Revision 2: Resit Paper

January 30, 2020

1. Let $A = [\sim (p \wedge q) \wedge r] \rightarrow (q \vee r)$

   (a) Construct a truth table for expression A. Hence, determine whether the expression A is a tautology, contradiction or contingency.

   i.

p	q	r	$p \wedge q$	$\sim (p \wedge q)$	$\sim (p \wedge q) \wedge r$	$q \vee r$	$[\sim (p \wedge q) \wedge r] \rightarrow (q \vee r)$
0	0	0	0	1	0	0	1
0	0	1	0	1	1	1	1
0	1	0	0	1	0	1	1
0	1	1	0	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	1	1	1
1	1	0	1	0	0	1	1
1	1	1	1	0	0	1	1

   ii. Expression A is a Tautology

   iii. Write the Principal Disjunctive Normal Form (PDNF) and Principal Conjunctive Normal Form (PCNF) of A and $\sim A$. (ASK TEACHER, can we just write t and c instead of everything?)

      A. PDNF of A: t

      B. PCNF of A: c, Does not exist.

      C. PDNF of $\sim A$: c, Does not exist.

      D. PCNF of $\sim A$: t

   (b) Let
   p : It is below freezing.
   q : It is snowing.

   i. Write the following proposition using p, q and logical connectives.
   Proposition: If it is not snowing, then it is below freezing.

      A. Answer: $\sim q \rightarrow p$

   ii. Find the converse, inverse and contrapositive of the answer in part (i). Then write your final answers in the most simplified form **without connective '$\rightarrow$'**

      A. Converse: $p \rightarrow \sim q$

$$p \rightarrow \sim q \equiv \boldsymbol{\sim p \vee \sim q}$$

B. Inverse: $q \rightarrow \sim p$

$$q \rightarrow \sim p \equiv \boldsymbol{\sim q} \vee \boldsymbol{\sim p}$$

C. Contrapositive: $\sim p \rightarrow q$

$$\begin{aligned} \sim p \rightarrow q &\equiv \sim (\sim p) \vee q \\ &\equiv \boldsymbol{p} \vee \boldsymbol{q} \end{aligned}$$

(c) Let the universe of discourse be the set of all integers p, q and r are denoted as follows:

i. $p(x) : x$ is odd
ii. $q(x) : x$ is divisible by 3
iii. $r(x) : x$ is divisible by 2
Rewrite the following statements formally using quantifiers, variables and connectives. Then determine their truth values. For each false statement, provide a counterexample.

A. If x is odd, then x is divisible by 3.
$p(x) \rightarrow q(x)$
False. Counterexample: 1 is odd but 1 is not divisible by 3.

B. There exist an even integer divisible by 3.
$\exists x \in \mathbb{Z}, \sim q(x) \vee r(x)$
True.

C. If x is divisible by 2, then x is even.
$r(x) \rightarrow \sim p(x)$
True. OPTIONAL: $\frac{x}{2} = k, x = 2k$, is even.

(d) Use diagram to check the validity of the following argument.
All healthy people eat an apple a day. (Unstated: Major premise)
Jenny eats an apple a day.(Unstated: Minor premise)
Therefore, Jenny is a healthy person. (Unstated: Conclusion)

i. Notes: Students may use the following notations, let :
   A. A : Set of people who eat an apple a day
   B. B : Set of people who are healthy person
   C. J : Jenny
ii. Answer
   A. The argument is INVALID
   B. It is also possible that Jenny eats an apple a day but is not a healthy person. It is also possible that Jenny eats an apple a day and is both a healthy and not healthy person (incomplete intersection)

2. Question 2

(a) Find the greatest common divisor of 509 and 1177 by using Euclidean algorithm. Hence, determine the least common multiple of 509 and 1177.

