Chapter 5 Functions

5.1 Introduction

5.2 Properties of Functions

5.3 Functions for Computer Science

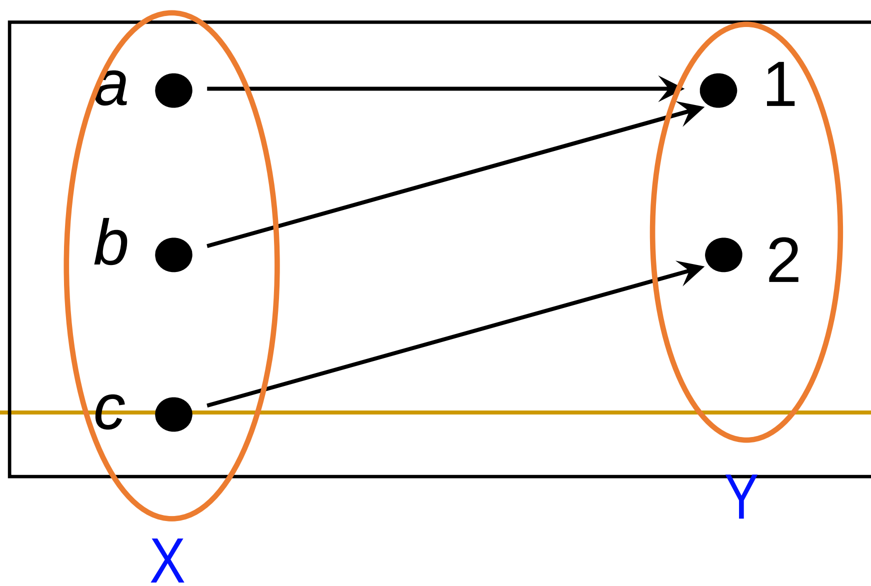
5.4 Permutations

5.1 Introduction

- Functions are binary relations in which further restrictions are imposed on the pairs which can occur. For each input, one output
- A function from a set *A* to a set *B* is a binary relation in which every element of *A* is associated with a uniquely specified element of *B*. only one
- Functions are also called as mappings or transformation.

5.1 Introduction (cont)

- In digraph terms a function is a relation such that there is precisely one arc leaving every element of *A*.
- E.g. The digraph representing the function from {*a*, *b*, *c*} to {1, 2} containing the pairs (*a*, 1), (*b*, 1) and (*c*, 2).



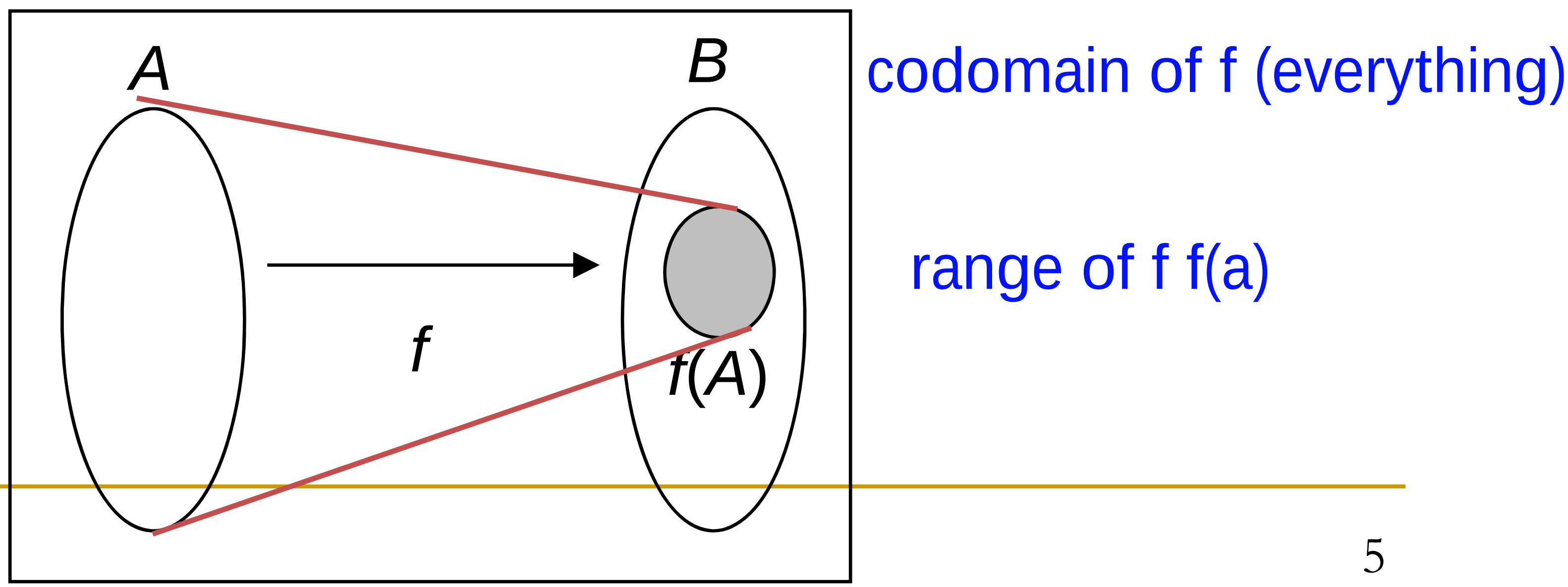
f(x) -> y

5.1 Introduction (cont)

- Let f be a function from a set A to a set B.
  - For each $a \in A$, there exists a uniquely determined $y \in B$ with $(x, y) \in f$, write as $y = f(x)$, and refer to $f(x)$ as the image of x under f.
  - Write as $f : A \rightarrow B$ to indicate that f is a function which transform, or maps, each element of A to a uniquely determined element of B.
  - A is called the domain of f, and B is the codomain of f.

5.1 Introduction (cont)

- The range of f is the set of images of all the elements of A under f, denoted by $f(A)$. Hence, $f(A) = \{f(x) : x \in A\}$.
- The Venn diagram provides a useful diagrammatic illustration of a function from a set A to a set B.



