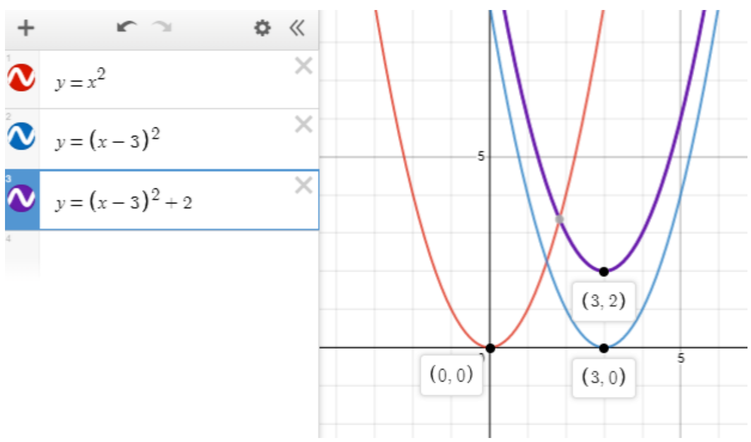
Calc 1 PYQ Sept-2018

September 9, 2019

1.

(a) $y = x^2 - 6x + 11$

$$
\begin{aligned}
y &= x^2 - 6x + 11 \\
y &= x^2 - 6x + (-3)^2 - (-3)^2 + 11 \\
&= (x-3)^2 - 9 + 11 \\
&= (x-3)^2 + 2
\end{aligned}
$$

i. 

(b) Evaluate the limits if exist

i. $\lim_{x\to 0} \frac{4\cos x - 4}{3x^2}$

$$
\begin{aligned}
\lim_{x\to 0} \frac{4\cos x - 4}{3x^2} &= \frac{4\cos 0 - 4}{30^2} \\
&= \frac{0}{0}
\end{aligned}
$$

A. Using L'Hospital's Rule

$$
\begin{aligned}
\lim_{x\to 0}\frac{4\cos x-4}{3x^2} &= \lim_{x\to 0}\frac{-4\sin x}{6x}\\
&= \lim_{x\to 0}\frac{-4\cos x}{6}\\
&= \lim_{x\to 0}\frac{-4\cos x}{6}\\
&= \frac{-4\cos 0}{6}\\
&= -\frac{2}{3}
\end{aligned}
$$

ii. $\lim_{x\to -1}\frac{\sqrt{x+10}-3}{2x+2}$

$$
\begin{aligned}
\lim_{x\to -1}\frac{\sqrt{x+10}-3}{2x+2} &= \lim_{x\to -1}\frac{\sqrt{x+10}-3}{2x+2}\cdot\frac{\sqrt{x+10}+3}{\sqrt{x+10}+3}\\
&= \lim_{x\to -1}\frac{x+10-9}{(2x+2)\left(\sqrt{x+10}+3\right)}\\
&= \lim_{x\to -1}\frac{x+1}{2(x+1)\left(\sqrt{x+10}+3\right)}\\
&= \lim_{x\to -1}\frac{1}{2\left(\sqrt{x+10}+3\right)}\\
&= \frac{1}{2\left(\sqrt{-1+10}+3\right)}\\
&= \frac{1}{2\left(\sqrt{9}+3\right)}\\
&= \frac{1}{2(6)}\\
&= \frac{1}{12}
\end{aligned}
$$

(c) $f(x)=\frac{x+5}{3x-3}, x\neq 1$. $g(x)=3x^2-2$

i. $(f \circ g)(x)$

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f\left(3x^2 - 2\right) \\
&= \frac{3x^2 - 2 + 5}{3(3x^2 - 2) - 3} \\
&= \frac{3x^2 + 3}{3(3x^2 - 2 - 1)} \\
&= \frac{3\left(x^2 + 1\right)}{3(3x^2 - 3)} \\
&= \frac{x^2 + 1}{3(x^2 - 1)}, x \neq \pm 1
\end{aligned}$$

ii. $\left(f^{-1} \circ g\right)(x)$

A. Find f^{-1}, Let $f(x) = y$

$$\begin{aligned}
y &= \frac{x + 5}{3x - 3} \\
y &= \frac{x + 5}{3(x - 1)} \\
3y(x - 1) &= x + 5 \\
3xy - 3 &= x + 5 \\
3xy - x &= 5 + 3 \\
x(3y - 1) &= 8 \\
x &= \frac{8}{3y - 1} \\
f^{-1}(x) &= \frac{8}{3x - 1}, x \neq \pm 1
\end{aligned}$$

