Calc 2: Tutorial 5

November 28, 2019

1. Solve the following differential equations: (CHECK WITH LECTURER)

   (a) $x^2 \frac{dy}{dx} + xy = x + 1$

      i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$
\begin{aligned}
x^2 \frac{dy}{dx} + xy &= x + 1 \\
\frac{1}{x^2}\left[x^2 \frac{dy}{dx} + xy\right] &= \frac{1}{x^2}\left[x + 1\right] \\
\boldsymbol{\frac{dy}{dx} + \frac{y}{x}} &\boldsymbol{= \frac{1}{x} + \frac{1}{x^2}, p\left(x\right) = \frac{1}{x}}
\end{aligned}
$$

      ii. Find the integrating factor $\mu\left(x\right) = e^{\int p(x)dx}$.

$$
\begin{aligned}
\mu\left(x\right) &= e^{\int \frac{1}{x}dx} \\
&= e^{\ln x} \\
\boldsymbol{\mu\left(x\right)} &\boldsymbol{= x}
\end{aligned}
$$

      iii. Multiply everything in the D.E. by $\mu\left(x\right)$, and verify that the left side becomes the product rule $\frac{d}{dx}\left[\mu\left(x\right)y\right]$ and write it as such.

$$
\begin{aligned}
x\left[\frac{dy}{dx} + \frac{y}{x}\right] &= x\left[\frac{1}{x} + \frac{1}{x^2}\right] \\
x\frac{dy}{dx} + y &= 1 + \frac{1}{x} \\
\frac{d}{dx}\left[yx\right] &= 1 + \frac{1}{x} \\
\int \frac{d}{dx}\left[yx\right]dx &= \int 1 + \frac{1}{x}dx \\
yx &= x + \ln x + c \\
y &= 1 + \frac{\ln x + c}{x} \\
&= 1 + \frac{\ln x}{x} + \frac{c}{x}
\end{aligned}
$$

(b) $\frac{dy}{dx} + y \cot x = \csc x$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

A. Already in correct standard form, $p(x) = \cot x$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$

$$\begin{aligned}\mu(x) &= e^{\int \cot x \, dx} \\ &= e^{\ln|\sin x|} \\ &= \sin x\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x) y]$ and write it as such.

$$\begin{aligned}\sin x\left[\frac{dy}{dx} + y\cot x\right] &= \sin x\left[\csc x\right] \\ \sin x\frac{dy}{dx} + y\sin x \cdot \frac{\cos x}{\sin x} &= \sin x\left[\frac{1}{\sin x}\right] \\ \sin x\frac{dy}{dx} + y\cos x &= 1 \\ \sin x\frac{dy}{dx} + y\cos x &= 1 \\ \frac{d}{dx}\left[y\sin x\right] &= 1 \\ \int \frac{d}{dx}\left[y\sin x\right]dx &= \int 1dx \\ y\sin x + c &= x + c \\ y &= \frac{x+c}{\sin x}\end{aligned}$$

(c) $x\frac{dy}{dx} = y - x^2e^{-x}$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned}x\frac{dy}{dx} &= y - x^2e^{-x} \\ \frac{dy}{dx} - \frac{y}{x} &= -xe^{-x}\end{aligned}$$

A. From the above, $p(x) = -\frac{1}{x}$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= e^{\ln x^{-1}} \\ \mu &= \frac{1}{x}\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)\,y]$ and write it as such.

$$\begin{aligned}
\frac{1}{x}\left[\frac{dy}{dx}-\frac{y}{x}\right] &= \frac{1}{x}\left[-xe^{-x}\right]\\
\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y &= -e^{-x}\\
\frac{d}{dx}\left[x^{-1}y\right] &= -e^{-x}\\
\int \frac{d}{dx}\left[x^{-1}y\right]dx &= -\int e^{-x}dx\\
x^{-1}y &= -(-1)\,e^{-x}+c\\
y &= x\left(e^{-x}+c\right)\\
y &= e^{-x}x+cx
\end{aligned}$$

(d) $x\frac{dy}{dx}+2y=\frac{\sin x}{x}$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{dy}{dx}+\frac{2y}{x}=\frac{\sin x}{x^2}$$

ii. Find the integrating factor $\mu(x)=e^{\int p(x)dx}$.

$$\begin{aligned}
\mu(x) &= e^{\int \frac{2}{x}dx}\\
&= e^{2\ln x}\\
\mu(x) &= x^2
\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)\,y]$ and write it as such.

$$\begin{aligned}
x^2\left[\frac{dy}{dx}+\frac{2y}{x}\right] &= x^2\left[\frac{\sin x}{x^2}\right]\\
x^2\frac{dy}{dx}+2xy &= \sin x\\
\frac{d}{dx}\left[x^2y\right] &= \sin x\\
\int \frac{d}{dx}\left[x^2y\right]dx &= \int \sin x\,dx+c\\
x^2y &= -\cos x+c\\
y &= \frac{c-\cos x}{x^2}
\end{aligned}$$

(e) $x\frac{dy}{dx}+3y=4x+3$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{1}{x}\left[x\frac{dy}{dx}+3y\right]=\frac{1}{x}\left[4x+3\right]$$
$$\frac{dy}{dx}+\frac{3y}{x}=4+\frac{3}{x}$$

A. From above $p\left(x\right)=\frac{3}{x}$

ii. Find the integrating factor $\mu\left(x\right)=e^{\int p(x)dx}$.