$$\begin{aligned}
1177 &= 509\,(2) + 159 \\
529 &= 159\,(3) + 52 \\
159 &= 32\,(4) + 31 \\
32 &= 31\,(1) + 1 \\
31 &= 1\,(31) + 0
\end{aligned}$$

$$\begin{aligned}
gcd\,(1177, 509) &= gcd\,(509, 159) \\
&= gcd\,(159, 32) \\
&= gcd\,(32, 31) \\
&= gcd\,(31, 1) \\
&= gcd\,(1, 0) \\
&= 1
\end{aligned}$$

$$\begin{aligned}
lcm\,(1177, 609)\,gcd\,(1177, 609) &= 1177 * 509 \\
lcm\,(1177, 609) &= \frac{1177 * 509}{gcd\,(1177, 609)} \\
&= \frac{1177 * 509}{1} \\
&= 599093
\end{aligned}$$

(b) Prove that the sum of any two even integers is even.

i. By definition of even integer,

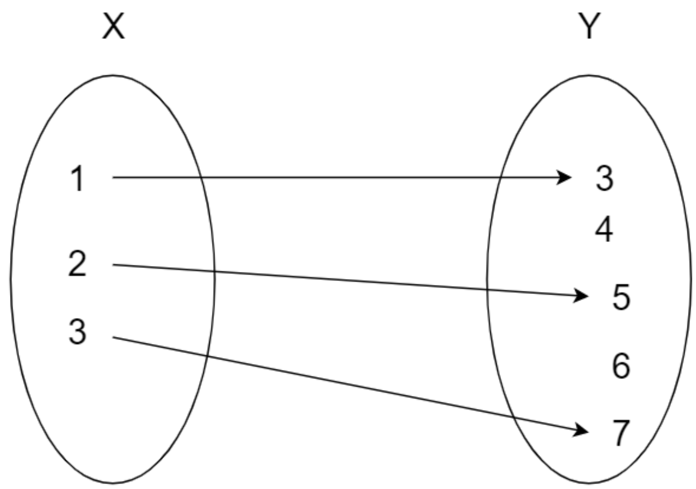
$$\begin{cases} x = 2a & , a \in \mathbb{Z} \\ y = 2b & , b \in \mathbb{Z} \end{cases}$$

ii. Then

$$\begin{aligned}
x + y &= 2a + 2b \\
&= 2\,(a + b)\,, a + b \in \mathbb{Z}
\end{aligned}$$

iii. Therefore $x + y$ is even, the sum of any two even integers is even.

(c) Given $f(x) = 2x+1$, a function from $X = \{1, 2, 3\}$ to $Y = \{3, 4, 5, 6, 7\}$. Find the domain and range of the function f. Hence, determine whether the function is everywhere defined or onto. Justify your answers.



i.
ii. $f(x) = 2x + 1$
iii. $Domain = \{1, 2, 3\} = X$
iv. $Range = \{3, 5, 7\} \neq Y$
v. Answer:
   A. f is everywhere defined.
vi. Explanation:
   A. Domain of f is X
   B. f is not onto Y

(d) Let A $= \{b, c, d, e, f, g\}$, $\rho_1 = (f, c, g)$ and $\rho_2 = (b, d, c, f, e)$ be permutations of A.

i. Compute $\rho_1 \circ \rho_2$.

$$\begin{aligned}
\rho_1 \circ \rho_2 &= \begin{pmatrix} b & c & d & e & f & g \\ b & g & d & e & c & f \end{pmatrix} \circ \begin{pmatrix} b & c & d & e & f & g \\ d & f & c & b & c & g \end{pmatrix} \\
&= \begin{pmatrix} b & c & d & e & f & g \\ d & c & g & b & e & f \end{pmatrix} \\
&= (b, d, g, f, e)
\end{aligned}$$

ii. Compute ρ_1^{-1} and write the result as the product of disjoint cycles.

$$\rho_1^{-1} = (g, c, f)$$

iii. Is ρ_2 even or odd permutation? Justify your answer.