B. Find the composition

$$\begin{aligned}
\left(f^{-1} \circ g\right)(x) &= f^{-1}(g(x)) \\
&= f^{-1}\left(3x^2 - 2\right) \\
&= \frac{8}{3(3x^2 - 2) - 1} \\
&= \frac{8}{9x^2 - 6 - 1} \\
\left(f^{-1} \circ g\right)(x) &= \frac{8}{9x^2 - 7}, x \neq \left(\pm \frac{\sqrt{7}}{9}\right)
\end{aligned}$$

(d) $$f(x) = \begin{cases} \frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1 & x < 1 \\ x^2 + 1 & x \geq 1 \end{cases}$$

i. $\lim_{x\to 1^-} f(x)$

$$\begin{aligned}\lim_{x\to 1^-} x+1 &= \lim_{x\to 1^-} 1+1\\ &= 2\end{aligned}$$

ii. $\lim_{x\to 1^+} f(x)$

$$\begin{aligned}\lim_{x\to 1^+} f(x) &= 1^2+1\\ &= 2\end{aligned}$$

iii. $\lim_{x\to 1} f(x) = 1$

iv. $f(1) = 1^2 + 1 = 2$

v. $f(x)$ is continuous at $x = 1$

2.

(a) $f'(x)$ from first principle if $f(x) = 3x^2 - 3$

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\\ &= \lim_{h\to 0} \frac{3(x+h)^2 - 3 - \left(3x^2 - 3\right)}{h}\\ &= \lim_{h\to 0} \frac{3\left(x^2 + 2hx + h^2\right) - 3 - 3x^2 + 3}{h}\\ &= \lim_{h\to 0} \frac{3x^2 + 6hx + 3h^2 - 3x^2}{h}\\ &= \lim_{h\to 0} \frac{h(6x+3h)}{h}\\ &= \lim_{h\to 0} 6x + 3h\\ f'(x) &= 6x\end{aligned}$$

(b) Differentiate with respect to x.

i. $f(x) = \frac{e^{2x}+3x}{\sqrt{x+1}}$

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\frac{e^{2x}+3x}{\sqrt{x+1}}\right] \\
&= \frac{\sqrt{x+1}\cdot\frac{d}{dx}\left[e^{2x}+3x\right]-\left(e^{2x}+3x\right)\cdot\frac{d}{dx}\left[\sqrt{x+1}\right]}{x+1} \\
&= \frac{\sqrt{x+1}\cdot\left(2e^{2x}+3\right)-\left(e^{2x}+3x\right)\cdot\frac{1}{2\sqrt{x+1}}}{x+1} \\
&= \frac{\sqrt{x+1}\cdot\left(2e^{2x}+3\right)-\frac{\left(e^{2x}+3x\right)}{2\sqrt{x+1}}}{x+1} \\
&= \frac{\frac{(x+1)\cdot\left(2e^{2x}+3\right)-\frac{1}{2}\left(e^{2x}+3x\right)}{\sqrt{x+1}}}{x+1} \\
&= \frac{(x+1)\cdot\left(2e^{2x}+3\right)-\frac{1}{2}\left(e^{2x}+3x\right)}{(x+1)^{\frac{3}{2}}} \\
&= \frac{2(x+1)\cdot\left(2e^{2x}+3\right)-\left(e^{2x}+3x\right)}{2(x+1)^{\frac{3}{2}}}
\end{aligned}$$

ii. $f(x) = \left(\sin^2 x\right)\left[\ln\left(x^4-1\right)\right]$

$$\begin{aligned}
f(x) &= (\sin x)^2\left[\ln\left(x^4-1\right)\right] \\
f'(x) &= 2(\sin x)(\cos x)\left[\ln\left(x^4-1\right)\right]+(\sin x)^2\left(\frac{4x^3}{x^4-1}\right) \\
&= \sin 2x\ln\left(x^4-1\right)+\sin^2 x\left(\frac{4x^3}{x^4-1}\right)
\end{aligned}$$