$$\begin{aligned}\mu\left(x\right)&=e^{\int\frac{3}{x}dx}\\&=e^{3\int\frac{1}{x}dx}\\&=e^{3\ln x}\\&=e^{\ln x^3}\\\mu\left(x\right)&=x^3\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu\left(x\right)$, and verify that the left side becomes the product rule $\frac{d}{dx}\left[\mu\left(x\right)y\right]$ and write it as such.

$$\begin{aligned}x^3\left[\frac{dy}{dx}+\frac{3y}{x}\right]&=x^3\left[4+\frac{3}{x}\right]\\x^3\frac{dy}{dx}+3x^2y&=4x^3+3x^2\\\frac{d}{dx}\left[x^3y\right]&=4x^3+3x^2\\\int\frac{d}{dx}\left[x^3y\right]dx&=\int\left(4x^3+3x^2\right)dx\\x^3y&=\frac{4x^4}{4}+\frac{3x^3}{3}+c\\x^3y&=x^4+x^3+c\\y&=\frac{x^4}{x^3}+\frac{x^3}{x^3}+\frac{c}{x^3}\\y&=x+1+\frac{c}{x^3}\end{aligned}$$

(f) $\frac{dy}{dx}=2y+e^{3x}$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned}\frac{dy}{dx}&=2y+e^{3x}\\\frac{dy}{dx}-2y&=e^{3x},p\left(x\right)=-2\end{aligned}$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}
\mu(x) &= e^{\int -2dx} \\
&= e^{\int -2dx} \\
&= e^{-\int 2dx} \\
\mu(x) &= e^{-2x}
\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\begin{aligned}
e^{-2x}\left[\frac{dy}{dx} - 2y\right] &= e^{-2x}\left[e^{3x}\right] \\
e^{-2x}\frac{dy}{dx} - 2e^{-2x}y &= e^{x} \\
\frac{dy}{dx}\left[e^{-2x}y\right] &= e^{x} \\
\int \frac{dy}{dx}\left[e^{-2x}y\right]dx &= \int e^{x}dx \\
e^{-2x}y &= e^{x} + c \\
y &= \frac{e^{x} + c}{e^{-2x}} \\
&= e^{3x} + ce^{-(-2x)} \\
y &= e^{3x} + ce^{2x}
\end{aligned}$$

(g) $x\frac{dy}{dx} = 2y + x^3 \ln x$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2y}{x} + x^2 \ln x \\
\frac{dy}{dx} - \frac{2y}{x} &= x^2 \ln x \\
\frac{dy}{dx} - \frac{2y}{x} &= x^2 \ln x, p(x) = -\frac{2}{x}
\end{aligned}$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}
\mu(x) &= e^{\int p(x)dx} \\
\mu(x) &= e^{\int -2x^{-1}dx} \\
&= e^{-2\int \frac{1}{x}dx} \\
&= e^{\ln x^{-2}dx} \\
&= x^{-2} \\
\mu(x) &= \frac{1}{x^2}
\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\frac{1}{x^2}\left[\frac{dy}{dx} - \frac{2y}{x}\right] = \frac{1}{x^2}\left[x^2 \ln x\right]$$
$$x^{-2}\frac{dy}{dx} - 2x^{-3}y = \ln x$$
$$\frac{dy}{dx}\left[x^{-2}y\right] = \ln x$$
$$\int \frac{dy}{dx}\left[x^{-2}y\right] dx = \int \ln x \, dx$$

A. Time to use Calc 1 skills, Integration by parts

$$\int u \, dv = uv - \int v du$$

Let $u = \ln x$ and $v' = 1$
$\frac{du}{dx} = \frac{1}{x}$, $v = x$

B. Finally we get

$$\begin{aligned}\int \ln x \, dx &= x \ln x - \int x \frac{1}{x} dx \\ &= x \ln x - \int 1 dx \\ &= x \ln x - x + c\end{aligned}$$

C. Continue plugging in

$$\begin{aligned}\int \frac{dy}{dx}\left[x^{-2}y\right] dx &= x \ln x - x + c \\ x^{-2}y &= x \ln x - x + c \\ y &= x^3 \ln x - x^3 + cx^2\end{aligned}$$

(h) $x\frac{dy}{dx} - 3y = x^4$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$x\frac{dy}{dx} - 3y = x^4$$
$$\frac{dy}{dx} - \frac{3y}{x} = x^3$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}
\mu(x) &= e^{\int -\frac{3}{x}dx} \\
&= e^{-3\int \frac{1}{x}dx} \\
&= e^{-3\ln x} \\
&= e^{\ln x^{-3}} \\
\mu(x) &= x^{-3}
\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\begin{aligned}
x^{-3}\left[\frac{dy}{dx} - \frac{3y}{x}\right] &= x^{-3}\left[x^3\right] \\
x^{-3}\frac{dy}{dx} - 3x^{-4}y &= 1 \\
\frac{d}{dx}\left[x^{-3}y\right] &= 1 \\
\int \frac{d}{dx}\left[x^{-3}y\right]dx &= \int 1dx \\
x^{-3}y &= x + c \\
y &= \frac{x}{x^{-3}} + \frac{c}{x^{-3}} \\
y &= x^4 + cx^3
\end{aligned}$$

2. Solve the following I.V.P.:

(a) $\frac{dy}{dx} = 9.8 - 0.196y, y(0) = 48$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{dy}{dx} + 0.196y = 9.8$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}
\mu(x) &= e^{\int 0.196dx} \\
&= e^{0.196\int 1dx} \\
&= e^{0.196x}
\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}\left[\mu\left(x\right)y\right]$ and write it as such.