$$\rho_2 = (b, e) \circ (b, f) \circ (b, c) \circ (b, d)$$

   A. The number of transpositions is 4

B. ρ_2 is an even permutation

3. Question 3

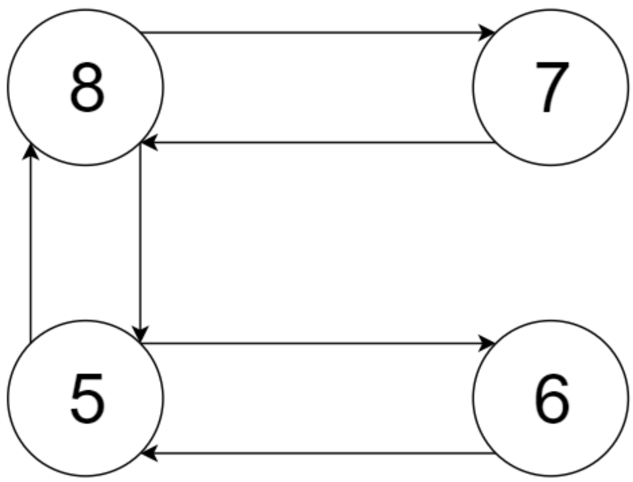
(a) Let R be the relation on $\{5,6,7,8\}$ given by $x\mathrm{R}y$ if and only if $|x–y| \geq$ 2.

i. List the ordered pairs belonging to the relation R.

$$R = \{(5,7), (5,8), (6,8), (7,5), (8,5), (8,6)\}$$

ii. Draw the digraph of R and represent R in matrix form.

A. Digraph



B. Matrix

$$M_R = \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

iii. Find the domain and range of R.

A.

$$Dom\,(R) = \{5,6,7,8\}$$
$$Ran\,(R) = \{5,6,7,8\}$$

iv. Compute the in-degree and out degree of each vertex.

A.

	5	6	7	8
InDegree	2	1	1	2
OutDegree	2	1	1	2

(b) Let A = {1, 2, 3, 4} and the relation on A is R = {(1, 1), (1, 2), (1, 3), (3, 1), (3, 3), (3, 4)}.

i. Determine whether the relation R is reflexive, irreflexive, symmetric, asymmetric, antisymmetric or transitive. Give a counterexample if the answer is "No".

A.

$$M_R = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

B. Not reflexive, $(2,2) \notin R$

C. Not irreflexive, $(1,1) \in R$

D. Not symmetric, $(1,2) \in R$ but $(2,1) \notin R$

E. Not asymmetric, $(1,3) \in R$ and $(3,1) \in R$

F. Not antisymmetric, $(1,3) \in R$ and $(3,1) \in R$ but $3 \neq 1$

G. Not transitive, $(1,3) \in R$ and $(3,4) \in R$ but $(1,4) \notin R$

ii. Is R an equivalence relation on A? Justify your answer.

A. R is not an equivalence relation on A, because R is not reflexive, not symmetric, and not transitive.

(c) Let $A = \{a, b, c, d\}$ and R and S be the relation on A described by the matrics

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

i. Compute $M_{S^{-1}}$, $M_{\bar{R}}$, $M_{R \cap S}$, $M_{S \circ R}$

A. $M_{S^{-1}}$

$$M_{S^{-1}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1}$$

$$M_{S^{-1}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

B. $M_{\bar{R}}$

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$M_{\bar{R}} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$M_{R \cap S} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

C. $M_{S \circ R}$

$$\begin{aligned} M_{S \circ R} &= M_R \cdot M_S \\ &= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ M_{S \circ R} &= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{aligned}$$

ii. Use Warshall's algorithm to compute the transitive closure of R.

A. W_0

$$W_0 = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & 1 & 0 & 1 \\ \mathbf{1} & 0 & 0 & 0 \\ \mathbf{0} & 0 & 1 & 1 \end{bmatrix}$$

$C_1 : \{b, c\}$
$R_1 : \{b, c\}$
ADD: $\{(b, b), (b, c), (c, b), (c, c)\}$

B. W_1

$$W_1 = \begin{bmatrix} 0 & \mathbf{1} & 1 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & \mathbf{1} & 1 & 1 \\ 0 & \mathbf{0} & 1 & 1 \end{bmatrix}$$

$C_1 : \{a, b, c\}$
$R_1 : \{a, b, c, d\}$

C. W_2

$$W_2 = \begin{bmatrix} 1 & 1 & \mathbf{1} & 1 \\ 1 & 1 & \mathbf{1} & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 1 \end{bmatrix}$$

