Calculus 1 Chapter 1 Notes

July 2, 2019

1 Examples

Functions are **one-to-one-relation** / **multiple-to-one relation**

1.1

1.2 Examples

1. $\mathbb{R}$
2. $\mathbb{R}^+$
3. $\mathbb{R}^- \cup \{0\} = \mathbb{R} \backslash \mathbb{R}^+ = \mathbb{R} - \mathbb{R}^+$
4. $\{x | 2 \leq x < 5\}$

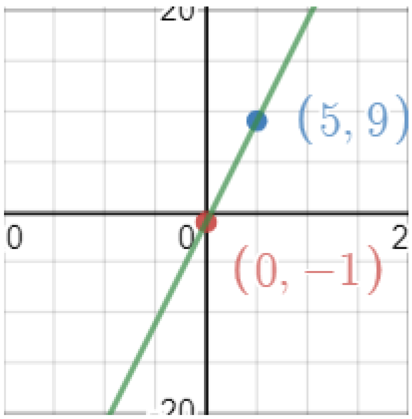
Notes

1. An arbitrary number in $D(f)$ is called an **independent variable**.
2. A number in the R(f) is called a **dependent variable**.

1.3 Examples

1. The graph of a function f is shown
   (a)
      i. $f(1) = 3$
      ii. $f(5) = -0.5$
   (b)
      i. $D_f = [0, 7]$
      ii. $R_f = [-2, 3]$

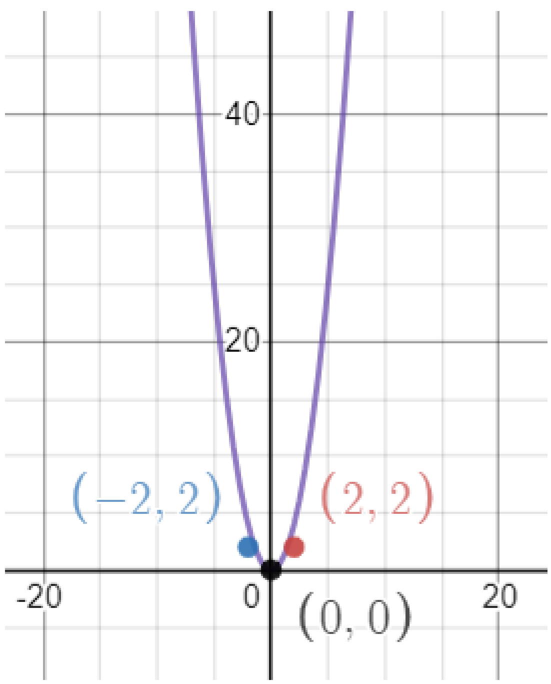
1.4 Example

1. 

   (a) $D_f = \mathbb{R}$

   (b) $R_f = \mathbb{R}$

2. $g(x) = x^2$ (normally need to find extreme points to draw the graph)

   (a) 

   (b) $D_f = \mathbb{R}$

   (c) $R_f = [0, +\infty]$

1.5 Example

F(x)	Domain (x)	Range (x)
y = x + 1	$x\epsilon\mathbb{R}$	$x\epsilon\mathbb{R}$
y = $\sqrt{x}$	$[0,+\infty)$	$0,+\infty)$
y = $\frac{1}{x}$	$x\epsilon\mathbb{R}$- $\{0\}$	$x\epsilon\mathbb{R}$-$\{0\}$

1.6 Example

1. $D_f = \mathbb{R}$
2.
   (a) $f(x) = x^2 + 4x - 6$

$$
\begin{aligned}
f(x) &= (x+2)^2 - (2)^2 - 6 \\
&= (x+2)^2 - 10
\end{aligned}
$$

   (b) Minimum point: -10
   (c) $[-10, \infty)$

1.7 Example

1.
   (a)
      i. $f(1) = 2$
      ii. $f(6) = 0$
   (b)
      i. $D(f) = [0, 6]$
      ii. $R(f) = [-1, 2]$

