Calculus 1: Tutorial 3

July 2, 2019

1. Evaluate the limit if it exists.

   (a) $\lim_{x\to 4}\left(5x^2-2x+3\right)$

   $$\begin{aligned}\lim_{x\to 4}\left(5x^2-2x+3\right) &= 5\left(4^2\right)-2\left(4\right)+3\\ &= 75\end{aligned}$$

   (b) $\lim_{x\to 2}\frac{2x^2+1}{x^2+6x-4}$

   $$\begin{aligned}\lim_{x\to 2}\frac{2x^2+1}{x^2+6x-4} &= \frac{2\left(2\right)^2+1}{\left(2\right)^2+6\left(2\right)-4}\\ &= 0.75\end{aligned}$$

   (c) $\lim_{u\to -2}\left(\sqrt{u^4+3u+6}\right)$

   $$\begin{aligned}\lim_{u\to -2}\left(\sqrt{u^4+3u+6}\right) &= \sqrt{\left(-2\right)^4+3\left(-2\right)+6}\\ &= \sqrt{16-6+6}\\ &= 4\end{aligned}$$

   (d)

   $$\begin{aligned}\lim_{x\to -4}\frac{x^2+5x+4}{x^2+3x-4} &= \frac{\left(-4\right)^2+5\left(-4\right)+4}{\left(-4\right)^2+3\left(-4\right)-4}\\ &= \frac{0}{16-12-4}\\ &= \frac{0}{0}\left(indeterminant\right)\end{aligned}$$

$$
\begin{aligned}
\lim_{x\to -4} \frac{x^2+5x+4}{x^2+3x-4} &= \lim_{x\to -4} \frac{\cancel{(x+4)}\,(x+1)}{\cancel{(x+4)}\,(x-1)} \\
&= \frac{-4+1}{-4-1} \\
&= \frac{-3}{-5} \\
\boldsymbol{\lim_{x\to -4} \frac{x^2+5x+4}{x^2+3x-4}} &\boldsymbol{= \frac{3}{5}}
\end{aligned}
$$

(e)

$$
\begin{aligned}
\lim_{x\to 1} \frac{x^3-1}{x^2-1} &= \lim_{x\to 1} \frac{x^3-1^3}{x^2-1} \\
&= \lim_{x\to 1} \frac{(x+(-1))\left(x^2-(x)(-1)+(-1)^2\right)}{(x-1)(x+1)} \\
&= \lim_{x\to 1} \frac{(x-1)\left(x^2+x+1\right)}{(x-1)(x+1)} \\
&= \lim_{x\to 1} \frac{\cancel{(x-1)}\left(x^2+x+1\right)}{\cancel{(x-1)}(x+1)} \\
&= \frac{\left((1)^2+(1)+1\right)}{((1)+1)} \\
\boldsymbol{\lim_{x\to 1} \frac{x^3-1}{x^2-1}} &\boldsymbol{= \frac{3}{2}}
\end{aligned}
$$

(f) $\frac{1}{2}$

(g) $\lim_{x\to 2} \frac{x^4-16}{x-2}$

$$
\begin{aligned}
\lim_{x\to 2} \frac{x^4-16}{x-2} &= \lim_{x\to 2} \frac{\left(x^2-4\right)\left(x^2+4\right)}{x-2} \\
&= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)\left(x^2+4\right)}{\cancel{x-2}} \\
&= \lim_{x\to 2} (x+2)\left(x^2+4\right) \\
&= (2+2)\left(2^2+4\right) \\
&= 32
\end{aligned}
$$

(h) 1

2. For the function f whose graph is given, state the value of the given quantity, if it exists. If it does not exist, explain why.

   (a) $\lim_{x\to 0} f(x)$

i. Yes, the value is 3.

(b) $\lim_{x\to 3^-} f(x)$

i. Yes, the value is 4.

(c) $\lim_{x\to 3^+} f(x)$

i. Yes, the value is 2.