$C_2 : \{a, b, c, d\}$
$R_2 : \{a, b, c, d\}$

D. W_3

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & \mathbf{1} \\ 1 & 1 & 1 & \mathbf{1} \\ 1 & 1 & 1 & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}$$

$C_3 : \{a, b, c, d\}$
$R_4 : \{b, c, d\}$

E. W_4

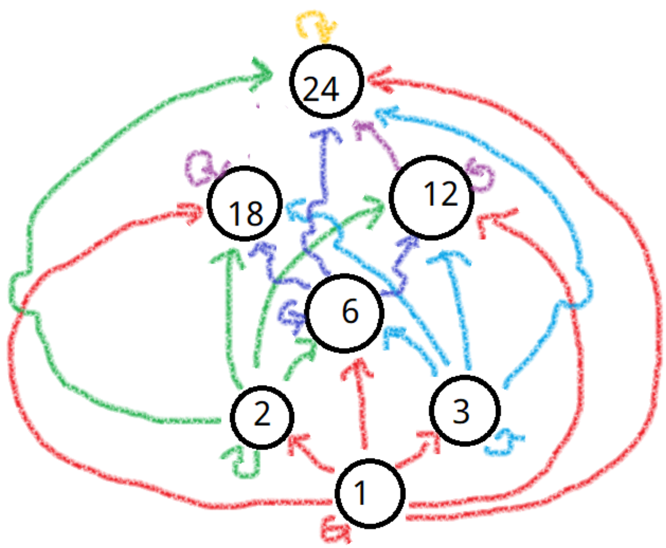
$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = M_{R^\infty}$$

F. The transitive closure of R is $\{(a, a), (a, b), (a, c), (a, d), ...\}$ (note, write everything down).

4. Question 4

(a)

i. (Note, please don't draw as framework for Hasse Diagram first, otherwise you'll kill yourself with hops)



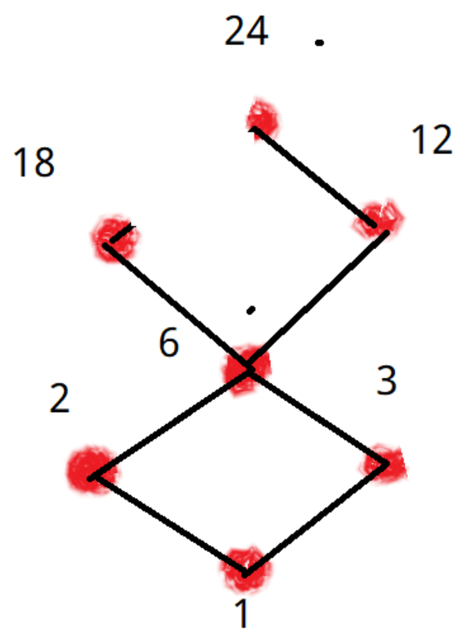
A.

B. All nodes have loops (reflexive)

C. If node a is connected to node b, and node b is connected to node c, then node a is connected to node c. (transitive)

D. If node a and node b is related, then node a is node b. (antisymmetric)

ii. Hasse Diagram



A.

iii.

A. Minimal elements: $\{1\}$

B. Maximal elements: $\{18, 24\}$

iv. L&G Element

A. Least element: 1

B. Greatest element: None

v. LUB & GLB of $\{2, 3, 6\}$

A. $LUB : 6$

B. $GLB : 1$

(b)

i. Start with LHS, if you want to start with RHS, go ahead

$$\begin{aligned}(x \wedge y \wedge z) \vee (x \wedge z) &\equiv xyz + xz \\ &\equiv xz\,(y+1) \\ &\equiv xz \\ &\equiv x \wedge z \\ &= RHS\end{aligned}$$

(c)

	z'	z'	z	z	
x'	0	0	1	1	y'
x'	0	0	1	1	y
x	1	0	0	1	y
x	0	1	1	0	y'
	w'	w	w	w'	

i. Simplified:

$$f\left(x,y,z,w\right)=x'z+w'xy+wxy'$$