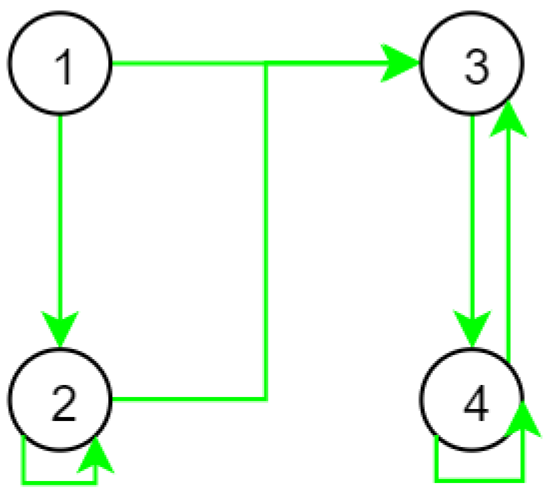
DM Tutorial 9

December 17, 2019

1. The following arrays describe a relation R on a set $A = \{1, 2, 3, 4\}$:
   VERT = [1, 2, 6, 4]
   TAIL = [1, 2, 2, 4, 4, 3, 4, 1]
   HEAD = [2, 2, 3, 3, 4, 4, 1, 3]
   NEXT = [8, 3, 0, 5, 7, 0, 0, 0]
   Compute both the digraph of R and the matrix M_R.



   (a)
   (b)

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

2. Let $A = B = \{1, 2, 3\}$ and let $R = \{(1, 1), (1, 2), (2, 3), (3, 1)\}$ and let $S = \{(2, 1), (3, 1), (3, 2), (3, 3)\}$. Let R and S be the relations from A to B. Compute

(a) $\bar{R}$

$$\bar{R} = \{(1,3),(2,1),(2,2),(3,2),(3,3)\}$$

(b) $R \cap S$

$$R \cap S = \{(3,1)\}$$

(c) $R \cup S$

$$R \cup S = \{(1,1),(1,2),(2,1),(2,3),(3,1),(3,2),(3,3)\}$$

(d) S^{-1}

$$S^{-1} = \{(1,2),(1,3),(2,3),(3,3)\}$$

3. Let $A = \{2,4,5,7\}$ and let R and S be the relations on A described by xRy if and only if $x+y$ is even and $M_S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. List the ordered pairs belonging to the following relations.

$$S = \{(1,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$$

(a) S^{-1}

$$S^{-1} = \{(2,1),(3,2),(4,2),(3,2),(3,3),(4,3)\}$$

(b) $S^{-1} \cap R$

$$R = \{(2,2),(2,4),(4,4),(4,2),(5,7),(7,5)\}$$

$$S^{-1} \cap R = \varnothing$$

(c) $\left(S^{-1} \circ R\right)^{-1}$

$$S^{-1} \circ R = \{(3,2),(3,4),(4,2),(4,4),(3,2),(3,4)\}$$

4. Let $A = \{1,2,3,4\}$ and $B = \{1,2,3\}$. The matrices M_R and M_S of the relation R and S be the relations from A to B are given by $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $M_S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Compute

(a) $M_{R\cup S}$

$$M_{R\cup S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(b) $M_{R\cap S}$

$$M_{R\cap S} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(c) $M_{R^{-1}}$

i. No inverse.

(d) $M_{\bar{S}}$

$$M_{\bar{S}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5. Let $A = \{a, b, c, d, e\}$ and let the equivalence relations R and S on A be given by

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(a) Compute

i. $M_{R\circ R}$

$$M_{R\circ R} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

ii. $M_{S\circ R}$

$$M_{S\circ R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

iii. $M_{R\circ S}$

$$M_{R\circ S} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

iv. $M_{S\circ S}$

$$M_{S\circ S} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(b) Compute the partition of A corresponding to $R \cap S$.

$$M_{R\cap S} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \{\{1\}, \{2,3\}, \{4\}, \{5\}\}$$

6. Let $A = \{1, 2, 3, 4\}$ and a relation R on A is R = {(1, 2), (1, 3), (2, 3), (3, 4)}. Find the reflexive closure and symmetric closure of R.