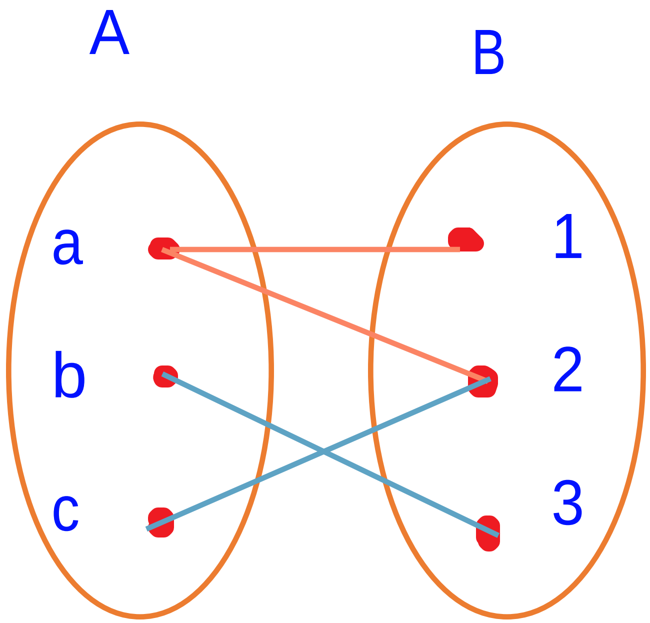
E.g.1

For each of the following relations between the sets $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Determine whether the relation gives a function from A to B.

i. $\{(a, 1), (a, 2), (b, 3), (c, 2)\}$ No

ii. $\{(a, 1), (b, 2), (c, 1)\}$ Yes

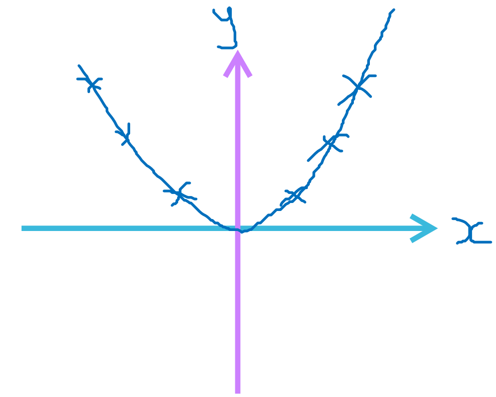
iii. $\{(a, 1), (c, 2)\}$ Yes



E.g.2

Which of the following relations are functions?

$\mathbb{Z}$



i. The relation on Z given by

$\{(\overset{x}{x}, \overset{y}{x^2}) : x \in Z\}$. $y=x^2$ Function

ii. The relation on R given by

$\{(x, y) : x = y^2\}$. Not a function

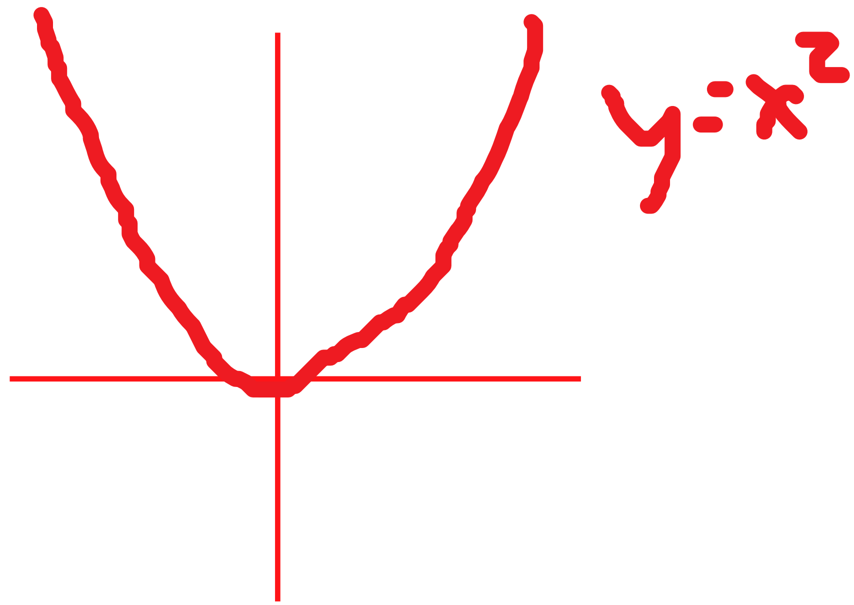
E.g.2

Which of the following relations are functions?

i. The relation on Z given by

$\{(x, x^2) : x \in Z\}$.

Function

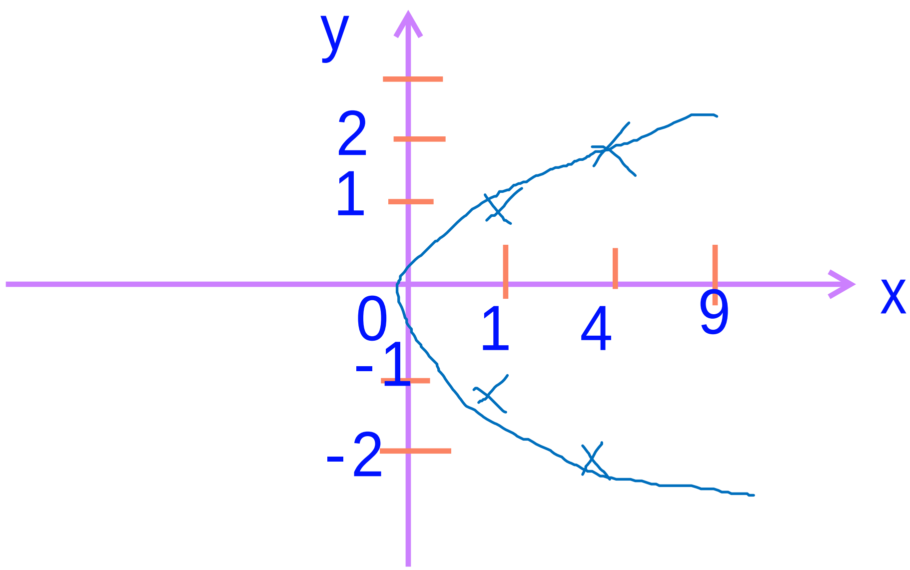


E.g.2

Which of the following relations are functions?

ii. The relation on R given by

$\{(x, y) : x = y^2\}$.