(c) $y^2-4y-2x+1=0$

$$\begin{aligned}
\frac{d}{dx}\left[y^2-4y-2x+1\right] &= 0 \\
2y\frac{dy}{dx}-4\frac{dy}{dx}-2 &= 0 \\
2y\frac{dy}{dx}-4\frac{dy}{dx} &= 2 \\
\frac{dy}{dx}(y-2) &= 1 \\
y' &= \frac{1}{y-2}
\end{aligned}$$

i. Equation of tangent at $(-2,1)$

$$\begin{aligned}
y' &= \frac{1}{1-2} \\
&= -1
\end{aligned}$$

A. Find equation

$$\begin{aligned} y - y_1 &= y'\,(x - x_1) \\ y - 1 &= -1\,(x - (-2)) \\ y &= -x - 2 + 1 \\ y &= -x - 1 \end{aligned}$$

(d) $x = \ln(2t+1)\,, y = 4t^2$

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt}\left[\ln(2t+1)\right] \\ &= \frac{1}{2t+1} \cdot 2 \\ \frac{dx}{dt} &= \frac{2}{2t+1} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}\left[4t^2\right] \\ \frac{dy}{dt} &= 8t \end{aligned}$$

i. Utilize chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} * \frac{dt}{dx} \\ &= 8t \cdot \frac{2t+1}{2} \\ &= 4t\,(2t+1) \\ &= 8t^2 + 4t \end{aligned}$$

ii. Substitute in t

$$\begin{aligned} \frac{dy}{dx}|_{t=1} &= 8\,(1) + 4\,(1) \\ &= 12 \end{aligned}$$

iii. Differentiate again (note, you cannot differentiate THEN chain for the second derivative, otherwise you will end up with $\frac{d^2y}{dt^2} * \frac{dt^2}{d^2x} = \frac{d^2y}{d^2x}$, which is obviously different from $\frac{d^2y}{dx^2}$). In this part, because our equations is in terms of t, we cannot $\frac{d}{dx}$, we must $\frac{d}{dt}$. Therefore, we must find a way to chain things together so that they become $\frac{d^2y}{dx^2}$after cancelling. So we know that we:

A. must $\frac{d}{dt}$ to get the d^2*something* on top, and since we are differentiating with respect to t, we must have dt on the bottom.

B. must do at least once, $\frac{dy}{d-something}$ to get one dy on top.

C. must do two $\frac{d-something}{dx}$ to get dx^2 on bottom.

D. Enter our solution:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dt} * \frac{dy}{dx} * \frac{dt}{dx} \\ &= \frac{d}{dt}\left(8t^2+4t\right) * \frac{2t+1}{2} \\ &= (16t+4) * \frac{2t+1}{2} \\ &= \frac{32t^2+16t+8t+4}{2} \\ &= 16t^2+12t+2\end{aligned}$$

E. Substitute in t

$$\begin{aligned}\frac{dy}{dx}|_{t=1} &= 16\left(1\right)+4 \\ &= 20\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2}|_{t=1} &= 16\left(1\right)^2+12\left(1\right)+2 \\ &= 30\end{aligned}$$

3.

(a) Newton-raphson method. Initial $x_0 = 0.8$. Find root of $x^3+2x-4 = 0$. 3 d.p.

i. Let $f\left(x\right) = x^3+2x-4$

$$\begin{aligned}f\left(x\right) &= x^3+2x-4 \\ f'\left(x\right) &= 3x^2+2\end{aligned}$$

ii. Recursive newton-raphson method

$$x_{n+1} = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)}$$

$$\begin{aligned}x_1 &= 1.2816 \\ x_2 &= 1.1851 \\ x_3 &= 1.1795 \\ x_4 &= 1.1795\end{aligned}$$

iii. Since x_3 and $x_4$agree to 3.d.p.. The root of the equation is 1.180 (3dp)

(b)

i. Define the terms
   A. $A = area = 486$
   B. $P = perimeter$
   C. $x = width$
   D. $y = height$

ii. Find the formulas to relate

$$A_{total} = xy$$

$$\begin{aligned} xy &= 486 \\ y &= \frac{486}{x} \end{aligned}$$

$$\begin{aligned} A_{print} &= (x-4)(y-6) \\ &= xy - 4y - 6x + 24 \end{aligned}$$

iii. Substitue $y = \frac{486}{x}$ into A_{print}

$$\begin{aligned} A_{print} &= x\left(\frac{486}{x}\right) - 4\left(\frac{486}{x}\right) - 6x + 24 \\ &= 486 - \frac{1944}{x} - 6x + 24 \end{aligned}$$

