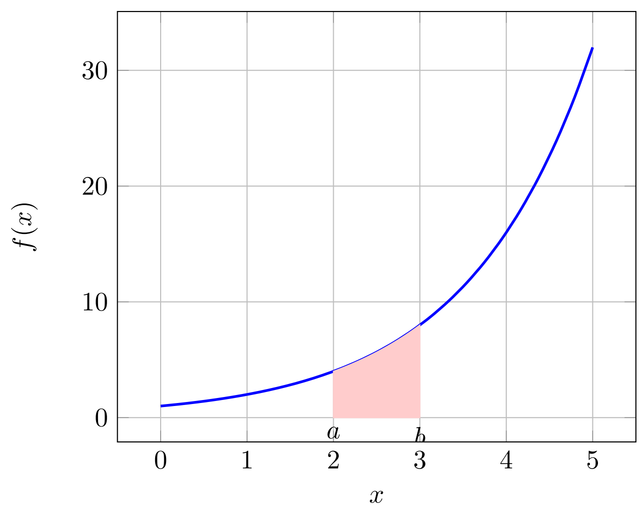
AAMS3123 Calc II - C1 Application of Integration

November 30, 2019

1 Area Under a Curve

$$Area = \int_a^b f(x)\,dx = \int_a^b f(y)\,dy$$



1.1 Example 1

Find the area bounded by the x-axis, the y-axis, the curve $y = e^x$ and the line $x = 2$

$$\begin{aligned}
A &= \int_0^2 f(x)\,dx \\
&= \int_0^2 e^x\,dx \\
&= \left[e^x\right]_0^2 \\
&= e^2 - e^0 \\
A &= e^2 - 1
\end{aligned}$$

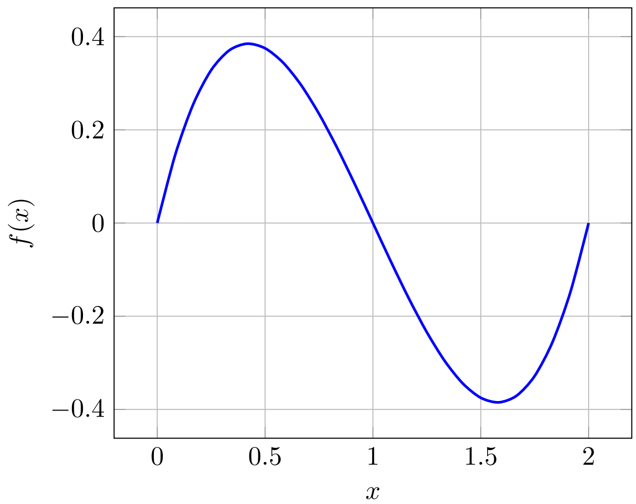
1.2 Example 2

Find the area between the curve $x = 4 - y^2$ and the y-axis.

$$\begin{aligned}
A &= \int_{-2}^2 4 - y^2 dy \\
&= \left[4y - \frac{y^3}{3}\right]_{-2}^2 \\
&= \left(4(2) - \frac{2^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right) \\
&= \left[4(2) - \frac{2^3}{3}\right] - \left[4(-2) - \frac{(-2)^3}{3}\right] \\
&= 10\frac{2}{3}
\end{aligned}$$

1.3 Example 3

Find the area enclosed between the curve $y = x(x-1)(x-2)$ and the x-axis



$$
\begin{aligned}
y &= x\left(x-1\right)\left(x-2\right)\\
&= \left(x^2-x\right)\left(x-2\right)\\
&= \left(x^3-2x^2-x^2+2x\right)\\
&= x^3-3x^2+2x
\end{aligned}
$$

$$
\begin{aligned}
A &= \int_0^1 f\left(x\right)dx - \int_1^2 f\left(x\right)dx\\
&= \int_0^1 x\left(x-1\right)\left(x-2\right)dx - \int_1^2 x\left(x-1\right)\left(x-2\right)dx\\
&= \int_0^1 x\left(x-1\right)\left(x-2\right)dx - \int_1^2 x\left(x-1\right)\left(x-2\right)dx\\
&= \frac{1}{2}
\end{aligned}
$$

1.4 Example 4 (Area between two curves)

$$Area = \int_a^b \left[f\left(x\right) - g\left(x\right)\right]dx$$

Calculate the area enclosed between the curves $y = 2x^2$ and $y^2 = 4x$.

1. Find the intersections

$$
\begin{aligned}
y &= 2x^2 \\
y &= \sqrt{4x} \\
&= 2\sqrt{x} \\
2x^2 &= 2\sqrt{x} \\
4x^4 &= 4x \\
x^4 &= x \\
x^4 - x &= 0 \\
x\left(x^3 - 1\right) &= 0 \\
x &= 0, x = 1
\end{aligned}
$$

2. Check which one is above, when $x = \frac{1}{2}$

$$
\begin{aligned}
y &= 2\left(\frac{1}{2}\right)^2 \\
&= \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
y &= 2\sqrt{\frac{1}{2}} \\
&= 1.414
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= y = 2\sqrt{x} \\
g(x) &= y = 2x^2
\end{aligned}
$$

3. They did not go below the $y-axis$, therefore, we can calculate immediately

$$
\begin{aligned}
A &= \int_0^1 2\sqrt{x} - 2x^2 dx \\
A &= \frac{2}{3}
\end{aligned}
$$

1.5 Example 5

Find the total area of the region between $y = 4x - x^2$ and $y = x$ from $x = 0$ to $x = 4$.

