Tutorial 6

January 17, 2020

1 Determine each of the following real numbers is rational or irrational.

1. 3.9602
   (a) Rational
2. $\frac{2}{5} - \frac{5}{6}$
   (a) $\frac{12}{30} - \frac{25}{30} = -\frac{13}{30} = 0.4\bar{3}$, Rational
3. 0.30303030 . . . (where the digits 30 are assumed to repeat forever)
   (a) Rational
   (b) $0.\overline{30}$

2 Justify your answer to each of the following questions, for the integers m and n.

1. Is $2m(2m + 2)$ divisible by 4?
   (a) If $2m\,(2m + 2)$ is divisible by 4, then it should be a multiple of 4.

$$\begin{aligned} 2m\,(2m+2) &= 2m \cdot 2\,(m+1) \\ &= 4m\,(m+1) \\ &= 4\left(m^2+m\right) \end{aligned}$$

   (b) $4\left(m^2 + m\right)$ is a multiple of 4, it is divisible by 4.
2. If $m = 4n + 3$, does 8 divides $m^2 - 1$?
   (a) Substitute in

$$\begin{aligned} (4n+3)^2 - 1 &= 16n^2 + 24n + 9 - 1 \\ &= 16n^2 + 24n + 8 \\ &= 8\left(2n^2 + 3n + 1\right) \end{aligned}$$

(b) $8\left(2n^2+3n+1\right)$ is a multiple of 8, 8 divides m^2-1.

3. Does 24|6?

   (a) Logic

$$24|6 = \frac{6}{24} = \frac{1}{4} \notin \mathbb{Z}$$
$$\therefore 24\nmid 6$$

4. Is (3m + 1)(3m + 2)(3m + 3) divisible by 3?

$$3\left(m+1\right)\left(3m+1\right)\left(3m+2\right)$$

$\therefore (3m+1)(3m+2)(3m+3)$ is divisible by 3.

3 Find integers q and r such that $n = dq + r, 0 \leq r < d$.

1. n = 100, d = 10

$$\begin{aligned} n &= dq + r \\ 100 &= 10q + r \\ q &= 10, r = 0 \end{aligned}$$

2. n = 9, d = 10

$$\begin{aligned} n &= dq + r \\ 9 &= 10q + r \\ q &= 0, r = 9 \end{aligned}$$

3. n = -57, d = 5

$$\begin{aligned} n &= dq + r \\ -57 &= 5q + r \\ q &= -12 \\ r &= 3 \end{aligned}$$

4 Evaluate

1. 59 div 3, 59 mod 3

$$\begin{aligned} 59 &= 3\left(19\right) + 2 \\ 59\text{div}3 &= q = 19 \\ 59 \mod 3 &= r = 2 \end{aligned}$$

2. 45 div 6, 45 mod 6

$$\begin{aligned} 45 &= 6\left(7\right) + 3 \\ 45\text{div}6 &= q = 7 \\ 45 \mod 6 &= r = 3 \end{aligned}$$

5 What are the floor and ceiling for the following values?

Value	Floor	Ceil
8	8	8
$\frac{25}{4}$	6	7
19.67	19	20
$-\frac{76}{12}$	-7	-6

6 Use the floor notation to express

1. 589 div 12 and 589 mod 12.

$$\begin{aligned} 589 &= 12\,(49) + 1 \\ 589 \text{div} 12 &= q = 49 \\ 589 \mod 12 &= r = 1 \end{aligned}$$

$$\begin{aligned} 589 \div 12 &= 49.9033 \\ \lfloor 49.9033 \rfloor &= 49 \end{aligned}$$

2. 123 div 5 and 123 mod 5.

$$\begin{aligned} 123 &= 5\,(24) + 3 \\ 123 \text{div} 5 &= q = 24 \\ 123 \mod 5 &= r = 3 \\ 123 \div 12 &= 10.25 \\ \lfloor 10.25 \rfloor &= 10 \end{aligned}$$

7 Write the following integers in standard factored form.

1. $702 = 13 \cdot 3^3 \cdot 2$
2. $112385 = 19 \cdot 13^2 \cdot 7 \cdot 5$
3. $(20!)^2$

$$\begin{aligned} (20!) &= 2^{18} * 3^8 * 5^4 * 7^2 * 11 * 13 * 17 * 19 \\ (20!)^2 &= 2^{36} * 3^{16} * 5^8 * 7^4 * 11^2 * 13^2 * 17^2 * 19^2 \end{aligned}$$

8 Prove the following statements by the method of direct proof.

Direct Proof: We use it to prove statements of the form "if p then q" or "p implies q" which we can write as $p \implies q$. The method of the proof is to takes an original statement p, which we assume to be true, and use it to show directly that another statement q is true. So a direct proof has the following steps:

- Find p and q
- Assume the statement p is true.
- Use what we know about p and other facts as necessary to deduce that another statement q is true, that is show $p \implies q$ is true.

