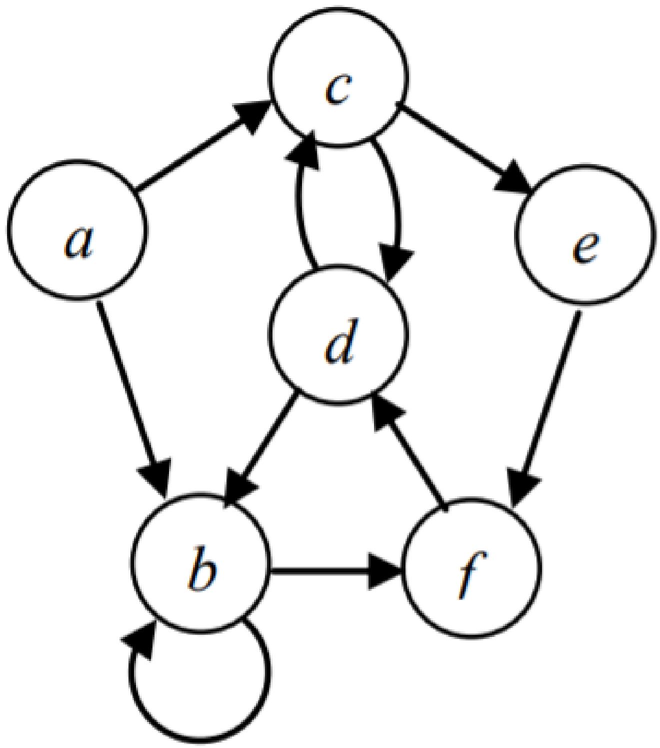
Tutorial 8

December 18, 2019

1. Let R be the relation whose digraph is given as follow:



(a) List all paths of length 1.

i. $R^1 = \{(a,c), (a,b), (b,b), (b,f), (c,e), (c,d), (d,c), (d,b), (e,f)\}$

(b) List all paths of length 3 starting from vertex a.

i. $R^3 = \{(a,b), (a,f), (a,d), (a,e)\}$

(c) Find a cycle starting at vertex d. **(DOUBLE CONFIRM)**

i. $\{d, c, d\}$

2. Determine whether the given relation on $A = \{1, 2, 3, 4\}$ is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive. Explain your answers

   (a) $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$

      i. Matrix-style

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

      ii. **Reflexive**, matrix have all 1's on its main diagonal.

      iii. **Not irreflexive.** $m_{11} = 1$

      iv. **Symmetric**. In the matrix, if $m_{ij} = 1$, then $m_{ji} = 1$

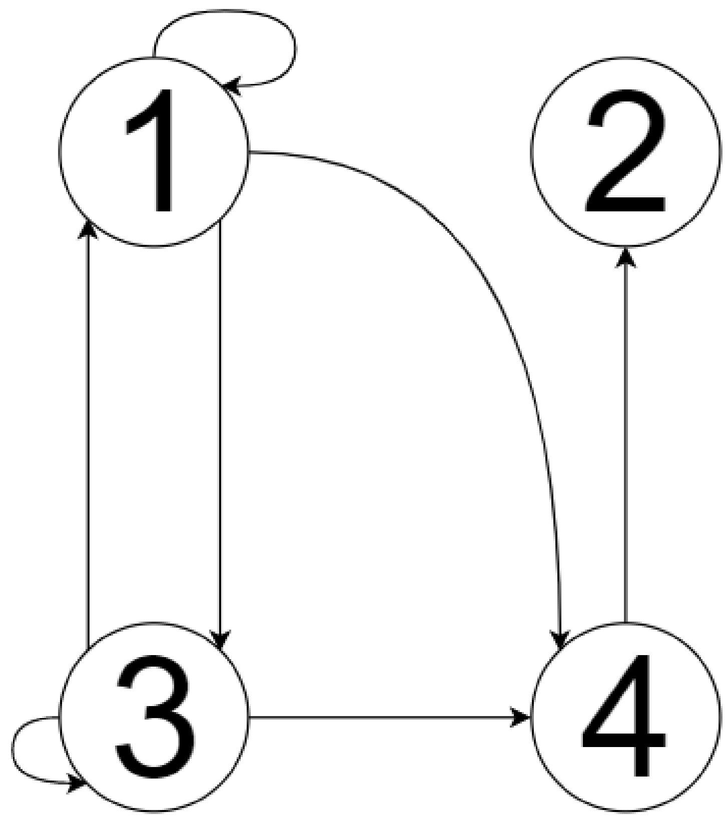
      v. **Not antisymmetric.**

      vi. **Transitive**. In the matrix, $(M_R)^2_{\odot} = M_R$

$$\begin{aligned} M_R \bigodot M_R &= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \end{aligned}$$