1.8 Example

1.
   (a)
      i. $D(f)$
         A. Rules:
         $\sqrt{3-x} \neq 0$
         $3 - x \geq 0$
         B. Therefore, $\sqrt{3-x} > 0$
         C. Calculation

$$
\begin{aligned}
3 - x &> 0 \\
x &< 3
\end{aligned}
$$

D. Conclusion

$$\boldsymbol{D\left(f\right)=\left(-\infty,3\right)}$$

ii. $R\left(f\right)$

A. $\left(2,+\infty\right)$

(b)

i. $D\left(f\right)$

A. Rules:
$x-1\neq0$
$x\neq1$

B. Conclusion
$D\left(f\right)=\mathbb{R}-\left\{1\right\}$

ii. $R\left(f\right)$

A. When x is very close to 1, its very close to ∞

B. When x is very close to ∞, very close to 0

C. $R\left(f\right)=\mathbb{R}-\left\{0\right\}$

1.9 Example

1.

(a) $x+2\geq0$

i. $x\geq-2$

ii. $\therefore D\left(f\right)=\left[-2,+\infty\right)$

(b)

i. Rules: $x^{2}-x\neq0$

ii. $x\left(x-1\right)\neq0$

$$x\neq0,x\neq1$$

iii. $\therefore D_{f}=\mathbb{R}-\left\{0,1\right\}$

1.10 Example

Vertical line test

If a function intersects a line $x=a$ twice, at $(a,b)$and (a,c) then the curve can't represent a function

1.11 Example

1.

(a) $f\left(0\right)$

$$\begin{aligned}f\left(0\right) & =1-x\\ & =1-0\\ & =1\end{aligned}$$

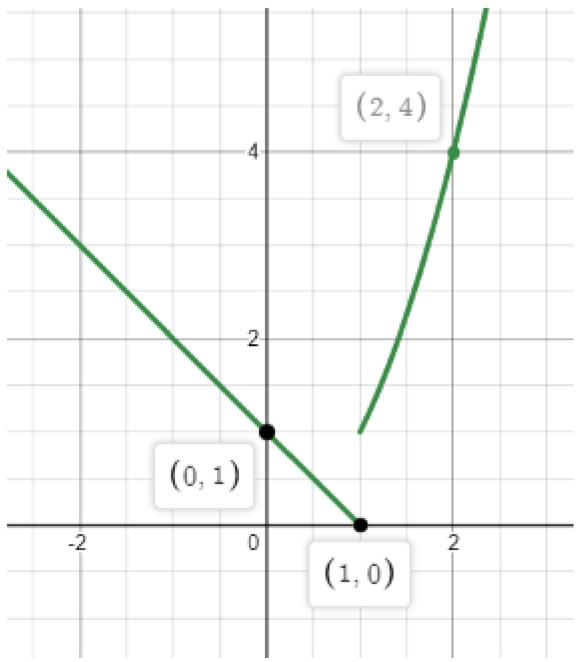
(b) $f(1)$

$$\begin{aligned} f(1) &= 1 - x \\ &= 1 - 1 \\ &\mathbf{= 0} \end{aligned}$$

(c) $f(2)$

$$\begin{aligned} f(2) &> 1 \\ &= 2^2 \\ &\mathbf{= 4} \end{aligned}$$

(d)



1.12 Example

Sketching the graph of absolute value functions

$$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$

Choose 0 and choose 1 to sub in

1.13 Example

Find a formula for the function f graphed in the figure

$$f\left(x\right)=\begin{cases}x & if\,0\leq x\leq1\\ 2-x & if\,1<x\leq2\\ 0 & if\,x>2\end{cases}$$

1.14 Example

1. Determine whether each of the following functions is even, odd or neither even nor odd.
   (a) $f(x) = x^5 + x$
      i. Check if the function is odd

$$\begin{aligned}f\left(-x\right)&=\left(-x\right)^{5}+\left(-x\right)\\&=-x^{5}-x\\&=-\left(x^{5}+x\right)\\&=-f\left(x\right)\end{aligned}$$