$$\begin{aligned}
e^{0.196x}\left[\frac{dy}{dx}+0.196y\right] &= e^{0.196x}\left[9.8\right] \\
e^{0.196x}\frac{dy}{dx}+e^{0.196x}0.196y &= 9.8e^{0.196x} \\
\frac{d}{dx}\left[ye^{0.196x}\right] &= 9.8e^{0.196x} \\
\int\frac{d}{dx}\left[ye^{0.196x}\right]dx &= \int\left(9.8e^{0.196x}\right)dx \\
ye^{0.196x} &= 9.8\cdot\left(\frac{1}{0.196}e^{0.196x}+c\right) \\
ye^{0.196x} &= \frac{9.8}{0.196}e^{0.196x}+9.8c \\
y &= \frac{9.8}{0.196\cdot e^{0.196x}}e^{0.196x}+\frac{9.8c}{e^{0.196x}} \\
&= \frac{9.8}{0.196}+\frac{9.8c}{e^{0.196x}} \\
y &= 50+\frac{9.8c}{e^{0.196x}}
\end{aligned}$$

iv. Plug in the values given, and solve for c

$$\begin{aligned}
y\left(0\right) &= 50+\frac{9.8c}{e^{0.196(0)}} \\
48 &= 50+\frac{9.8c}{1} \\
c &= \frac{48-50}{9.8} \\
&= -\frac{10}{49}
\end{aligned}$$

v. Substitute c back into the original equation

$$\begin{aligned}
y &= 50+\frac{9.8\left(-\frac{10}{49}\right)}{e^{0.196x}} \\
&= 50+\frac{9.8\left(-\frac{10}{49}\right)}{e^{0.196x}} \\
&= 50-\frac{2}{e^{0.196x}} \\
y &= 50-2e^{-0.196x}
\end{aligned}$$

(b) $x\frac{dy}{dx}+y=\frac{x}{x+1}, y\left(1\right)=1$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{dy}{dx}+\frac{y}{x}=\frac{1}{x+1}$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}\mu(x) &= e^{\int \frac{1}{x}dx} \\ &= e^{\ln x} \\ \mu(x) &= x\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)\, y]$ and write it as such.

$$\begin{aligned}x\left[\frac{dy}{dx} + \frac{y}{x}\right] &= x\left[\frac{1}{x+1}\right] \\ x\frac{dy}{dx} + y &= \frac{x}{x+1} \\ \frac{d}{dx}[xy] &= \frac{x}{x+1} \\ \int \frac{d}{dx}[xy]\, dx &= \int \frac{x}{x+1}dx \\ xy &= \int x(x+1)^{-1}\, dx\end{aligned}$$

A. On first sight, integration by parts seems to be pretty good, but if you try it out you're in for a world of pain. A combination of long division and u substitution works best in this case. Because:
   1. Numerator and denominator same power

B. Do long division

$$\begin{aligned}\int \frac{x}{(x+1)}dx &= \int 1 - \frac{1}{x+1}dx \\ &= \int 1dx - \int \frac{1}{x+1}dx \\ \int \frac{x}{(x+1)}dx &= x - \ln(x+1) + c\end{aligned}$$

C. Move everything to the right place

$$\begin{aligned}xy &= x - \ln(x+1) + c \\ y &= \frac{x - \ln(x+1) + c}{x} \\ &= 1 - \frac{1}{x}\ln(x+1) + \frac{c}{x}\end{aligned}$$

iv. Plug in the values given, and solve for c

$$\begin{aligned}y(1) &= 1 - \frac{1}{1}\ln(1+1) + \frac{c}{1} \\ 1 &= 1 - \ln(2) + c \\ c &= \ln(2)\end{aligned}$$

v. Substitute c back into the original equation

$$\begin{aligned} y &= 1 - \frac{1}{x}\ln(x+1) + \frac{1}{x}\ln(2) \\ &= 1 - \frac{1}{x}(\ln(x+1) - \ln(2)) \\ &= 1 - \frac{1}{x}\ln\frac{x+1}{2} \\ &= 1 + \frac{1}{x}\ln\left(\frac{x+1}{2}\right)^{-1} \\ y &= 1 + \frac{1}{x}\ln\frac{2}{x+1} \end{aligned}$$

(c) $x\frac{dy}{dx} + 3y = x^2 - 4x + 3, y(1) = 0$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1

$$\frac{dy}{dx} + \frac{3}{x}y = x - 4 + \frac{3}{x}$$

.

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned} \mu(x) &= e^{\int p(x)dx} \\ \mu(x) &= e^{\int \frac{3}{x}dx} \\ &= e^{3\ln x} \\ &= e^{\ln x^3} \\ \mu(x) &= x^3 \end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\begin{aligned} \frac{dy}{dx} + \frac{3}{x}y &= x - 4 + \frac{3}{x} \\ x^3\left[\frac{dy}{dx} + \frac{3}{x}y\right] &= x^3\left[x - 4 + \frac{3}{x}\right] \\ x^3\frac{dy}{dx} + 3x^2y &= x^4 - 4x^3 + 3x^2 \\ \frac{d}{dx}\left[x^3y\right] &= x^4 - 4x^3 + 3x^2 \\ \int \frac{d}{dx}\left[x^3y\right]dx &= \int x^4 - 4x^3 + 3x^2dx \\ x^3y &= \frac{x^5}{5} - x^4 + x^3 + c \\ y &= \frac{x^2}{5} - x + 1 + \frac{c}{x^3} \end{aligned}$$

iv. Plug in the values given, and solve for c

$$\begin{aligned} y(1) &= \frac{1^2}{5} - 1 + 1 + \frac{c}{1^3} \\ 0 &= \frac{1}{5} + c \\ c &= -\frac{1}{5} \end{aligned}$$

v. Substitute c back into the original equation

$$y = \frac{x^2}{5} - x + 1 - \frac{1}{5x^3}$$

3. **Separable Equations:**
   Solve the following differential equations:

