Calc I - Tutorial 6

July 23, 2019

1.

(a) $f(t) = \sqrt[3]{1+\tan t}$

$$\begin{aligned} f(t) &= \sqrt[3]{1+\tan t} \\ &= (1+\tan t)^{\frac{1}{3}} \\ f'(t) &= \frac{1}{3}(1+\tan t)^{-\frac{2}{3}} \cdot \sec^2 t \\ &= \frac{\sec^2 t}{3\sqrt[3]{(1+\tan t)^2}} \end{aligned}$$

(b) $y = 4\sec 5x$

$$\begin{aligned} \frac{dy}{dx} &= 4(\sec 5x \tan 5x) \cdot 5 \\ &= 20(\sec 5x \tan 5x) \end{aligned}$$

(c) $y = e^{-5x}\cos 3x$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(e^{-5x}\right) \cdot \cos 3x + e^{-5x} \cdot \frac{d}{dx}(\cos 3x) \\ &= -5e^{-5x} \cdot \cos 3x + e^{-5x} \cdot -3\sin 3x \\ &= -e^{-5x}(5\cos 3x + 3\sin 3x) \end{aligned}$$

(d) $y = 10^{1-x^2}$

$$\begin{aligned} \frac{dy}{dx} &= 10^{1-x^2}\ln(10) \cdot -2x \\ &= -2x \cdot \ln(10) \cdot 10^{1-x^2} \end{aligned}$$

(e) $y = \tan^2(3\theta)$

$$
\begin{aligned}
\frac{dy}{dx} &= (\tan 3\theta)^2 \\
&= 2(\tan 3\theta) \cdot \frac{d}{dx}(\tan 3\theta) \\
&= 2(\tan 3\theta) \cdot \sec^2 3\theta \cdot 3 \\
&= 6 \tan 3\theta \sec^2 3\theta
\end{aligned}
$$

2. $f(x) = 2\cos x + \sin^2 x$

$$
\begin{aligned}
f(x) &= 2\cos x + (\sin x)^2 \\
f'(x) &= 2(-\sin x) + 2(\sin x) \cdot \frac{d}{dx}(\sin x) \\
&= -2\sin x + 2\sin x \cdot \cos x \\
\boldsymbol{f'(x)} &= \boldsymbol{-2\sin x + \sin 2x}; \text{Note:} 2\sin x \cos x = \sin 2x \\
f''(x) &= -2(\cos x) + (\cos 2x)(2) \\
\boldsymbol{f''(x)} &= \boldsymbol{2(\cos 2x - \cos x)}
\end{aligned}
$$

3.

(a) $f(x) = (x-1)^2, point\ (2,1)$

i. Find the equation of the tangent line

A. Find the gradient, m

$$
\begin{aligned}
f'(x) &= 2(x-1) \cdot 1 \\
&= 2(x-1)
\end{aligned}
$$

$$
\begin{aligned}
f'(2) &= 2(2-1) \\
\boldsymbol{m} &= \boldsymbol{2}
\end{aligned}
$$

B. Find the $y-intercept$, c

$$
\begin{aligned}
y &= mx + c \\
y &= 2x + c
\end{aligned}
$$

Substitute $x = 2, y = 1$

$$
\begin{aligned}
1 &= 2(2) + c \\
c &= 1 - 4 \\
\boldsymbol{c} &= \boldsymbol{-3}
\end{aligned}
$$

C. Form the equation of the tangent line, $y = mx + c$

$$\boldsymbol{y = 2x - 3}$$

ii. Find the equation of the normal line

A. Find the gradient, m_n (Note, m_n is perpendicular to the m_t)

$$\begin{aligned} m_t m_n &= -1 \\ m_n &= -\frac{1}{m_t} \end{aligned}$$

m_t is from tangent line part above, where $m_t = 2$

$$m_n = -\frac{1}{2}$$

B. Find the $y - intercept, c$

$$\begin{aligned} y &= mx + c \\ y &= -\frac{1}{2}x + c \end{aligned}$$

Substitute $x = 2, y = 1$

$$\begin{aligned} &1 = -\frac{1}{2}(2) + c \\ &c = 1 + 1 \\ &\boldsymbol{c = 2} \end{aligned}$$

