbol{u})(\theta-\hat{\boldsymbol{\theta}})\to 0$. This leads to the following persistency of excitation condition.[1]

If $\exists\alpha_0,\alpha_1,\delta>0$ such that

$$\begin{aligned}\alpha_0\boldsymbol{I}\ge&\int_{t_0}^{t_0+\delta}\boldsymbol{bf}(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\boldsymbol{f}(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))^T\boldsymbol{b}^T\,d\tau\\ \ge&\alpha_1\boldsymbol{I},\quad\text{for all } t_0\end{aligned}$$

then $\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\to 0$.

Now we show how $\boldsymbol{P}$ and $\boldsymbol{L}$ satisfying (12), (13) can be computed in a systematic manner. We propose to use the LMI solution techniques of Section B. However, it remains to prove that the nonlinear, nonstrict inequality (12), (13) (with respect to $\boldsymbol{P}$ and $\boldsymbol{L}$) can be recast into a linear, strict inequality in order for the interior point methods to be directly applicable [12], [13]. In the following, we discuss these two issues: nonlinearity and nonstrictness.

Nonlinearity: Inequality (12) is first rewritten as

$$\begin{aligned}&\boldsymbol{A}^T\boldsymbol{P}-\boldsymbol{C}^T\boldsymbol{X}^T+\boldsymbol{PA}-\boldsymbol{XC}+(\gamma_1+\gamma_2\gamma_3\|\boldsymbol{b}\|)\boldsymbol{PP}\\ &\quad+(\gamma_1+\gamma_2\gamma_3)\boldsymbol{I}<\boldsymbol{O}\end{aligned}\tag{17}$$

where a coordinate transformation $\boldsymbol{X}=\boldsymbol{PL}$ is used. The resulting inequality (17) is equivalent to (12) since $\boldsymbol{P}>\boldsymbol{O}$, and the coordinate transformation is one-to-one. The convex, nonlinear inequality (17) is converted to a convex, linear inequality using the Schur complement as described in Section B (10), (11). Note that (18), as shown at the bottom of the page, is linear and convex with respect to its variables $\boldsymbol{P}$ and $\boldsymbol{X}$.

[1] This condition is analogous to the one in [3].

$$\begin{bmatrix}\boldsymbol{A}^T\boldsymbol{P}-\boldsymbol{C}^T\boldsymbol{X}^T+\boldsymbol{PA}-\boldsymbol{XC}+(\gamma_1+\gamma_2\gamma_3)\boldsymbol{I} & (\gamma_1+\gamma_2\gamma_3\|\boldsymbol{b}\|)^{1/2}\boldsymbol{P}\\ (\gamma_1+\gamma_2\gamma_3\|\boldsymbol{b}\|)^{1/2}\boldsymbol{P} & -\boldsymbol{I}\end{bmatrix}<\boldsymbol{O}\tag{18}$$

$$\begin{bmatrix}\boldsymbol{A}^T\boldsymbol{P}-\boldsymbol{C}^T\boldsymbol{X}^T+\boldsymbol{PA}-\boldsymbol{XC}+(\gamma_1+\gamma_2\gamma_3+\alpha^2)\boldsymbol{I} & (\gamma_1+\gamma_2\gamma_3\|\boldsymbol{b}\|)^{1/2}\boldsymbol{P}\\ (\gamma_1+\gamma_2\gamma_3\|\boldsymbol{b}\|)^{1/2}\boldsymbol{P} & -\boldsymbol{I}\end{bmatrix}<\boldsymbol{O}\tag{20}$$

Nonstrictness: Equality constraint (13) can be easily eliminated to make the resulting LMI strict. Since $\boldsymbol{b}$ and $\boldsymbol{C}$ are known *a priori*, we can find a set of matrices $\{\boldsymbol{P}_i\}$ which form a basis for $\boldsymbol{P}$ such that $\boldsymbol{b}^T\boldsymbol{P}\boldsymbol{C}^{\perp} = \boldsymbol{O}$. This set of $\boldsymbol{P}_i$ is then used to rewrite (18) in the form (9) and solve for the coefficients.

Once the nonlinearity and nonstrictness issues have been addressed, the interior point method is used to solve the resulting strict LMI and to compute $\boldsymbol{P}$ and $\boldsymbol{L}^2$ if there exists a feasible solution.

In the remainder of this paper, we discuss three important issues on the implementation of the adaptive observer: computation of $\boldsymbol{C}^{\perp}$ and $\boldsymbol{C}_1\tilde{\boldsymbol{x}}$, rate of convergence, and region of stability.