1. If n is any odd integers, then $(-1)^n = -1$.
   (a) Find p and q
      i. Let p be the statement that n is any odd integers
      ii. Let q be the statement that $(-1)^n = -1$
   (b) Assume the statement p is true.
      i. Assume that n is any odd integer
   (c) Use what we know about p and other facts as necessary to deduce that another statement q is true, that is show $p \implies q$ is true.
      i. Then by definition $n = 2k + 1$ for some integer k
      ii. Proof

$$
\begin{aligned}
(-1)^{2k+1} &= (-1)^{2k} \cdot (-1)^1 \\
&= -1\left((-1)^2\right)^k \\
&= -1\,(1)^k \\
(-1)^{2k+1} &= -1
\end{aligned}
$$

      iii. Therefore, $(-1)^{2k+1} = -1$.

2. If r is any rational number, then $2r^2 - r + 1$ is rational.
   (a) Find p and q
      i. Let p be the statement that r is any rational number
      ii. Let q be the statement that $2r^2 - r + 1$ is rational
   (b) Assume the statement p is true.
      i. Assume that r is a rational number.
   (c) Use what we know about p and other facts as necessary to deduce that another statement q is true, that is show $p \implies q$ is true.

i. Then by definition $r = \frac{a}{b}, a, b \in \mathbb{Z}, b \neq 0$.

ii. Proof

$$\begin{aligned} 2r^2 - r + 1 &= 2\left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right) + 1 \\ &= \frac{2a^2}{b^2} - \frac{a}{b} + 1 \\ &= \frac{2a^2 - ab + b^2}{b^2} \end{aligned}$$

iii. Since $2a^2 - ab + b^2$ and b^2 are integers, $2r^2 - r + 1$ is rational (Extra note: whether improper or proper doesn't matter).

3. For integers a, b and c. If $a|b$, then $a|bc$.

(a) Find p and q

i. Let p be the statement that $a|b$

ii. Let q be the statement that $a|bc$

(b) Assume the statement p is true.

i. Assume that $a|b$

(c) Use what we know about p and other facts as necessary to deduce that another statement q is true, that is show $p \implies q$ is true.

i. If $a|b$, then $\frac{b}{a} = k$, where k is an integer

ii. Proof

$$\begin{aligned} a|bc &= \frac{bc}{a} \\ &= kc \end{aligned}$$

iii. Since k is an integer, and c is an integer. Their product must be an integer. Therefore, $a|bc$.

9 Disprove the following statements.

1. For all integers n, if n is prime then $(-1)^n = -1$.

(a) Find p and q

i. Let p be the statement that n is prime

ii. Let q be the statement that $(-1)^n = -1$

(b) Assume the statement p is true.

i. Assume n is prime.

(c) Use what we know about p and other facts as necessary to find a counterexample, and disprove the statement.

i. If $n = 2$

A. Let p be the statement that n is prime
B. Let q be the statement that $(-1)^n = -1$

ii. Then $(-1)^2 = 1$
iii. $1 \neq -1$
iv. Therefore, the statement is disproven.

2. For all integers m, if $m > 2$, then $m^2 - 4$ is composite.

(a) Find p and q

i. Let p be the statement that $m > 2$
ii. Let q be the statement that $m^2 - 4$ is composite (AKA not prime)

(b) Assume the statement p is true.

i. Assume that $m > 2$

(c) Use what we know about p and other facts as necessary to find a counterexample, and disprove the statement.

i. If $m = 3$

$$3^2 - 4 = 5$$

ii. But 5 is not a composite number, therefore, the statement is disproven

3. If m and n are positive integers and mn is a perfect square, then m and n are perfect square.

(a) Find p and q

i. Let p be the statement that m and n are positive integers and mn is a perfect square
ii. Let q be the statement that m and n are perfect square

(b) Assume the statement p is true.

i. Assume that m and n are positive integers and mn is a perfect square

(c) Use what we know about p and other facts as necessary to find a counterexample, and disprove the statement.

i. If $m = 8, n = 2$, $mn = 16$
ii. 16 is a perfect square $4 * 4$, but 8 and 2 are not perfect square.
iii. The statement is disproven

10 Prove the following statements by contradiction.

Prove by contradiction: Establish validity, by showing the opposite leads to a contradiction.

1. The proposition to be proved, P, is assumed to be false. That is, $\neg P$ is true.

2. It is then shown that $\neg P$ implies two mutually contradictory assertions, Q and $\neg Q$.

3. Since Q and $\neg Q$ cannot both be true, the assumption that P is false must be **wrong**, so P must be **true**.

1. For all integers n, if n is even, then $n^2 + n + 1$ is odd.
   Negation: There exists integers n, if n is even, then $n^2 + n + 1$ is not odd.

   (a) Find p and q

      i. $p : n$ is even

      ii. $q : n^2 + n + 1$ is odd

   (b) The proposition to be proved, P, is assumed to be false. That is, $\neg P$ is true.

      i. Assume p is false, therefore n is NOT even.