1. Find intersection

$$
\begin{aligned}
4x - x^2 &= x \\
x^2 + x - 4x &= 0 \\
x^2 - 3x &= 0 \\
x(x - 3) &= 0 \\
x &= 0, 3
\end{aligned}
$$

2. Check which on top

$$\begin{aligned} y|_{x=1} &= 4\left(1\right) - 1 \\ &= 3 \end{aligned}$$

$$\begin{aligned} y|_{x=1} &= 1 \\ f\left(x\right) &= 4x - x^2 \\ g\left(x\right) &= x \end{aligned}$$

3. Find the integral

$$\begin{aligned} A &= \int_0^{43} 4x - x^2 - xdx + \int_3^4 x - \left(4x - x^2\right) dx \\ &= 1.833 + 4.5 \\ &= 6.333 \\ &= \frac{19}{3} \end{aligned}$$

1.6 Example 6

The point of intersection in the positive quadrant of the graph of $y = 4x$ and the graph of $y = \frac{1}{x}$ is A, and B is the point of intersection of the graph of $y = \frac{1}{x}$ with the graph of $y = x^2$. Find the area of the region OAB bounded by portions of the graphs, where O is the origin.

1. Find the intersection points

   (a) Intersection point, A

$$\begin{aligned} 4x &= \frac{1}{x} \\ 4x^2 &= 1 \\ x &= \sqrt{\frac{1}{4}} \\ &= \pm\frac{1}{2} \\ &= \frac{1}{2}\left(positive\, quadrant\right) \end{aligned}$$

$$\begin{aligned} y|_{x=\frac{1}{2}} &= 4\left(\frac{1}{2}\right) \\ &= 2 \end{aligned}$$

$$A = \left(\frac{1}{2}, 2\right)$$

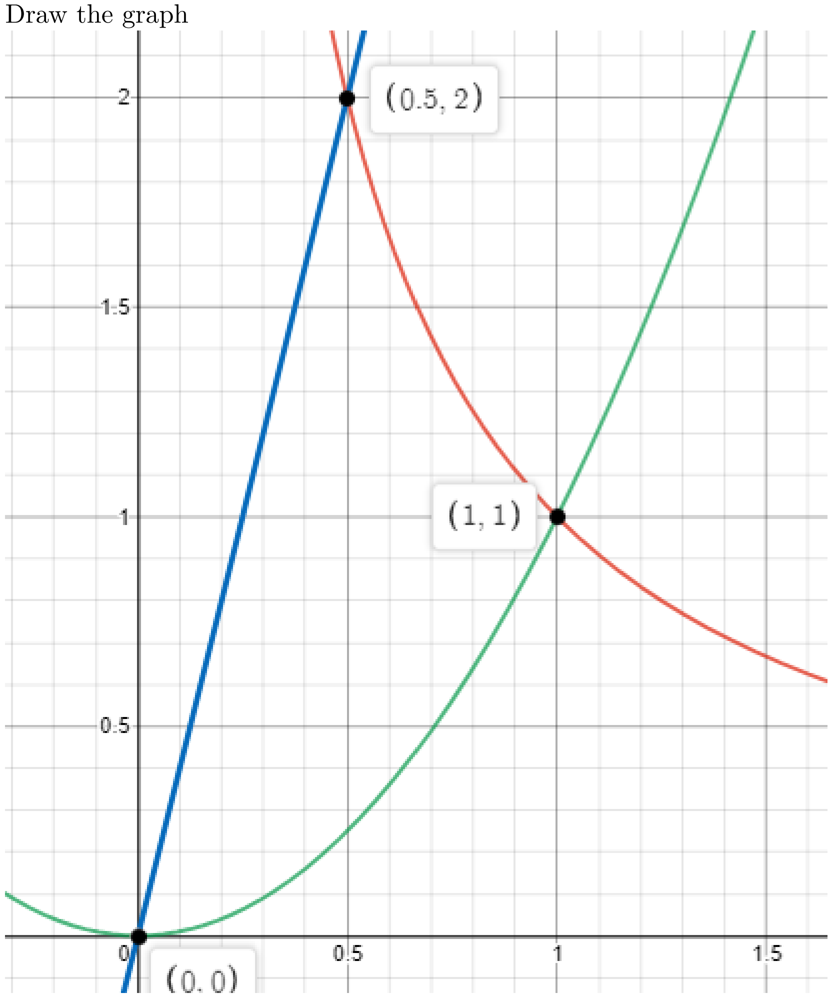
(b) Intersection point, B

$$\begin{aligned}\frac{1}{x} &= x^2 \\ x^3 &= 1 \\ x &= 1\end{aligned}$$

$$\begin{aligned}y|_{x=1} &= \frac{1}{1} \\ &= 1\end{aligned}$$

$$B = (1, 1)$$

2. Draw the graph



3. Find the area within

$$
\begin{aligned}
A &= \int_0^{0.5} \left(4x - x^2\right) dx + \int_{0.5}^1 \left(\frac{1}{x} - x^2\right) dx \\
&= 0.85981
\end{aligned}
$$

Parametric Equations

$$A = \int_\alpha^\beta g\left(t\right) f'\left(t\right) dt$$

1. Explanation: Cut the area under the graph into tiny rectangles. For each tiny rectangle, multiply the $y/f\left(x\right)$ value with the small change in x, $\Delta x \approx f'\left(t\right)$ to get the area of every individual rectangle. Then, add them all up to get the total area.

1.7 Example 7 (Parametric Equations)

A curve has parametric equation $x = at^2$, $y = 2at$.Find the area bounded by the curve, the $x - axis$, and the coordinates at $t = 1$ and $t = 2$.

$$
\begin{aligned}
A &= \int_1^2 2at \cdot \frac{d}{dt}\left[at^2\right] dt \\
&= \int_1^2 2at \cdot 2at\, dt \\
&= \int_1^2 4a^2t^2 dt \\
&= a^2 \int_1^2 4t^2 dt \\
&= \frac{28}{3} a^2
\end{aligned}
$$