Computation of $\boldsymbol{C}^{\perp}$ and $\boldsymbol{C}_1\tilde{\boldsymbol{x}}$: Suppose $\boldsymbol{C} = \boldsymbol{U\Sigma V}^T \in \Re^{m\times n}$ is the SVD of $\boldsymbol{C}$ and that $r = \operatorname{rank}(\boldsymbol{C})$. If we have the $\boldsymbol{U}$ and $\boldsymbol{V}$ partitionings

$$\boldsymbol{U} = [\underset{r}{\boldsymbol{U}_r} \quad \underset{m-4}{\tilde{\boldsymbol{U}}_r}], \quad \boldsymbol{V} = [\underset{r}{\boldsymbol{V}_r} \quad \underset{n-r}{\tilde{\boldsymbol{V}}_r}]$$

then $\tilde{\boldsymbol{V}}_r\tilde{\boldsymbol{V}}_r^T = \boldsymbol{C}^{\perp}$ [the orthogonal projection on to null $(\boldsymbol{C})$].

$\boldsymbol{C}_1\tilde{\boldsymbol{x}}$ can be computed from $\boldsymbol{C}\tilde{\boldsymbol{x}}$ if there exists a coordinate transformation matrix $\boldsymbol{T}$ such that $\boldsymbol{C}_1 = \boldsymbol{TC}$. With the pseudo-inverse of $\boldsymbol{C}(\boldsymbol{C}^{\dagger}), \boldsymbol{T}$ can be computed as $\boldsymbol{C}_1\boldsymbol{C}^{\dagger}$. This can be verified in the following way:

$$\boldsymbol{TC} = \boldsymbol{C}_1\boldsymbol{C}^{\dagger}\boldsymbol{C} = \boldsymbol{C}_1\boldsymbol{V}_r\boldsymbol{V}_r^T = \boldsymbol{C}_1$$

where the last equality comes from the fact that $\boldsymbol{V}_r\boldsymbol{V}_r^T$ is the orthogonal projection on to $\operatorname{range}(\boldsymbol{C}^T)$ (row span of $\boldsymbol{C}$ that includes row span of $\boldsymbol{C}_1$) [16].

Rate of Convergence: It is worthwhile to note that the LMI approach in (17) and (18) can be easily recast in order to design an observer that guarantees a certain convergence rate for the adaptive observer. Satisfying the inequality

$$\begin{aligned}\boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{C}^T\boldsymbol{X}^T + \boldsymbol{PA} - \boldsymbol{XC} + (\gamma_1 + \gamma_2\gamma_3\|\boldsymbol{b}\|)\boldsymbol{PP}\\ + (\gamma_1 + \gamma_2\gamma_3)\boldsymbol{I} < -\alpha^2\boldsymbol{I}\end{aligned} \tag{19}$$

ensures a minimum convergence rate of α. Inequality (19) can be recast into the LMI (20), as shown at the bottom of the previous page.

Region of Stability: Assume that the Lipschitz constants γ_1 and γ_2 for $\boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{u})$ and $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})$ are defined over the following local region ($\|\boldsymbol{x}\| < \epsilon$ and $\|\hat{\boldsymbol{x}}\| < \epsilon$):

$$\|\boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{\Phi}(\hat{\boldsymbol{x}},\boldsymbol{u})\| < \gamma_1\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|$$

and

$$\|\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{f}(\hat{\boldsymbol{x}},\boldsymbol{u})\| < \gamma_2\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|.$$

If the operating region is such that $\|\boldsymbol{x}\| < \epsilon$ and $\|\hat{\boldsymbol{x}}\| < \epsilon$ are always satisfied, then we have

$$\begin{aligned}&\dot{V} < 0, \quad V = \tilde{\boldsymbol{x}}^T\boldsymbol{P}\tilde{\boldsymbol{x}}\\ &V > 0, \dot{V} < 0 \Rightarrow V(t) < V(0).\end{aligned}$$

As a result

$$\begin{aligned}&\lambda_{\min}(\boldsymbol{P})\|\tilde{\boldsymbol{x}}\|^2 \le V(t) \le V(0)\\ &\|\tilde{\boldsymbol{x}}(t)\| \le \sqrt{\frac{V(0)}{\lambda_{\min}(\boldsymbol{P})}} \le \|\tilde{\boldsymbol{x}}(0)\|\sqrt{\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}}.\end{aligned}$$