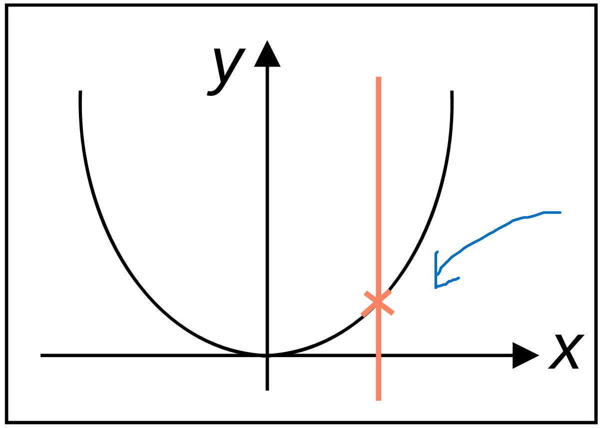
This is not a function from X to Y

5.1 Introduction (cont)

- When dealing with a function $f : A \to B$ where A and B are infinite sets of numbers, we can use the more traditional mathematical idea of graphing a function to give a geometric picture of the function.

5.1 Introduction (cont)

- The graph of the function $f : R \to R$ given by $f(x) = x^2$:

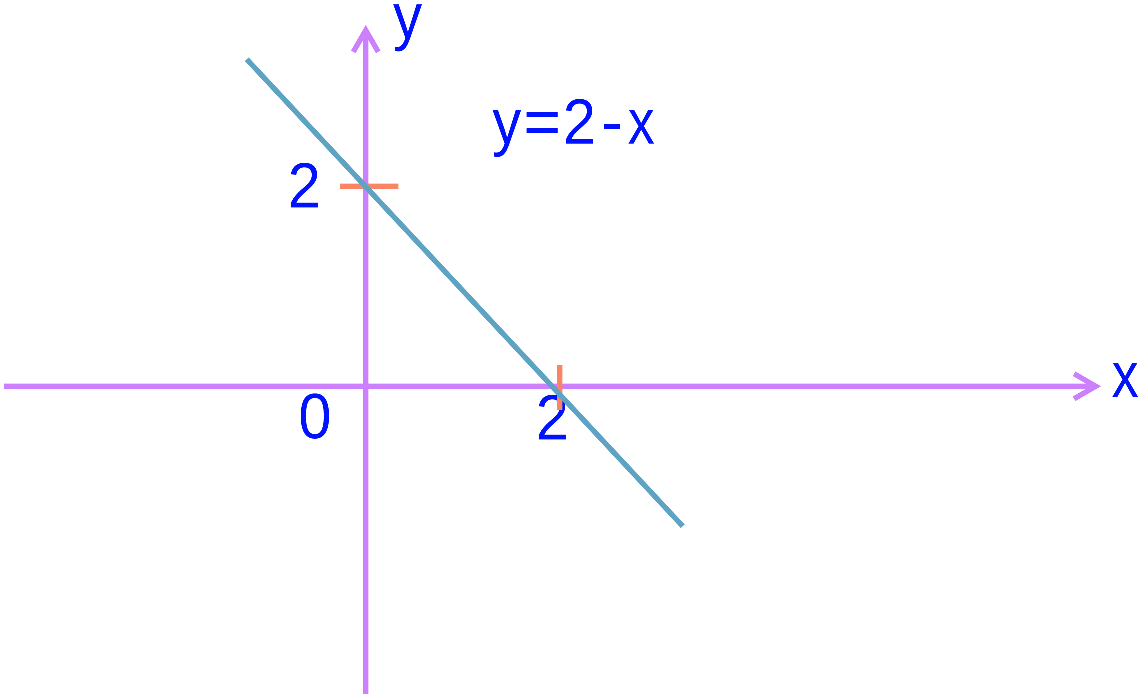


only one

- The horizontal axis (x-axis) denotes the domain R and the vertical axis (y-axis) denotes the codomain R.
- The curve consists of those points (x, y) in Cartesian product $R \times R$ for which $y = f(x)$.

E.g.3

Sketch the graph of the function $f : R \rightarrow R$ given by $f(x) = 2 - x$. Find the image of x under f when $x = 2$ and $x = 3$.

