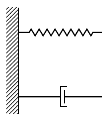
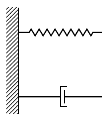
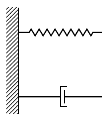
## **Mass-Spring-Damper system**

The modelling of the problem can be carried out applying the well-known approach used to carry out modal analyses and to derive the Frequency Response Function (FRF; e.g., the mechanical impedance or the dynamic stiffness) of a system studying its free vibration [1]. Each sample can be seen as a mass-spring-damper system with a Single Degree of Freedom (SDoF). However, if the test is carried out while the sample is attached on the pavement, and if a load plate is used, the problem can be solved finding the solution of the matrix equation of motion of a Multiple Degree of Freedom (MDoF) system, which can be written as follows:

(1)

where *[M]* is the mass matrix, *[C]* is the damping matrix, *[K]* is the stiffness matrix, *ẍ(t)*, *ẋ(t)*, and *x(t)* are the acceleration, the velocity, and the displacement of the system, while *f(t)* is the force applied to the system.

The following figure shows a 3DoF mass-damper-stiffness system, and the free body diagrams of each mass of the system.



*m3*

*m2*

*m1*

*k1 c1*

*k2 c2*

*k3 c3*

*f(t)*

Legend:

mi = masses of the i-th part of the system [kg];

ki = stiffness of the i-th part of the system [N/m];

ci = damping of the i-th part of the system [dimensionless];

ẍi, ẋi, xi = acceleration, velocity, and displacement of the i-th part of the system.

*m1*

*m2*

*m3*

*f*

*k1 (x1- x2) c1 (ẋ1- ẋ2)*

*m1 ẍ*

*m2 ẍ*

*m3 ẍ*

*k1 (x2- x1) c1 (ẋ2- ẋ1)*

*k2 (x2- x3) c2 (ẋ2- ẋ3)*

*k2 (x3- x2) c2 (ẋ3- ẋ2)*

*k3 x3 c3 ẋ3*

*x1(t)*

*x2(t)*

*x3(t)*

**Fig. 1.** a) Three Degree of Freedom (DoF) mass-damper-stiffness system, and (b) the related free body diagrams.

By considering the body diagrams mentioned above and employing a Newton's second law, the equation of motion of each mass are the following:

 (2)

To find the FRF of the system from the differential equations above, the Laplace Transform can be used [1]. Assuming that all initial conditions are zero and applying the Laplace Transform, the differential equations become the following algebraic equations in the variable *s* (a.k.a., the Laplace variable):

 (3)

The system of equations (3) can be expressed in matrix form (*Ax = b*) as follows:

. (4)

When the accelerometer is attached on the top of the first mass (m1), the acceleration *ẍ1(t)* is measured. Hence, the system of equation (4) must be solved to derive the response of the 3DoF system in terms of displacement X1. If the Cramer’s rule is used, the term X1 is obtained solving the following equation:

 , (5)

where the numerator was obtained substituting the first column of the matrix *A* with the vector *b*, and the denominator, Δ, is the determinant of the matrix *A*.

Consequently, the transfer function of the 3 DoF system (*G(s)*) is:

, (6)

where the quantities *Δ* and *C* are the following:

 (7)

 (8)

Finally, the *G(s)* becomes the FRF function (i.e., the dynamic stiffness *DS(s)* of the 3DoF system measured on the top of mass m1) when only the imaginary part of the *s* operator is used [1]. Hence, because of the fact that *s = j ω* [2], where *j* is the imaginary part and *ω = 2πf* (i.e., the angular frequency, which is function of the frequency, *f* ), the terms s4, s3, s2, and s become *ω4, j3ω3, -ω2,* and *jω,* respectively, and the variation of the Dynamic Stiffness (*DS(f)*) with the frequency can be obtained.

# References

[1] J. He, Z.-F. Fu, Modal analysis of a damped MDoF system, in: Modal Anal., 2001. doi:10.1016/b978-075065079-3/50006-1.

[2] L. Gu, L. Chen, W. Zhang, H. Ma, T. Ma, Mesostructural Modeling of Dynamic Modulus and Phase Angle Master Curves of Rubber Modified Asphalt Mixture, Materials (Basel). (2019). doi:10.3390/ma12101667.