      Since $f\left(-x\right)=-f\left(x\right)$, the function is **odd**

   (b) $g(x) = 1 - x^4$
      i. Check if the function is even

$$\begin{aligned}f\left(-x\right)&=1-\left(-x\right)^{4}\\&=1-x^{4}\\&=f\left(x\right)\end{aligned}$$

      Since $f\left(-x\right)=f\left(x\right)$, the function is **even**

   (c) $h(x) = 2x–x^2$
      i. Check if the function is even & odd

$$\begin{aligned}f\left(-x\right)&=2\left(-x\right)-\left(-x\right)^{2}\\&=-2-x^{2}\\&=-\left(2+x^{2}\right)\\&\neq f\left(x\right)or\,-f\left(x\right)\end{aligned}$$

      Since the function is neither even nor odd, it is **neither even nor odd**

      ii. **Note: Need to memorize symmetry functions**

1.15 Example

1.15.1 Notes

1.4 Increasing and Decreasing Functions

A function f is called **increasing** on an interval I if: $f(x1) < f(x2)$ whenever $x1 < x2$ in I

It is called **decreasing** on I if: $f(x1) > f(x2)$ whenever $x1 < x2$ in I.

1.15.2 Example Questions

The function f(x) = x 2 is decreasing on the interval (-∞,0) and increasing on the interval [0,∞)

Is the function increasing on $\{0\}$,even though the $\frac{d}{dx}$is 0? Answer: No, but usually we just include it along with another interval to make it easier

1.16 Example

1.16.1 Notes

1. $(f+g)(x) = f(x) + g(x)$ domain $= A \cap B$
2. $B(f–g)(x) = f(x)–g(x)$ domain $= A \cap B$
3. $B(fg)(x) = f(x)g(x)$ domain $= A \cap B$
4. $B(\frac{f}{g})(x) = g(x)f(x)$ domain $= x\epsilon A \cap B|g(x) \neq 0$

1.16.2 Example

If $f(x) = x$ and $g(x) = \sqrt{4-x^2}$, find the functions $f+g$,$f-g$,fg andf/g. Find the domain of each function.

$f(x) = \sqrt{x}\&\ g(x) = \sqrt{4-x^2}$

1. $f+g$

$$f+g = \sqrt{x} + \sqrt{4-x^2}$$

   (a) $D(f) = [0, +\infty)$

   (b) $D(g)$

$$\begin{aligned} 4-x^2 &> 0 \\ x^2-4 &< 0 \\ (x-2)(x+2) &< 0 \\ \therefore \boldsymbol{D(g) = -2 < x < 2} \end{aligned}$$

   (c) $D(f+g) = D(f) \cap D(g)$

   (d)

$$\boldsymbol{D(f \cap g) = -2 < x < 2}$$

1.17

1.17.1 Notes

1.6 Composition of Functions

Given two functions f and g, the composite function fg (also called the composition of f and g is defined by $(fg)(x) = f(g(x))$ or $fg(x)$.

1.17.2 Exercise

1. If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$

   (a) $f \circ g$

$$\begin{aligned} f \circ g &= f\left(g\left(x\right)\right) \\ &= f\left(x-3\right) \\ \boldsymbol{f \circ g} &\boldsymbol{= (x-3)^2} \end{aligned}$$

   (b) $g \circ f$

$$\begin{aligned} g \circ f &= g\left(f\left(x\right)\right) \\ &= g\left(x^2\right) \\ \boldsymbol{g \circ f} &\boldsymbol{= x^2 - 3} \end{aligned}$$

1.18 Example

1. If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$, find each function and its domain.

   (a) $f \circ g$

      i. Calculation

$$\begin{aligned} \left(f \circ g\right)\left(x\right) &= f\left(g\left(x\right)\right) \\ &= f\left(\sqrt{2-x}\right) \\ &= \sqrt{\sqrt{2-x}} \\ &= \sqrt[4]{2-x} \end{aligned}$$

      ii. Domain

$$\begin{aligned} D_{f \circ g} &= \{x | 2 - x \geq 0 \cap x \geq 0\} \\ &= \{x | x \leq 2\} \\ \boldsymbol{D_{f \circ g}} &\boldsymbol{= (-\infty, 2]} \end{aligned}$$