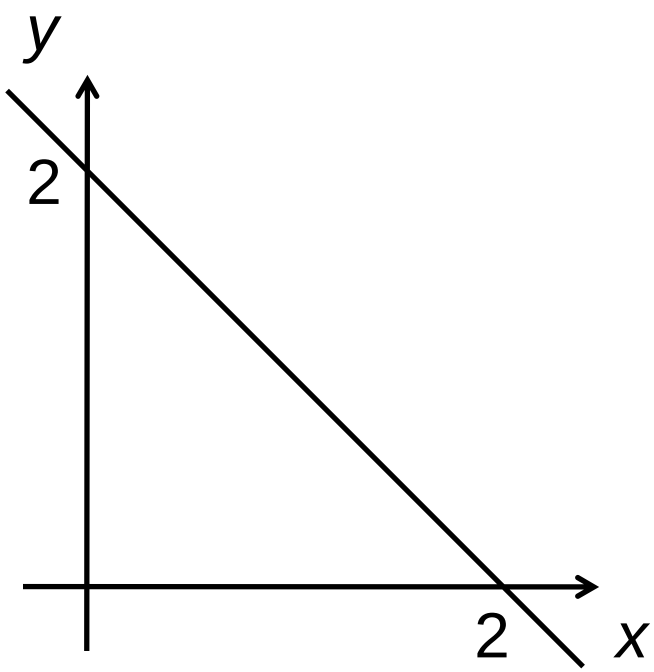


f(2)=2-2=0

f(3)=2-3=-1

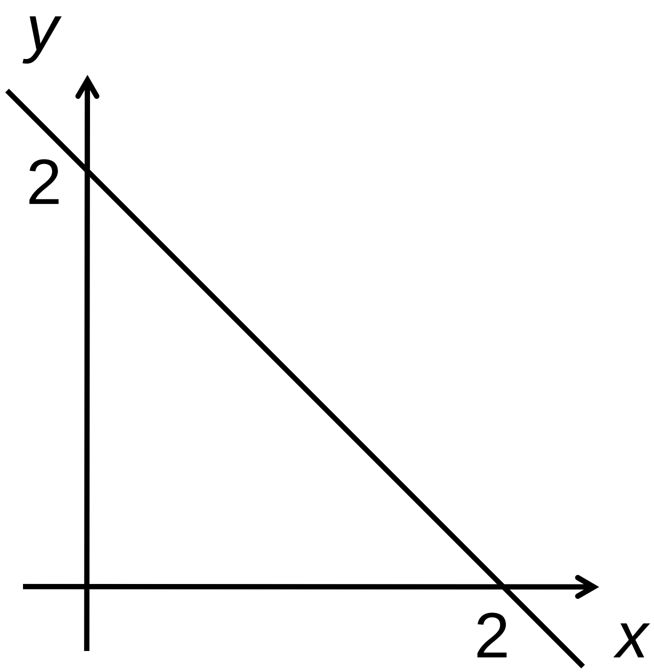
E.g.3

Sketch the graph of the function $f : R \to R$ given by $f(x) = 2 - x$. Find the image of x under f when $x = 2$ and $x = 3$.



E.g.3

Sketch the graph of the function $f : R \to R$ given by $f(x) = 2 - x$. Find the image of x under f when $x = 2$ and $x = 3$.



$x = 2, f(2) = 0$

$x = 3, f(3) = -1$

5.2 Properties of Functions

- Let $f : A \to B$ be a function.
  - f is an injective (or one-to-one) function if

    $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ for all $a_1, a_2 \in A$;

    or $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ i.e. different inputs give different outputs.
  - f is everywhere defined if $\mathrm{Dom}(f) = A$.
  - f is a surjective (or onto) function if the range of f coincides with the codomain of f, i.e. each element in the codomain is a value of the function; f(A)=B

    or for every $b \in B$, there exists an $a \in A$ with $b = f(a)$.

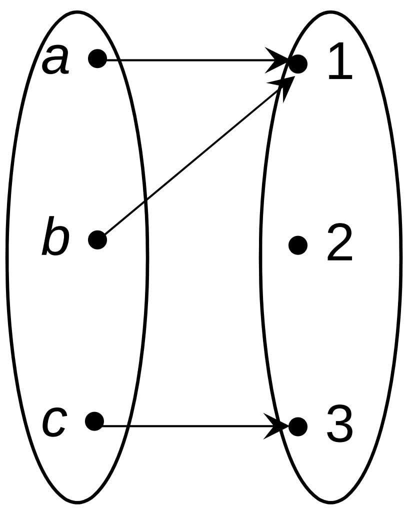
5.2 Properties of Functions (cont)

- *f* is a bijective function if *f* is both injective and surjective.
- *f* is a one-to-one correspondence between *A* and *B* if *f* is everywhere defined, injective, and surjective.

E.g.4

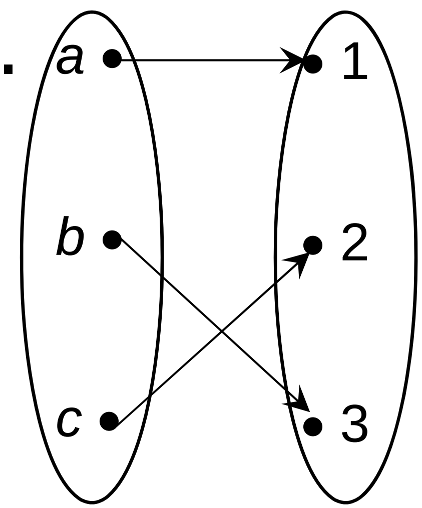
Decide which of the functions below is injective or surjective. Which are bijective?

i.



not injective
not surjective

ii.



Bijective

f(A) = B

f is surjective

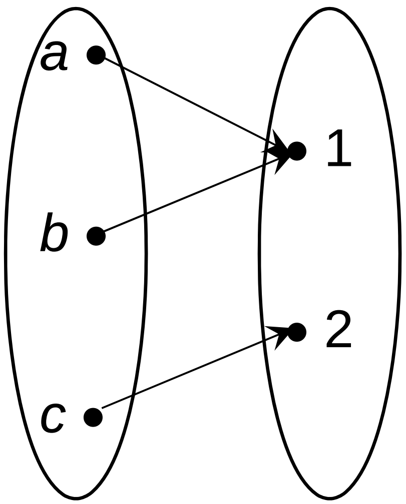
f(a)=1=f(b)
But a$\neq$b, f is not injective

2 $\in$ B but 2$\notin$ f(n). Is not surjective

$\therefore$ f is not bijective

E.g.4 (cont)

iii.



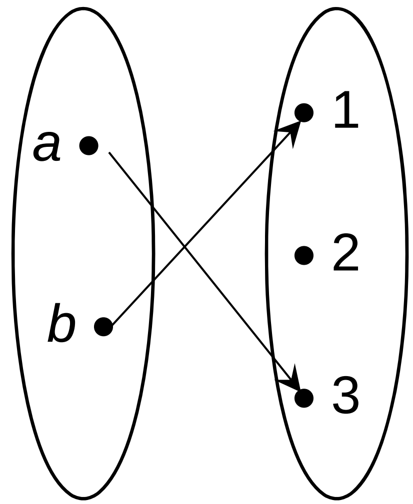
f(a)=1=f(b)

but a $\neq$ b. f is not injective.

f(A) = B

f is surjective.

iv.



f is injective.

