

Calculus L2 - LIMITS AND CONTINUITY

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1 Example 2.1

1.

(a) $\lim_{x \rightarrow 2} 5 = 5$

(b) $\lim_{x \rightarrow 2} (4x - 5)$

$$\begin{aligned}\lim_{x \rightarrow 2} (4x - 5) &= 4(2) - 5 \\ &= 3\end{aligned}$$

(c) $\lim_{x \rightarrow 0} (2x^2 + 3x - 5)$

$$\lim_{x \rightarrow 0} (2x^2 + 3x - 5) = -5$$

(d) $\lim_{x \rightarrow 0} (x + 1)(x - 1)$

$$\lim_{x \rightarrow 0} (x + 1)(x - 1) = -1$$

(e) $\lim_{x \rightarrow 2} \left(\frac{x-1}{x^2-1} \right)$

$$\lim_{x \rightarrow 2} \left(\frac{x-1}{x^2-1} \right) = \frac{1}{3}$$

(f) $\lim_{x \rightarrow 1} \left(\frac{x-1}{x^2-1} \right)$

$$\begin{aligned}\lim_{x \rightarrow 2} \left(\frac{x-1}{x^2-1} \right) &= \lim_{x \rightarrow 1} \left(\frac{x-1}{x^2-1} \right) \\ &= \frac{0}{0} \text{ (indeterminant) }\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 2} \left(\frac{x-1}{x^2-1} \right) &= \lim_{x \rightarrow 1} \left(\frac{\cancel{x-1}}{(\cancel{x-1})(x+1)} \right) \\ &= \lim_{x \rightarrow 1} \left(\frac{1}{(x+1)} \right)\end{aligned}$$

$$\lim_{x \rightarrow 2} \left(\frac{x-1}{x^2-1} \right) = \frac{1}{2}$$

$$(g) \lim_{x \rightarrow 3} \left(\frac{x^2 - x - 6}{x - 3} \right)$$

$$\lim_{x \rightarrow 3} \left(\frac{x^2 - x - 6}{x - 3} \right) = \frac{0}{0} \text{ (indeterminant)}$$

$$\begin{aligned} \lim_{x \rightarrow 3} \left(\frac{x^2 - x - 6}{x - 3} \right) &= \lim_{x \rightarrow 3} \left(\frac{(x-3)(x+2)}{x-3} \right) \\ &= \lim_{x \rightarrow 3} (x+2) \\ &= 5 \end{aligned}$$

$$(h) \lim_{x \rightarrow -1} \left(\frac{x^2 + 4x + 3}{x + 1} \right)$$

$$\lim_{x \rightarrow -1} \left(\frac{x^2 + 4x + 3}{x + 1} \right) = \frac{0}{0} \text{ (indeterminate)}$$

$$\begin{aligned} \lim_{x \rightarrow -1} \left(\frac{x^2 + 4x + 3}{x + 1} \right) &= \lim_{x \rightarrow -1} \left(\frac{(x+3)(x+1)}{x+1} \right) \\ &= -1 + 3 \\ &= 2 \end{aligned}$$

$$(i) \lim_{x \rightarrow 8} (\sqrt{x+1})$$

$$\begin{aligned} \lim_{x \rightarrow 8} (\sqrt{x+1}) &= \sqrt{8+1} \\ &= \sqrt{9} \\ &= 3 \end{aligned}$$

$$(j) \lim_{x \rightarrow 1} \left(\frac{7+6x}{3x-2} \right)^4$$

$$\begin{aligned} \lim_{x \rightarrow 1} \left(\frac{7+6x}{3x-2} \right)^4 &= \left(\frac{7+6(1)}{3(1)-2} \right)^4 \\ &= \left(\frac{13}{1} \right)^4 \\ &= 28561 \end{aligned}$$

$$(k) \lim_{x \rightarrow 0} \left(\frac{x}{1 - \sqrt{1+x}} \right)$$

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{x}{1 - \sqrt{1+x}} \right) &= \lim_{x \rightarrow 0} \left(\frac{x}{1 - \sqrt{1+x}} \right) \cdot \left(\frac{1 + \sqrt{1+x}}{1 + \sqrt{1+x}} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{x(1 + \sqrt{1+x})}{1^2 - (1+x)} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{\cancel{x}(1 + \sqrt{1+x})}{-\cancel{x}} \right) \\ &= \lim_{x \rightarrow 0} (-(1 + \sqrt{1+x})) \\ &= -2 \end{aligned}$$

$$(l) \lim_{x \rightarrow 0} \left(\frac{\sqrt{1+x^2}-1}{x^2} \right)$$

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{\sqrt{1+x^2}-1}{x^2} \right) &= \lim_{x \rightarrow 0} \left(\frac{\sqrt{1+x^2}-1}{x^2} \right) * \left(\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}+1} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{1+x^2-1^2}{x^2(\sqrt{1+x^2}+1)} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{\cancel{x^2}}{\cancel{x^2}(\sqrt{1+x^2}+1)} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{1}{\sqrt{1+x^2}+1} \right) \\ &= \frac{1}{\sqrt{1+0^2}+1} \\ &= \frac{1}{2} \end{aligned}$$