   (b) $g \circ f$

      i. $g\left(f\left(x\right)\right) = g\left(\sqrt{x}\right) = \sqrt{2 - \sqrt{x}}$

      ii. Domain: $x \geq 0 \cap 2 - \sqrt{x} \geq 0$

$$\begin{aligned} x \geq 0 \cap \sqrt{x} \leq 2 \\ x \geq 0 \cap x \leq 4 \\ \boldsymbol{Domain = \{x | 0 \leq x \leq 4\} = [0, 4]} \end{aligned}$$

   (c) $f \circ f$

i. Calculation

$$\begin{aligned} f^2(x) &= \sqrt{\sqrt{x}} \\ &= \sqrt[4]{x} \end{aligned}$$

ii. Domain

$$x \geq 0 \cap x \geq 0 \cap x \geq 0 = x \geq 0$$

iii.

$$\boldsymbol{Domain = \{x|x \geq 0\} = [0.\infty)}$$

(d) $g \circ g$

i. Calculation

$$g^2(x) = \sqrt{2 - \sqrt{2-x}}$$

ii. Domain

iii.

$$\begin{aligned} 2 - x \geq 0 \cap 2 - \sqrt{2-x} \geq 0 \\ x \leq 2 \cap \sqrt{2-x} \leq 2 = x \leq 2 \cap x \geq -2 \\ \boldsymbol{Domain = [-2, 2]} \end{aligned}$$

1.19 Example

Find $f \circ g \circ h$ if $f(x) = \frac{x}{x+1}$, $g(x) = x^{10}$, and $h(x) = x + 3$.

$$\begin{aligned} f \circ g \circ h(x) &= f(g(h(x))) \\ &= f(g(x+3)) \\ &= f\left((x+3)^{10}\right) \\ \boldsymbol{f \circ g \circ h(x)} &= \boldsymbol{\frac{(x+3)^{10}}{(x+3)^{10}+1}} \end{aligned}$$

1.20 Example

1.20.1 Notes

1.7 Transformations of Functions (Graph sketching)

1. **Shifting Vertical and Horizontal Shifts**

   (a) Suppose $c > 0$

   i. $y = f(x) + c$, shift c units upwards

   ii. $y = f(x) - c$, shift c units downwards

   iii. $y = f(x - c)$, shift c units to the left (happens later)

iv. $y = f(x+c)$, shift $c$units to the right (happens earlier)

2. **Stretching and Reflecting**

   (a) Suppose $c > 1$

      i. Vertical

         A. $y = cf(x)$, stretch vertically by a factor of c ("result" of $f(x)$ is amplified/scaled up)

         B. $y = \frac{1}{c}f(x)$, compress/squish vertically by a factor of c ("result" of $f(x)$ is dampened/scaled down)

      ii. Horizontal

         A. $y = f(cx)$, compress horizontally by a factor of c ("source" of x is amplified, "result" arrives "earlier")

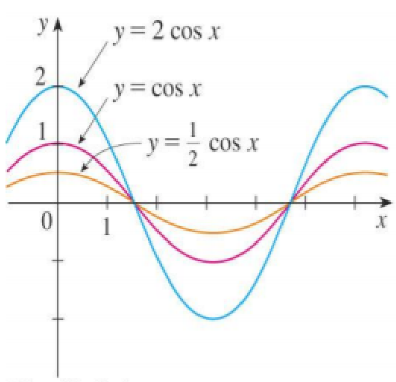
         B. $y = f(\frac{x}{c})$, stretch horizontally by a factor of c ("source" of x is dampened, "result" arrives "later").