2 $\in$ B but 2 f(A)

f is not surjective

f is not bijective

E.g.5

Let $A = \{1, 2, 3\}$, $B = \{x, y, z\}$, $C = \{\pi, e\}$, and $D = \{\theta, \alpha, \beta, \gamma\}$. Determine whether each of the following function is one-to-one, onto, and everywhere defined.

injective

surjective

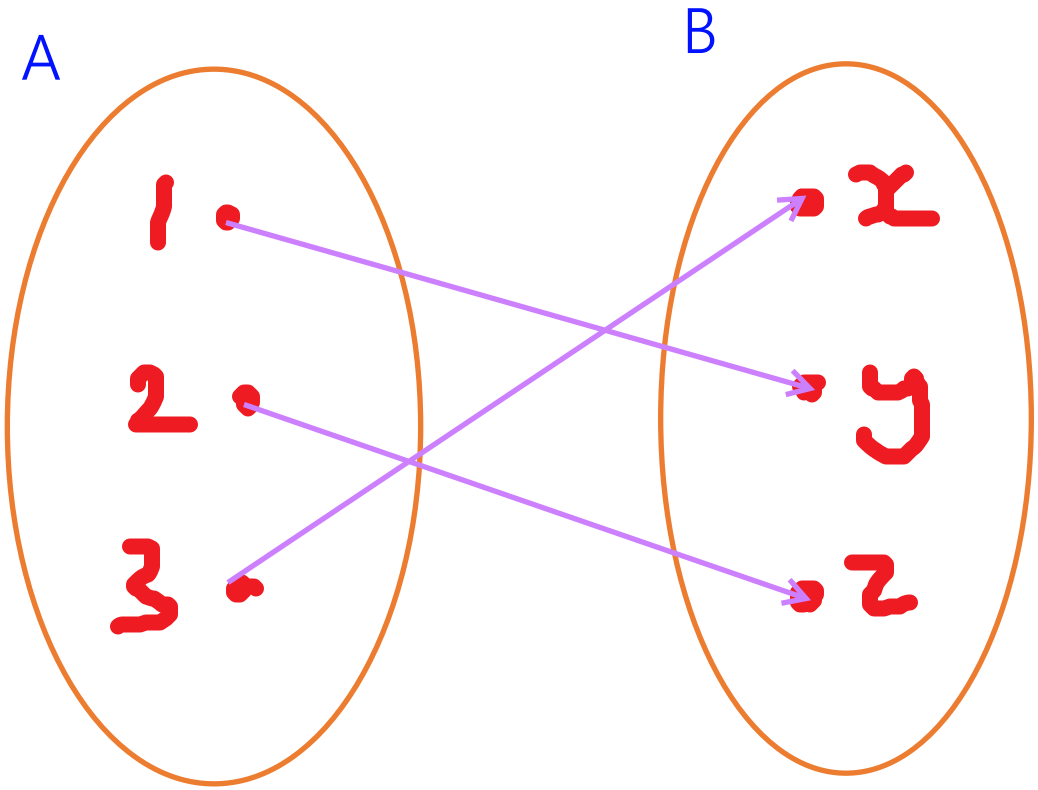
i. $f : A \to B$; $f = \{(1, y), (2, z), (3, x)\}$

ii. $g: A \to D$; $g = \{(1, \alpha), (2, \theta), (3, \gamma)\}$

iii. $h: B \to C$; $h = \{(x, e), (y, e), (z, \pi)\}$

iv. $k : D \to B$; $k = \{(\theta, x), (\alpha, y), (\beta, x)\}$.

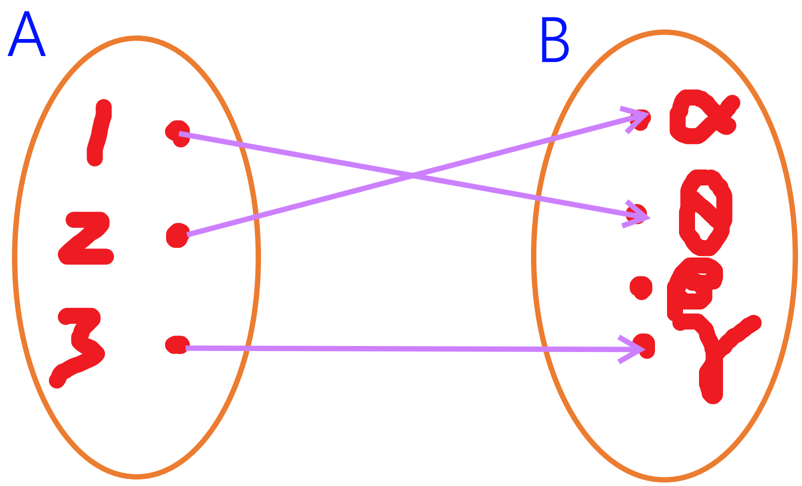
i. $f : A \rightarrow B$; $f = \{(1, y), (2, z), (3, x)\}$