(d) $\lim_{x\to 3} f(x)$

i. No, this is because $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$, as a result, the limit does not exist.

(e) $f(3)$

i. Yes, 3.

3. Use the Squeeze Theorem to show that $\lim_{x\to 0} \sqrt{x^3+x^2} sin \frac{\pi}{x} = 0$

(a) Start with $sin(x)$

$$-1 \leq sin(x) \leq 1$$

(b) As long as we avoid $x = 0$, since limit doesn't care about a specific point.

$$-1 \leq sin\left(\frac{\pi}{x}\right) \leq 1$$

(c) Multiply everything by $\sqrt{x^3+x^2}$

$$-\sqrt{x^3+x^2} \leq \sqrt{x^3+x^2} sin\left(\frac{\pi}{x}\right) \leq \sqrt{x^3+x^2}$$

(d) Find the limits of the outer function

$$\begin{aligned}\lim_{x\to 0} -\sqrt{x^3+x^2} &= -\sqrt{0} \\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0} \sqrt{x^3+x^2} &= \sqrt{0} \\ &= 0\end{aligned}$$

(e) By the squeeze theorem

$$\lim_{x\to 0} \sqrt{x^3+x^2} sin\left(\frac{\pi}{x}\right) = 0$$

4.

$$1 \leq f(x) \leq x^2+2x+2$$

$$\lim_{x\to -1} 1 = 1$$

$$\begin{aligned}\lim_{x\to -1} x^2+2x+2 &= (-1)^2 + 2(-1) + 2 \\ &= 1-2+2 \\ &= 1\end{aligned}$$

(a) By the squeeze theorem,

$$\lim_{x \to -1} f(x) = 1$$

5. Find the limit, if it exists. If the limit does not exist, explain why.

(a) $\lim_{x \to 2} \frac{|x-2|}{x-2}$; $\frac{0}{0}$

i. Find the absolute value function

$$|x-2| = \begin{cases} x-2 & x \geq 2 \\ -(x-2) & x < 2 \end{cases}$$

ii. Find the limits

$$\lim_{x \to 2^-} \frac{|x-2|}{x-2} = \lim_{x \to 2^-} \frac{-(x-2)}{x-2}$$
$$= -1$$

$$\lim_{x \to 2^+} \frac{|x-2|}{x-2} = \lim_{x \to 2^+} \frac{x-2}{x-2}$$
$$= 1$$

iii. Conclusion

A. Since $\lim_{x \to 2^-} \frac{|x-2|}{x-2} \neq \lim_{x \to 2^+} \frac{|x-2|}{x-2}$, the limit D.N.E.

(b) $\lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{|x|}\right)$

i. Start with the absolute value function

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

ii. Find the limit

$$\lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{|x|}\right) = \lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{-x}\right)$$
$$= \lim_{x \to 0^-} \left(\frac{1}{x} + \frac{1}{x}\right)$$
$$= \lim_{x \to 0^-} \left(\frac{2}{x}\right)$$
$$= -\infty$$

(c) $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right)$

i. Start with the absolute value function

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

ii. Find the limit

$$\begin{aligned}\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right) &= \lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x}\right)\\ &= \lim_{x\to 0^+}(0)\\ &= 0\end{aligned}$$

iii. ALTERNATIVE ANSWER (given by lecturer)

$$\begin{aligned}\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right) &= \frac{1}{0.000000001}-\frac{1}{|0.000000001|}\\ &= 0\end{aligned}$$

6.

(a) Sketch the graph

(b)

i. $\lim_{x\to 1^+} g(x) = 0$
ii. $\lim_{x\to 1} g(x) = 0$
iii. $\lim_{x\to 0} g(x) = 1$
iv. $\lim_{x\to 1^{-1}} g(x) = 1$
v. $\lim_{x\to -1^+} g(x) = 0$
vi. $\lim_{x\to 1} g(x) = D.N.E$

7.

