Cross Spectral Analysis of Nonstationary Processes

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Abstract—This paper is concerned with the generalization of stationary cross spectral analysis methods to a class of nonstationary processes, specifically, the class of semistationary finite energy processes possessing sample functions that are of finite energy almost surely. A new quantity called the time–frequency coherence (TFC) is defined and demonstrates that its properties are analogous to those possessed by the stationary coherence function. The problem of estimating the TFC by using elements of Cohen's class of joint time-frequency representations is investigated. In particular, it is shown that the only admissible estimators are those based on the class of time-frequency smoothed periodograms. Thus the familiar procedure of segmentation and (smoothed) short-time Fourier analysis cannot be improved upon (within the framework considered here) by the use of the higher resolution nonparametric time-frequency methods. Procedures for selection of the appropriate estimators and a possible application are suggested.

I. Introduction

THE PROBLEM of cross spectral analysis of two random processes will be considered. The results presented represent an attempt to deal with the generalization of the cross spectral analysis of weakly stationary processes to a class of nonstationary processes. As has been argued [2], [3] the assumption of stationarity of a random process is, in general, inadequate for dealing with measuring the properties of physical systems. Thus, methods based on this assumption are generally inappropriate for cross spectral analysis of signals derived from measurements made on physical systems. To overcome these difficulties, a new quantity that is termed the time–frequency coherence (TFC) is defined, and the problem of constructing suitable estimators for this quantity is addressed. Although the TFC may be defined for the general class of harmonized processes as introduced by Loeve [4], this paper shall restrict attention to the subclass of finite energy processes to avoid the mathematical difficulties of dealing with distributions. There is, however, no reason why the material presented here cannot be extended to the general harmonizable case.

Manuscript received June 10, 1988; revised November 24, 1989. This work was presented in part at the IASTED International Symposium of Signal Processing and Applications, Brisbane, Australia, 1987. Part of this work was completed when the authors were visiting the Center for Water Research, Univ. of Western Australia, Nedlands, 6009 WA, Australia.

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IEEE Log Number 9034932.

Let $x(t)$ and $y(t)$, $t \in T \subseteq R$, be harmonizable random processes, meaning [2] covariance functions of x and y have the representations

$$K_x(s,t) = E\{x(s)x^*(t)\} = \int_{\mathbb{R}^2} e^{2\pi i(s\mu + t\lambda)} d^2F_x(\mu,\lambda)$$

$$K_y(s,t) = E\{y(s)y^*(t)\} = \int_{\mathbb{R}^2} e^{2\pi i(s\mu + t\lambda)} d^2F_y(\mu,\lambda) \tag{1}$$

where F_x and F_y are measures of bounded variation on $\mathbb{R}^2$. As previously mentioned, it will be assumed that x and y are of finite energy, i.e.,

$$E_x = \int_{\mathbb{R}} E\{|x(t)|^2\}\, dt < \infty \tag{2}$$

and similarly for y. Then K_x and K_y are absolutely integrable and thus possess the bounded Fourier transforms

$$f_x(\mu,\lambda) = \int_{\mathbb{R}^2} e^{-2\pi i(s\mu + t\lambda)} K_x(s,t)\, ds\, dt \tag{3}$$

and similarly for y. Of course weakly stationary processes are not of finite energy, so strictly speaking we are not "generalizing" the stationary coherence by focussing on this class. However, in any practical analysis, it is necessary to consider sample realizations of bounded support, so the stationary case may be practically treated within the framework presented here. The evolutive (or Wigner–Ville) spectrum (ES) of x is then defined by

$$S_x(t,f) = \int_{\mathbb{R}} e^{-2\pi i f\tau} K_x(t+\tau/2, t-\tau/2)\, d\tau, \tag{4}$$

and similarly for y. As pointed out in [2] the variables t and f in (4) may be unambiguously interpreted as time and frequency, thus permitting S_x to be regarded as a time-frequency representation of the process $x(t)$. By the Schwartz inequality, the cross covariance of x and y

$$K_{xy}(s,t) = E\{x(s)y^*(t)\} \tag{5}$$

is also absolutely integrable, and thus the cross evolutive spectrum (XES) may be defined as

$$S_{xy}(t,f) = \int_{\mathbb{R}} e^{-2\pi i f\tau} K_{xy}(t+\tau/2, t-\tau/2)\, d\tau. \tag{6}$$