C. Form the equation of the normal line to the tangent, $y = mx + c$

$$\boldsymbol{y = -\frac{1}{2}x + 2}$$

(b) $h(x) = \sqrt{1 - 4x}, point\ (-2, 3)$

i. Find the equation of the tangent line

A. Find the gradient, m

$$\begin{aligned} h(x) &= (1 - 4x)^{\frac{1}{2}} \\ h'(x) &= \frac{1}{2}(1 - 4x)^{-\frac{1}{2}} \cdot (-4) \\ &= -2(1 - 4x)^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} h'(2) &= -(-2)(1 - 4(-2))^{-\frac{1}{2}} \\ m &= -\frac{2}{\sqrt{1+8}} \\ &= -\frac{2}{\sqrt{9}} \\ \boldsymbol{m} &= \boldsymbol{-\frac{2}{3}} \end{aligned}$$

B. Find the $y - intercept$, c

$$\begin{aligned} y &= mx + c \\ y &= -\frac{2}{3}x + c \end{aligned}$$

Substitute $x = -2, y = 3$

$$\begin{aligned} 3 &= -\frac{2}{3}(-2) + c \\ c &= 3 - \frac{4}{3} \\ \boldsymbol{c} &= \mathbf{\frac{5}{3}} \end{aligned}$$

C. Form the equation of the tangent line, $y = mx + c$

$$y = -\frac{2}{3}x + \frac{5}{3}$$

ii. Find the equation of the normal line

A. Find the gradient, m_n (Note, m_n is perpendicular to the m_t)

$$\begin{aligned} m_t m_n &= -1 \\ m_n &= -\frac{1}{m_t} \end{aligned}$$

m_t is from tangent line part above, where $m_t = -\frac{2}{3}$

$$\begin{aligned} m_n &= -\frac{1}{\left(-\frac{2}{3}\right)} \\ &= \frac{3}{2} \end{aligned}$$

B. Find the $y - intercept, c$

$$\begin{aligned} y &= mx + c \\ y &= \frac{3}{2}x + c \end{aligned}$$

Substitute $x = -2, y = 3$

$$\begin{aligned} 3 &= \frac{3}{2}(-2) + c \\ c &= 3 + 3 \\ \boldsymbol{c} &= \mathbf{6} \end{aligned}$$

C. Form the equation of the normal line to the tangent, $y = mx + c$

$$\boldsymbol{y = \frac{3}{2}x + 6}$$

4.

$$\begin{aligned} x &= \sin t \\ y &= \cos 2t \\ t &= \frac{\pi}{6} \end{aligned}$$

(a) Differentiate the two equations separately, and find the gradient of each part

i. $x = \sin t$

$$\frac{dx}{dt} = \cos t$$

A. Substitute $t = \frac{\pi}{6}$

$$\begin{aligned} \frac{dx}{dt} &= \cos\left(\frac{\pi}{6}\right) \\ &= \frac{1}{2} \end{aligned}$$

ii. $y = \cos 2t$

$$\begin{aligned} \frac{dy}{dt} &= -\sin 2t \cdot 2 \\ &= -2\sin 2t \end{aligned}$$

A. Substitute $t = \frac{\pi}{6}$

$$\begin{aligned} \frac{dy}{dt} &= -2\sin\left(\frac{\pi}{3}\right) \\ &= -2\left(\frac{1}{2}\right) \\ &= -1 \end{aligned}$$

(b) Find the gradient of the tangent line $\frac{dy}{dx}$ by using chain-rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} * \frac{dt}{dx} \\ &= -1 * \frac{1}{\frac{dx}{dt}} \\ &= -1 * 2 \\ \frac{dy}{dx} &= -2 \end{aligned}$$

(c) Convert the coordinates from parametric to cartesian format

$$\begin{aligned}
x &= \sin t \\
&= \sin\left(\frac{\pi}{6}\right) \\
\boldsymbol{x} &= \mathbf{\frac{1}{2}} \\
y &= \cos 2t \\
t &= \cos\left(\frac{\pi}{3}\right) \\
\boldsymbol{y} &= \mathbf{\frac{1}{2}}
\end{aligned}$$

i. $t = \frac{\pi}{6} \Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right)$

(d) Find the equation of the tangent line (this method bypasses the need to find c)

$$\begin{aligned}
y - y_1 &= m\,(x - x_1) \\
y - \frac{1}{2} &= -2\left(x - \frac{1}{2}\right) \\
y &= -2x + 1 + \frac{1}{2} \\
\boldsymbol{y} &= \boldsymbol{-2x} + \mathbf{\frac{3}{2}}
\end{aligned}$$