   (a) $\frac{dy}{dx} = \frac{e^{2x}}{4y^3}$

   i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\begin{aligned} \frac{dy}{dx} &= \frac{e^{2x}}{4y^3} \\ 4y^3 \frac{dy}{dx} &= e^{2x} \end{aligned}$$

   ii. Integrate both sides

$$\begin{aligned} \int 4y^3 \frac{dy}{\cancel{dx}} \cancel{dx} &= \int e^{2x} dx \\ 4 \int y^3 dy &= \int e^{2x} dx \\ 4\left(\frac{y^4}{4} + c\right) &= \frac{1}{2} e^{2x} + c \\ y^4 + c &= \frac{1}{2} e^{2x} + c \end{aligned}$$

   iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

   A. DO NOT forget to include C, constant of integration

$$\begin{aligned} y^4 &= \frac{1}{2} e^{2x} + c \\ y &= \pm \sqrt[4]{\frac{1}{2} e^{2x} + c} \end{aligned}$$

   (b) $\frac{dy}{dx} = \frac{xy}{2 \ln y}$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\frac{dy}{dx} = \frac{xy}{2\ln y}$$
$$2\ln y \frac{dy}{dx} = xy$$
$$\frac{2}{y}\frac{dy}{dx}\ln y = x$$

ii. Integrate both sides

$$\int \frac{2}{y}\frac{dy}{dx}\ln y\,dx = \int x\,dx$$
$$2\int \frac{1}{y}\ln y\,dy = \frac{x^2}{2} + c$$
$$2\left(\frac{\ln^2 y}{2} + c\right) = \frac{x^2}{2} + c \text{check below for elaboration}$$
$$\ln^2 y + c = \frac{x^2}{2} + c$$

A. Use $u - substitution$, let $u = \ln y$

$$\frac{du}{dy} = \frac{1}{y}$$
$$du = \frac{1}{y}dy$$

$$\int \frac{1}{y}\ln y\,dy = \int u\,du$$
$$= \frac{u^2}{2} + c$$
$$= \frac{(\ln y)^2}{2} + c$$
$$\int \frac{1}{y}\ln y\,dy = \frac{\ln^2 y}{2} + c$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

B.

$$\ln^2 y + c = \frac{x^2}{2} + c$$
$$\ln y = \pm\sqrt{\frac{x^2}{2} + c}$$
$$y = e^{\pm\sqrt{\frac{x^2}{2}+c}}$$

(c) $\frac{dx}{dt} + e^{t+x} = 0$
Before-we-start: This question is a lil' tricky. Usually, we have x as our parameter, but this has t as parameter. Therefore, we need to make it $x(t) = ...$ instead.

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\begin{aligned}\frac{dx}{dt} + e^{t+x} &= 0\\ \frac{dx}{dt} &= -e^{t+x}\\ &= -e^{t} \cdot e^{x}\\ \frac{1}{e^{x}} dx &= -e^{t} dt\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}\int e^{-x} dx &= -\int e^{t} dt\\ -e^{-x} + c &= -\left(e^{t} + c\right)\\ -e^{-x} &= -e^{t} - c\\ e^{-x} &= e^{t} + c\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)
   A. DO NOT forget to include C, constant of integration

$$\begin{aligned}\ln e^{-x} &= \ln\left(e^{t} + c\right)\\ -x &= \ln\left(e^{t} + c\right)\\ x &= -\ln\left(e^{t} + c\right)\end{aligned}$$

(d) $2\sqrt{xy}\frac{dy}{dx} = 1, x, y > 0$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\begin{aligned}2\sqrt{xy}\frac{dy}{dx} &= 1\\ \sqrt{xy} dy &= \frac{1}{2} dx\\ \sqrt{x}\sqrt{y} dy &= \frac{1}{2} dx\\ y^{\frac{1}{2}} dy &= \frac{1}{2} x^{-\frac{1}{2}} dx\end{aligned}$$

ii. Integrate both sides

$$\int y^{\frac{1}{2}}\,dy = \int \frac{1}{2}x^{-\frac{1}{2}}\,dx$$

$$\frac{y^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{1}{2}\left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c\right)$$

$$\frac{2}{3}y^{\frac{3}{2}} + c = \frac{1}{2}\left(2x^{\frac{1}{2}} + c\right)$$

$$\frac{2}{3}y^{\frac{3}{2}} + c = x^{\frac{1}{2}} + c$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

$$\frac{2}{3}y^{\frac{3}{2}} = x^{\frac{1}{2}} + c$$

$$y^{\frac{3}{2}} = \frac{3}{2}\left(x^{\frac{1}{2}} + c\right)$$

$$y^{\frac{3}{2}} = \frac{3}{2}\sqrt{x} + c$$

(e) $\frac{dy}{dx} = \frac{xy}{x+2}$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\frac{dy}{dx} = \frac{xy}{x+2}$$

$$\frac{1}{xy}dy = \frac{1}{x+2}dx$$

$$\frac{1}{x}\cdot\frac{1}{y}dy = \frac{1}{x+2}dx$$

$$\frac{1}{y}dy = \frac{x}{x+2}dx$$

$$y^{-1}dy = \frac{x}{x+2}dx$$

ii. Integrate both sides

$$\int y^{-1}dy = \int \frac{x}{x+2}dx$$

A. For first part,

$$\int y^{-1}dy = \ln y + c$$

B. For second part, ideally you want to use long division + u-substitution. Just something learned from above.

$$\begin{aligned}\int \frac{x}{x+2} dx &= \int 1 - \frac{2}{x+2} dx \\ &= \int 1 dx - 2 \int \frac{1}{x+2} dx\end{aligned}$$