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Evidently, $S_x(t,\cdot)$ is bounded and uniformly continuous for almost all $t \in T$ and $S_x(\cdot,f) \in L_2(T)$, the set of all Lebesgue measurable functions that are square integrable on T, for all $f \in \mathbb{R}$. Similar comments hold for S_x and S_{xy}.

Define the joint time–frequency support of x and y by the subset of $\mathbb{R}^2$

$$\Theta_{xy} = \operatorname{supp}(S_{xy}) = \mathrm{Cl}\{(t,f) \in \mathbb{R}^2: |S_{xy}(t,f)| > 0\} \quad (7)$$

with $\Theta_x = \Theta_{xx}$ and $\Theta_y = \Theta_{yy}$. Here Cl denotes closure in the usual Euclidean topology on $\mathbb{R}^2$. The time–frequency coherence (TFC) of x and y is defined by

$$C_{xy}(t,f) = \frac{S_{xy}(t,f)}{\left[S_x(t,f)S_y(t,f)\right]^{1/2}} \quad (8)$$

for all $(t,f) \in \Theta_x \cap \Theta_y$. In general the ES is not nonnegative, and thus the formulation in (8) will not be well-defined. A semistationary process $x(t)$ is one for which for all $t_0 \in T$, given $\epsilon > 0$, there is $\delta > 0$ called the width of stationary at t_o with error ϵ, and a $K_o \in L_1(\mathbb{R})$ dependent on ϵ and t_o such that

$$|K_x(t+\tau/2, t-\tau/2) - K_0(\tau)| < \epsilon, \quad \text{for all } |t - t_0| + |\tau| < \delta. \quad (9)$$

Flandrin [5] has shown that all semistationary processes possess nonnegative evolutive spectra, and thus (8) will then be well defined. Subsequent discussion will thus be limited to the semistationary, finite energy processes.

Since the ES reduces to the usual power spectral density when the processes are weakly stationary, the formulation (8) reduces to the stationary coherence function (SCF) [6] and [7]. The TFC possesses the following properties that allow it to be used and interpreted in much the same way as the stationary coherence. First, the magnitude of the TFC is bounded by unity. To see this consider $S_{xy}(t,f)$ for each fixed (t,f). This defines a pseudo inner product $S_{xy}(t,f) = \langle x,y\rangle_{(t,f)}$ on $L_2(T)$, i.e., an inner product that induces a pseudo metric $d(x,y) = \langle x-y, x-y\rangle^{1/2}_{(t,f)}$. Thus the result follows directly by the Schwartz inequality. Second, if x and y are uncorrelated at time t, i.e., $K(t+\tau/2, t-\tau/2) = 0$ for all τ, then clearly $S_{xy}(t,\cdot) = 0$ and thus $C_{xy}(t,\cdot) = 0$. Third, suppose x and y are linearly related, i.e.,

$$y(t) = \int_{\mathbb{R}} h(\tau)x(t-\tau)\,d\tau \quad (10)$$

where $h \in L_2(\mathbb{R})$ represents the impulse response of a linear shift-invariant system; then

$$C_{xy}(t,f) = \frac{S_x(t,f) * 2h^*(2t)e^{4\pi i f t}}{\left[(S_x(t,f) * W_h(t,f))S_x(t,f)\right]^{1/2}} \quad (11)$$

where the asterisk denotes convolution in the t variable, W_h is the Wigner–Ville Distribution (WVD) of h, h^* denotes the conjugate of h, and

$$W_h(t,f) = \int_{\mathbb{R}} e^{-2\pi i f\tau} h(t+\tau/2)h^*(t-\tau/2)\,d\tau. \quad (12)$$

