Calc II - Tutorial 4

November 29, 2019

1. Questions

   (a) Verify that all members of the family $y = \frac{1}{x+C}$ are solutions to the D.E. $y' = -y^2$.

      i. Differentiate y

      $$\begin{aligned} y' &= \frac{d}{dx}\left[(x+C)^{-1}\right] \\ &= -(x+C)^{-2} \\ &= -\left(\frac{1}{x+C}\right)^2 \\ y' &= -y^2 \end{aligned}$$

      ii. Thus, all members of $y = \frac{1}{x+C}$ is the solution to the D.E. $y' = -y^2$

   (b) Find a solution of the initial-value problem

   $$y' = -y^2, y(0) = 0.5$$

      i. From the top, $y(x) = \frac{1}{x+c}$, at $y(0)$:

      $$\begin{aligned} y(0) &= \frac{1}{0+c} \\ 0.5 &= \frac{1}{c} \\ c &= \frac{1}{\frac{1}{2}} \\ &= 2 \end{aligned}$$

      ii. Obtain the final solution

      $$y = \frac{1}{x+2}$$

2. Question

(a) Verify that all members of the family $y = \left(C - x^2\right)^{-\frac{1}{2}}$ are solutions to the D.E. $y' = xy^3$.

$$\begin{aligned}
y' &= \frac{d}{dx}\left[\left(C - x^2\right)^{-\frac{1}{2}}\right] \\
&= -\frac{1}{2}\left(C - x^2\right)^{-\frac{3}{2}} \cdot -2x^3 \\
&= \left(C - x^2\right)^{-\frac{3}{2}} x^3 \\
&= \left[\left(C - x^2\right)^{-\frac{1}{2}}\right]^3 x^3 \\
y' &= xy^3
\end{aligned}$$

(b) Find a solution of the initial-value problem

$$y' = xy^3, y\left(0\right) = 2$$

i. Answer

$$\begin{aligned}
y\left(0\right) &= 2 \\
C^{-\frac{1}{2}} &= 2 \\
\frac{1}{2} &= \sqrt{c} \\
c &= \frac{1}{4} \left(\text{cannot be negative, cause } \sqrt{-n} = undefined\right)
\end{aligned}$$

ii. $\therefore y = \left(\frac{1}{4} - x^2\right)^{-\frac{1}{2}}$

3. Show that $y = x^{-\frac{3}{2}}$ is a solution to the D.E. $4x^2y'' + 12xy' + 3y = 0$ for $x > 0$

$$\begin{aligned}
y' &= -\frac{3}{2}x^{-\frac{5}{2}} \\
y'' &= \frac{15}{4}x^{-\frac{7}{2}}\ (LHS)
\end{aligned}$$

$$\begin{aligned}
4x^2y'' + 12xy' + 3y &= 0 \\
4x^2y'' + 12x\left(-\frac{3}{2}x^{-\frac{5}{2}}\right) + 3\left(x^{-\frac{1}{2}}\right) &= 0 \\
4x^2y'' - 18x^{-\frac{3}{2}} + 3x^{-\frac{1}{2}} &= 0 \\
4x^2y'' - 15x^{-\frac{3}{2}} &= 0 \\
4x^2y'' &= 15x^{-\frac{3}{2}} \\
y'' &= \frac{15}{4}x^{-\frac{1}{2}}
\end{aligned}$$

(a) Therefore, $y = x^{-\frac{1}{2}}$ is a solution to the D.E. for $x > 0$

4. Question

(a) Show that $y = \frac{3}{4} + \frac{C}{x^2}$ is the general solution to the D.E. $2xy' + 4y = 3$.

$$\begin{aligned} y &= \frac{3}{4} + Cx^{-2} \\ y' &= -2Cx^{-3} \\ y' &= -2Cx^{-3} \end{aligned}$$

$$\begin{aligned} 2xy' + 4y &= 3 \\ 2x\left(-2Cx^{-3}\right) + 4y &= 3 \\ -4Cx^{-2} + 4y &= 3 \\ 4y &= 3 + 4Cx^{-2} \\ y &= \frac{3}{4} + \frac{C}{x^2} \end{aligned}$$

(b) Find the solution of the I.V.P. $2xy' + 4y = 3,\ y(1) = -4$.

$$\begin{aligned} -4 &= \frac{3}{4} + \frac{C}{1} \\ -4 - \frac{3}{4} &= C \\ C &= -\frac{19}{4} \end{aligned}$$

i. $\therefore y = \frac{3}{4} - \frac{19}{4x^2}$

5. Use Euler's Method with step size 0.5 to compute the approximate y-value y_1, y_2, y_3 and y_4 of the solution of the I.V.P. $y' = y - 2x, y(1) = 0$.

n	Xn	$y_n = y_{n-1} + 0.5\left(y_{n-1} - 2x_{n-1}\right)$	y_n
0	1	0	0
1	1.5	$y_n = 0 + 0.5\left(0 - 2\left(1\right)\right) = 0.5\left(-2\right)$	-1
2	2	$y_n = -1 + 0.5\left(-1 - 2\left(1.5\right)\right)$	-3
3	2.5	$y_n = -3 + 0.5\left(-3 - 2\left(2\right)\right)$	-6.5
4	3	$y_n = -6.5 + 0.5\left(-6.5 - 2\left(2.5\right)\right)$	-12.25