1.8 Example 8

If $x = a\sin\theta$, $y = b\cos\theta$, find the area under the curve between $\theta = 0$ and $\theta = \pi$

$$
\begin{aligned}
A &= \int_0^\pi b\cos\theta \cdot \frac{d}{d\theta}\left[a\sin\theta\right] d\theta \\
&= \int_0^\pi b\cos\theta \cdot \frac{d}{d\theta}\left[a\sin\theta\right] d\theta \\
&= \int_0^\pi b\cos\theta \cdot \frac{d}{d\theta}\left[a\sin\theta\right] d\theta \\
&= \int_0^\pi b\cos\theta \cdot a\cos\theta d\theta \\
&= ab\int_0^\pi \cos^2\theta d\theta \\
&= ab\int_0^\pi \cos^2\theta d\theta \\
&= \frac{\pi}{2}ab
\end{aligned}
$$

2 Volume of Revolution (Cylindrical Shells Method)

$$
\begin{aligned}
V &= \int_a^b \pi\left[f(x)\right]^2 dx \\
&= \int_a^b \pi y^2 dx \\
&= \int_c^d \pi x^2 dy
\end{aligned}
$$

1. Explaination:
   (a) Comparing it to calculating the volume of a cylinder, from a to b:
      i. Cylinder: $V = \int_a^b \pi r^2 dx$
   (b) For the first equation, remember that πr^2 is the area of a circle. So, how do you find, the volume of a cylinder? Easy, you take the area of the circle πr^2, multiplied by the height of the circle, h. This means that if the cylinder is 1000 units long, you will multiply by 1000, hence adding 1000 cylinders, with area of πr^2 together. This is the same thing, but instead of a cylinder, you now have a "curvy cylinder". Because the area of the circle constantly changes, you have to recompute the area of the circle at every point.
   (c) The formula is simply saying, add together ($\int$) all the circles $\pi\left[f(x)\right]^2$, where $f(x)$ is the radius of the circle, from a to b ($\int_a^b$).

2.1 Example 1

The area between a curve $y = x\,(2-x)$, the x-axis, the lines $x = 0$ and $x = 2$ is rotated about the x-axis through 360° , find the volume generated.

$$\begin{aligned} y &= x\,(2-x) \\ V &= \pi \int_0^2 \left[x\,(2-x)\right]^2 dx \\ &= \pi \int_0^2 \left[x\,(2-x)\right]^2 dx \\ &= \pi \left(\frac{32}{3} - 16 + \frac{32}{5}\right) \\ &= \frac{16}{15}\pi \end{aligned}$$

2.2 Example 2

Find the volume generated when the region bounded by $y = x^2$, the y-axis and $y = 1$ is rotated about the y-axis.

$$\begin{aligned} V &= \int_0^1 \pi \left[y^{\frac{1}{2}}\right]^2 dy \\ &= \int_0^1 \pi y dy \\ &= \pi \int_0^1 y\, dy \\ &= \frac{1}{2}\pi \end{aligned}$$

2.3 Example 3

Error 404: Not found. No, seriously, its missing from the notes. So I'll leave the answer for your imagination.

2.4 Example 3

Find the volume of the solid generated by rotating the region bounded by the curve with equation $y = \frac{1}{2}\left(e^x + e^{-x}\right)$, the $x - axis$ and the lines $x = \pm 1$ about the $x - axis$

$$\begin{aligned} V &= \int_{-1}^{1} \pi \left[\frac{1}{2}\left(e^{x} + e^{-x}\right)\right]^2 dx \\ &= \pi \int_{-1}^{1} \frac{1}{4}\left(e^{x} + e^{-x}\right)^2 dx \\ &= \pi \frac{4e^2 - 1 + e^4}{4e^2} \\ &= 2.813\pi \end{aligned}$$

2.5 Example 5

1. Extra Notes: If area is bounded by 2 curves $y = f(x)$ and $y = g(x)$.

$$V = \int_{a}^{b} \pi \left[(f(x))^2 - (g(x))^2\right] dx$$

$$V = \int_{c}^{d} \pi \left[f(y)^2 - (g(y))^2\right] dx$$

2. Explanation: Kinda the same as the previous formula $V = \int_a^b \pi \left[f(x)\right]^2 dx$, just now is a little more stuff. Basically, the previous one, in longer form should be

$$V = \int_{a}^{b} \pi \left[f(x)\right]^2 - 0^2 dx$$

but now we don't have the luxury of having another equation with 0 in it.

Calculate the volume obtained by rotating the area enclosed by the curves $y^2 = 4x$, and $y = 2x^2$ (i) about the x-axis, (ii) about the y-axis.

1. about the x-axis

   (a) Find the intersections

$$\begin{aligned} y^2 &= 4x \\ y &= \sqrt{4x} \\ y &= 2\sqrt{x} \end{aligned}$$

$$\begin{aligned} 2\sqrt{x} &= 2x^2 \\ \sqrt{x} &= x^2 \\ x &= x^4 \\ x^4 - x &= 0 \\ x\left(x^3 - 1\right) &= 0 \\ x &= 0, x = 1 \end{aligned}$$

(b) Check which one is on top

$$x = \frac{1}{4}$$

$$\begin{aligned} y &= 2\sqrt{\frac{1}{4}} \\ &= \pm 1 \\ &= 1 \, (\text{since we rotate 360 degrees positive or negative doesn't matter}) \end{aligned}$$

$$\begin{aligned} y &= 2\left(\frac{1}{4}\right)^2 \\ &= 2\left(\frac{1}{16}\right) \\ &= \frac{1}{8} \end{aligned}$$

$$\begin{aligned} f(x) &= 2\sqrt{x} \\ g(x) &= 2x^2 \end{aligned}$$

(c) Find the volume

$$\begin{aligned} \int_0^1 \pi \left[2\sqrt{x} - 2x^2\right] dx &= \pi \int_0^1 4x - \left(2x^2\right)^2 dx \\ &= \pi \frac{6}{5} \\ &= \frac{6}{5}\pi \end{aligned}$$

2. about the $y - axis$

(a) Find the intersections (in terms of y)

$$\begin{aligned} x &= \frac{y^2}{4} \\ x^2 &= \frac{y^4}{16} \\ x^2 &= \frac{y}{2} \end{aligned}$$

$$\begin{aligned} \frac{y^4}{16} &= \frac{y}{2} \\ y^4 &= 8y \\ y^4 - 8y &= 0 \\ y\left(y^3 - 8\right) &= 0 \\ y &= 0, 2 \end{aligned}$$