iv. Maximize A_{print}

$$\frac{dA}{dx} = \frac{1944}{x^2} - 6$$

$$\begin{aligned} \frac{1944}{x^2} - 6 &= 0 \\ \frac{1944}{x^2} &= 6 \\ 1944 &= 6x^2 \\ x^2 &= \frac{1944}{6} \\ &= 324 \\ x &= 18cm \end{aligned}$$

v. Find y

$$\begin{aligned} y &= \frac{486}{18} \\ &= 27cm \end{aligned}$$

vi. Maximized dimensions

$$18cm * 27cm$$

(c) $y = x^3 - 12x^2$

$$\frac{dy}{dx} = 3x^2 - 24x$$

$$\begin{aligned} 3x^2 - 24x &= 0 \\ 3x(x-8) &= 0 \\ x &= 0, 8 \end{aligned}$$

i. $\frac{d^2y}{dx^2} = 6x - 24$

A. When $x = 0$

$$\begin{aligned} \frac{d^2y}{dx^2} &= -24 \\ &= max\, point \end{aligned}$$

When $x = 8$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 48 - 24 \\ &= 24 > 0 \\ &= min\, point \end{aligned}$$

ii. Coordinates

A. When $x = 0$

$$y = 0$$

B. When $x = 8$

$$\begin{aligned} y &= x^3 - 12x^2 \\ &= -256 \end{aligned}$$

iii. Maximum point: $(0, 0)$

iv. Minimum point $(8, -256)$

v. Coordinates of inflection point

$$\begin{aligned} \frac{d^2y}{dx^2} &= 6x - 24 \\ 6x - 24 - 0 \\ x &= \frac{24}{6} \\ x &= 4 \end{aligned}$$

A. When $x = 4$

$$\begin{aligned} y &= (4)^3 - 12\,(4)^2 \\ &= -128 \end{aligned}$$

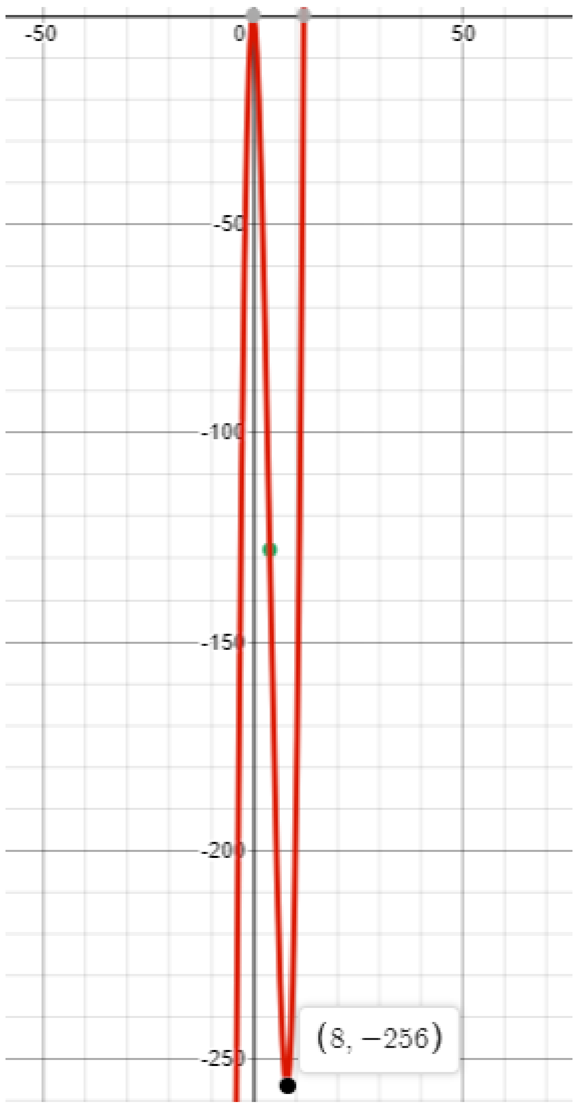
B. Inflection point

$$(4, -128)$$

vi. Concave up (range of x): $(-\infty, 4)$

vii. Concave down (range of x): $(4, +\infty)$

viii. Sketch the curve

A. 