Examination of the expression (11) indicates that the TFC will not, in general, reduce to unity for linearly related processes. However, if the duration of the impulse response $h(t)$ is small compared to the stationary width δ (9) at time t_o, then y will be semistationary with stationarity width close to δ. Then, approximately at $t = t_o$, $W_h(t,f) * S_x(t,f) \cong S_x(t_o,f)|H(f)|^2$, and $S_x(t,f) * 2h^*(2t)e^{4\pi i f t} \cong S_x(t_o,f)H^*(f)$ and thus $C_{xy}(t_o,f) \cong H^*(f)/|H(f)|$. This argument may be made rigorous [3], although the proof is tedious and is not repeated here.

This discussion indicates that the TFC has useful properties for cross spectral analysis, namely that it may be interpreted as a time and frequency dependent correlation coefficient. Values near zero over a wide-frequency range indicate that the processes are nearly uncorrelated at that time, while values of magnitude near unity suggest that the processes are related by a linear filtering that does not disturb the semistationary structure excessively. It would not be expected that the TFC magnitude is exactly unity for all time under the linear relationship (10) since if the duration of $h(t)$ is too large, there will be mixing between the segments of $x(t)$, which have differing statistical properties.

II. Estimation of the Time–Frequency Coherence

In this section, a class of estimators for the TFC is introduced. These estimators are based on Cohen's joint time–frequency representations [8], [9] which is the class of all L_2 convolutions with the Wigner–Ville distribution of the process realisations. For the class of estimators of the ES to be well defined, it is necessary for the process under consideration x, to possess finite energy almost surely (a.s.) or equivalently (a.e.) $x \in L_2(\mathbb{R})$ almost surely. The Wigner–Ville distribution (WVD) of x is the random quantity

$$W_x(t,f) = \int_{\mathbb{R}} e^{-2\pi i f\tau} x(t+\tau/2)x^*(t-\tau/2)\,d\tau \quad (13)$$

and given $\Phi \in L_2(\mathbb{R}^2)$, the time-frequency distribution (TFD) of x is the two-dimensional (2-D) convolution

$$\rho_x(t,f;\Phi) = \Phi(t,f) ** W_x(t,f). \quad (14)$$

Since $W_x \in L_2(\mathbb{R}^2)$ a.s. then ρ_x is bounded a.s. Denote by Π the class of all $\Phi \in L_2(\mathbb{R}^2)$ that are real valued.

Martin [2] has shown that $E\{W_x(t,f)\} = S_x(t,f)$ which motivates an estimation procedure for the ES based on sample realisations of the process TFD. The ES estimator (ESE) is thus defined to be $\hat{S}_x(t,f) = \rho_x(t,f;\Phi)$, where Φ is determined by the properties of the process. The selection of Φ will be considered subsequently. The TFC estimator (TFCE) is defined by

$$\hat{C}_{xy}(t,f) = \frac{\hat{S}_{xy}(t,f)}{\left[\hat{S}_x(t,f)\hat{S}_y(t,f)\right]^{1/2}} \quad (15)$$

where

$$\hat{S}_{xy}(t,f) = \rho_{xy}(t,f;\Phi) = \Phi(t,f) ** W_{xy}(t,f). \quad (16)$$

Here W_{xy} is the cross WVD (XWVD) of x and y defined by

$$W_{xy}(t,f) = \int_{\mathbb{R}} e^{-2\pi i f \tau} x(t+\tau/2) y^*(t-\tau/2)\, d\tau. \quad (17)$$

The TFCE is defined for all $(t,f) \in \hat{\Theta}_x \cap \hat{\Theta}_y$ where $\hat{\Theta}_x$ and $\hat{\Theta}_y$ denote the supports of the ESE defined analogously to (7). It is now shown that the only choices of Φ for which the TFCE is meaningful are those corresponding to the positive class of TFDs, i.e., those TFDs that are nonnegative a.s. The subset of Π generating the positive class of TFDs will be denoted by Π^+.