Now, $\|\hat{\boldsymbol{x}}\| \le \|\tilde{\boldsymbol{x}}\| + \|\boldsymbol{x}\|$. In order to guarantee that $\|\hat{\boldsymbol{x}}\| < \epsilon, \boldsymbol{x}$ must satisfy that $\|\tilde{\boldsymbol{x}}\| + \|\boldsymbol{x}\| < \epsilon$ and therefore

$$\|\boldsymbol{x}\| < \epsilon - \|\tilde{\boldsymbol{x}}(0)\|\sqrt{\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}}.$$

This is the region of stability of the observer. If the nonlinearities are global, the adaptive observer is also globally stable ($\epsilon = \infty$).

[2] In fact, $\boldsymbol{P}$ and $\boldsymbol{X}$ are first computed.

IV. Concluding Remarks

The work presented in this paper represents a systematic algorithm for adaptive observer synthesis for nonlinear systems. A numerically efficient interior point method is used to solve an inequality for obtaining observer gains. The method will check for the feasibility of a (numerical) solution to an inequality and then provide observer gains which ensure that state estimates converge to the correct values. A nonlinear function involving the parameter estimation errors converges to zero. Convergence of all the parameters, however, depends on persistence of excitation.

The practical use of this type of adaptive observer has been demonstrated experimentally for an automotive suspension application in [17]. In [17], this type of adaptive observer was used to estimate the parameters, which were then subsequently used in closed-loop control.

References

[1] A. Isidori, *Nonlinear Control Systems: An Introduction*. New York: Springer-Verlag, 1989.
[2] A. J. Krener and A. Isidori, "Linearization by output injection and nonlinear observers," *Syst. Contr. Lett.*, vol. 3, pp. 47–52, 1983.
[3] K. S. Narendra and A. Annaswamy, *Stable Adaptive Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1989.
[4] S. Sastry and A. Isidori, "Adaptive control of linearizable systems," *IEEE Trans. Automat. Contr.*, vol. 34, no. 11, pp. 1123–1131, 1989.
[5] R. Marino, "Adaptive observers for single-ouput nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 35, no. 9, pp. 1054–1058, 1990.
[6] R. Marino and P. Tomei, "Adaptive observers for a class of multi-output nonlinear systems," *Int. J. Adaptive Contr. Signal Processing*, vol. 6, pp. 353–365, 1992.
[7] P. V. Kokotovic, "The joy of feedback: Nonlinear and adaptive," Bode Prize Lecture, Amer. Contr. Conf., 1991.
[8] M. Krstic and P. V. Kokotovic, "Observer-based schemes for adaptive nonlinear state feedback control," *Int. J. Contr.*, vol. 59, no. 6, pp. 1373–1381, 1994.
[9] I. Kanellakopoulos, P. V. Kokotovic, and A. S. Morse, "A toolkit for nonlinear feedback design," *Syst. Contr. Lett.*, vol. 18, pp. 83–92, 1992.
[10] K. Yi and J. K. Hedrick, "Observer based identification of nonlinear system parameters," *ASME J. Dynamic Syst., Measurement Contr.*, 1994.
[11] S. Raghavan, "Observers and compensators for nonlinear systems, with application to flexible-joint robots," Ph.D. dissertation, Univ. California, Berkeley, 1992.
[12] Y. Nesterov and A. Nemirovskii, "Interior-point polynomial methods in convex programming," vol. 13 of *Studies in Applied Mathematics*. Philadelphia, PA: SIAM, 1994.
[13] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, "Linear matrix inequalities in system and control theory," vol. 15 of *Studies in Applied Mathematics*. Philadelphia, PA: SIAM, 1994.
[14] S. Wu and S. Boyd, "Design and implementation of a parser/solver for semidefinite programs with matrix structure," in *Proc. Joint Conf. Contr. Appl. Intelligent Contr. Computer Aided Contr. Syst. Design*, Dearborn, MI, Sept. 1996, pp. 240–245.
[15] R. Rajamani, "Observers for nonlinear systems, with application to active automotive suspensions," Ph.D. dissertation, Univ. California, Berkeley, 1993.
[16] G. Golub and C. Van Łoan, *Matrix Computations*. Baltimore, MD: Johns Hopkins Univ. Press, 1989, pp. 75–76.
[17] R. Rajamani and J. K. Hedrick, "Adaptive observer for active automotive suspensions," in *Proc. American Contr. Conf.*, San Francisco, CA, June 1993, pp. 706–710.