2 Example 2.2

Since $\lim_{x \rightarrow 0} (u(x))$ is between the functions

$$\begin{aligned} \lim_{x \rightarrow 0} \left(1 - \frac{x^2}{4} \right) &\leq \lim_{x \rightarrow 0} u(x) \leq \lim_{x \rightarrow 0} \left(1 + \frac{x^2}{2} \right) \\ 1 &\leq u(x) \leq 1 \end{aligned}$$

Since $u(x)$ is between the two functions, **by squeeze theorem**, $\lim_{x \rightarrow 0} (u(x)) =$

1

3 Example 2.3

For the limit to exist at $x = 0$,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} (f(x))$$

$$\begin{aligned}\lim_{x \rightarrow 0^-} (f(x)) &= \lim_{x \rightarrow 0^-} (0) \\ &= 0\end{aligned}$$

$$\lim_{x \rightarrow 0^+} (f(x)) = \lim_{x \rightarrow 0^+} (1)$$

\therefore Since $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} (f(x))$, the limit $\lim_{x \rightarrow 0} (f(x))$ does not exist

4 Example 2.4

4.1 (a) $\lim_{x \rightarrow 0} (f(x))$

$$\begin{aligned}\lim_{x \rightarrow 0^-} (f(x)) &= \lim_{x \rightarrow 0^-} (x + 2) \\ &= 2\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 0^+} (f(x)) &= \lim_{x \rightarrow 0^+} (x + 2) \\ &= 2\end{aligned}$$

\therefore Since $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} (f(x))$, the limit $\lim_{x \rightarrow 0} (f(x))$ is 2

4.2 (b) $\lim_{x \rightarrow 1} (f(x))$

$$\begin{aligned}\lim_{x \rightarrow 1^-} (f(x)) &= \lim_{x \rightarrow 1^-} (x + 2) \\ &= 1 + 2 \\ &= 3\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 1^+} (f(x)) &= \lim_{x \rightarrow 1^+} (2x^2) \\ &= 2(1)^2 \\ &= 2\end{aligned}$$

\therefore Since $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} (f(x))$, the limit $\lim_{x \rightarrow 0} (f(x))$ does not exist

5 Example 2.5

To prove that the limit does not exist, $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} (f(x))$, where a is a number

1. Form the piecewise defined function (optional, just to make it easier)

$$f(x) = \begin{cases} \frac{-(x)}{x} = -1 & x < 0 \\ \frac{x}{x} = 1 & x \geq 0 \end{cases}$$

2. Prove that $\lim_{x \rightarrow 0} \left(\frac{|x|}{x} \right)$ D.N.E.

$$\begin{aligned} \lim_{x \rightarrow 0^-} \left(\frac{|x|}{x} \right) &= \lim_{x \rightarrow 0^-} (-1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} \left(\frac{|x|}{x} \right) &= \lim_{x \rightarrow 0^+} (1) \\ &= 1 \end{aligned}$$

3. Draw inference

\therefore Since $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} (f(x))$, the limit $\lim_{x \rightarrow 0} (f(x))$ does not exist

6 Example 2.6

Note: *D.N.E* means does not exist.

6.1 $\lim_{x \rightarrow 2^-} (g(x)) = 3$

6.2 $\lim_{x \rightarrow 2^+} (g(x)) = 1$

6.3 $\lim_{x \rightarrow 2} (g(x)) = D.N.E.$

6.4 $\lim_{x \rightarrow 5^-} (g(x)) = 2$

6.5 $\lim_{x \rightarrow 5^+} (g(x)) = 2$

6.6 $\lim_{x \rightarrow 5} (g(x)) = 2$

6.7 $g(5) = 1$

7 Example 2.7

Answer already given

8 Example 2.8

$$f(x) = \begin{cases} x^2 + 1 & x < 0 \\ \cos x & x > 0 \end{cases}$$

Determine continuity

1. $x = -1$

(a) Condition: $f(-1)$ must exist

$$\begin{aligned} f(-1) &= (-1)^2 + 1 \\ &= 2 \end{aligned}$$

(b) Condition 2: $\lim_{x \rightarrow -1} (f(x))$ must exist

$$\begin{aligned} \lim_{x \rightarrow -1} (x^2 + 1) &= (-1)^2 + 1 \\ &= 2 \end{aligned}$$