Theorem 1: The support $\hat{\Theta}_{xy}$ of the cross ESE is contained in the intersection $\hat{\Theta}_x \cap \hat{\Theta}_y$ of the ESE supports a.s. if and only if $\Phi \in \Pi^+$. Thus the TFCE is a.s. bounded (by unity) if and only if $\Phi \in \Pi^+$.

Proof: See Appendix I. □

Corollary: Let $\{x_i\}$ $i = 1,\cdots,N$ be a finite sequence of a.s. finite energy processes and let

$$x(t) = \sum_{i=1}^{N} x_i(t). \quad (18)$$

Then if

$$\Phi \in \Pi^+, \hat{\Theta}_x \subseteq \bigcup_{i=1}^{N} \hat{\Theta}_{x_i} \text{ a.s.}$$

Proof: It may be readily verified that

$$\rho_x(t,f;\Phi) = \sum_{i=1}^{N} \rho_{x_i}(t,f;\Phi) + 2\,\mathrm{Re} \sum_{i=1,\, j<i}^{N} \rho_{x_i,x_j}(t,f;\Phi).$$

Thus,

$$\begin{aligned}
\hat{\Theta}_x &\subseteq \bigcup_{i=1}^{N} \hat{\Theta}_{xi} \bigcup_{i=1} \bigcup_{j<i} \mathrm{supp}\{\mathrm{Re}\,\rho_{x_i,x_j}(\cdot,\cdot;\Phi)\} \\
&\subseteq \bigcup_{i=1}^{N} \hat{\Theta}_{x_i} \bigcup_{i=1}^{N} \bigcup_{j<i} \hat{\Theta}_{x_i,x_j} \\
&\subseteq \bigcup_{i=1}^{N} \hat{\Theta}_{x_i} \bigcup_{i=1}^{N} \bigcup_{j<i} \hat{\Theta}_{x_i} \cap \hat{\Theta}_{x_j} \text{ (by Theorem 1)} \\
&\subseteq \bigcup_{i=1}^{N} \hat{\Theta}_{x_i}.
\end{aligned}$$

□

Comment: There has been considerable discussion in the literature concerning the removal and suppression of interference effects in TFDs caused by the inherent bilinearity of the formulation (14) (see e.g., [24]). This corollary shows that the positive class of TFDs, while in general retaining interference effects, confine them to the supports of the individual signal TFDs. Thus they are generally not as easily observed and do not complicate the interpretation of the resulting TFD. A converse of this result would be particularly interesting, allowing a class of test signals to be used to test for positivity of the TFD. However, the authors have not yet been able to produce a converse using the previous set theoretic formulation.

III. The Class of Positive TFDs

In this section, the class Π^+ of positive TFDs will be characterized. It is stressed that the particular TFD class $\rho_x(\cdot,\cdot;\Phi)$ must be positive for all $x \in L_2(\mathbb{R})$ and thus the parameterization Φ cannot directly depend on x. This does not prevent Φ being chosen with some *a priori* properties of the class of processes under analysis in mind, as long as there is no direct (i.e., functional) dependence. Thus the class of positive distributions defined by Cohen [10], [11] are not admissible in the scheme proposed here, since they do not fall in the class of distributions that are bilinear in the signal x being realisations of a given a.s. finite energy process.

Denote by D the subset of Π consisting of Wigner–Ville Distributions (WVD). It is easy to verify that D is a subset of Π^+ since

$$\rho_x(t,f;W_h) = |P_x^h(t,f)|^2 \geq 0 \quad (19)$$

where $P_x^h(t,f)$ is the (windowed) periodogram

$$P_x^h(t,f) = \int_{\mathbb{R}} e^{-2\pi i f \tau} x(t+\tau) h(\tau)\, d\tau. \quad (20)$$

The following theorem characterises the class of all positive TFDs as 2-D convolutions of periodograms of the form (19) with nonnegative real valued functions.