(b) Find the volume

$$\int_0^2 \pi \left[\frac{y^4}{16} - \frac{y}{2}\right] dy = \pi \int_0^2 \left[\frac{y^4}{16} - \frac{y}{2}\right] dy$$
$$= -\frac{3}{5}\pi$$

i. Looks like we got negative area, which doesn't sounds right, but this is simply because we mixed up the order, subtracting a bigger number from a smaller number, the actual should be $\frac{3}{5}\pi$

2.6 Example

Find the points of intersection of the curves $y^2 = x^3$ and $y^2 = 2 - x$ and find the volume of the solid generated by rotating the region enclosed by the two curves about the x-axis.

1. Find the points of intersection

$$x^3 = 2 - x$$
$$x^3 + x - 2 = 0$$
$$x = 1$$

2. Remember the region enclosed by two curves ABOUT the x-axis (Now would be a good time to draw out the diagram)

$$V = \int_0^1 \pi x^3 dx + \int_1^2 \pi (2 - x)\, dx$$
$$= \pi \left(\int_0^1 x^3 dx + \int_1^2 (2 - x)\, dx \right)$$
$$V = \frac{3}{4}\pi$$

2.7 Example

1. Notes: Finding volume of parametric equations, with:
   (a) $x = f(t)$
   (b) $y = g(t)$
2. Same, cylindrical shells method

$$V = \pi \int_\alpha^\beta [g(t)]^2 f'(t)\, dt$$

3. Explanation, in case you're wondering, the original form is this:

$$V = \int_\alpha^\beta \pi [g(t)]^2 f'(t)\, dt$$

(a) Remember the πr^2? The π is just π, the r is $g(t)$, which is y (since $y = g(t)$), and since $f(x)$ or y is essentially the radius of the circle at x (drawing a line up x till you meet the $f(x)$).

(b) The $f'(t)$ is simply the best estimate to the small change in $f(t)$, which is also approximately equal to Δx

The parametric equations of a curve are $x = 3t^2, y = 3t - t^2$ Find the volume generated when the plane figure bounded by the curve, the x-axis and the coordinates corresponding to $t = 0, t = 2$, rotates about the $x - axis$.

$$\begin{aligned} V &= \pi \int_0^2 \left[3t - t^2\right]^2 \frac{d}{dt}\left(3t^2\right) dt \\ &= \pi \int_0^2 \left[3t - t^2\right]^2 (6t)\, dt \\ &= \frac{248\pi}{5} \end{aligned}$$

2.8 Example

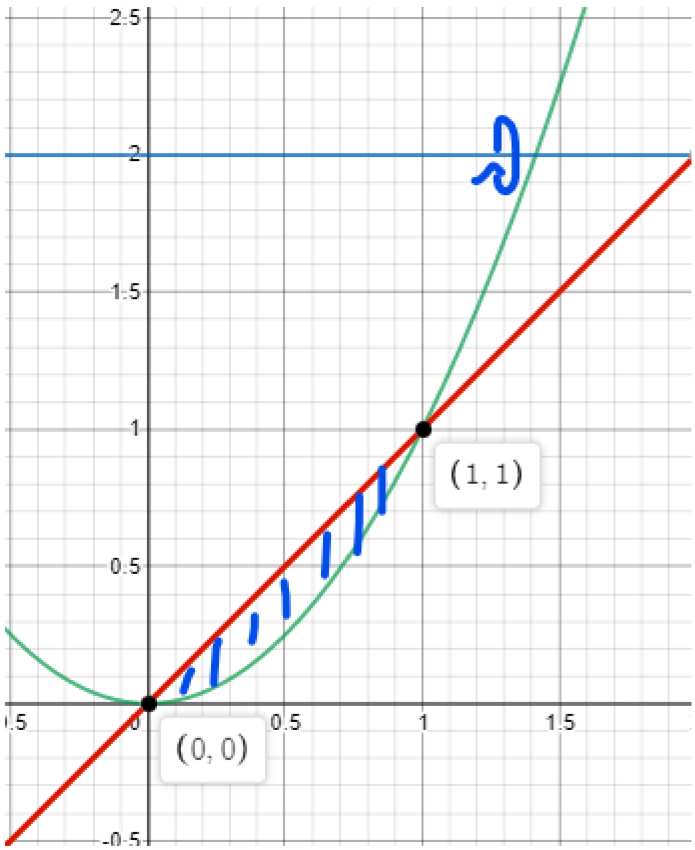
Find the volume of the solid obtained by rotating the region enclosed by the curves $y = x$ and $y = x^2$ **about the line** $y = 2$.

1. Note: In this case, the entire graph is "shifted above the x-axis by 2 units". Therefore, you gotta "bring it down".

2. Find the intersection point

$$\begin{aligned} x &= x^2 \\ x^2 - x &= 0 \\ x(x-1) &= 0 \\ x &= 0, x = 1 \end{aligned}$$

3. Figure out the area



$$V = \int_0^1 \pi \left[\left(x^2-2\right)^2 - (x-2)^2\right] dx$$
$$= \frac{8}{15}\pi$$

2.9 Example Find the volume of the solid obtained by rotating the region enclosed by the curves $y = x$ and $y = x^2$ about the line $x = -1$.