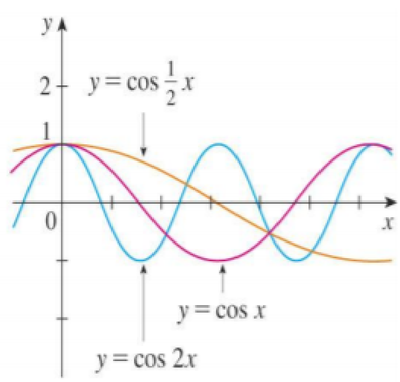
      iii. Reflection

         A. $y = -f(x)$, reflect about the x-axis. (the result is negated, whatever value you plug, will return the original result, but negated)

         B. $y = f(-x)$, reflect about the y-axis. (the source is negated, whatever value you plug, will return the value when a negated version of your value is plugged)

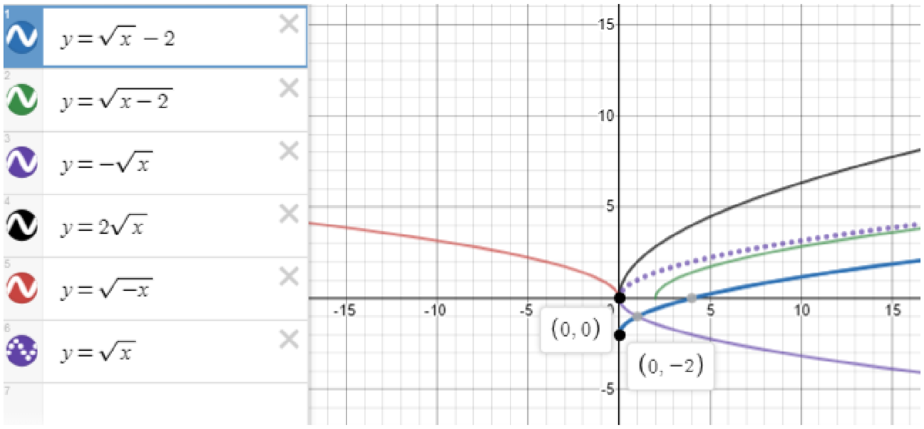
1.20.2 Example





1.21 Example

Sketch the graph of $y = \sqrt{x}$, and hence use the transformations to sketch graph of $y = \sqrt{x} - 2$, $y = \sqrt{x-2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, $y = \sqrt{-x}$ and $y = \sqrt{x}$.

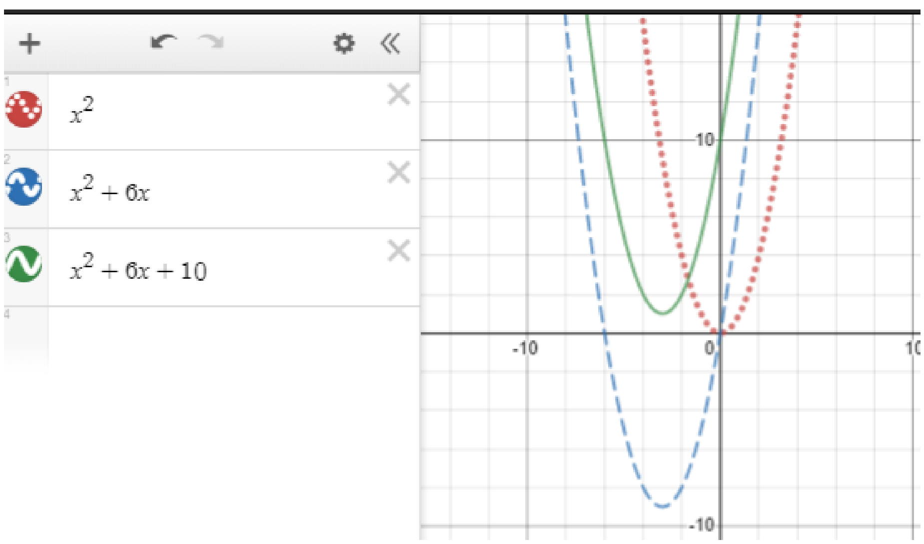


1.22 Example

Sketch the graph of the function $f(x) = x2 + 6x + 10$.