Theorem 2: $\Phi \in \Pi^+$ if and only if Φ has the form

$$\Phi(t,f) = q(t,f) ** W_h(t,f), \quad (21)$$

where $q \geq 0$ and either $q(t,f) = \delta(t,f)$ (corresponding to $\Phi \in D$) or $q \in L_1(\mathbb{R}^2)$. Here $\delta(t,f)$ is the 2-D Dirac function.

Proof: See Appendix II. □

Comment: The class Π^+ of functions of the form (21) are the Fourier transforms of the class known as generalised ambiguity functions introduced in [11]. Thus $\Phi \in \Pi^+$ if and only if its Fourier transform can be expressed as $\phi(n,\tau) = Q(n,\tau) A_h(n,\tau)$ a.e., where Q is a 2-D correlation function (i.e., those functions that possess a.e. nonnegative 2-D Fourier transforms) and A_h is the ambiguity function of some $h \in L_2(\mathbb{R})$. A partial result was also given in [13].

The degenerate class D consisting of WVDs plays an important role in estimation of the TFC for if $\Phi \in D$, the TFCE magnitude is unity for all processes x, as the following theorem shows. This means that the degenerate class is inadmissible for proper estimation of the TFC.

Theorem 3: Let $\Phi \in D$ then $|\hat{C}_{xy}(t,f)| = 1$, for all $x, y \in L_2(\mathbb{R})$.

Proof: Suppose $\Phi = W_h$, then

$$\hat{C}_{xy}(t,f) = \frac{A_{xh}(f,t)A_{yh}(f,t)^*}{|A_{xh}(f,t)||A_{yh}(f,t)|} \quad (22)$$

where A_{xh} is the cross ambiguity function

$$A_{xh}(\theta,\tau) = \int_{\mathbb{R}} e^{-2\pi i t\theta} x(t+\tau/2)h^*(t+\tau/2)\,dt \quad (23)$$

yielding the desired result. □

IV. Choice of the Estimation Window

To design appropriate estimators of the TFC, it is necessary to choose a suitable window function $\Phi \in \Pi^+ - D$, i.e., a smoothed periodogram. In [3], [14], [15] considerable attention has been given to the problem of selecting appropriate windows. The general approach is summarized next.

1) Perform a joint segmentation of the process realisations under study using a linear modeling (ARMA) technique [16], [17] or methods based on information theoretic criteria [18]. This gives a sequence of intervals $[t_i, t_{i+1})$ on which the processes are semistationary with specified accuracy ϵ as in (9). This defines the interval on which the analysis for the estimation of $C_{xy}((t_i + t_{i+1})/2, f)$ takes place.

2) The Karhunen–Loeve expansions (KLE) of the processes may be determined on the interval i, using the eigenstructure of the estimated approximating stationary covariance K_o (9) and the resulting dominant eigenvalues used to identify the most significant principal components in the ES at that time. The desired bias/variance/resolution tradeoffs may then be determined as suggested in [14], and a suitable smoothing q selected.

3) The procedure is iterated by adjusting the interval and the number of covariance lags in K_o on the interval.

4) Repeat for other intervals.

In the Gaussian case, a numerical method has been proposed by the authors [15] to determine the one-dimensional (1-D) probability densities for the ESE and other related quantities, and this method may be used for determination of Φ in the Gaussian case.

This procedure is a refinement of the procedure applied in the stationary case where averages of overlapping periodograms have been used [19], [20]. In this case, under the assumption that each periodogram is determined on statistically independent segments (i.e., no overlap) and the process is stationary on each interval with Gaussian statistics, the TFCE magnitude squared is a multiple correlation, and the probability density is known. The study of this idealised case provides a guide for the more complicated case considered here. In particular, the TFCE magnitude illustrates a strong bias towards unity, particularly for a small number of averaging segments. However the method may be useful for detecting regions in the time-frequency plane where there is little coherence, provided an adequate number of averages may be performed.