f is one-to-one
f is onto
f is everywhere defined

ii. $g: A \to D$; $g = \{(1, \alpha), (2, \theta), (3, \gamma)\}$

Change f to g



f is one-to-one

$\beta \in D$ but $\beta \notin f(n)$

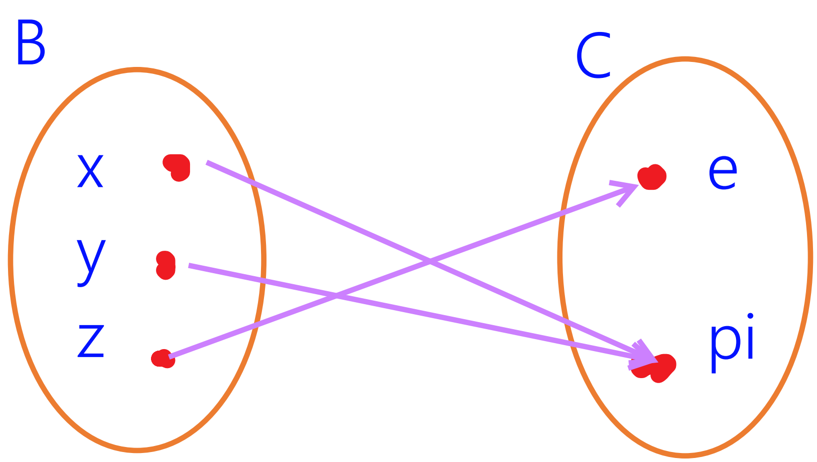
f is not onto

f is everywhere defined

Note: beta is due to definition of D earlier

iii. $h: B \to C$; $h = \{(x, e), (y, e), (z, \pi)\}$

Change f to h



$f(x) = e = f(y)$, $but\ x \neq y$

f is not one-to-one

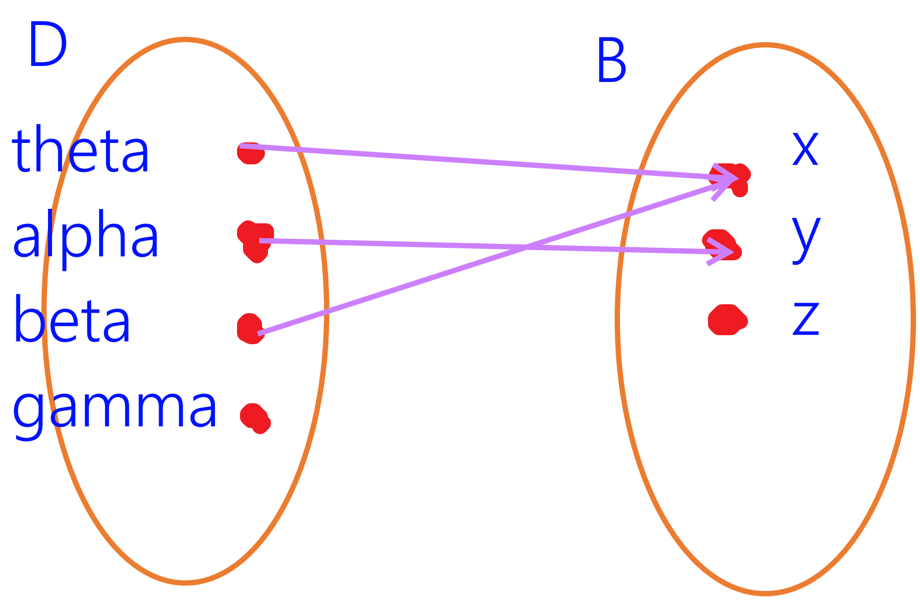
$f(B) = C$

f is onto

f is everywhere defined

$A = \{1, 2, 3\}$, $B = \{x, y, z\}$, $C = \{\pi, e\}$, and $D = \{\theta, \alpha, \beta, \gamma\}$

iv. $k : D \to B$; $k = \{(\theta, x), (\alpha, y), (\beta, x)\}$



Change f to A

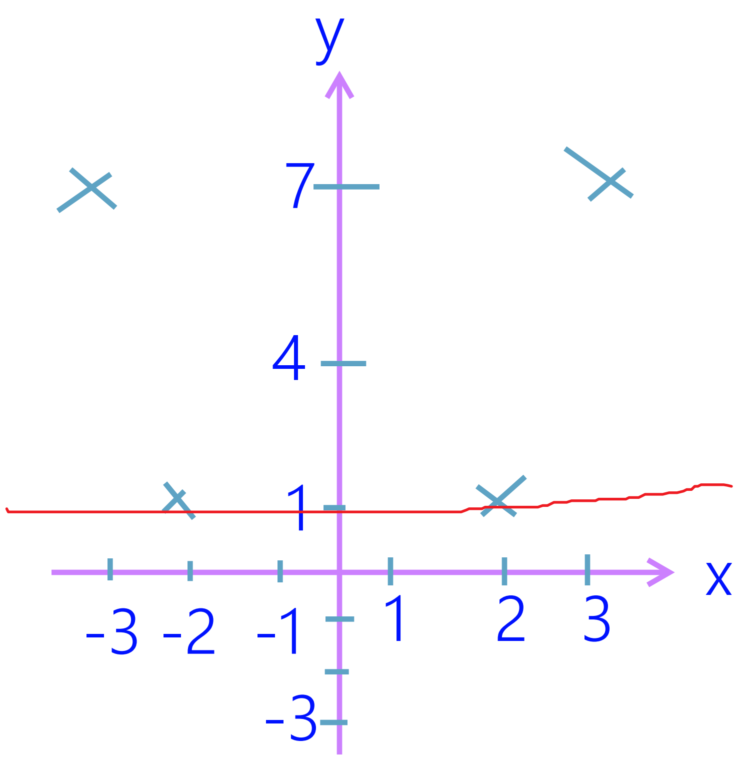
$f(\theta) = x = f(\beta)$, $but\, \theta \neq \beta$
f is not one-to-one

$Z \in B, but\, Z \notin f(D)$

f is not onto,
$f(\gamma)$ is not defined,
f is NOT everywhere defined

E.g.6

Show that the function $h : \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = x^2$ is neither injective nor surjective.