Steps:

1. Start with x^2, a parabola, the graph should be a ∪-shaped curve, that cuts $x = 0$
2. Add on a $6x$ to the equation, make the graph cut both $x = 0$ and $x = -6$ (they are the roots), and the equation now is $x^2 + 6x$
3. Tack on a constant of 10, shift the equation up by 10 to make the equation $f(x)$



Lecturer example: 1. convert to standard form

$$
\begin{aligned}
y &= x^2 + 6x + 10 \\
&= (x+3)^2 - (3)^2 + 10 \\
&= (x+3)^2 + 1
\end{aligned}
$$

At this point, we can move $x^2$to $(x+3)^2$and then add 1 to the $f(x)$ or the end result to arrive at the solution.

1.23 Example

1. Sketch the graphs of the following functions:
   (a) $y = sin2x$
      i. Steps
         A. Start with $y = sin(x)$
         B. Increase the oscillation by twice the amount $(2x)$, because whatever value plugged in will be amplified/scaled up to twice the original, before the function starts processing the value, the function is now $y = sin(2x)$
      ii. Graph
         A.
   (b) $y = 1 - sin(x)$
      i. Steps
         A. Start with $sin(x)$
         B. Reverse the entire oscillation, basically flip the entire wiggly thing by 180°, you now have $-sin(x)$
         C. Finally, shift the equation up by 1 unit, move all the points on the $x - axis$,on $x = 1$, on $x = -1$ by 1 unit, then draw a smooth curve connecting them. You now have $y = -sin(x) + 1$ or $y = 1 - sin(x)$
      ii. Graph

1.24 Notes & Examples

1.24.1 Notes

1.8 Inverse Functions

A function f is called a one-to-one function if it never takes on the same value twice i.e.

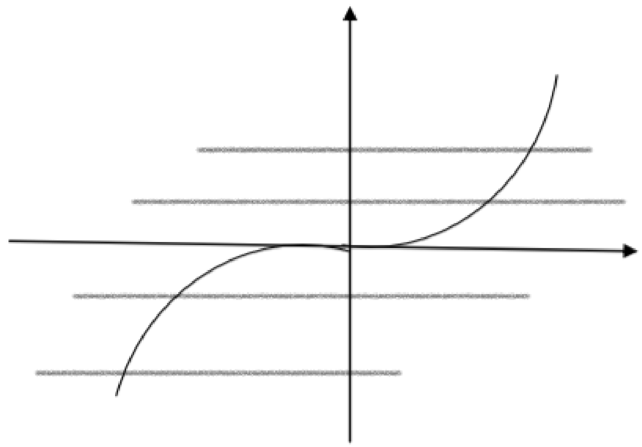
$$f(x_1) \neq f(x_2)$$

whenever $x_1 \neq x_2$

Horizontal line test: A function is **one-to-one** if and only if no horizontal line intersects the graph more than once.

1.24.2 Exercises

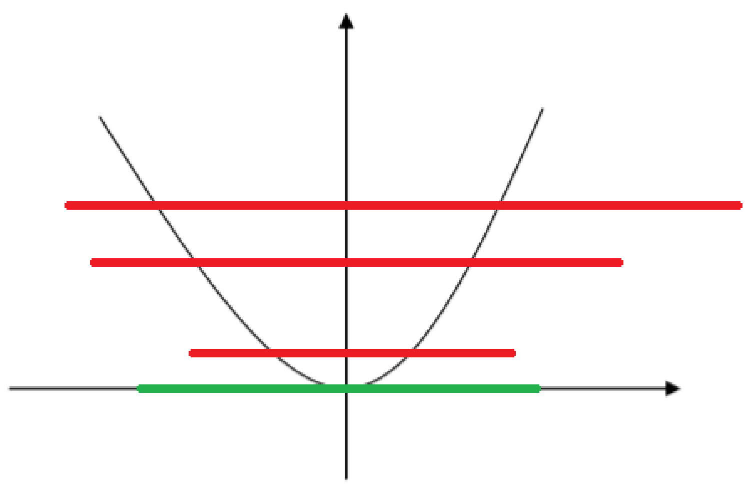
Example 1.24 . Is the function$f(x) = x3$ one-to-one?