4. Evaluate integrals

   (a) $\int_3^4 \frac{2x dx}{(x+4)(x-2)}$

      i. Partial fractions

$$\frac{2x}{(x+4)(x-2)} = \frac{A}{(x+4)} + \frac{B}{(x-2)}$$
$$2x = A(x-2) + B(x+4)$$
$$A(x-2) + B(x+4) - 2x = 0$$

         A. When $x = 2$

$$A(2-2) + B(2+4) - 2(2) = 0$$
$$6B = 4$$
$$B = \frac{4}{6}$$
$$B = \frac{2}{3}$$

         B. When $x = -4$

$$A(-4-2) - 2(-4) = 0$$
$$-6A + 8 = 0$$
$$A = \frac{8}{6}$$
$$= \frac{4}{3}$$

         C. Find equation

$$\frac{2x}{(x+4)(x-2)} = \frac{4}{3(x+4)} + \frac{2}{3(x-2)}$$

         D. Differentiate

$$\begin{aligned}
\int_3^4 \frac{2x dx}{(x+4)(x-2)} &= \int_3^4 \frac{4}{3(x+4)} + \frac{2}{3(x-2)} dx \\
&= \frac{4}{3}\int_3^4 \frac{1}{(x+4)} dx + \frac{2}{3}\int_3^4 \frac{1}{(x-2)} dx \\
&= \frac{4}{3}\left[\ln(x+4)\right]_3^4 + \frac{2}{3}\left[\ln(x-2)\right]_3^4 \\
&= \frac{4}{3}(\ln 8 - \ln 7) + \frac{2}{3}(\ln 2 - \ln 1) \\
&= \frac{4}{3}\ln\frac{8}{7} + \frac{2}{3}\ln 2 \\
&= 0.6401
\end{aligned}$$

(b) $\int_0^{\frac{\pi}{4}} (1-\sin 2x)^{\frac{3}{2}} \cos 2x dx$, Substitution

$$\begin{aligned} u &= 1-\sin 2x \\ \frac{du}{dx} &= -2\cos 2x \\ dx &= -\frac{du}{2\cos 2x} \end{aligned}$$

i. Find bounds with respect to u

A. When $x = \frac{\pi}{4}$

$$\begin{aligned} u &= 1-\sin 2\left(\frac{\pi}{4}\right) \\ &= 1-\sin\frac{\pi}{2} \\ &= 0 \end{aligned}$$

When $x = 0$

$$\begin{aligned} u &= 1-\sin 0 \\ &= 1 \end{aligned}$$

ii. Substitute inside

$$\begin{aligned} \int_0^{\frac{\pi}{4}} (1-\sin 2x)^{\frac{3}{2}} \cos 2x dx &= \int_1^0 u^{\frac{3}{2}} \cos 2x \left(-\frac{du}{2\cos 2x}\right) \\ &= -\frac{1}{2}\int_1^0 u^{\frac{3}{2}} du \\ &= -\frac{1}{2}\left[\frac{3}{2}u^{\frac{1}{2}}\right]_1^0 \\ &= -\frac{1}{2}\left(-\frac{3}{2}\right) \end{aligned}$$

$$\int_0^{\frac{\pi}{4}} (1-\sin 2x)^{\frac{3}{2}} \cos 2x dx = \frac{3}{4}$$

(c) Trapezium rule

$$\int_0^2 \ln\left(x^3+5\right) dx, n=5, \Delta x = \frac{2-0}{5} = 0.4, 2.decimal.places$$

i. Find the points

x	$f(x) = \ln\left(x^3+5\right)$	$f(x)$
0		1.605
0.4	1.622	
0.8	1.707	
1.2	1.906	
1.6	2.208	
2.0		2.565
SUM	7.443	4.17

ii. Trapezium rule

$$\begin{aligned}\int_0^2 \ln\left(x^3+5\right) dx &\approx \frac{0.4}{2}\left(4.17+2\left(7.443\right)\right)\\ &= 3.8112\\ &= 3.81\,(2\text{dp})\end{aligned}$$

(d) Simpson's rule $n=6$, $\int_3^6 \sqrt{7x^3+2}dx$. Correct to $2d.p.$. $\Delta x = \frac{6-3}{6} = 0.5$

i. Find the points, $f\left(x\right)=\sqrt{7x^3+2}$

x	Odd $f\left(x\right)$	Even $f\left(x\right)$	First/Last
3.0			13.820
3.5	17.381		
4.0		21.213	
4.5	25.255		
5.0		29.614	
5.5	34.155		
6			38.91
SUM	76.791	50.827	52.73

ii. Simpson's rule

$$\begin{aligned}\int_0^2 \ln\left(x^3+5\right) dx &\approx \frac{0.5}{3}\left(52.73+4\left(76.791\right)+2\left(50.827\right)\right)\\ &= 76.92\\ &= 76.92\,(2\text{dp})\end{aligned}$$