1. Find intersection

$$x = x^2$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0, 1$$

2. Convert equations to $x = ...$ Remember, the radius now stretches along the $x - axis$ unlike the last question. Therefore, you need to find radius x.

$$x = y$$

$$x = \sqrt{y}$$

3. Find the volume. Note: shifted "down", so you gotta "shift back up"

$$\int_0^1 \pi \left[(x - (-1))^2 - (x - (-1))^2 \right] dx = \pi \int_0^1 \left[(\sqrt{y} + 1)^2 - (y + 1)^2 \right] dy$$
$$= \frac{\pi}{2}$$

3 Arc Length Of Curves

A smooth curve with parametric equations $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, the arc length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

3.1 Example

Find the length of the curve $x = 2\cos^3\vartheta, y = 2\sin^3\vartheta$ between the points corresponding to $\vartheta = 0$ and $\vartheta = \frac{\pi}{2}$

$$\frac{dx}{d\vartheta} = 2 \cdot 3 (\cos\vartheta)^2 (-\sin\vartheta)$$
$$\frac{dx}{d\vartheta} = -6\cos^2\vartheta \sin\vartheta$$

$$\frac{dy}{d\vartheta} = 2 \cdot 3 (\sin\vartheta)^2 (\cos\vartheta)$$
$$= 6\sin^2\vartheta\cos\vartheta$$

$$
\begin{aligned}
L &= \int_0^{\frac{\pi}{2}} \sqrt{\left(-6\cos^2\vartheta\sin\vartheta\right)^2 + \left(6\sin^2\vartheta\cos\vartheta\right)^2} d\vartheta \\
&= \int_0^{\frac{\pi}{2}} \sqrt{36\cos^4\vartheta\sin^2\vartheta + 36\sin^4\vartheta\cos^2\vartheta} d\vartheta \\
&= \int_0^{\frac{\pi}{2}} \sqrt{36\cos^2\vartheta\sin^2\vartheta\left(\cos^2\vartheta + \sin^2\vartheta\right)} d\vartheta \\
&= \int_0^{\frac{\pi}{2}} 6\cos\vartheta\sin\vartheta\sqrt{1} d\vartheta \quad \text{Note: } \sin^2\theta + \cos^2\theta = 1 \\
&= \int_0^{\frac{\pi}{2}} 3\left(2\cos\vartheta\sin\vartheta\right) d\vartheta \\
&= 3\int_0^{\frac{\pi}{2}} \sin\left(2\vartheta\right) d\vartheta \quad \text{Note:} 2\cos\theta\sin\theta = 1 \\
&= 3\left[-\frac{1}{2}\cos\left(2\vartheta\right)\right]_0^{\frac{\pi}{2}} \\
&= 3 \cdot 1 \\
&= 3
\end{aligned}
$$

Alternatively, substitution rule:

$$
L = 6\int_0^{\frac{\pi}{2}} \cos\vartheta\sin\vartheta d\vartheta
$$

Let $u = \sin\vartheta$

$$
\begin{aligned}
\frac{du}{d\vartheta} &= \cos\vartheta \\
du &= \cos\vartheta d\vartheta
\end{aligned}
$$

$$
u|_{\vartheta=\frac{\pi}{2}} = 1, u|_{\vartheta=0} = 0
$$

$$
\begin{aligned}
L &= 6\int_0^1 u du \\
&= 6 \cdot \frac{1}{2} \\
L &= 3
\end{aligned}
$$

3.2 Example

Find the length of the curve $x = \vartheta - \sin\vartheta$, $y = 1 - \cos\vartheta$ between ....

$$
\begin{aligned}
\frac{dx}{d\vartheta} &= 1 - \cos\vartheta \\
\frac{dy}{d\vartheta} &= -(-\sin\vartheta) \\
&= \sin\vartheta
\end{aligned}
$$

1. Therefore,

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\vartheta}\right)^2 + \left(\frac{dy}{d\vartheta}\right)^2} d\vartheta \\
&= \int_a^b \sqrt{(1-\cos\vartheta)^2 + (\sin\vartheta)^2} d\vartheta \\
&= \int_a^b \sqrt{1 - 2\cos\vartheta + \cos^2\vartheta + \sin^2\vartheta} d\vartheta \\
&= \int_a^b \sqrt{1 - 2\cos\vartheta + 1} d\vartheta \\
&= \int_0^{2\pi} \sqrt{2 - 2\cos\vartheta} d\vartheta \\
&= \int_0^{2\pi} \sqrt{2\left(1 - \cos\vartheta\right)} d\vartheta \\
&= \int_0^{2\pi} \sqrt{2}\sqrt{1 - \cos\vartheta} d\vartheta \\
&= \sqrt{2}\int_0^{2\pi} \sqrt{1 - \cos\vartheta} d\vartheta \\
&= \sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2\left(\frac{\vartheta}{2}\right)} d\vartheta \\
&= \sqrt{2}\int_0^{2\pi} \sqrt{2}\sqrt{\sin^2\left(\frac{\vartheta}{2}\right)} d\vartheta \\
&= 2\int_0^{2\pi} \left|\sin\left(\frac{\vartheta}{2}\right)\right| d\vartheta \\
&= 2\int_0^{2\pi} \sin\left(\frac{\vartheta}{2}\right) d\vartheta
\end{aligned}
$$

2. Let $u = \frac{\vartheta}{2}$

$$
\begin{aligned}
\frac{du}{d\vartheta} &= \frac{1}{2} \\
du &= \frac{1}{2} d\vartheta
\end{aligned}
$$

3. Find endpoints

   (a) When $\vartheta = 2\pi$

$$u = \frac{2\pi}{2}$$
$$= \pi$$

4. Substitution

$$L = 2\int_0^{2\pi} \sin\left(\frac{\vartheta}{2}\right) d\vartheta$$
$$= 2\int_0^{\pi} \sin u \, 2du$$
$$= 4\int_0^{\pi} \sin u \, du$$
$$= 4\left(-\cos\pi + \cos 0\right)$$
$$= 4\left(1+1\right)$$
$$= 8$$

3.3 Example

3.3.1 Notes (Length of a curve, $f(x)$)

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$
$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

3.3.2 Example

Find the length of the curve $y^2 = x^3$ between $x = 0$ and $x = 4$.

$$y = x^{\frac{3}{2}}$$
$$L = \int_0^4 \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2} dx$$
$$= \int_0^4 \sqrt{1 + \frac{9}{4}x dx}$$
$$= \int_0^4 \sqrt{1 + \frac{9}{4}x dx}$$

