Calc II - T3

October 27, 2019

1. Find the average value of the function on the given interval:

   (a) $f(x) = \cos x$, $\left[0, \frac{\pi}{2}\right]$

      i. Answer

$$\begin{aligned} M &= \frac{1}{\frac{\pi}{2} - 0} \int_0^{\frac{\pi}{2}} \cos x \, dx \\ &= \frac{2}{\pi} \left[\sin x\right]_0^{\frac{\pi}{2}} \\ &= \frac{2}{\pi} \left(1 - (0)\right) \\ M &= \frac{2}{\pi} \end{aligned}$$

   (b) $f(t) = te^{-t^2}$, $[0, 5]$

      i. Answer

$$\begin{aligned} M &= \frac{1}{5 - 0} \int_0^5 te^{-t^2} dt \\ &= \frac{e^{25} - 1}{10e^{25}} \\ M &= 0.1 \left(1 - e^{-25}\right) \end{aligned}$$

   (c) $f(r) = \frac{3}{(1+r)^2}$, $[1, 6]$

      i. Answer

$$\begin{aligned} M &= \frac{1}{6 - 1} \int_1^6 \frac{3}{(1+r)^2} dr \\ &= \frac{1}{5} \int_1^6 \frac{3}{(1+r)^2} dr \\ &= \frac{3}{14} \end{aligned}$$

      ii. Note, to solve $\int_1^6 \frac{3}{(1+r)^2} dr$,

A. Let $u = 1 + r$
B. $\frac{du}{dr} = 1$
C. Find bounds

$$\begin{aligned} u|_{r=6} &= 1 + 6 \\ &= 7 \end{aligned}$$

$$\begin{aligned} u|_{r=1} &= 1 + 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \int_1^6 \frac{3}{(1+r)^2} dr &= \int_2^7 \frac{3}{u^2} du \\ &= \int_2^7 3u^{-2} du \end{aligned}$$

(d) $f(y) = \ln y, [1, 3]$
i. Answer

$$\begin{aligned} M &= \frac{1}{3-1} \int_1^3 \ln y \, dy \\ &= \frac{1}{2} \int_1^3 \ln y \, dy \\ &= \frac{1}{2} (3 \ln (3) - 2) \\ M &= \frac{3}{2} \ln 3 - 1 \end{aligned}$$

ii. Extra, to find integral for $\int_1^3 \ln y \, dy$, integrate by parts
A. Let $u = \ln y$
B. Let $v' = 1$

$$\begin{aligned} \frac{du}{dy} &= \frac{1}{y} \\ du &= \frac{1}{y} dy \end{aligned}$$

$$\begin{aligned} v &= \int 1 \\ &= y \end{aligned}$$

$$\begin{aligned} \int_1^3 \ln y \cdot 1 dy &= y \ln y|_1^3 - \int_1^3 y \frac{1}{y} dy \\ &= [y \ln y - y]_1^3 \\ &= 3 \ln 3 - 2 \end{aligned}$$

(e) $y = 3x^2 + 5x - 7, [-2, 3]$

i. Answer

$$\begin{aligned}
M &= \frac{1}{3-(-2)} \int_{-2}^{3} 3x^2 + 5x - 7dx \\
&= \frac{1}{5} \int_{-2}^{3} 3x^2 + 5x - 7dx \\
&= \frac{5}{2} \\
M &= 2.5
\end{aligned}$$

(f) $y = xe^{-\frac{x}{a}}, [0, a]$

i. Answer

$$\begin{aligned}
M &= \frac{1}{a-0} \int_{0}^{a} xe^{-\frac{x}{a}} dx \\
&= \frac{1}{a} \int_{0}^{a} xe^{-\frac{x}{a}} dx \\
&= \frac{(e-2)\,a}{e} \\
M &= a - \frac{2a}{e}
\end{aligned}$$

2. Find the values of b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

(a) Answer

$$\begin{aligned}
3 &= \frac{1}{b-0} \int_{0}^{b} 2 + 6x - 3x^2 dx \\
3b &= \int_{0}^{b} 2 + 6x - 3x^2 dx \\
&= \left[2x + 3x^2 - x^3\right]_0^b \\
3b &= 2b + 3b^2 - b^3 \\
b^3 - 3b^2 + 3b - 2b &= 0 \\
b^3 - 3b^2 + b &= 0 \\
b\left(b^2 - 3b + 1\right) &= 0 \\
b &= 0, \frac{3 \pm \sqrt{5}}{2}
\end{aligned}$$

(b) Since $b \neq 0$, otherwise the entire equation evaluates to 0,

(c) $b = \frac{3 \pm \sqrt{5}}{2}$

3. The marginal revenue $R'(x)$ from selling x items is $90-0.02x$.The revenue from the sales of the first 100 items is \$8800. What is the revenue from the sales of the first 200 items?

   (a) Answer

      i.

$$\begin{aligned}\int_{100}^{200} R'(x)\,dx &= \int_{100}^{200} 90-0.02x\,dx \\ &= 8700\end{aligned}$$

   (b) Total figure

$$\begin{aligned}8700 &= R(200)-R(100) \\ &= R(200)-8800 \\ R(200) &= \$17500\end{aligned}$$

4. The demand function for a certain commodity is $p = 5 - \frac{x}{10}$. Find the consumer surplus when the sales level is 30.