V. Conclusion

The problem of cross analysis of finite energy semistationary random processes has been considered. A new quantity termed the time–frequency coherence (TFC) has been defined, and its properties have been shown to be similar to those of the stationary coherence function (i.e., a correlation coefficient that depends on time and frequency). In particular, the TFC is zero if the processes are uncorrelated at a certain time, and the TFC attains its maximum modulus of unity if the processes are related via a linear time-invariant filter of sufficiently short length compared to the stationarity width. A procedure for estimation of the TFC based on Cohen's class of time-frequency distributions (TFD) has been introduced, and it has been shown that for the TFC estimate to be well behaved, in particular, bounded almost surely by unity, it is necessary and sufficient to choose the positive elements of Cohen's class. The weighting windows that are associated with these positive TFDs are convolutions of Wigner–Ville distributions (WVD) with nonnegative functions. This convolution is shown to be necessary otherwise TFC estimates are produced that are of unity modulus for all process realisations. Some brief discussion is directed to the problem of selection of the window functions, a problem that the authors have addressed elsewhere. The method has also found application in an oceanography signal processing situation where it is desired to determine the correlation as a function of time (depth) and frequency between two time series records, each being the output of a temperature microstructure probe that is passed up through a water column. These data series are highly nonstationary, and the TFC may be used to ascertain the regions of low coherence, and to estimate dominant eddy size, an important physical parameter [21], [22].

Acknowledgment

The authors acknowledge the assistance of the Defence Science and Technology Organisation Australia and the Australian Research Council. The authors would also like to thank Dr. J. Chabrowski, Department of Mathematics, University of Queensland, and the anonymous reviewers for helpful suggestions. The authors also thank the Center for Water Research, University of Western Australia.

Appendix I
Proof of Theorem 1

In proving Theorem 1, it is beneficial to introduce a linear isomorphism between the set of TFDs and the set of all self-adjoint operators on $L_2(\mathbb{R}^2)$.

Theorem A1: Let $\Phi \in \Pi$ then there is a bounded, self-adjoint operator S_Φ determined uniquely by Φ such that

$$\rho_{xy}(t,f;\Phi) = \langle S_\Phi T_{(t,f)}x, T_{(t,f)}y \rangle, \tag{A1}$$

where $\langle \cdot,\cdot \rangle$ denotes the L_2 inner product

$$\langle x,y \rangle = \int_{\mathbb{R}} x(t) y^*(t)\, dt. \tag{A2}$$

Here $T_{(t,f)}$ denotes the time–frequency translation operator

$$[T_{(t,f)}x](u) = x(u+t)e^{2\pi i f u}. \tag{A3}$$

Furthermore, the mapping $\Phi \to S_\Phi$ is a linear isometry.

Proof: Observe firstly that $\rho_{xy}(t,f;\Phi) = \rho_{x'y'}(0,0;\Phi)$, where $x'(u) = [T_{(t,f)}x](u)$ and similarly for y'. Thus it suffices to consider ρ_{xy} at the origin,

$$\begin{aligned}\rho_{xy}(0,0;\Phi) &= \int_{\mathbb{R}^3} \phi(n,\tau) e^{-2\pi i n u} x(u+\tau/2) y^*(u-\tau/2)\, dn\, du\, d\tau \\ &= \int_{\mathbb{R}^2} \phi(n,\tau) A_{xy}(n,\tau)\, dn\, d\tau \\ &= \langle \phi, A_{xy}^* \rangle \end{aligned}\tag{A4}$$

where A_{xy} denotes the cross ambiguity function (23), and ϕ is the 2-D Fourier transform of Φ. Thus ρ_{xy} is a sesquilinear form on $L_2(\mathbb{R})$. To see that this form is bounded consider

$$|\langle \phi, A_{xy}^* \rangle| \le \|\phi\| \|A_{xy}\| = \|\phi\| \|x\| \|y\| \tag{A5}$$

which follows from the Schwartz inequality and Moyal's formula

$$\langle A_{uv}, A_{xy} \rangle = \langle u,x \rangle \langle v,y \rangle^*. \tag{A6}$$