$h(1) = 1 = h(-1)$, but $1 \neq -1$

$\therefore h$ is not injective

$h(x) \geq 0$

$-1 \notin h(\mathbb{Z})$

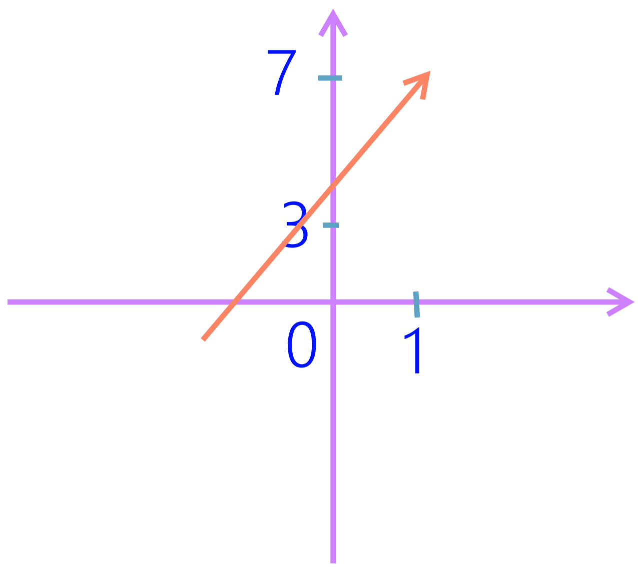
but $-1 \in \mathbb{Z}$

h is not surjective

E.g.7

$\mathbb{R}$ $\mathbb{R}$

Show that the function k : R → R given by $k(x) = 4x + 3$ is bijective.



$$k(a) = k(b)$$
$$4a + 3 = 4b + 3$$
$$4a = 4b$$
$$a = b$$

$\therefore k$ is injective.
Let $c \in \mathbb{R}$, $k(x) = c$

$$4x + 3 = c$$
$$4x = c - 3$$
$$x = \frac{c-3}{4}$$

This mean for any value in $\mathbb{R}$, we can find a value in x.

$$k(\mathbb{R}) = \mathbb{R}$$

$\therefore k$ is surjective.
$\therefore k$ is **bijective.**

5.3 Functions for Computer Science

- Let A be a subset of the universal set $U = \{u_1, u_2, u_3, \ldots, u_n\}$. The characteristics function of A is defined as a function from U to $\{0, 1\}$ by the following:
  - $f_A(u_i) = \begin{cases} 1 & \text{if } u_i \in A \\ 0 & \text{if } u_i \notin A \end{cases}$
  - If $A = \{4, 7, 9\}$ and $U = \{1, 2, 3, \ldots, 10\}$, then $f_A(2) = 0$, $f_A(4) = 1$, $f_A(7) = 1$, and $f_A(12)$ is undefined.
  - f_A is everywhere defined and onto, but not one-to-one.

5.3 Functions for Computer Science (cont)

- A Family of mod-n functions, one for each positive integer n, $f_n(m) = m \pmod{n}$, is a function from the nonnegative integers to the set $\{0, 1, 2, 3, ..., n - 1\}$.
  - For a fixed n, any nonnegative integer z can be written as $z = kn + r$ with $0 \leq r < n$.

    Then $f_n(z) = r$, which can also written as $z \equiv r \pmod{n}$.
  - Each member of the mod function family is everywhere defined and onto, but not one-to-one.

5.3 Functions for Computer Science (cont)

- The floor function, which is defined for rational numbers as $f(q) = \lfloor q \rfloor$ is the largest integer less than or equal to q.
  - E.g. $\lfloor 1.5 \rfloor = 1$, $\lfloor -3 \rfloor = -3$, $\lfloor -2.7 \rfloor = -3$
- The ceiling function, which is defined for rational numbers as $c(q) = \lceil q \rceil$ is the smallest integer greater than or equal to q.
  - E.g. $\lceil 1.5 \rceil = 2$, $\lceil -3 \rceil = -3$, $\lceil -2.7 \rceil = -2$

5.3 Functions for Computer Science (cont)

- Many common algebraic functions are used in computer science, often with domains restricted to subsets of integers.
  - Any polynomial with integer coefficients, p, can be used to define a function on Z as follows:

    If $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and $z \in$ Z, then $f(z)$ is the value of p evaluated at z.

5.3 Functions for Computer Science (cont)

- Let $A = B = Z^+$ and let $f : A \to B$ be defined by $f(z) = 2^z$, which is called as base 2 exponential function. Other bases can be used to define similar functions.
- Let $A = B = R$ and let $f_n : A \to B$ be defined for each positive integer $n > 1$ as $f_n(x) = \log_n x$, the logarithm to the base n of x. Usually bases 2 and 10 are used.

5.3 Functions for Computer Science (cont)

- The domains and codomains in the function need not be sets of numbers.
  - Let A be a finite set and define l: $A \to Z$ as $l(w)$ is the length of the string w.
  - Let B be a finite subset of the universal set U and define $pow(B)$ to be the power set of B. Then pow is a function from V, the power set of U, to the power set of V.

5.3 Functions for Computer Science (cont)

- Let $A = B =$ the set of 2×2 matrices with real number entries and let $t(\mathbf{M}) = \mathbf{M}^T$, the transpose of **M**. Then t is everywhere defined, onto, and one-to-one.
- For elements of $Z^+ \times Z^+$, $g(z_1, z_2)$ is defined to be GCD (z_1, z_2) and $m(z_1, z_2)$ to be LCM (z_1, z_2). Then g and m are a function from $Z^+ \times Z^+$ to Z^+.

5.3 Functions for Computer Science (cont)

- A Boolean function is a set from A to B, where B = {True, False}.

  Let $P(x)$: x is even and $Q(y)$: y is odd.

  Then P and Q are functions from Z to B.

  The predicate $R(x, y)$: x is even or y is odd is a Boolean function of two variables from Z $\times$ Z to B.

5.4 Permutations

- A bijection from a set A to itself.
  - E.g. Let $A = \{1, 2, 3\}$. Then all the permutations of A are

(1,2), (2,1), (3,3)

$$I_A \overset{A}{\underset{A}{=}} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$3! = 3 \cdot 2 \cdot 1 = 6$$

5.4 Permutations (cont)

- Theorem 1

  If $A = \{a_1, a_2, \ldots, a_n\}$ is a set containing n elements, then there are $n!$ permutations on A.

- The composition of two permutations, $p_1 \circ p_2$ is another permutation, usually referred to as the product of these permutations.

