Calc 1 : Tutorial 10

September 10, 2019

1. By using suitable substitution, find the following indefinite integral.

   (a) $\int \frac{x}{x^2+1} dx$

      i. Let $u = x^2 + 1$

$$\begin{aligned} u &= x^2 + 1 \\ du &= 2x\,dx \\ dx &= \frac{du}{2x} \end{aligned}$$

      ii. Subsitute inside

$$\begin{aligned} \int \frac{x}{x^2+1} dx &= \int \frac{x}{u} \frac{du}{2x} \\ &= \frac{1}{2} \int \frac{1}{u} du \\ &= \frac{1}{2} \ln(u) + c \\ &= \frac{1}{2} \ln\left(x^2+1\right) + c \end{aligned}$$

   (b)

   (c) $\int x e^{x^2} dx$

      i. Let $u = x^2$

$$\begin{aligned} du &= 2x\,dx \\ dx &= \frac{du}{2x} \end{aligned}$$

      ii. Substitute in

$$\begin{aligned} \int x e^u \frac{du}{2x} &= \int e^u \frac{du}{2} \\ &= \frac{1}{2} \int e^u du \\ &= \frac{1}{2} e^u + c \\ &= \frac{1}{2} e^{x^2} + c \end{aligned}$$

(d)

(e) $\int e^x \sqrt{1+e^x} dx$

i. Let $u = 1 + e^x$

$$\begin{aligned} \frac{du}{dx} &= e^x \\ du &= e^x dx \\ dx &= \frac{du}{e^x} \end{aligned}$$

ii. Substiute in

$$\begin{aligned} \int e^x \sqrt{u} \frac{du}{e^x} &= \int \sqrt{u} du \\ &= \int u^{\frac{1}{2}} du \\ &= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \\ \int e^x \sqrt{1+e^x} dx &= \frac{2}{3} \left(1+e^x\right)^{\frac{3}{2}} + c \end{aligned}$$

(f) $\int \frac{\sec x \tan x}{1+2\sec x} dx$

$$\int \sec x \tan x \left(1+2\sec x\right)^{-1} dx$$

i. Let $u = 1 + 2\sec x$

$$\begin{aligned} \frac{du}{dx} &= 2\sec x \tan x \\ dx &= \frac{du}{2\sec x \tan x} \end{aligned}$$

ii. Substitute in

$$\begin{aligned} \int \sec x \tan x \left(1+2\sec x\right)^{-1} dx &= \int \sec x \tan x \left(1+2\sec x\right)^{-1} \frac{du}{2\sec x \tan x} \\ &= \frac{1}{2} \int u^{-1} du \\ &= \frac{1}{2} \ln u + c \\ &= \frac{1}{2} \ln\left(1+2\sec x\right) + c \end{aligned}$$

2.

(a) $\int \frac{2x+1}{(x-3)^6} dx; u = x - 3$

i. Find dx

$$\frac{du}{dx} = 1$$
$$du = dx$$

ii. Substitute in

$$\begin{aligned}
\int \frac{2\left(x+3-3\right)+1}{u^6} du &= \int \frac{2\left(u+3\right)+1}{u^6} du \\
&= \int \frac{2u+6+1}{u^6} du \\
&= \int \frac{2u}{u^6} du + \int \frac{6}{u^6} du + \int \frac{1}{u^6} du \\
&= \int 2u^{-5} du + \int 6u^{-6} du + \int u^{-6} du \\
&= \left(-\frac{2}{4}u^{-4}\right) + 6\left(-\frac{1}{5}u^{-5}\right) + \left(-\frac{1}{5}u^{-5}\right) + c \\
&= -\frac{1}{2u^4} - \frac{6}{5u^5} - \frac{1}{5u^{-5}} + c \\
&= -\frac{1}{2\left(x-3\right)^4} - \frac{7}{5\left(x-3\right)^5} + c
\end{aligned}$$

(b)