Let $u = x + 2$,

$$\begin{aligned}\frac{du}{dx} &= 1 \\ du &= dx\end{aligned}$$

$$\begin{aligned}\int 1 dx - 2 \int \frac{1}{x+2} dx &= x - 2 \int \frac{1}{u} du + c \\ &= x - 2 \ln u + c \\ &= x - 2 \ln (x+2) + c\end{aligned}$$

C. Find the final integration

$$\int y^{-1} dy = \int \frac{x}{x+2} dx$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

$$\begin{aligned}\ln y + c &= x - 2 \ln (x+2) + c \\ \ln y &= x - \ln (x+2)^2 + c \\ y &= e^{x - \ln (x+2)^2 + c} \\ &= \frac{e^{x+c}}{e^{\ln (x+2)^2}} \\ &= \frac{e^x \cdot e^c}{(x+2)^2} \\ y &= \frac{Ae^x}{(x+2)^2}, A = e^c\end{aligned}$$

(f) $y\left(x^2 - 1\right) \frac{dy}{dx} = 1$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\begin{aligned}
y\left(x^2-1\right)\frac{dy}{dx} &= 1\\
\left(x^2-1\right)\frac{dy}{dx} &= \frac{1}{y}\\
\frac{1}{\left(x^2-1\right)}\frac{dx}{dy} &= y\\
\frac{1}{\left(x^2-1\right)}dx &= ydy
\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}
\int \frac{1}{\left(x^2-1\right)}dx &= \int ydy\\
\int \frac{1}{\left(x^2-1\right)}dx &= \frac{y^2}{2}+c\\
\int \frac{1}{(x-1)(x+1)}dx &= \frac{y^2}{2}+c\\
\int \frac{1}{2(x-1)}-\frac{1}{2(x+1)}dx &= \frac{y^2}{2}+c\\
\frac{1}{2}\left(\int \frac{1}{(x-1)}dx-\int \frac{1}{(x+1)}dx\right) &= \frac{y^2}{2}+c\\
\int \frac{1}{(x-1)}dx-\int \frac{1}{(x+1)}dx &= y^2+c\\
\ln(x-1)-\ln(x+1) &= y^2+c
\end{aligned}$$

A. Partial fractions

$$\begin{aligned}
\frac{1}{(x-1)(x+1)} &= \frac{A}{(x-1)}+\frac{B}{(x+1)}\\
1 &= A(x+1)+B(x-1)
\end{aligned}$$

B. When $x=-1$

$$\begin{aligned}
1 &= B(-2)\\
B &= -\frac{1}{2}
\end{aligned}$$

C. When $x=1$

$$\begin{aligned}
1 &= A(2)\\
A &= \frac{1}{2}
\end{aligned}$$

D. Therefore

$$\frac{1}{(x-1)(x+1)} = \frac{1}{2(x-1)}-\frac{1}{2(x+1)}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

$$\ln \frac{(x-1)}{(x+1)} = y^2 + c$$
$$y^2 = \ln \frac{(x-1)}{(x+1)} - 2c$$
$$y = \sqrt{\ln \frac{(x-1)}{(x+1)} + c}$$

B. Let $A = e^c$

$$y = \sqrt{\ln \frac{(x-1)}{(x+1)} + \ln A}$$
$$y = \sqrt{\ln \frac{A(x-1)}{(x+1)}}$$

(g) $\frac{dx}{dt} = \frac{4\sin t + 6\cos 2t}{x}$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\frac{dx}{dt} = \frac{4\sin t + 6\cos 2t}{x}$$
$$xdx = 4\sin t + 6\cos 2t\,dt$$

ii. Integrate both sides

$$\int xdx = \int 4\sin t + 6\cos 2t\,dt$$
$$\frac{x^2}{2} + c = 4(-\cos t + c) + 6\int \cos 2t\,dt$$
$$= 4(-\cos t + c) + 6\left(\frac{1}{2}\sin(2t) + c\right)$$
$$\frac{x^2}{2} + c = -4\cos t + 3\sin(2t) + c$$

A. To integrate $\cos 2t$, let $u = 2t$

$$\frac{du}{dt} = 2$$
$$du = 2dt$$

$$\int \cos 2t dt = \frac{1}{2} \int \cos u \, du$$
$$= \frac{1}{2} (\sin u + c)$$
$$\int \cos 2t dt = \frac{1}{2} \sin (2t) + c$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)
   A. DO NOT forget to include C, constant of integration

$$\frac{x^2}{2} + c = -4 \cos t + 3 \sin (2t) + c$$
$$\frac{x^2}{2} = -4 \cos t + 3 \sin 2t + c$$
$$x^2 = -8 \cos t + 6 \sin 2t + c$$
$$x = \sqrt{-8 \cos t + 6 \sin 2t + c}$$