Now by the Riesz representation theorem [23], there is a bounded linear operator S_Φ on $L_2(\mathbb{R})$ such that

$$\rho_{xy}(0,0;\Phi) = \langle S_\Phi x, y \rangle. \tag{A7}$$

Since $\rho_{xy}(0,0;\Phi) = \rho_{yx}^*(0,0;\Phi)$, S_Φ is self adjoint. To see S_Φ depends continuously on ϕ (and thus Φ) consider

$$\begin{aligned}\|S_\Phi\| &= \sup\{|\langle S_\Phi x, y \rangle| : \|x\| = \|y\| = 1\} \\ &= \sup\{|\langle \phi, A_{xy}^* \rangle| : \|x\| = \|y\| = 1\} \\ &\le \sup\{\|\phi\| \|x\| \|y\| : \|x\| = \|y\| = 1\} \\ &= \|\phi\|. \end{aligned}\tag{A8}$$

The supremum is attained for

$$\phi = A_{xy}^*, \text{ so } \|S_\Phi\| = \|\phi\| = \|\Phi\|. \qquad \square$$

We may now proceed to the proof of Theorem 1 after the following preliminary lemma.

Lemma A1: Let $\mathscr{H}$ be a Hilbert space over the complex field, and let S be a self-adjoint operator on S. Then, if for each $x \in \mathscr{H}$, $\langle Sx, x \rangle = 0$ implies $\langle Sx, y \rangle = 0$, for all $y \in \mathscr{H}$, then S is a sign definite operator and conversely. (A sign definite operator S is one for which $\langle Sx, x \rangle$ is of one sign for all $x \in \mathscr{H}$).

Proof: Suppose S is sign definite, then [23] there is a bounded, self adjoint operator A such that $S = \pm AA$. Then $0 = \langle Sx, x \rangle = \pm \langle Ax, Ax \rangle = \pm \|Ax\|^2$. Thus $Ax = 0$, and $\langle Sx, y \rangle = \pm \langle Ax, Ay \rangle = 0$, for all $y \in \mathscr{H}$.

Conversely, assume S is not sign definite, then there are $y, z \in \mathscr{H}$ such that $\langle Sy, y \rangle > 0$ and $\langle Sz, z \rangle < 0$. Let $x_\lambda = \lambda z + (1-\lambda)y$. By the continuity of the inner product and S, there is a $\mu \in (0,1)$ such that $\langle Sx_\mu, x_\mu \rangle = 0$. Now there must exist a $y_o \in \mathscr{H}$ such that $\langle Sx_\mu, y_o \rangle \ne 0$, for suppose otherwise, then $\langle Sx_\mu, y \rangle = 0$ for all $y \in \mathscr{H}$, and

$$0 = \mu Sy + (1-\mu)Sz, \tag{A9}$$

which implies

$$0 = \mu \langle Sy, y \rangle + (1-\mu) \langle Sz, y \rangle. \tag{A10}$$

Similarly,

$$0 = \mu \langle Sy, z \rangle + (1-\mu) \langle Sz, z \rangle. \tag{A11}$$

Using the self adjointness of S and $0 < \mu < 1$, (A10) and (A11) yield

$$0 = \mu^2 \langle Sy, y \rangle - (1-\mu^2) \langle Sz, z \rangle \tag{A12}$$

which is a contradiction since the right-hand side of (A12) is strictly positive. This completes the proof. $\square$