   (b) $R = \{(1, 3), (1, 1), (3, 1), (1, 2), (3, 3), (4, 4)\}$

      i. Digraph-style



ii. **Not reflexive**. $(4,4) \notin R$.
iii. **Not irreflexive.** $(1,1) \in R$
iv. **Not symmetric**. $(1,4) \in R$ but $(4,1) \notin R$
v. **Not asymmetric**. $(1,3) \in R$ and $(3,1) \in R$
vi. **Not antisymmetric**. $(1,3), (3,1) \in R$, but $3 \neq 1$, or just **have loops**
vii. **Not transitive.** $(1,4), (4,2) \in R$ but $(1,2) \notin R$

(c) $R = \varnothing$

i. Tikam style
ii.

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= M^T$$

iii. R is not reflexive since $(1,1) \notin R$,

iv. R is irreflexive since $(1,1), (2,2), (3,3), (4,4) \notin R$

v. R is symmetric

vi. R is asymmetric. No loops

vii. R is antisymmetric

viii. R is transitive

(d) $R = \{(1,2), (1,3), (3,1), (1,1), (3,3), (3,2), (1,4), (4,2), (3,4)\}$

i. Tikam style version 2

ii. Not reflexive. $(2,2) \notin R$

iii. R is not irreflexive since $(1,1), (3,3)$

iv. Not symmetric. $(1,2) \in R$ but $(2,1) \notin R$.

v. Not asymmetric. $(1,3) \in R$ and $(3,1) \in R$

vi. Not antisymmetric. $(1,3) \in R$ and $(3,1) \in R$ but $1 \neq 3$

vii. R is transitive.

3. Let $A = \{w, x, y, z\}$. Determine whether the relation R whose matrix M_R is given below is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive. Explain your answers

> **Determining transitivity through matrix multiplication**: Basically, the original M_r tells you about the number of $1-step$ paths from x to x. By multiplying each row of M_r by each column of M_r, you will find how many 2-step paths there are. So say you multiply 1st row times first column, you'll get the 2-step paths for $(1,1)$ (note: $(row, column)$). For a matrix to be transitive, all elements with a $1-step$ path must also have a $2-step$ path. Note: Since we only care about "presence" of links, not "how many", its safe to just write '1' if there is to speed up calculation.

(a) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

i. **Not reflexive.** $(1,1) \notin R$

ii. **Irreflexive**. The main diagonal are all 0's.

iii. **Symmetric.** If $m_{ij} = 1$, then $m_{ji} = 1$

iv. **Not asymmetric.** $m_{12} = m_{21} = 1$

v. **Not antisymmetric.** $m_{12} = m_{21} = 1$ but $1 \neq 2$.

vi. **Not Transitive.**

$$(M_R)^2_{\odot} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$M_{R23} = 1$ but $M_{R_{32}} = 0$.

(b) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

i. **Not reflexive, not irreflexive.** $M_{00} = 0$**,** but $M_{33} = 1$

ii. **Not symmetric, not asymmetric.** $M_{10} = 1$,but $M_{01} = 0$. $M_{32} = 1, M_{23} = 1$.

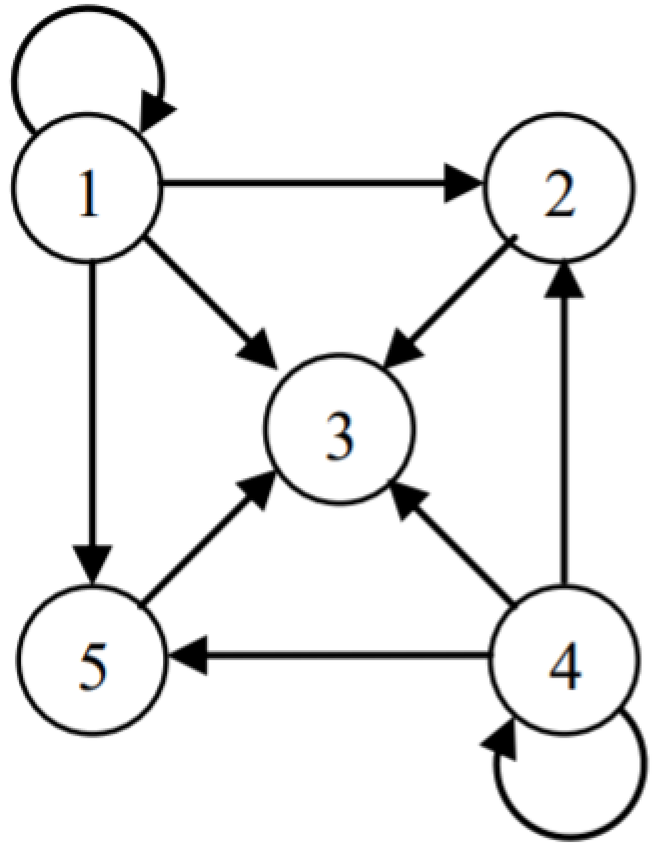
iii. **Not antisymmetric.** $M_{32} = 1, M_{23} = 1$. But $2 \neq 3$.