Answer: Yes, because at no point does x^3fails the horizontal line test.

1.25 Example

Is the function g(x) = x2 one-to-one?



Answer: **No,** because the graph fails the horizontal line test at all points except when $x = 0$.

1.26 Notes & Examples

Definition: Let f be a one-to-one function with domain A and range B. Then its inverse function f^{-1}has domain B and range A, and is defined by

$$f^{-1}\left(y\right)=x\Leftrightarrow f\left(x\right)=y$$

for any y in B.
If f is not one-to-one, then f^{-1} would not be uniquely defined.
Note:

1. $f^{-1}\left(x\right)\neq\frac{1}{f(x)}/\left[f\left(x\right)\right]^{-1}$
2. $f^{-1}\left(f\left(x\right)\right)=x$ for every x in A.
3. $f\left(f^{-1}\left(x\right)\right)=x$ for every x in B.

1.26.1 Example

If$f(1)=5$, $f(3)=7$, and $f(8)=-10$, find $f^{-1}(7)$, $f^{-1}\left(5\right)$ and $f^{-1}\left(-10\right)$.
Note: the inverse is simply the reverse, or the input of the function

$$\begin{aligned}f^{-1}\left(7\right)&=3\\f^{-1}\left(5\right)&=1\\f^{-1}\left(-10\right)&=8\end{aligned}$$

1.27 Example and Notes

How to Find the Inverse of a One-to-One Function 1. Write y = f(x). 2. Solve this equation for x in terms of y (if possible). 3. Interchange x and y. The resulting equation is y = f –1 (x)

1.27.1 Find the inverse function of f(x) $=x^{3}$

$$\begin{aligned}y&=x^{3}\\\sqrt[3]{y}&=x\\f^{-1}\left(x\right)&=\sqrt[3]{x}\end{aligned}$$

1.28 Example

Find the inverse function of $f(x)=5x^{3}+2$

$$
\begin{aligned}
y &= 5x^3 + 2 \\
x^3 &= \frac{y-2}{5} \\
x &= \left(\frac{y-2}{5}\right) \\
f^{-1}(x) &= \left(\frac{x-2}{5}\right)^{\frac{1}{3}}
\end{aligned}
$$

1.29 Example

Find the inverse of these functions. (a) $h(x) = \frac{1}{x} - 3$ (b) $g(x) = \frac{3}{x-1}$

1.

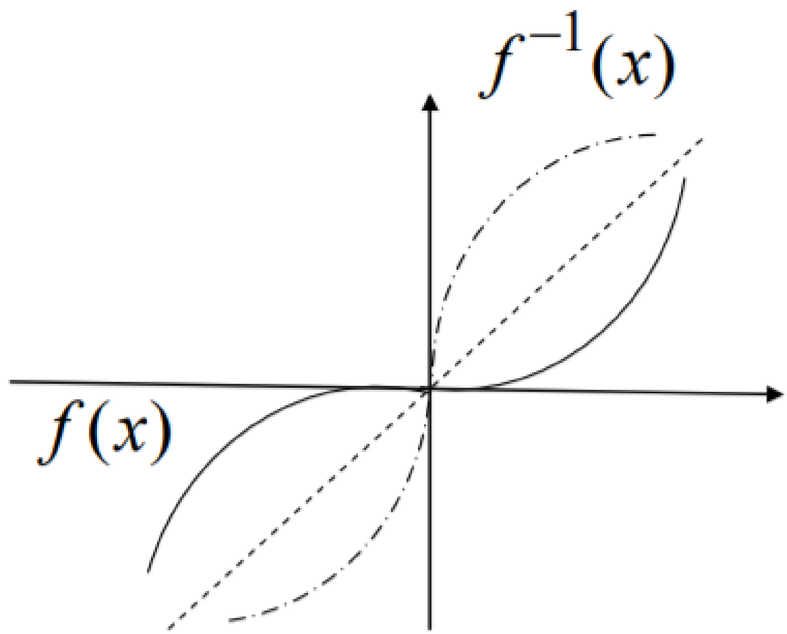
   (a) $h(x) = \frac{1}{x} - 3$

$$
\begin{aligned}
y &= \frac{1}{x} - 3 \\
\frac{1}{y+3} &= x \\
f(x) &= \frac{1}{x+3}, x \neq 0
\end{aligned}
$$