6. Use Euler's Method with step size 0.2 to estimate $y(1.4)$, where $y(x)$ is the solution of the initialvalue problem $y' = x - xy, y(1) = 0$.

n	Xn	$y_n = y_{n-1} + 0.2\left(x_{n-1} - x_{n-1}y_{n-1}\right)$	y_n
0	1	0	0
1	1.2	$y_n = 0 + 0.2\left(1 - 1\left(0\right)\right)$	0.2
2	1.4	$y_n = 0.2 + 0.2\left(1.2 - 1.2\left(0.2\right)\right)$	0.392

7. Use Euler's Method with step size 0.1 to estimate $y(1)$, where $y(x)$ is the solution of the initial-value problem $y' = 1–xy, y(0) = 0$.

n	Xn	$y_n = y_{n-1} + 0.1\,(1 - x_{n-1}y_{n-1})$	y_n
0	0	0	0
1	.1	$0 + 0.1\,(1 - 0 \cdot 0)$	0.1
2	.2	$0.1 + 0.1\,(1 - (0.1) \cdot (0.1))$	0.199
3	.3	$0.199 + 0.1\,(1 - (0.2) \cdot (0.199))$	0.2950
4	.4	$0.2950 + 0.1\,(1 - (0.3) \cdot (0.2950))$	0.3862
5	.5	$0.3862 + 0.1\,(1 - (0.4) \cdot (0.3862))$	0.4708
6	.6	$0.4708 + 0.1\,(1 - (0.5) \cdot (0.4708))$	0.5473
7	.7	$0.5473 + 0.1\,(1 - (0.6) \cdot (0.5473))$	0.6145
8	.8	$0.6145 + 0.1\,(1 - (0.7) \cdot (0.6145))$	0.6715
9	.9	$0.6715 + 0.1\,(1 - (0.8) \cdot (0.6715))$	0.7178
10	1	$0.7178 + 0.1\,(1 - (0.9) \cdot (0.7178))$	0.7532

8. Use Euler's Method with step size 0.1 to estimate $y(0.5)$, where $y(x)$ is the solution of the initial-value problem $y' + 2y = 2 - e^{-4x}, y(0) = 1$.

$$y' = 2 - 2y - e^{-4x}$$

n	Xn	$y_n = y_{n-1} + h\left(y'_{n-1}\right)$	y_n
0	0	1	1
1	.1	$1 + 0.1\left(2 - 2\,(1) - e^{-4(0)}\right)$	0.9
2	.2	$0.8530 + 0.1\left(2 - 2\,(0.8530) - e^{-4(.1)}\right)$	0.8530
3	.3	$0.8530 + 0.1\left(2 - 2\,(0.8530) - e^{-4(.2)}\right)$	0.8375
4	.4	$0.8375 + 0.1\left(2 - 2\,(0.8375) - e^{-4(.3)}\right)$	0.8399
5	.5	$0.8399 + 0.1\left(2 - 2\,(0.8399) - e^{-4(.4)}\right)$	0.8517

9. Use Euler's Method with step size 0.1 to estimate $y(0.5)$, where $y(x)$ is the solution of the initial-value problem $y' = y + xy, y\,(0) = 1$.

n	Xn	$y_n = y_{n-1} + h\left(y'_{n-1}\right)$	y_n
0	0	1	1
1	.1	$y_n = 1 + 0.1\,(1 + (0)\,(1))$	1.1
2	.2	$1.1 + 0.1\,(1.1 + (.1)\,(1.1))$	1.221
3	.3	$1.221 + 0.1\,(1.221 + (.2)\,(1.221))$	1.3675
4	.4	$1.3675 + 0.1\,(1.3675 + (.3)\,(1.3675))$	1.5453
5	.5	$1.5453 + 0.1\,(1.5453 + (.4)\,(1.5453))$	1.7616

10. Question

    (a) Use Euler's Method with step size 0.2 to estimate $y(1.4)$, where $y(x)$ is the solution of the initial-value problem $y' + 3y = x^2, y\,(1) = 1$

n	Xn	$y_n = y_{n-1} + 0.2\left(x_{n-1}^2 - 3y_{n-1}\right)$	y_n
0	1	1	1
1	1.2	$y_n = 1 + 0.2\,(1 - 3\,(1))$	0.6
2	1.4	$0.6 + 0.2\left(1.2^2 - 3\,(0.6)\right)$	0.528

(b) Repeat part (a) with step size 0.1.

n	Xn	$y_n = y_{n-1} + 0.1\left(x_{n-1}^2 - 3y_{n-1}\right)$	y_n
0	1	1	1
1	1.1	$1 + 0.1\,(1 - 3\,(1))$	0.8
2	1.2	$0.8 + 0.1\left(1.1^2 - 3\,(0.8)\right)$	0.681
3	1.3	$0.681 + 0.1\left(1.2^2 - 3\,(0.681)\right)$	0.6207
4	1.4	$0.6207 + 0.1\left(1.3^2 - 3\,(0.6207)\right)$	0.6035