iv. **Not transitive.**

$$(M_R)^2_{\odot} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

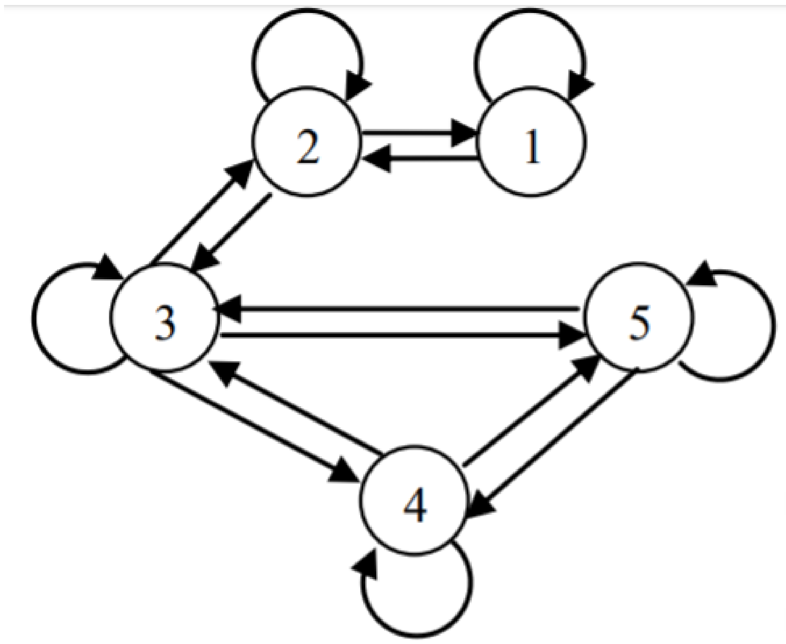
A. Since $(M_R)_{21} = 1$ but $(M_R)^2_{21} = 0$.

4. Let $A = \{1, 2, 3, 4, 5\}$. Determine whether the relation R whose digraph is given below is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive. Explain your answers.



(a)

i. **Not reflexive.** $(2,2) \notin R$.
ii. **Not irreflexive.** $(1,1) \in R$
iii. **Not symmetric.** $(2,3) \in R$, but $(3,2) \notin R$
iv. **Not asymmetric.** $(1,1),(4,4) \in R$
v. **Antisymmetric.** For all (a,b), if aRb then $a = b$.
vi. **Transitive.** For all (a,b,c), if aRc, then aRb.



(b)

5. Let $A = \{a, b, c\}$. Determine whether the relation R whose matrix M_R is given is an equivalence relation. If yes, find A/R.