1. $u-substitution$, $u=1+\frac{9}{4}x$

$$\frac{du}{dx}=\frac{9}{4}$$
$$\frac{4}{9}du=dx$$

2. Find endpoints

$$\begin{aligned} u|_{x=4} &= 1+\frac{9}{4}(4) \\ &= 1+9 \\ &= 10 \end{aligned}$$

$$\begin{aligned} u|_{x=0} &= 1+\frac{9}{4}(0) \\ &= 1 \end{aligned}$$

3. Find new equation

$$\begin{aligned} L &= \int_0^4 \sqrt{1+\frac{9}{4}x}dx \\ &= \int_1^{10} \sqrt{u}\frac{4}{9}du \\ &= \frac{4}{9}\int_1^{10} u^{\frac{1}{2}}du \\ L &= \frac{4\left(20\sqrt{10}-2\right)}{27} \\ L &\approx 9.073 \end{aligned}$$

3.4 Example

Find the length of the curve $y=\frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}}$ between $x=1$ and $x=4$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{2}{3}\cdot\frac{3}{2}\left(x^2+1\right)^{\frac{1}{2}}\cdot 2x \\ &= 2x\left(x^2+1\right)^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
L &= \int_1^4 ds \\
&= \int_1^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx \\
&= \int_1^4 \sqrt{1+4x^2\left(x^2+1\right)} dx \\
&= \int_1^4 \sqrt{1+4x^4+4x^2} dx \\
&= \int_1^4 \sqrt{4x^4+4x^2+1} dx \\
&= \int_1^4 \sqrt{\left(2x^2+1\right)^2} dx \\
&= \int_1^4 2x^2+1 dx \\
L &= 45
\end{aligned}
$$

4 Surface Area of Revolution

4.0.1 Formulas

These formulas, come from the formula to calculuate the surface area of frustums, a.k.a. $2\pi rl$

1. $y = f(x), a \leq x \leq b$. About $x - axis$

$$
S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2} dx
$$

$$
S = 2\pi \int_a^b y(t) \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} dt
$$

2. $x = g(y), a \leq y \leq b$. About $y - axis$.

$$
S = 2\pi \int_a^b x\sqrt{1+\left(\frac{dx}{dy}\right)^2} dy
$$

$$
S = 2\pi \int_a^b x(t) \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} dt
$$

4.1 Example

Find the area generated when the arc of the curve $y = \sqrt{x}$ between $x = 0$ and $x = 4$ rotates about the x-axis.

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2x^{\frac{1}{2}}}\end{aligned}$$

$$\begin{aligned}S &= 2\pi\int_0^4 \sqrt{x}\sqrt{1+\left(\frac{1}{2x^{\frac{1}{2}}}\right)^2}dx \\ &= 2\pi\int_0^4 \sqrt{x}\sqrt{1+\frac{1}{4x}}dx \\ &= 2\pi\int_0^4 \sqrt{x+\frac{1}{4}}dx\end{aligned}$$

1. Let $u = x + \frac{1}{4}$

$$du = dx$$

2. Find endpoints
   (a) $u|_{x=0} = \frac{1}{4}$
   (b) $u|_{x=4} = 4 + \frac{1}{4} = 4\frac{1}{4}$
3. Make new equation

$$\begin{aligned}S &= 2\pi\int_{\frac{1}{4}}^{\frac{17}{4}} u^{\frac{1}{2}}\,du \\ &= \frac{\pi\left(17\cdot 17^{\frac{1}{2}}-1\right)}{6} \\ &\approx 11.515\pi\,unit^2\end{aligned}$$

4.2 Example

Find the area generated when the arc of the curve $y^2 = x$ between $x = 0$ and $x = 2$ rotates about the x-axis. (Note: same as above question except different bounds)

1. Find endpoints
   (a) $u|_{x=0} = \frac{1}{4}$
   (b) $u|_{x=2} = 2 + \frac{1}{4} = \frac{9}{4}$

2. Make new equation

$$\begin{aligned} S &= 2\pi \int_{\frac{1}{4}}^{\frac{9}{4}} u^{\frac{1}{2}} du \\ &= \frac{13\pi}{3} \\ &= 4\frac{1}{3}\pi \, unit^2 \end{aligned}$$

4.3 Example

Find the surface area generated when the arc of the curve $y = 3t^2$, $x = 3t - t^3$ between $t = 0$ and $t = 4$ rotates about the x-axis through 2π.

$$\begin{aligned} \frac{dy}{dt} &= 6t \\ \frac{dx}{dt} &= 3 - 3t^2 \end{aligned}$$

$$\begin{aligned} S &= 2\pi \int_a^b y\left(t\right) \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} dt \\ &= 2\pi \int_a^b 3t^2 \sqrt{\left(6t\right)^2 + \left(3 - 3t^2\right)^2} dt \\ &= 2\pi \int_a^b 3t^2 \sqrt{36t^2 + 9t^4 - 18t^2 + 9} dt \\ &= 2\pi \int_a^b 3t^2 \sqrt{9t^4 + 18t^2 + 9} dt \\ &= 2\pi \int_a^b 3t^2 \sqrt{9\left(t^4 + 2t^2 + 1\right)} dt \\ &= 2\pi \int_a^b 3t^2 \sqrt{9\left(t^2 + 1\right)^2} dt \\ &= 2\pi \int_a^b 9t^2 \left(t^2 + 1\right) dt \\ &= 18\pi \int_a^b t^4 + t^2 dt \end{aligned}$$

1. $a = 0, b = 4$

$$\begin{aligned} S &= 18\pi \int_0^4 t^4 + t^2 dt \\ &= \frac{20352\pi}{5} \\ &= 4070.4\pi \end{aligned}$$