(h) $\frac{dy}{dx} = e^{-y} (2x - 4)$
   i. Write D.E. as $g(y) \, dy = f(x) \, dx$

$$\frac{1}{e^{-y}} dy = (2x - 4) \, dx$$
$$e^y dy = (2x - 4) \, dx$$

   ii. Integrate both sides

$$\int e^y dy = \int (2x - 4) \, dx$$
$$e^y + c = \frac{2x^2}{2} - 4x + c$$
$$e^y + c = x^2 - 4x + c$$

   iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)
   A. DO NOT forget to include C, constant of integration

$$e^y + c = x^2 - 4x + c$$
$$e^y = x^2 - 4x + c$$
$$\ln e^y = \ln \left(x^2 - 4x + c\right)$$
$$y = \ln \left(x^2 - 4x + c\right)$$

(i) $\sec x \frac{dy}{dx} = e^{y + \sin x}$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\begin{aligned}\cos x \frac{dx}{dy} &= \frac{1}{e^{y+\sin x}}\\ \cos x dx &= e^{-y} \cdot e^{-\sin x}\\ \frac{\cos x}{e^{-\sin x}} dx &= e^{-y} dy\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}\int e^{-y} dy &= \int \cos x \cdot e^{\sin x} dx\\ -e^{-y} + c &= \int \cos x \cdot e^{\sin x} dx\\ &= \int \cos x \cdot e^{\sin x} dx\\ e^{y} &= -e^{\sin x} + c \text{ (check below)}\end{aligned}$$

A. Integrating $\int \cos x \cdot e^{\sin x} dx$ probably needs $u-$integration. Let $u = \sin x$

$$\begin{aligned}\frac{du}{dx} &= \cos x\\ du &= \cos x dx\end{aligned}$$

B. Therefore,

$$\begin{aligned}\int \cos x e^{\sin x} dx &= \int e^{u} du\\ &= e^{u} + c\\ &= e^{\sin x} + c\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

$$\begin{aligned}e^{-y} &= c - e^{\sin x}\\ \ln e^{-y} &= \ln\left(c - e^{\sin x}\right)\\ y &= -\ln\left(c - e^{\sin x}\right)\end{aligned}$$

(j) $\frac{dy}{dx} = \frac{3x^2+4x-4}{2y-4}$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$2y - 4\, dy = 3x^2 + 4x - 4\, dx$$

ii. Integrate both sides

$$\int 2y - 4\,dy = \int 3x^2 + 4x - 4\,dx$$
$$y^2 - 4y + c = x^3 + 2x^2 - 4x + c$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

$$y^2 - 4y = x^3 + 2x^2 - 4x + c$$
$$y^2 - 4y - \left(x^3 + 2x^2 - 4x + c\right) = 0$$

B. $a = 1, b = -4, c = -\left(x^3 + 2x^2 - 4x + c\right)$

$$\begin{aligned} y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-(x^3 + 2x^2 - 4x + c))}}{2(1)} \\ &= \frac{4 \pm \sqrt{4^2 + 4(x^3 + 2x^2 - 4x + c)}}{2} \\ &= 2 \pm \frac{\sqrt{4(4 + x^3 + 2x^2 - 4x + c)}}{2} \\ &= 2 \pm \frac{2\sqrt{4 + x^3 + 2x^2 - 4x + c}}{2} \\ &= 2 \pm \sqrt{4 + x^3 + 2x^2 - 4x + c} \end{aligned}$$

4. Solve the following I.V.P.:

(a) $y' = \frac{y\cos x}{1+y^2}, y(0) = 1$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\frac{dy}{dx} = \frac{y\cos x}{1 + y^2}$$
$$\left(1 + y^2\right) dy = y\cos x\,dx$$
$$\frac{\left(1 + y^2\right)}{y} dy = \cos x\,dx$$

ii. Integrate both sides

$$\frac{\left(1+y^2\right)}{y}dy = \cos x\, dx$$
$$\int \frac{1}{y} + \frac{y^2}{y}dy = \int \cos x\, dx$$
$$\int \frac{1}{y}dy + \int ydy = \sin x + c$$
$$\ln y + \frac{y^2}{2} + c = \sin x + c$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$). Note in some cases (like this, it is impossible at our current level)

A. DO NOT forget to include C, constant of integration

$$\ln y + \frac{y^2}{2} + c = \sin x + c$$
$$\ln y + \frac{y^2}{2} = \sin x + c$$

iv. Substitute in the values given to find C

$$\ln 1 + \frac{1^2}{2} = \sin(0) + c$$
$$\frac{1}{2} = 0 + c$$
$$c = \frac{1}{2}$$

v. Substitute C back into the equation, to find the solution for the I.V.P.

$$\ln y + \frac{y^2}{2} = \sin x + \frac{1}{2}$$

(b) $x + 2y\sqrt{x^2+1}\frac{dy}{dx} = 0, y(0) = 1$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$x + 2y\sqrt{x^2+1}\frac{dy}{dx} = 0$$
$$2y\sqrt{x^2+1}\frac{dy}{dx} = -x$$
$$2y\frac{dy}{dx} = -\frac{x}{\sqrt{x^2+1}}$$
$$2ydy = -\frac{x}{\sqrt{x^2+1}}dx$$

ii. Integrate both sides

$$\begin{aligned}\int 2y dy &= \int -\frac{x}{\sqrt{x^2+1}} dx \\ y^2 + c &= -\int x\left(x^2+1\right)^{-\frac{1}{2}} dx \\ y^2 + c &= -\left(x^2+1\right)^{\frac{1}{2}} + c \text{ (Check below)}\end{aligned}$$