   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

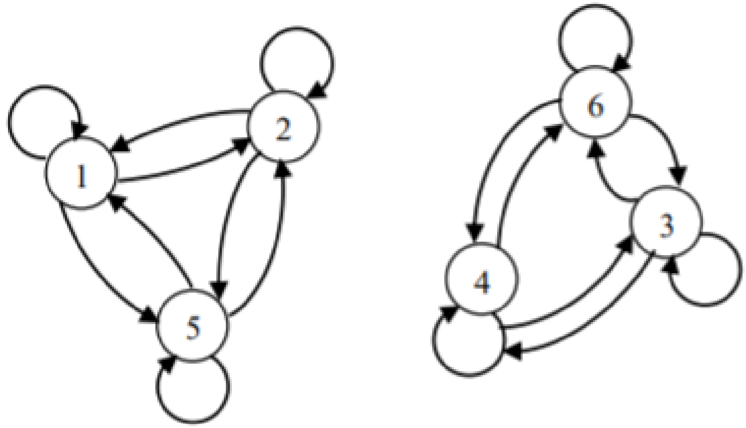
      i. Equivalence relation

      ii. $A/R = \{\{b, c\}, \{a\}\}$

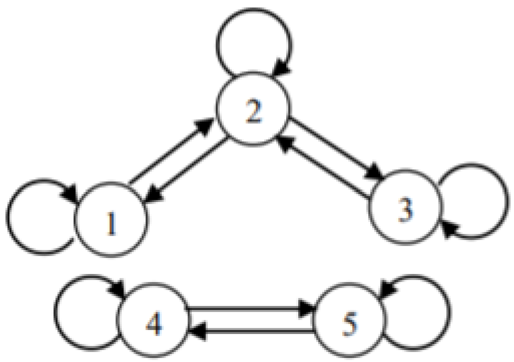
   (b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

      i. Not an equivalent relation

6. Determine whether the relation R whose digraph is given as below is an equivalence relation. If yes, find A/R.



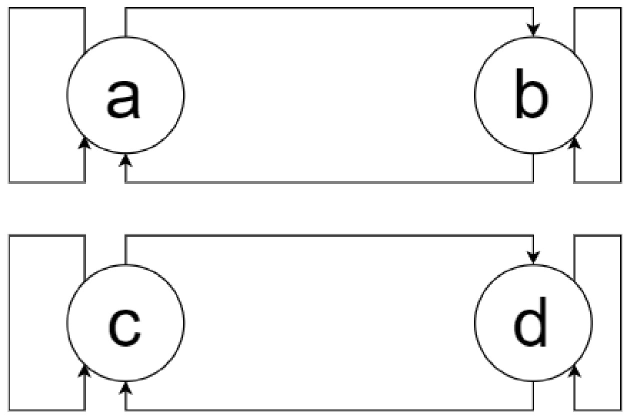
(a)



(b)

7. Determine whether the following relation R on the set A is an equivalence relation. If yes, find A/R

   (a) $A = \{a, b, c, d\}$, $R = \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d)\}$

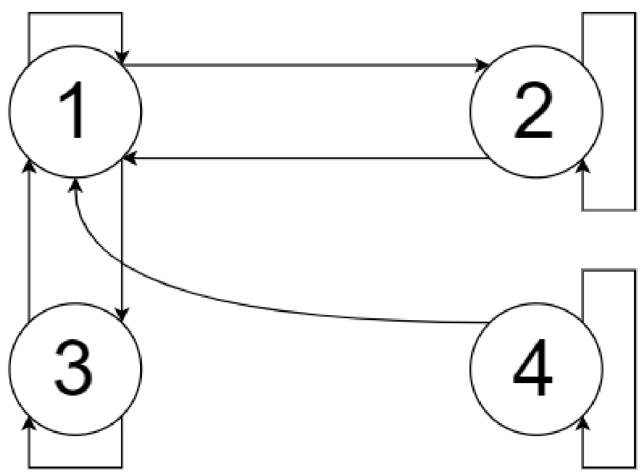


i.

ii. Equivalence relation

iii. $A/R = \{\{a, b\}, \{c, d\}\}$

(b) $A = \{1, 2, 3, 4\}$,$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 3), (1, 3), (4, 1), (4, 4)\}$



i.

ii. Not equivalence relation

A. R is reflexive

B. R is not symmetric $(1, 4) \in A$ but $(4, 1) \notin A$

C. R is not transitive since $(3, 1), (1, 2) \in A$ but $(3, 2) \notin A$

(c) $A = \{2, 3, 5, 6, 8\}$, xRy iff $3|(x–y)$.

i. $R = \{(2, 2), (3, 3), (5, 5), (6, 6), (8, 8), (3, 6), (6, 3), (8, 2), (2, 8), (8, 5), (5, 8)\}$

ii. R is reflexive

iii. R is symmetric

iv. R is transitive.

v. equivalence relation

vi. $A|R = \{\{2,5,8\},\{3,6\}\}$

(d) $A = \{1,2,3,4,5\}$, $xRy \iff x = y \pmod 2$

i. $R = y \mod 2$

ii. $R = \{(1,1),(1,3),(1,5),(2,2),(2,4),(3,1),(3,3),(3,5),(4,2),(4,4),(5,1),(5,3),(5,5)\}$

iii. R is reflexive, $(2,2) \notin R$

iv. R is symmetric, $(1,3) \in R$ but not $(3,1)$.

v. R is transitive.

vi. Equivalent relation

vii. $A|R = \{\{1,3,5\},\{2,4\}\}$