(c) $\int x^3\sqrt{1+x^2}dx; u = \sqrt{1+x^2}$

i. Find dx

$$\begin{aligned}
\frac{du}{dx} &= \frac{1}{2}\left(1+x^2\right)^{-\frac{1}{2}}\left(2x\right) \\
&= \frac{x}{\sqrt{1+x^2}} \\
du &= \frac{x}{\sqrt{1+x^2}} dx \\
dx &= \frac{\left(1+x^2\right)^{\frac{1}{2}}}{x} du
\end{aligned}$$

$$\begin{aligned}
u &= \sqrt{1+x^2} \\
u^2 &= 1+x^2 \\
x &= \sqrt{u^2-1}
\end{aligned}$$

ii. Substitute in

$$\begin{aligned}
\int x^3\sqrt{1+x^2}dx &= \int x^3\sqrt{1+x^2}\frac{\left(1+x^2\right)^{\frac{1}{2}}}{x}du \\
&= \int x^2\left(1+x^2\right)du \\
&= \int \left(u^2-1\right)u^2du \\
&= \int u^4-u^2du \\
&= \frac{u^5}{5}-\frac{u^3}{3}+c \\
&= \frac{\left(1+x^2\right)^{\frac{5}{2}}}{5}-\frac{\left(1+x^2\right)^{\frac{3}{2}}}{3}+c
\end{aligned}$$

(d)

(e) $\int \cos^2 x\sin^3 xdx; u=\cos x$

i. Find dx

$$\begin{aligned}
\frac{du}{dx} &= -\sin x \\
dx &= -\frac{du}{\sin x}
\end{aligned}$$

ii. Find x in terms of u

$$u=\cos x$$

iii. Substitute in

$$\begin{aligned}
\int \cos^2 x\sin^3 x\,dx &= \int u^2\sin^3 x\left(-\frac{du}{\sin x}\right) \\
&= -\int u^2\sin^2 x\,du \\
&= -\int u^2\left(1-\cos^2(x)\right)\,du \\
&= -\int u^2\left(1-u^2\right)du \\
&= -\int u^2-u^4du \\
&= -\left[\frac{u^3}{3}-\frac{u^5}{5}\right]+c \\
&= -\frac{\cos^3 x}{3}-\frac{\cos^5 x}{5}+c
\end{aligned}$$

3.

(a) $\int \frac{x-9}{(x+5)(x-2)} dx$

i. Part-ify the equation

$$\begin{aligned}\frac{x-9}{(x+5)(x-2)} &= \frac{A}{(x+5)} + \frac{B}{(x-2)} \\ &= \frac{A(x-2)+B(x+5)}{(x+5)(x-2)} \\ x-9 &= A(x-2)+B(x+5)\end{aligned}$$

$$A(x-2)+B(x+5)-x+9=0$$

A. When $x=2$

$$\begin{aligned}A(2-2)+B(2+5)-2+9&=0 \\ 7B+7&=0 \\ B&=-\frac{7}{7} \\ &=-1\end{aligned}$$

B. When $x=-5$

$$\begin{aligned}A(-5-2)+B(-5+5)-(-5)+9&=0 \\ -7A+5+9&=0 \\ -7A&=-14 \\ A&=2\end{aligned}$$

C. Form the final equation

$$\begin{aligned}\frac{x-9}{(x+5)(x-2)} &= \frac{A}{(x+5)} + \frac{B}{(x-2)} \\ &= \frac{2}{(x+5)} - \frac{1}{(x-2)}\end{aligned}$$

ii. Integrate the equations

$$\begin{aligned}\int \frac{x-9}{(x+5)(x-2)} dx &= \int \frac{2}{(x+5)} - \frac{1}{(x-2)} dx \\ &= 2\int \frac{1}{x+5} dx - \int \frac{1}{x-2} dx \\ \int \frac{x-9}{(x+5)(x-2)} dx &= 2\ln|(x+5)| - \ln|(x-2)| + c\end{aligned}$$

(b)

(c) $\int \frac{2x^2+14x+10}{2x^2+9x+4} dx$

i. Hmm, since the numerator has same power as denominator, we know its not a "proper fraction" yet. So lets do some fancy math, which is "long division"

$$
\begin{array}{r|l}
 & 1 \\
2x^2+9x+4 & 2x^2 + 14x + 10 \\
 & 2x^2 + 9x + 4 \\
\hline
 & 5x + 6
\end{array}
$$

A.

B. Now we have $1 + \frac{5x+6}{2x^2+9x+4}$, which is much better

ii. Partial-ify it (just the second part)

$$\frac{5x+6}{2x^2+9x+4} = \frac{5x+6}{(x+4)(2x+1)}$$

$$\frac{5x+6}{(x+4)(2x+1)} = \frac{A}{(x+4)} + \frac{B}{(2x+1)}$$
$$5x+6 = A(2x+1) + B(x+4)$$
$$A(2x+1) + B(x+4) - 5x - 6 = 0$$

A. When $x = -\frac{1}{2}$

$$\begin{aligned} A(2x+1) + B(x+4) - 5x - 6 &= 0 \\ A(0) + B\left(\frac{7}{2}\right) - 5\left(-\frac{1}{2}\right) - 6 &= 0 \\ B\left(\frac{7}{2}\right) &= 6 - \frac{5}{2} \\ &= \frac{7}{2} \\ B &= 1 \end{aligned}$$