   (c) It is then shown that $\neg P$ implies two mutually contradictory assertions, Q and $\neg Q$.

      i. According to definition, if n is even, then $n = 2k$, k is an integer.

      ii. Substitute inside

$$\begin{aligned} n^2 + n + 1 &= (2k)^2 + 2k + 1 \\ &= 4k^2 + 2k + 1 \\ &= 2\left(2k^2 + k\right) + 1 \end{aligned}$$

         A. Let $m = 2k^2 + k$

$$n^2 + n + 1 = 2m + 1\,(odd)$$

   (d) Since Q and $\neg Q$ cannot both be true, the assumption that P is false must be **wrong**, so P must be **true**.

      i. Since the contradiction is false, the statement must be true.

2. The product of any nonzero rational number and any irrational number is irrational.
   Negation: The product of at least one non-zero rational number and at least one irrational number is rational.

   (a) The proposition to be proved, P, is assumed to be false. That is, $\neg P$ is true.

      i. Assume x is an irrational number. The product of x and a rational $\frac{a}{b}$ is a rational $\frac{c}{d}$. Where a, b, c, d are integers. a, b, d cannot be 0.

ii. According to the statement, $x \cdot \frac{a}{b} = \frac{c}{d}$

(b) It is then shown that $\neg P$ implies two mutually contradictory assertions, Q and $\neg Q$.

i. By division

$$\begin{aligned} x &= \frac{c}{d} \div \frac{a}{b} \\ &= \frac{c}{d} \cdot \frac{b}{a} \\ &= \frac{cb}{da} \end{aligned}$$

ii. Since division are closed under multiplication, cb, and da are integers. Making $\frac{cb}{da}$ a rational number by definition.

iii. This contradicts with the statement that x is an irrational number.

(c) Since Q and $\neg Q$ cannot both be true, the assumption that P is false must be **wrong**, so P must be **true**.

i. Since the assumption of the contradiction is wrong, therefore, the product of any nonzero rational number and any irrational number is irrational.

11 Prove the following statements by contraposition.

Proof by Contrapositive: Proof by contrapositive takes advantage of the logical equivalence between "P implies Q" and "Not Q implies Not P". For example, the assertion "If it is my car, then it is red" is equivalent to "If that car is not red, then it is not mine". So, to prove "If P, Then Q" by the method of contrapositive means to **prove "If Not Q, Then Not P"**.

1. For all integers a, b and c, if a does not divide the product of b and c, then a does not divide b.

(a) Form the contrapositive of the given statement.

i. For all integers a, b, and c, if a divides b, then a divides the product of b and c.

(b) Assume the statement $\sim q$ is true.

i. Assume that for all integers a, b, c, a divides b. Therefore, $a|b = k$.

$$\frac{b}{a} = k, k \in \mathbb{Z}$$

(c) Use what we know about $\sim q$ and other facts as necessary to deduce that another statement $\sim p$ is true, that is show $\sim q \implies \sim p$ is true. Hence, we can deduce that $p \implies q$ is also true.

$$\frac{b}{a} \cdot c = kc$$
$$\frac{bc}{a} = kc$$

i. Since c is an integer, and by definition the product of 2 integers are integers. Therefore, a divides the product of b and c. **The statement is true.**

2. For all irrational number x, $\sqrt{x}$ is irrational.

(a) Form the contrapositive of the given statement.

i. If x is an irrational number, then $\sqrt{x}$ is irrational.

ii. **Contrapositive**: If $\sqrt{x}$ is rational, then x is an rational number.

(b) Assume the statement $\sim q$ is true.

i. Assume that $\sqrt{x}$ is rational. This means that

$$\sqrt{x} = \frac{a}{b}, \text{where} a, b \in \mathbb{Z}, b \neq 0$$

(c) Use what we know about $\sim q$ and other facts as necessary to deduce that another statement $\sim p$ is true, that is show $\sim q \implies \sim p$ is true. Hence, we can deduce that $p \implies q$ is also true.

i. $x = \frac{a^2}{n^2} \rightarrow$ x is rational

ii. Since x is a rational number. The contrapositive is true. Hence, by law of logical equivalence, **the statement is true.**

12 Use the Euclidean algorithm to find the greatest common divisor of the following pair of integers.

1. 27 and 72

$$27 = 72\,(0) + 27$$
$$72 = 27\,(2) + 18$$
$$27 = 18\,(1) + 9$$
$$18 = 9\,(2) + 0$$

$$gcd\,(9, 0) = 9$$

2. 3510 and 672

$$gcd\,(3510, 672)$$

$$\begin{aligned}3510 &= 672\,(5) + 150\\672 &= 150\,(4) + 72\\150 &= 72\,(2) + 6\\72 &= 6\,(12) + 0\end{aligned}$$

(a) $gcd\,(3510, 672) = 6$

13 Find the least common multiple for the above pair of integers.

1. 27 and 72

$$\begin{aligned}lcm\,(27, 72) &= \frac{a \cdot b}{gcd\,(a, b)}\\&= \frac{27 \cdot 72}{9}\\&= 216\end{aligned}$$

2. 3510 and 672

$$\begin{aligned}lcm\,(3510, 672) &= \frac{a \cdot b}{gcd\,(a, b)}\\&= \frac{3510 \cdot 672}{6}\\&= 393120\end{aligned}$$