(a)

i. $\lim_{x\to 1^+} F(x)$

$$\begin{aligned}\lim_{x\to 1^+} F(x) &= \lim_{x\to 1^+}\frac{x^2-1}{|x-1|}\\ &= \lim_{x\to 1^+}\frac{x^2-1}{x-1}\\ &= \lim_{x\to 1^+}\frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}\\ &= \lim_{x\to 1^+}(x+1)\\ &= 2\end{aligned}$$

ii. $\lim_{x\to 1^-} F(x)$

$$\begin{aligned}\lim_{x\to 1^-} F(x) &= \lim_{x\to 1^-} \frac{x^2-1}{|x-1|}\\ &= \lim_{x\to 1^-} \frac{x^2-1}{-(x-1)}\\ &= \lim_{x\to 1^-} \frac{\cancel{(x-1)}(x+1)}{-\cancel{(x-1)}}\\ &= \lim_{x\to 1^-} -(x+1)\\ &= -(1+1)\\ &= -2\end{aligned}$$

(b) No, since $\lim_{x\to 1^+} F(x) \neq \lim_{x\to 1^-} F(x)$.

(c) $f(x) = \frac{x^2-1}{|x-1|}$

i. Plot the points and join them, or, you can find a piecewise defined function

ii. Factorize out to find the answer

$$f(x) = \begin{cases} x+1 & x>1 \\ -(x+1) & x<1 \end{cases}$$

(No in between)

8. From the graph of f given below, state the values of x at which f is discontinuous, and state the intervals on which f is continuous.

(a) Continuous: $[-4.-2)$, $(-2,2)$, $[2,4)$, $(4,6)$, $(6,8)$

(b) Discontinuous points: $x|x\epsilon\{-2,2,4,6\}$

9. Notes: For $f(x)$ to be continuous at $x=a$, the following must be satisfied

$$\begin{aligned}&1.\lim_{x\to a} f(x)\ exist\\ &2.f(a)\ exists\\ &3.\lim_{x\to a} f(x) = f(a)\end{aligned}$$

(a) $f(x) = \begin{cases} \frac{1}{x-1} & if\, x\neq 1 \\ 2 & if\, x=1 \end{cases} \quad a=1$

i. Answer: Because $\lim_{x\to 1} f(x)$does not exist

ii. Graph

(b) $f(x) = \begin{cases} 1+x^2 & if\, x<1 \\ 4-x & if\, x\geq 1 \end{cases} \quad a=1$

i. Answer: Because $\lim_{x\to1^+} f(x) = 2, \lim_{x\to1^+} 4{-}1 = 3, \lim_{x\to1^-} f(x) \neq \lim_{x\to1^+} f(x)$

ii. Graph

10.

(a) At $x = 0$

$$\begin{aligned}\lim_{x\to0^-} f(x) &= \lim_{x\to0^-} x+2\\ &= 2\end{aligned}$$

$$\begin{aligned}\lim_{x\to0^+} f(x) &= \lim_{x\to0^-} e^x\\ &= e^0\\ &= 1\end{aligned}$$

i. Conclusion, since $\lim_{x\to0^-} f(x) \neq \lim_{x\to0^+} f(x)$,$\lim_{x\to0} f(x)$ does not exist. The function is not continuous at $x = 0$

(b) At $x = 1$

$$\begin{aligned}\lim_{x\to1^-} f(x) &= \lim_{x\to1^-} e^x\\ &= e\end{aligned}$$

$$\begin{aligned}\lim_{x\to1^+} f(x) &= \lim_{x\to1^+} 2-x\\ &= 2-1\\ &= 1\end{aligned}$$

i. Conclusion, since $\lim_{x\to1^-} f(x) \neq \lim_{x\to1^+} f(x)$,$\lim_{x\to1} f(x)$does not exist. Therefore, the function is not continuous at $x = 1$