5.4 Permutations (cont)

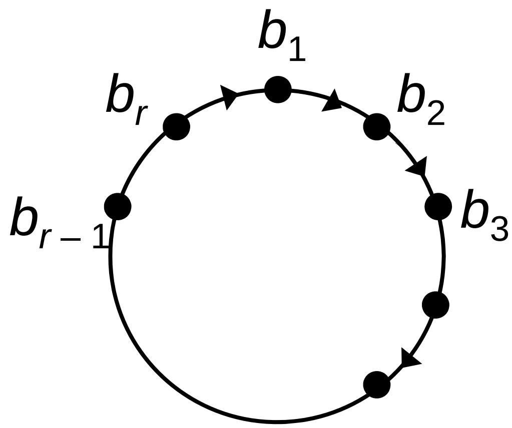
- Let $b_1, b_2, \ldots, b_r$ be r distinct elements of the set $A = \{a_1, a_2, \ldots, a_n\}$. The permutation $p: A \to A$ defined by

$$p(b_1) = b_2$$
$$p(b_2) = b_3$$
$$\vdots$$
$$p(b_{r-1}) = b_r$$
$$p(b_r) = b_1$$
$$p(x) = x, \text{ if } x \in A, x \notin \{b_1, b_2, \ldots, b_r\}$$

is called a cyclic permutation of length r, or a cycle of length r, and denoted by $(b_1, b_2, \ldots, b_r)$.

5.4 Permutations (cont)

- The cyclic permutation p can be written by starting with any b_i, $1 \le i \le r$, and moving in a clockwise direction.



- E.g. Let $A = \{1, 3, 5\}$. The cycle $(1, 3, 5)$ denotes the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$.

5.4 Permutations (cont)

- The notation for a cycle does not indicate the number of elements in the set A. Thus (3, 2, 1, 4) could be a permutation of the set {1, 2, 3, 4} or of {1, 2, 3, 4, 5, 6, 7, 8}.
- The cycle on a set A is of length 1 if and only if it is the identity permutation, I_A.

5.4 Permutations (cont)

- Two cycles of a set *A* are said to be disjoint if no element of *A* appears in both cycles.
  - E.g. Let *A* = {1, 2, 3, 4, 5, 6}. Then the cycles (1, 2, 5) and (3, 4, 6) are disjoint whereas (1, 2, 5) and (2, 4, 6) are not.
- Theorem 2

  A permutation of a finite set that is not the identity or a cycle can be written as a product of disjoint cycles of length ≥ 2.

5.4 Permutations (cont)

- When a permutation is written as a product of disjoint cycles, the product is unique except for the order of the cycles.

E.g.8

Let A = {1, 2, 3} and $p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

Find

i. p_1^{-1} 1 go to 2, 2 go to 3, 3 go to 1 $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

ii. $p_3 \circ p_2$ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$= (1 \; 3)$

cycle form

E.g.9

Let $A = \{1, 2, 3, 4, 5, 6\}$. Compute

i. $(4, 1, 3, 5) \circ (5, 6, 3)$

STA

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 4 & 6 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 4 & 1 & 6 & 5 \end{pmatrix} = (1,3,4) \circ (5,6) \ \#$$

ii. $(5, 6, 3) \circ (4, 1, 3, 5)$.

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 4 & 6 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 3 & 1 & 4 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 1 & 4 & 3 \end{pmatrix} = (1,5,4) \circ (3,6)$$

E.g.10

Write the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 6 & 5 & 2 & 1 & 8 & 7 \end{pmatrix}$

of the set A = {1, 2, 3, 4, 5, 6, 7, 8} as a product of disjoint cycles.

(1, 3, 6) o (2,4,5) o (7,8)

5.4.1 Even and Odd Permutations

- A cycle of length 2 is called a transposition, that is $p = (a_i, a_j)$, where $p(a_i) = a_j$ and $p(a_j) = a_i$.
- If $p = (a_i, a_j)$ is a transposition of A, then $p \circ p = I_A$, the identity permutation of A.
- Every cycle can be written as a product of transpositions. In fact

  $(b_1, b_2, \ldots, b_{k+1})$

  $= (b_1, b_{k+1}) \circ (b_1, b_k) \circ \ldots \circ (b_1, b_3) \circ (b_1, b_2)$.

  - $(1, 2, 3, 4, 5) = (1, 5) \circ (1, 4) \circ (1, 3) \circ (1, 2)$

5.4.1 Even and Odd Permutations (cont)

- Corollary 1

  Every permutation of a finite set with at least two elements can be written as a product of transpositions which need not be disjoint.

- Every cycle can be written as a product of transpositions in many different ways.
  - $(1, 2, 3) = (1, 3)\circ(1, 2) = (2, 1)\circ(2, 3)$ even

    $= (1, 3)\circ(3, 1)\circ(1, 3)\circ(1, 2)\circ(3, 2)\circ(2, 3)$

all are even

5.4.1 Even and Odd Permutations (cont)

- Theorem 3

  A permutation of a finite set can be written as a product of an even number of transpositions, then it can never be written as a product of an odd number of transpositions, and conversely.

5.4.1 Even and Odd Permutations (cont)

- A permutation of a finite set is called even if it can be written as a product of an even number of transpositions, and it is called odd if it can be written as a product of an odd number of transpositions.
  - The product of two even permutations is even.
  - The product of two odd permutations is even.
  - The product of an even and an odd permutation is odd.

5.4.1 Even and Odd Permutations (cont)

- Theorem 4

  Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite set with n elements, $n \geq 2$. There are $\frac{n!}{2}$ even permutations and $\frac{n!}{2}$ odd permutations.

E.g.11

Write the permutation in E.g.10 as a product of transpositions and determine whether it is an even or an odd permutation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 6 & 5 & 2 & 1 & 8 & 7 \end{pmatrix}$$

$$= (1,3,6) \circ (2,4,5) \circ (7,8)$$

$$= (1,6) \circ (1,3) \circ (2,5) \circ (2,4) \circ (7,8)$$

5 transpositions

therefore, it is an odd permutation

E.g.12

Determine whether the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 5 & 7 & 6 & 3 & 1 \end{pmatrix}$$

is even or odd.

$= (1,2,4,7) \circ (3,5,6)$

$= (1,7) \circ (1,4) \circ (1,2) \circ (3,6) \circ (3,5)$

5 permutations

therefore, it is an odd permutations