4.4 Example

Find the surface area generated when the arc of the curve $x = r\cos\theta$, $y = r\sin\theta$ where r is a constant between $\theta = 0$ and $\theta = \pi$ rotates about the x-axis through 2π.

$$\frac{dx}{d\theta} = r\left(-\sin\theta\right)$$
$$\frac{dx}{d\theta} = -r\sin\theta$$

$$\frac{dy}{d\theta} = r\cos\theta$$

$$\begin{aligned}
S &= 2\pi\int_a^b y\left(t\right)\sqrt{\left(\frac{dy}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2}dt \\
&= 2\pi\int_0^\pi r\sin\theta\sqrt{\left(r\cos\theta\right)^2+\left(-r\sin\theta\right)^2}d\theta \\
&= 2\pi r\int_0^\pi \sin\theta\sqrt{r^2\cos^2\theta+r^2\sin^2\theta}d\theta \\
&= 2\pi r\int_0^\pi \sin\theta\sqrt{r^2\left(\cos^2\theta+\sin^2\theta\right)}d\theta \\
&= 2\pi r\int_0^\pi \sin\theta\sqrt{r^2\cdot 1}d\theta \\
&= 2\pi r\int_0^\pi \sin\theta r d\theta \\
&= 2\pi r^2\int_0^\pi \sin\theta d\theta \\
&= 2\pi r^2\int_0^\pi \sin\theta d\theta \\
&= 2\pi r^2\left(2\right) \\
&= 4\pi r^2
\end{aligned}$$

5 Mean Values

$$M = \frac{1}{b-a}\int_a^b f\left(x\right)dx$$

5.1 Example

Find the mean value of $y = 3x^2 + 4x + 1$ between $x = -1$ and $x = 2$

$$\begin{aligned}
M &= \frac{1}{2-(-1)}\int_{-1}^{2} 3x^2 + 4x + 1 dx \\
&= \frac{1}{3}\int_{-1}^{2} 3x^2 + 4x + 1 dx \\
M &= 6
\end{aligned}$$

5.2 Example

Find the mean value of $y = 3\sin 5t + 2\cos 3t$ between $t = 0$ and $t = \pi$

$$\begin{aligned}
M &= \frac{1}{\pi - 0}\int_{0}^{\pi} 3\sin 5t + 2\cos 3t\, dt \\
&= \frac{1}{\pi}\left[5 \cdot 3\cos 5t + 3 \cdot 2\left(-\sin 3t\right)\right]_0^{\pi} \\
&= \frac{1}{\pi}\left[15\cos 5t - 6\sin 3t\right]_0^{\pi} \\
M &= \frac{6}{5\pi}
\end{aligned}$$

5.3 Example

Find the mean value of $f(x) = \frac{x}{2x+3}$ on $[0, 1]$.

$$\begin{aligned}
M &= \frac{1}{1-0}\int_{0}^{1} \frac{x}{2x+3} dx \\
&= \int_{0}^{1} \frac{x}{2x+3} dx
\end{aligned}$$

1. Let $u = 2x + 3$

$$\begin{aligned}
\frac{du}{dx} &= 2 \\
dx &= \frac{du}{2} \\
x &= \frac{u-3}{2}
\end{aligned}$$

2. Find endpoints

$$\begin{aligned}
u|_{x=0} &= 3 \\
u|_{x=1} &= 2 + 3 \\
u|_{x=1} &= 5
\end{aligned}$$

3. Make equation

$$\begin{aligned}
M &= \int_3^5 \frac{u-3}{2\left(2\left(\frac{u-3}{2}\right)+3\right)}\frac{du}{2} \\
&= \int_3^5 \frac{u-3}{2u}\cdot\frac{du}{2} \\
&= \int_3^5 \frac{u-3}{4u}\cdot du \\
&= \int_3^5 \frac{u-3}{4u}\cdot du \\
&= \int_3^5 \frac{u-3}{4u} du \\
&= \frac{1}{4}\int_3^5 1-\frac{3}{u} du \\
&= \frac{1}{4}\left(2-3\left(\ln\left(5\right)-\ln\left(3\right)\right)\right) \\
&= 0.1169
\end{aligned}$$

5.4 Example

Find the mean value of $y=\frac{1}{\sqrt{1-x^2}}, \left[0,\frac{1}{2}\right]$.

$$\begin{aligned}
M &= \frac{1}{\frac{1}{2}-0}\cdot\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}dx \\
&= 2\cdot\left[\sin^{-1} x\right]_0^{\frac{1}{2}} \\
&= 2\left(\frac{\pi}{6}\right) \\
&= \frac{\pi}{3} \\
&= 1.047
\end{aligned}$$

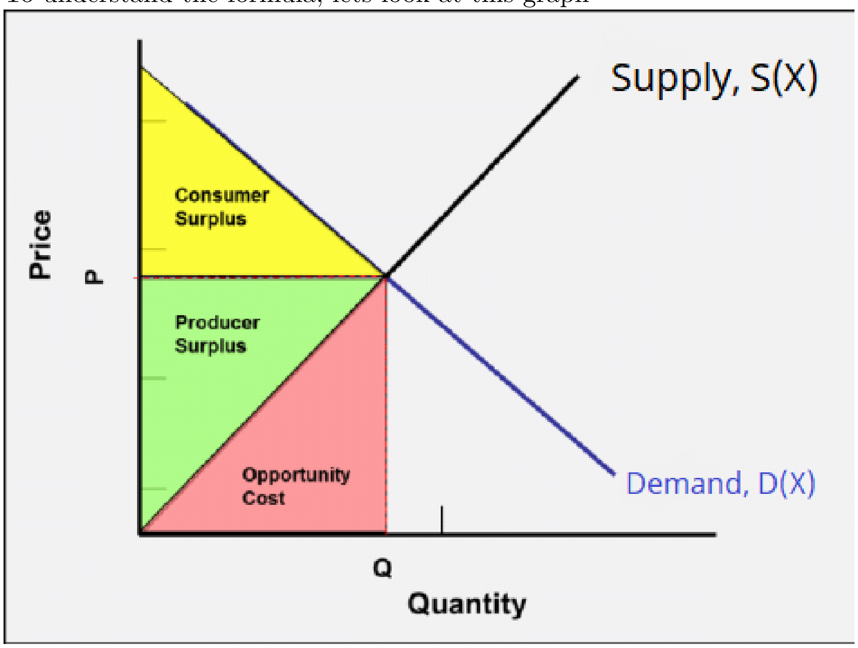
6 Consumer & Producer Surplus

6.0.1 Formula

$$\begin{aligned}
CS &= \int_0^{\bar{x}} D\left(x\right) dx - \overline{px} \\
&= \int_0^{\bar{x}} \left[D\left(x\right)-\bar{p}\right] dx
\end{aligned}$$