   (a) Reflexive closure

   $$R \cup \Delta = \{(1,1), (2,2), (3,3), (4,4), (1,2), (1,3), (2,3), (3,4)\}$$

   (b) Symmetric closure

   $$R \cup R^{-1} = \{(2,1), (3,1), (3,2), (4,3), (1,2), (1,3), (2,3), (3,4)\}$$

7. Let A = {1, 2, 3, 4}. For the relation R whose matrix is given, find the matrix of the transitive closure by using Warshall's algorithm. (Note: in exam, you don't actually have to draw so many matrices, its just for my own personal clarity)

   (a) $M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

      i. W_0

      $$W_0 = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & 0 & 1 & 0 \\ \mathbf{0} & 0 & 0 & 1 \\ \mathbf{0} & 0 & 0 & 0 \end{bmatrix}$$

         A. $C_1 = 2$, $R_1 = 2$
         B. $ADD : (2, 2)$

      ii. W_1

      $$W_1 = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ 0 & \mathbf{0} & 0 & 1 \\ 0 & \mathbf{0} & 0 & 0 \end{bmatrix}$$

         A. $C_2 = \{1, 2\}$
         B. $R_2 = \{1, 2, 3\}$
         C. $ADD : \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$

      iii. W_2

      $$W_2 = \begin{bmatrix} 1 & 1 & \mathbf{0} & 0 \\ 1 & 1 & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ 0 & 0 & \mathbf{0} & 0 \end{bmatrix}$$

         A. $C_3 = \{1\}$
         B. $R_3 = \{1, 2, 4\}$
         C. $ADD : \{(1,1), (2,1), (4,1)\}$

iv. W_3

$$W_3 = \begin{bmatrix} 1 & 1 & 0 & \mathbf{0} \\ 1 & 1 & 1 & \mathbf{0} \\ 1 & 1 & 0 & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

A. $C_4 = \{3\}$

B. $R_4 = \{1\}$

C. $ADD : \{(1,3)\}$

v. $W_4(M_{R^\infty})$, transitive closure

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(b) $M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

i. W_0

$$W_0 = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{1} \\ 0 & 1 & 1 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

A. $R_0 = \{2\}$

B. $C_0 = \{2,3,4\}$

C. $ADD : \{(2,2), (2,3), (2,4)\}$

ii. W_1

$$W_1 = \begin{bmatrix} 0 & 1 & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 1 & \mathbf{1} & 1 \\ 0 & 0 & \mathbf{0} & 1 \end{bmatrix}$$

A. $R_1 = \{2,3,4\}$

B. $C_1 = \{2,3\}$

C. $ADD : \{(2,2), (2,3), (3,2), (3,3), (4,2), (4,3)\}$

iii. W_2

$$W_2 = \begin{bmatrix} 0 & 1 & \mathbf{0} & 0 \\ 0 & 1 & \mathbf{1} & 1 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 1 & \mathbf{1} & 1 \end{bmatrix}$$

A. $R_2 = \{2,3,4\}$

B. $C_2 = \{2,3,4\}$

C. Add $\{(2,4),(3,4),(4,4)\}$ (Basically $W_3 = W_2$)

iv. W_3

$$W_3 = \begin{bmatrix} 0 & 1 & 0 & \mathbf{0} \\ 0 & 1 & 1 & \mathbf{1} \\ 0 & 1 & 1 & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}$$

A. $R_3 = \{4\}$

B. $C_3 = \{2, 3, 4\}$

C. Nothing new again, $W_4 = W_3$

v. W_4, or transitive closure, or M_{R^∞}

$$M_{R^\infty} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

8. Let A = {1, 2, 3, 4} and let R and S be relations on A described by

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Use Warshall's algorithm to compute the transitive closure of $R \cup S$

$$M_{R \cup S} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

(a) Start of Warshall, W_0

$$W_0 = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & 1 & 0 & 0 \\ \mathbf{0} & 1 & 1 & 0 \\ \mathbf{0} & 1 & 1 & 1 \end{bmatrix}$$

i. $R_1 = \{1, 2, 4\}$, $C_1 = \{1\}$

ii. $ADD : \{(1,1),(2,1),(4,1)\}$

(b) W_1

$$W_1 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

i. $R_2 = \{1, 2, 3, 4\}$

ii. $C_2 = \{1, 2\}$

iii. $ADD : \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2),(4,1),(4,2)\}$

(c) W_2

$$W_2 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

i. $R_3 = \{3, 4\}$

ii. $C_3 = \{1, 2, 3, 4\}$

iii. $ADD : \{(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$

(d) W_3

$$W_3 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

i. $R_4 = \{1, 2, 3, 4\}$

ii. $C_4 = \{1, 3, 4\}$

iii. $ADD : \{(1,1),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,3),(3,4),(4,1),(4,3),(4,4)\}$

(e) W_4, or the transitive closure, or M_{R^∞}

$$M_{R^\infty} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$