   (b) $g(x) = \frac{3}{x-1}$

$$
\begin{aligned}
g(x) &= \frac{3}{x-1} \\
y &= \frac{3}{x-1} \\
\frac{1}{y} &= \frac{x-1}{3} \\
\frac{3}{y} &= x - 1 \\
\frac{3}{y} + 1 &= x \\
f(x) &= 1 + \frac{3}{x}, x \neq 0
\end{aligned}
$$

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.



1.30 Example

Sketch the graph of the function $y = 3 - 2^x$ and determine its domain and range.

1. Start with $f(x) = 2^x$
2. Reflect the graph vertically, with respect to the $x - axis$
3. Push the graph up by 3 units

Domain: $\mathbb{R}$
Range: $(-\infty, 3)$

1.31 Example

Graph the function $y = \frac{1}{2}e^{-x}$, and state the domain and range.

1. Start with e^x
2. Flip the graph horizontally to get the inverse function e^{-x}
3. Scale down the graph vertically by a factor of $\frac{1}{2}$ with respect to the $y - axis$

Domain: $\mathbb{R}$
Range: $(-1, \infty)$

1.32 Example

Sketch the graph of the function $y = \ln(x - 2)–1$.

$$y = \ln(x - 2) - 1$$

1. Start with $\ln(x)$
2. "Slow down"/ shift the graph right by 2 units, so the graph becomes $\ln(x - 2)$
3. Shift the entire graph down by 1 units, so the graph becomes $\ln(x - 2) - 1$

1.33 Example

Sketch the curve defined by the parametric equations x = t2 2t, y = t+1. Find the Cartesian equation of the curve.

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4

Cartesian equation

$$t = y - 1$$

$$\begin{aligned} x &= (y-1)^2 - 2(y-1) \\ &= y^2 - 2y + 1 - 2y + 2 \\ x &= y^2 - 4y + 3 \end{aligned}$$

1.34 Example

What curve is represented by the parametric equations x = cost, y = sint, 0 t 2 ?

$$\begin{aligned} x &= cos(t) \\ t &= cos^{-1}(x) \end{aligned}$$

$$\begin{aligned} y &= sin\left(t\right) \\ &= sin\left(cos^{-1}\left(x\right)\right) \\ \boldsymbol{y} &\boldsymbol{= tan\left(x\right)} \end{aligned}$$

Alternatively:

$$\begin{aligned} x^2 &= cos^2\left(t\right) \\ y^2 &= sin^2\left(t\right) \\ cos^2\left(t\right) + sin^2\left(t\right) &= 1 \\ x^2 + y^2 &= 1 \end{aligned}$$

This is a circle with the center at $(0, 0)$ and radius of 1.

1.35 Example

1.

t	x=$sin\left(t\right)$	y=$sin^2\left(t\right)$
0	0	0
$\frac{\pi}{2}$	1.0	0.84
π	0.00	0.00
$\frac{3}{2}\pi$	-1.00	-0.84
2π	0.00	0.00

2. Sketch the curve

3.

$$\begin{aligned} x &= sin\left(t\right) \\ t &= sin^{-1}\left(x\right) \\ y &= sin^2\left(sin^{-1}x\right) \\ y &= sin\left(x\right) \end{aligned}$$

1.36 Example

1.

   (a) $f\left(x\right) = 5^x$: Exponential function

   (b) $g\left(x\right) = x^5$:Power function/polynomial function with degree 5

   (c) $h\left(x\right) = \frac{1+x}{1-\sqrt{x}}$: Rational function/Algebraic function

   (d) $u\left(t\right) = 1 - t + 5t^4$: Polynomial function with degree 4

2 Exam Tips

Domain is frequently asked, codomain seldom asked