Proof of Theorem 1: Suppose firstly that $\hat{\Theta}_x \cap \hat{\Theta}_y \supset \hat{\Theta}_{xy}$ a.s. then for each $x \in L_2(\mathbb{R})$, $\rho_x(0,0;\Phi) = 0$ implies $\rho_{xy}(0,0;\Phi) = 0$ for all $y \in L_2(\mathbb{R})$. By Theorem A1, there is a bounded self adjoint operator S_Φ such that $\rho_{xy}(0,0;\Phi) = \langle S_\Phi x, y \rangle$. Thus for each $x \in L_2(\mathbb{R})$, $\langle S_\Phi x, x \rangle = 0$ implies $\langle S_\Phi x, y \rangle = 0$ for all $y \in L_2(\mathbb{R})$. Thus, by Lemma A2 S_Φ is sign definite. Furthermore, S_Φ must be positive definite since

$$\int_{\mathbb{R}^2} \rho_x(t,f;\Phi)\, dt\, df > 0 \tag{A13}$$

for all $\Phi \in \Pi$, and all nonzero $x \in L_2(\mathbb{R})$. Thus $\Phi \in \Pi^+$.

Conversely, if $\Phi \in \Pi^+$, then the associated operator S_Φ is positive definite, and the result that $\hat{\Theta}_x \cap \hat{\Theta}_y \supset \hat{\Theta}_{xy}$ a.s. follows from Lemma A2. To show now that the TFCE is a.s. bounded by unity if and only if $\Phi \in \Pi^+$, firstly suppose $\Phi \in \Pi^+$ and fix $(t,f) \in \mathbb{R}^2$. Then $\rho_{xy}(t,f;\Phi)$ is a pseudo inner-product (i.e., $\rho_{(x-y),(x-y)}(t,f;\Phi)$ is a pseudo metric [23]) and the bound follows directly from the Schwartz inequality. Suppose now that $\Phi \notin \Pi^+$ then by the previous argument there is (without loss of generality) a $(t_o, f_o) \in \hat{\Theta}_{xy} - \hat{\Theta}_x$. Take any sequence $(t_n, f_n) \to (t_o, t_o)$ and $C_{xy}(t_n, f_n) \to \infty$, by the continuity of ρ_{xy}. This completes the proof. $\square$

Appendix II
Proof of Theorem 2

Cohen has shown [11] that the cross TFD (16) has the unique (up to scale) decomposition

$$\rho_{xy}(t,f;\Phi) = \int_{\mathbb{R}^2} q(t-\tau, f-\theta) A_{xh}(\theta,\tau) A_{yh}^*(\theta,\tau)\, d\theta\, d\tau \tag{A14}$$

where A_{xh} is the cross ambiguity function

$$A_{xh}(\theta,\tau) = \int_{\mathbb{R}} e^{-2\pi i t\theta} x(t+\tau/2) h^*(t+\tau/2)\, dt. \tag{A15}$$

The dependence on Φ is contained in both a and q. First assume $q = \delta$, then from (A14)

$$\rho_x(t,f;\Phi) = |A_{xh}(f,t)|^2 = |P_x^h(t,f)|^2 \ge 0. \tag{A16}$$

Suppose now $q \ge 0$, then

$$\rho_x(t,f;\Phi) = \int_{\mathbb{R}^2} q(t-\tau, f-\theta) |P_x^h(t,f)|^2\, d\theta\, d\tau \ge 0. \tag{A17}$$

Conversely, assume $\rho_x(t,f;\Phi) \ge 0$, then q must be either the delta function, or positive a.e., for suppose there is a subset S of $\mathbb{R}^2$ of positive measure where $q(S) < 0$. Then we can select x to be a sufficiently large sum of Gabor wavelets [12] chosen so that P_{x^h} is concentrated around points of S and such that $\rho_x(t, f; \Phi) < 0$ for some $(t, f) \in S$ [3], thus contradicting the positivity. □

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