B. When $x = -4$

$$\begin{aligned} A(2x+1) + B(x+4) - 5x - 6 &= 0 \\ A(2(-4)+1) + B(-4+4) - 5(-4) - 6 &= 0 \\ -7A + 14 &= 0 \\ -7A &= -14 \\ A &= \frac{-14}{-7} \\ &= 2 \end{aligned}$$

iii. Get the full equation (oh and do not forget that "whole" part earlier)

$$\begin{aligned}\frac{2x^2+14x+10}{2x^2+9x+4} &= 1+\frac{A}{(x+4)}+\frac{B}{(2x+1)}\\ &= 1+\frac{2}{(x+4)}+\frac{1}{(2x+1)}\end{aligned}$$

iv. Integrate it and get the answer

$$\int \frac{2x^2+14x+10}{2x^2+9x+4}dx = \int 1+\frac{2}{(x+4)}+\frac{1}{(2x+1)}dx$$
$$\int \frac{2x^2+14x+10}{2x^2+9x+4}dx = x+2\ln(x+4)+\frac{1}{2}\ln(2x+1)+c$$

(d)

4.

(a) $\int \ln 2x\, dx$

i. Find u and v', again F-A-N-C-Y math

$$\begin{aligned}u &= \ln 2x\\ v' &= 1\end{aligned}$$

ii. Find u' and v

$$\begin{aligned}u' &= \frac{2}{2x}\\ &= \frac{1}{x}\end{aligned}$$

$$\begin{aligned}v &= \int v' dx\\ &= x\end{aligned}$$

iii. Integrate by parts

$$\begin{aligned}\int uv' &= uv - \int u'v\, dx\\ \int \ln 2x\, dx &= x\ln 2 - \int \frac{1}{x}\cdot x\, dx\\ &= x\ln 2 - \int 1\, dx\\ &= x\ln 2 - x + c\end{aligned}$$

5. $\int (\ln x)^n\, dx$

(a) Okay so lets just flex our formula first, we know that

$$\int uv' = uv - \int u'v\,dx$$

i. and $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

ii. So, lets just let

$$\int uv' = \int (\ln x)^n\,dx$$

$$uv = x(\ln x)^n$$
$$\int u'v\,dx = -n\int (\ln x)^{n-1}\,dx$$

iii. and so from this, we figured out that we need to let:

A. u be $(\ln x)^n$

B. and v' be 1

iv. Now we need to find u' and v

$$\begin{aligned} u' &= n(\ln x)^{n-1} * \frac{1}{x} \\ &= \frac{n}{x}(\ln x)^{n-1} \end{aligned}$$

$$\begin{aligned} v &= \int v' dx \\ &= x \end{aligned}$$

v. Let's dump inside the formula

$$\begin{aligned} \int (\ln x)^n\,dx &= x(\ln x)^n - \int \frac{n}{x}(\ln x)^{n-1}\,x\,dx \\ &= x(\ln x)^n - \int n(\ln x)^{n-1}\ dx \\ &= x(\ln x)^n - n\int (\ln x)^{n-1}\ dx \end{aligned}$$

(b) Now let's fine $\int (\ln x)^3 \, dx$

$$\begin{aligned}
\int (\ln x)^3 \, dx &= x (\ln x)^3 - 3 \int (\ln x)^{3-1} \, dx + c \\
&= x (\ln x)^3 - 3 \int (\ln x)^2 \, dx + c \\
&= x (\ln x)^3 - 3 \left(x (\ln x)^2 - 2 \int (\ln x)^{2-1} \right) + c \\
&= x (\ln x)^3 - 3 \left(x (\ln x)^2 - 2 \int (\ln x)^1 \right) + c \\
&= x (\ln x)^3 - 3x (\ln x)^2 + 6 \int (\ln x)^1 + c \\
&= x (\ln x)^3 - 3x (\ln x)^2 + 6 \left(x (\ln x)^n - n \int (\ln x)^{n-1} \, dx \right) + c \\
&= x (\ln x)^3 - 3x (\ln x)^2 + 6x \ln x - 6 \int 1 \, dx + c \\
&= x (\ln x)^3 - 3x (\ln x)^2 + 6x \ln x - 6x + c \\
&= x \left[(\ln x)^3 - 3 (\ln x)^2 + 6 \ln x - 6 + c \right]
\end{aligned}$$