A. Integrate $-\int x\left(x^2+1\right)^{-\frac{1}{2}} dx$, I'm thinking about u-substitution. Let $u = x^2+1$

$$\begin{aligned}\frac{du}{dx} &= 2x \\ du &= 2x\,dx\end{aligned}$$

$$\begin{aligned}-\int x\left(x^2+1\right)^{-\frac{1}{2}} dx &= -\frac{1}{2}\int \left(x^2+1\right)^{-\frac{1}{2}} \cdot 2x\,dx \\ &= -\frac{1}{2}\int u^{-\frac{1}{2}} du \\ &= -\frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) \\ &= -u^{\frac{1}{2}} + c \\ -\int x\left(x^2+1\right)^{-\frac{1}{2}} dx &= -\left(x^2+1\right)^{\frac{1}{2}} + c\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

$$\begin{aligned}y^2 + c &= -\left(x^2+1\right)^{\frac{1}{2}} + c \\ y^2 &= -\left(x^2+1\right)^{\frac{1}{2}} + c\end{aligned}$$

iv. Substitute in the values given to find C, $y(0) = 1$

$$\begin{aligned}1^2 &= -\left((0)^2+1\right)^{\frac{1}{2}} + c \\ 1 &= -1 + c \\ c &= 2\end{aligned}$$

v. Substitute C back into the equation, to find the solution for the I.V.P.

$$y^2 = -\left(x^2+1\right)^{\frac{1}{2}} + 2$$

(c) $\frac{dy}{dt} = te^y, y(1) = 0$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\frac{1}{e^y}dy = t\,dt$$

ii. Integrate both sides

$$\int e^{-y}dy = \int t\,dt$$
$$-e^{-y} + c = \frac{t^2}{2} + c$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)
   A. DO NOT forget to include C, constant of integration

$$\begin{aligned}
-e^{-y} + c &= \frac{t^2}{2} + c \\
-e^{-y} &= \frac{t^2}{2} + c - c \\
e^{-y} &= -\frac{t^2}{2} + c \text{ (Const. - cons. = cons. (even 0))} \\
-y \ln e &= \ln\left(-\frac{t^2}{2} + c\right) \\
y &= -\ln\left(-\frac{t^2}{2} + c\right) \\
&= \ln \frac{1}{-\frac{t^2}{2} + c} \\
y &= \ln \frac{1}{-\frac{1}{2}t^2 + c}
\end{aligned}$$

iv. Substitute in the values given to find C

$$\begin{aligned}
0 &= \ln \frac{1}{-\frac{1}{2}1^2 + c} \\
\ln \frac{1}{-\frac{1}{2} + c} &= 0 \\
\frac{1}{-\frac{1}{2} + c} &= e^0 \\
-\frac{1}{2} + c &= 1 \\
c &= 1 + \frac{1}{2} \\
c &= \frac{3}{2}
\end{aligned}$$

v. Substitute C back into the equation, to find the solution for the I.V.P.

$$\begin{aligned} y &= \ln \frac{1}{-\frac{t^2}{2} + \frac{3}{2}} \\ &= \ln \frac{1}{\frac{-t^2+3}{2}} \\ y &= \ln \frac{2}{3 - t^2} \end{aligned}$$

(d) $t\left(t-1\right)\frac{dx}{dt} = x\left(x+1\right), x\left(2\right) = 2$

i. Write D.E. as $g\left(y\right)dy = f\left(x\right)dx$

$$\begin{aligned} t\left(t-1\right)\frac{dx}{dt} &= x\left(x+1\right) \\ \frac{dx}{x\left(x+1\right)} &= \frac{dt}{t\left(t-1\right)} \end{aligned}$$

ii. Integrate both sides

$$\begin{aligned} \int \frac{1}{x\left(x+1\right)} dx &= \int \frac{1}{t\left(t-1\right)} dt \\ \int \frac{1}{x\left(x+1\right)} dx &= \int \frac{1}{t\left(t-1\right)} dt \end{aligned}$$

A. For both parts, we need to use partial fractions, lets start with $\frac{1}{x(x+1)}$

$$\begin{aligned} \frac{1}{x\left(x+1\right)} &= \frac{A}{x} + \frac{B}{x+1} \\ &= \frac{A\left(x+1\right) + Bx}{x\left(x+1\right)} \\ \frac{x\left(x+1\right)}{x\left(x+1\right)} &= A\left(x+1\right) + Bx \\ 1 &= A\left(x+1\right) + Bx \end{aligned}$$

When $x = 0$

$$\begin{aligned} 1 &= A\left(1\right) + 0 \\ A &= 1 \end{aligned}$$

When $x = -1$

$$\begin{aligned} 1 &= A\left(-1+1\right) + B\left(-1\right) \\ 1 &= -B \\ B &= -1 \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{x(x+1)} &= \frac{A}{x} + \frac{B}{x+1}\\ &= \frac{1}{x} - \frac{1}{x+1}\end{aligned}$$

Start integrating:

$$\begin{aligned}\int \frac{1}{x} - \frac{1}{x+1}dx &= \int \frac{1}{x}dx - \int \frac{1}{x+1}dx\\ &= \ln(x) - \ln(x+1) + c\end{aligned}$$

B. Lets integrate the other one, $\int \frac{1}{t(t-1)}dt$
Again, partial fractions first.

$$\begin{aligned}\frac{1}{t(t-1)} &= \frac{A}{t} + \frac{B}{t-1}\\ 1 &= A(t-1) + Bt\end{aligned}$$

When $t = 1$

$$B = 1$$

When $t = 2, B = 1$

$$\begin{aligned}1 &= A(2-1) + 1(2)\\ 1 &= A + 2\\ A &= -1\end{aligned}$$

Substitute back in

$$\begin{aligned}\frac{1}{t(t-1)} &= \frac{A}{t} + \frac{B}{t-1}\\ &= -\frac{1}{t} + \frac{1}{t-1}\end{aligned}$$