5. Find $\frac{dy}{dx}$ by implicit differentiation,

(a) $x^2 - 2xy + y^3 = c$

$$\begin{aligned}
x^2 - 2xy + y^3 &= c \\
\frac{dy}{dx}\left(x^2 - 2xy + y^3\right) &= \frac{dy}{dx}\,(c) \\
2x - 2\left(1 \cdot y + x \cdot \frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} &= 0 \\
2x - 2y + 2x\frac{dy}{dx} + 3y^2\frac{dy}{dx} &= 0 \\
2x\frac{dy}{dx} + 3y^2\frac{dy}{dx} &= 2x - 2y \\
\frac{dy}{dx}\left(2x + 3y^2\right) &= 2\,(x - y) \\
\frac{dy}{dx} &= \frac{2\,(x - y)}{(2x + 3y^2)}
\end{aligned}$$

(b) $\sqrt{1+x^2y^2} = 2xy$

$$\begin{aligned}
1+x^2y^2 &= (2xy)^2 \\
1+x^2y^2 &= 4x^2y^2 \\
\frac{dy}{dx}\left(1+x^2y^2\right) &= \frac{dy}{dx}\left(4x^2y^2\right) \\
2x*y^2+x^2*2y*\frac{dy}{dx} &= 4\left(2x\cdot y^2+x^2\cdot 2y\cdot\frac{dy}{dx}\right) \\
2xy^2+x^2 2y\frac{dy}{dx} &= 8xy^2+x^2 8y\frac{dy}{dx} \\
2xy^2-8xy^2 &= 8x^2y\frac{dy}{dx}-2x^2y\frac{dy}{dx} \\
-6xy^2 &= \frac{dy}{dx}\left(6x^2y\right) \\
\frac{dy}{dx} &= \frac{6x^2y}{-6xy^2} \\
\frac{dy}{dx} &= -\frac{x}{y}
\end{aligned}$$

(c) $\sin x + \cos y = \sin x \cos y$

$$\begin{aligned}
\frac{dy}{dx}(\sin x+\cos y) &= \frac{dy}{dx}(\sin x\cos y) \\
\cos x-\sin y\cdot\frac{dy}{dx} &= \cos x\cdot\cos y+\sin x\cdot(-\sin y)\cdot\frac{dy}{dx} \\
\cos x-\cos x\cos y &= -\sin x\sin y\frac{dy}{dx}+\sin y\frac{dy}{dx} \\
\cos x\,(1-\cos y) &= \frac{dy}{dx}(-\sin x\sin y+\sin y) \\
\frac{dy}{dx} &= \frac{\cos x\,(1-\cos y)}{(-\sin x\sin y+\sin y)} \\
&= \frac{\cos x\,(1-\cos y)}{\sin y\,(1-\sin x)} \\
&= \frac{\cos x\,(\cos y-1)}{\sin y\,(\sin x-1)}
\end{aligned}$$

6.

(a) $y = \left(\sin^{-1} x\right)^2$

$$\begin{aligned} y &= \left(\sin^{-1} x\right)^2 \\ \frac{dy}{dx} &= 2\left(\sin^{-1} x\right) \cdot \frac{dy}{dx}\left[\left(\sin^{-1} x\right)\right] \\ &= 2\left(\sin^{-1} x\right) \cdot \frac{1}{\sqrt{1-x^2}} \\ \frac{dy}{dx} &= \frac{2\sin^{-1} x}{\sqrt{1-x^2}} \end{aligned}$$

(b) $y = \tan^{-1}\left(2x^2+1\right)$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left[\tan^{-1}\left(2x^2+1\right)\right] \cdot \frac{d}{dx}\left[\left(2x^2+1\right)\right] \\ &= \frac{1}{1+\left(2x^2+1\right)^2} \cdot 4x \\ &= \frac{4x}{1+\left(4x^4+4x^2+1\right)} \\ &= \frac{4x}{4x^4+4x^2+2} \\ \frac{dy}{dx} &= \frac{2x}{2x^4+2x^2+1} \end{aligned}$$

7.