$$
\begin{aligned}
PS &= \overline{px} - \int_0^{\bar{x}} S(x)\,dx \\
&= \int_0^{\bar{x}} \left[\bar{p} - S(x)\right] dx
\end{aligned}
$$

1. To understand the formula, lets look at this graph



   (a) As you can see, the **consumer surplus,** CS, is referring to the extra money the consumers are saving. For most of the eager customers, you can see that they saved quite a bit of money (however, the more patient they get, the less they save). As the price decreases, more people save more money (or the yellow triangle region).

      i. So, the consumer surplus, is simply the area above the equilibrium price (where both sides do not gain profit).

   (b) Whereas, the **producer surplus**, PS, is referring to the extra money that the producers are getting. To understand this, look at the **opportunity cost**. For the first few supplies, they are extremely optimized (using the best land, best distance, best transport, best workers). As the supplies increase, the optimization decreases, hence increasing the opportunity cost for every extra product they want to produce. To break even, the demand must be equal to the supplier. Therefore, by selling it at the equilibrium price, you can see that a lot of the suppliers managed to get a healthy profit, except for the ones supplying for the last few stocks (but they did break even).

i. So, the produce surplus, is simply the area below the equilibrium price.

6.1 Example

The demand function for a certain make of 10-speed bicycle is given by $p = D(x) = -0.001x^2 + 250$ where p is the unit price in dollars and x is the quantity demanded in units of a thousand. The supply function for these bicycles is given by $p = S(x) = 0.0006x^2 + 0.02x + 100$ where p stands for the unit price in dollars and x stands for the number of bicycles that the supplier will put on the market, in units of a thousand. Determine the consumers' surplus if the market price of a bicycle is set at the equilibrium price.

1. Find the equilibrium

$$\begin{aligned}
D(x) &= S(x) \\
-0.001x^2 + 250 &= 0.0006x^2 + 0.02x + 100 \\
0.0016x^2 + 0.02x - 150 &= 0 \\
2x^2 + 25x - 187500 &= 0 \\
x = 300, x &= -\frac{625}{2}\,(ignored)
\end{aligned}$$

2. Now we know

$$\begin{aligned}
\bar{x} &= 300 \\
\bar{p} &= D(300) \\
&= -0.001(300)^2 + 250 \\
\bar{p} &= 160
\end{aligned}$$

3. Next we find consumer surplus

$$\begin{aligned}
CS &= \int_0^{\bar{x}} [D(x) - \bar{p}]\,dx \\
&= \int_0^{300} \left(-0.001x^2 + 250 - 160\right) dx \\
&= \int_0^{300} \left(-0.001x^2 + 90\right) dx \\
&= 18000\$
\end{aligned}$$

6.2 Example

Lost in the woods, literally. In the notes, there is no example 2. The numbering jumped to 3.

6.3 Example

The demand function for a certain make of exercise bicycle that is sold exclusively through cable television is $p = D(x) = \sqrt{9 - 0.02x}$ where p is the unit price in hundreds of dollars and x is the quantity demanded per week. The corresponding supply function is given by $p = S(x) = \sqrt{1 + 0.02x}$ where p has the same meaning as before and x is the number of exercise bicycles the supplier will make available at price p. Determine the consumers' surplus if the unit price is set at the equilibrium price.

1. Find the equilibrium

$$\begin{aligned}
\sqrt{9 - 0.02x} &= \sqrt{1 + 0.02x} \\
9 - 0.02x &= 1 + 0.02x \\
8 &= 0.04x \\
x &= \frac{8}{0.04} \\
\bar{x} &= 200
\end{aligned}$$

$$\begin{aligned}
p &= S(200) \\
&= \sqrt{1 + 0.02(200)} \\
\bar{p} &= \sqrt{5}
\end{aligned}$$

2. Determine consumer surplus

$$\begin{aligned}
CS &= \int_0^{200} \left(\sqrt{9 - 0.02x} - \sqrt{5}\right) dx \\
&= \int_0^{200} \sqrt{9 - 0.02x} dx - \int_0^{200} \sqrt{5} dx \\
&= \int_0^{200} \sqrt{9 - 0.02x} dx - \int_0^{200} \sqrt{5} dx
\end{aligned}$$

3. Utilize $u - \textit{substitution}$

   (a) Let $u = 9 - 0.02x$

$$\begin{aligned}
u &= 9 - 0.02x \\
\frac{du}{dx} &= -0.02 \\
dx &= -\frac{du}{0.02}
\end{aligned}$$

   (b) Find the bounds

      i. $u|_{x=0} = 9$

ii. $u|_{x=200} = 9 - 0.02\,(200) = 9 - 4 = 5$

4. Make new equation

$$\begin{aligned} CS &= \int_0^{200} \sqrt{9 - 0.02x}dx - \int_0^{200} \sqrt{5}dx \\ &= -\frac{1}{0.02} \int_9^5 u^{\frac{1}{2}}\,du - \int_0^{200} \sqrt{5}dx \\ &= -\frac{1}{0.02} \left(-18 + \frac{10 \cdot\ 5^{\frac{1}{2}}}{3} \right) - 200\sqrt{5} \\ &= 80.1084 \end{aligned}$$