Integrate:

$$\begin{aligned}\int -\frac{1}{t} + \frac{1}{t-1}dt &= \int -\frac{1}{t}dt + \int \frac{1}{t-1}dt\\ &= -\ln(t) + \ln(t-1) + c\end{aligned}$$

C. Combine the entire equation

$$\int \frac{1}{x\left(x+1\right)} dx = \int \frac{1}{t\left(t-1\right)} dt$$
$$\ln\left(x\right) - \ln\left(x+1\right) + c = -\ln\left(t\right) + \ln\left(t-1\right) + c$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y\left(x\right)$)

A. DO NOT forget to include C, constant of integration

$$\begin{aligned}
\ln\left(x\right) - \ln\left(x+1\right) + c_1 &= -\ln\left(t\right) + \ln\left(t-1\right) + c_2 \\
\ln\left(x\right) - \ln\left(x+1\right) + c_1 - c_1 &= -\ln\left(t\right) + \ln\left(t-1\right) + c_2 - c_1 \\
\ln\left(x\right) - \ln\left(x+1\right) &= -\ln\left(t\right) + \ln\left(t-1\right) + c \\
\ln\frac{x}{x+1} &= \ln\frac{\left(t-1\right)}{t} + c \\
\ln\frac{x}{x+1} &= \ln\frac{\left(t-1\right)}{t} + c \\
\ln\frac{x}{x+1} &= \ln\frac{\left(t-1\right)}{t} + \ln e^c \\
\ln\frac{x}{x+1} &= \ln\frac{e^c\left(t-1\right)}{t} \\
\frac{x}{x+1} &= \frac{e^c\left(t-1\right)}{t} \\
1 - \frac{1}{x+1} &= \frac{e^c\left(t-1\right)}{t} \\
\frac{1}{x+1} &= 1 - \frac{e^c\left(t-1\right)}{t} \\
\frac{1}{x+1} &= \frac{t - e^c\left(t-1\right)}{t} \\
x+1 &= \frac{t}{t - e^c\left(t-1\right)} \\
x &= \frac{t}{t - e^c\left(t-1\right)} - 1 \\
&= \frac{t - \left(t - e^c\left(t-1\right)\right)}{t - e^c\left(t-1\right)} \\
&= \frac{e^c\left(t-1\right)}{t - e^c\left(t-1\right)} \\
x &= \frac{c\left(t-1\right)}{t + c\left(1-t\right)}
\end{aligned}$$

iv. Substitute in the values given to find C

$$
\begin{aligned}
x &= \frac{c\,(t-1)}{t+c\,(1-t)} \\
2 &= \frac{c\,(2-1)}{2+c\,(1-2)} \\
2 &= \frac{c}{2-c} \\
4-2c &= c \\
c &= \frac{4}{3}
\end{aligned}
$$

v. Substitute C back into the equation, to find the solution for the I.V.P.

$$
\begin{aligned}
x &= \frac{\frac{4}{3}\,(t-1)}{t+\frac{4}{3}\,(1-t)} \\
&= \frac{\frac{4}{3}\,(t-1)}{t+\frac{4}{3}\,(1-t)} \\
&= \frac{\frac{4}{3}t-\frac{4}{3}}{t+\frac{4}{3}-\frac{4}{3}t} \\
&= \frac{\frac{4t-4}{3}}{\frac{-t+4}{3}} \\
x &= \frac{4\,(t-1)}{4-t}
\end{aligned}
$$

(e) $\frac{dx}{dt} = e^{x+t}, x\,(0) = a$

i. Write D.E. as $g\,(y)\,dy = f\,(x)\,dx$

$$
\begin{aligned}
\frac{dx}{dt} &= e^{x+t} \\
\frac{dx}{dt} &= e^{x}e^{t} \\
e^{-x}dx &= e^{t}dt
\end{aligned}
$$

ii. Integrate both sides

$$
\begin{aligned}
\int e^{-x}dx &= \int e^{t}dt \\
-e^{-x}+c &= e^{t}+c
\end{aligned}
$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y\,(x)$)

A. DO NOT forget to include C, constant of integration

$$\begin{aligned}
-e^{-x} &= e^{t} + c - c \\
-e^{-x} &= e^{t} + c - c \\
\ln\left(-e^{-x}\right) &= \ln\left(e^{t} + c\right) \\
\ln(-1) + \ln\left(e^{-x}\right) &= \ln\left(e^{t} + c\right) \\
\ln(-1) - x &= \ln\left(e^{t} + c\right) \\
-x &= \ln\left(e^{t} + c\right) - \ln(-1) \\
x &= -\ln\frac{\left(e^{t} + c\right)}{-1} \\
&= -\ln\left(-e^{t} - c\right) \\
&= -\ln\left(-e^{t} + c\right) \\
&= -\ln\left(-e^{t} + c\right)
\end{aligned}$$

iv. Substitute in the values given to find C

$$\begin{aligned}
a &= -\ln\left(-e^{0} + c\right) \\
&= -\ln(-1 + c) \\
a &= -\ln(-1 + c) \\
\ln(-1 + c) &= -a \\
-1 + c &= e^{-a} \\
c &= e^{-a} + 1
\end{aligned}$$

v. Substitute C back into the equation, to find the solution for the I.V.P.

$$\begin{aligned}
x &= -\ln\left(-e^{t} + e^{-a} + 1\right) \\
x &= -\ln\left(1 + e^{-a} - e^{t}\right)
\end{aligned}$$