   (a) Answer

$$\bar{x} = 30$$

$$\begin{aligned}\bar{p} &= 5 - \frac{30}{10} \\ &= 2\end{aligned}$$

$$\begin{aligned}CS &= \int_0^{30} \left[5 - \frac{x}{10} - 2\right] dx \\ &= \int_0^{30} 3 - \frac{x}{10} dx \\ &= \$45\end{aligned}$$

5. The demand function for a certain make of cartridge typewriter ribbon is given by $p = -0.01x^2 - 0.1x + 6$, where p is the unit price in dollars and x is the quantity demanded each week, measured in units of a thousand. Determine the consumer surplus if the market price is set at \$4 per cartridge.

   (a) Determine demand.

      i. $p = -0.01x^2 - 0.1x + 6$, $\bar{p} = 4$

      ii.

$$\begin{aligned}4 &= -0.01x^2 - 0.1x + 6 \\ -0.01x^2 - 0.1x + 2 &= 0 \\ \bar{x} &= -20, 10\end{aligned}$$

iii. Reject -20 because quantity demanded cannot be negative, therefore $\bar{x} = 10$

(b) Determine consumer surplus

$$\begin{aligned} CS &= \int_0^{10} \left[-0.01x^2 - 0.1x + 6 - 4\right] dx \\ &= 11.67\$ \end{aligned}$$

6. It is known that the quantity demanded of a certain make of portable hair dryer is x hundred units per week, and the corresponding wholesale unit price is $p = \sqrt{225 - 5x}$ dollars. Determine the consumer surplus if the wholesale market price is set at \$10 per unit.

   (a) Answer

   i. $p = \sqrt{225 - 5x}$, $p = 10$

   ii. Formula for CS: $CS = \int_0^{\bar{x}} \left[D\left(x\right) - \bar{p}\right] dx$

   iii. Determine the quantity demanded, $\bar{x}$

$$\begin{aligned} p &= 10 \\ 10 &= \sqrt{225 - 5x} \\ 100 &= 225 - 5x \\ 100 - 225 &= -5x \\ \frac{125}{5} &= x \\ \bar{x} &= 25 \end{aligned}$$

   iv. CS

$$\begin{aligned} CS &= \int_0^{25} \sqrt{225 - 5x} - 10dx \\ &= \frac{200}{3} \\ &= \$66.67 \end{aligned}$$

7. For a commodity and pure competition, the number of units produced and the price per unit are determined as the coordinates of the point of intersection of the supply and demand curves. Given the demand curve $p = 50 - \frac{x}{20}$ and the supply curve $p = 20 + \frac{x}{10}$, find the consumer surplus and producer surplus.

   (a) Find the intersection

$$\begin{aligned} 50 - \frac{x}{20} &= 20 + \frac{x}{10} \\ 1000 - x &= 400 + 2x \\ 600 &= 3x \\ \bar{x} &= 200 \end{aligned}$$

(b) Find equilibrium

i. Equilibrium price, $\bar{p}$

$$\begin{aligned}\bar{p} &= 50 - \frac{200}{20}\\ &= 50 - 10\\ \bar{p} &= 40\end{aligned}$$

(c) Consumer surplus

$$\begin{aligned}CS &= \int_0^{200} \left[50 - \frac{x}{20} - 40\right] dx\\ &= \int_0^{200} 10 - \frac{x}{20} dx\\ \boldsymbol{CS} &= \mathbf{1000\$}\end{aligned}$$

(d) Produceer surplus

$$\begin{aligned}PS &= \int_0^{200} 40 - \left[20 + \frac{x}{10}\right] dx\\ &= \int_0^{200} 20 - \frac{x}{10} dx\\ \boldsymbol{PS} &= \mathbf{2000\$}\end{aligned}$$

8. The quantity demanded x (in units of a hundred) of the Mikado miniature cameras per week is related to the unit price p (in dollars) by $p = -0.1x^2 - x + 40$ and the quantity x (in units of a hundred) that the suppliers is willing to make available in the market is related to the unit price p (in dollars) by $p = 0.1x^2 + 2x + 20$. If the market price is set at the equilibrium price, find the consumer surplus and the producer surplus.

(a) Find the intersection point

$$\begin{aligned}&-0.1x^2 - x + 40 = 0.1x^2 + 2x + 20\\ &0.2x^2 + 3x - 20 = 0\\ &x^2 + 15x - 100 = 0\\ &(x + 20)(x - 5) = 0\\ &\bar{x} = -20\,(ignored)\,, 5\end{aligned}$$

(b) Find the equilibrium price

$$\begin{aligned}\bar{p} &= 0.1\,(5)^2 + 2\,(5) + 20\\ &= 32.5\$\end{aligned}$$

(c) Find the consumer surplus

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\begin{aligned}
CS &= \int_0^5 -0.1x^2 - x + 40 - 32.5dx \\
&= 20.83\$
\end{aligned}
$$

(d) Find the producer surplus

$$
\begin{aligned}
PS &= \int_0^5 32.5 - \left(0.1x^2 + 2x + 20\right) dx \\
&= \int_0^5 32.5 - 0.1x^2 - 2x - 20dx \\
PS &= \$33.33
\end{aligned}
$$