(a) Find the equation for the slope, $\frac{dy}{dx}$

$$\frac{dy}{dx}\left[x^2+y^2+3xy-11\right] = 0$$

$$2x + 2y\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} = 0$$

$$\begin{aligned} 2x+3y &= -\left(2y\frac{dy}{dx} + 3x\frac{dy}{dx}\right) \\ -\left(2x+3y\right) &= \frac{dy}{dx}\left(2y+3x\right) \\ \frac{dy}{dx} &= -\frac{2x+3y}{2y+3x} \end{aligned}$$

(b) Find the gradient at the point of the slope where $x = 1, y = 2$

$$\begin{aligned} \frac{dy}{dx} &= -\frac{2\,(1)+3\,(2)}{2\,(2)+3} \\ &= -\frac{2+6}{4+3} \\ &= -\frac{8}{7} \end{aligned}$$

(c) Find the gradient of the normal

$$\begin{aligned} m_n m_t &= -1 \\ m_n &= -\frac{1}{m_t} \\ &= \frac{7}{8} \end{aligned}$$

(d) Find the equation of the normal

$$\begin{aligned} y - y_1 &= m\left(x - x_1\right) \\ y - 2 &= \frac{7}{8}\left(x - 1\right) \\ y &= \frac{7}{8}x - \frac{7}{8} + 2 \\ &= \frac{7}{8}x + \frac{9}{8} \\ y &= \frac{1}{8}\left(7x + 9\right) \end{aligned}$$

8. Differentiate the following function (NO ANSWER)

(a) $f\left(x\right) = \cos\left(\ln x\right)$

$$\begin{aligned} f'\left(x\right) &= -\sin\left(\ln x\right) \cdot \frac{1}{x} \\ &= -\frac{1}{x}\sin\left(\ln x\right) \end{aligned}$$

(b) $f\left(t\right) = \frac{1+\ln t}{1-\ln t}$

$$\begin{aligned} f'\left(t\right) &= \frac{dt}{dx}\left[\frac{1+\ln t}{1-\ln t}\right] \\ &= \frac{\left(1-\ln t\right)\frac{d}{dx}\left(1+\ln t\right) - \left(1+\ln t\right)\frac{d}{dx}\left(1-\ln t\right)}{\left(1-\ln t\right)^2} \\ &= \frac{\left(1-\ln t\right)\frac{1}{t} + \left(1+\ln t\right)\frac{1}{t}}{\left(1-\ln t\right)^2} \\ &= \frac{\frac{1}{t}\left(1-\ln t+1+\ln t\right)}{\left(1-\ln t\right)^2} \\ &= \frac{2}{t\left(1-\ln t\right)^2} \end{aligned}$$

9. $y = \ln\left(\sec x + \tan x\right)$

(a) y'

$$\begin{aligned}
y' &= \frac{1}{(\sec x + \tan x)} * \frac{d}{dx}(\sec x + \tan x) \\
&= \frac{1}{(\sec x + \tan x)} * \left(\sec x \tan x + \sec^2 x\right) \\
&= \frac{1}{\cancel{(\sec x + \tan x)}} * \sec x \cancel{(\tan x + \sec x)} \\
\boldsymbol{y'} &\boldsymbol{= \sec x}
\end{aligned}$$

(b) y''

$$\begin{aligned}
y'' &= \frac{d}{dx}[y'] \\
&= \frac{d}{dx}[\sec x] \\
\boldsymbol{y''} &\boldsymbol{= \sec x \tan x}
\end{aligned}$$

10. Find the equation of the tangent lines to $y = \frac{1}{x} \cdot \ln x$

(a) Find the equation of the slope to the tangent line

$$\begin{aligned}
\frac{dy}{dx} &= -x^{-2} \cdot \ln x + x^{-1} \cdot \frac{1}{x} \\
&= -\frac{\ln x}{x^2} + \frac{1}{x^2} \\
&= \frac{1 - \ln x}{x^2}
\end{aligned}$$

(b) Find the equation of the tangent line at $(1, 0)$

i. Find the slope to the tangent line, $\frac{dy}{dx}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1 - \ln(1)}{1^2} \\
&= \frac{1 - 0}{1} \\
&= 1
\end{aligned}$$

ii. Find the equation to the tangent line

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 0 &= 1(x - 1) \\
\boldsymbol{y} &\boldsymbol{= x - 1}
\end{aligned}$$

(c) Find the equation of the tangent line at $\left(e, \frac{1}{e}\right)$

i. Find the slope to the tangent line, $\frac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1-\ln(e)}{e^2}\\ &= \frac{1-1}{e^2}\\ &= 0\end{aligned}$$

ii. Find the equation to the tangent line

$$\begin{aligned}y-\frac{1}{e} &= 0\,(x-e)\\ &= 0\\ \boldsymbol{y} &\boldsymbol{=} \boldsymbol{\frac{1}{e}}\end{aligned}$$