(c) Condition 3: $\lim_{x \rightarrow -1} (f(x)) = f(-1)$

(d) Since all the three conditions are satisfied, the function is continuous at $x = -1$

2. $x = 0$

(a) Condition: $f(0)$ must exist

$$f(0) = \text{undefined}$$

(b) Since the first condition test failed, the function is not continuous at $x = 0$

9 Example 2.9

$$1. f(x) = \begin{cases} x^2 + 1 & x < 1 \\ x^3 - x + 2 & x \geq 1 \end{cases}$$

(a) Condition 1: Check if $f(1)$ exist

$$\begin{aligned} f(1) &= (1)^3 - (1) + 2 \\ &= 1 - 1 + 2 \\ &= 2 \end{aligned}$$

(b) Condition 2: Check if $\lim_{x \rightarrow 1} f(x)$ exist

$$\begin{aligned}\lim_{x \rightarrow 1^-} f(x) &= (1) + 1 \\ &= 2\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 1^+} f(x) &= 1 - 1 + 2 \\ &= 2\end{aligned}$$

i. \therefore Since $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} (f(x))$, the limit $\lim_{x \rightarrow 1} (f(x)) = 2$.

(c) Condition 3: $\lim_{x \rightarrow 1} f(x) = f(1)$

i. PASS

(d) Inference

i. $f(x)$ is continuous at $x = 1$

$$2. f(x) = \begin{cases} x^2 + 1 & x < 1 \\ 4 & x = 1 \\ x^3 - x + 2 & x > 1 \end{cases}$$

(a) Condition 1: Check if $f(1)$ exist

$$f(1) = 4$$

(b) Condition 2: Check if $\lim_{x \rightarrow 1} f(x)$ exist

$$\begin{aligned}\lim_{x \rightarrow 1^-} f(x) &= (1)^2 + 1 \\ &= 2\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 1^+} f(x) &= 1 - 1 + 2 \\ &= 2\end{aligned}$$

i. \therefore Since $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} (f(x))$, the limit $\lim_{x \rightarrow 1} (f(x)) = 2$.

(c) Condition 3: $\lim_{x \rightarrow 1} f(x) = f(1)$

i. FAIL.

$$\lim_{x \rightarrow 1} f(x) \neq f(1)$$

(d) Inference

i. $f(x)$ is NOT continuous at $x = 1$

10 Example 2.10

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & x \neq 3 \\ 4 & x = 3 \end{cases}$$

Determine whether the $f(x)$ is continuous at $x = 3$.

1. Condition 1: $f(3)$ must exist

$$f(3) = 4$$

2. Condition 2: $\lim_{x \rightarrow 3} (f(x))$ must exist

$$\begin{aligned} \lim_{x \rightarrow 3} (f(x)) &= \lim_{x \rightarrow 3} \left(\frac{x^2 - 9}{x - 3} \right) \\ &= \frac{0}{0} \text{ (indeterminate)} \\ &= \lim_{x \rightarrow 3} \left(\frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} \right) \\ &= \lim_{x \rightarrow 3} ((x+3)) \\ \lim_{x \rightarrow 3} (f(x)) &= 6 \end{aligned}$$

3. Condition 3: $\lim_{x \rightarrow 3} (f(x) = f(3))$

$$\lim_{x \rightarrow 3} (f(x) \neq f(3))$$

4. Inference

(a) $f(x)$ is NOT continuous at $x = 3$

11 Example 2.11

$$g(x) = \begin{cases} x^2 + ax - 3 & x < 1 \\ 0 & x = 1 \\ 3x + b & x > 1 \end{cases}$$

continuous at $x = 1$.

1. Since the function is continuous at $x = 1$:

(a) $f(1) = 0$

(b) $\lim_{x \rightarrow 1} f(x) = f(1) = 0$

$$\begin{aligned}
\lim_{x \rightarrow 1^-} (f(x)) &= f(1) \\
(1)^2 + a(1) - 3 &= 0 \\
a - 2 &= 0 \\
\mathbf{a} &= \mathbf{2} \\
\lim_{x \rightarrow 1^+} (f(x)) &= f(1) \\
3(1) + b &= 0 \\
\mathbf{b} &= \mathbf{-3}
\end{aligned}$$

12 Example 2.12

Find $\lim_{x \rightarrow 3^+} \left(\frac{2x}{x-3} \right)$ and $\lim_{x \rightarrow \mathbf{3}^-} \left(\frac{2x}{x-3} \right)$.

Note: There's an errata in the questions: the bolded part is the correction.

$$\begin{aligned}
\lim_{x \rightarrow 3^+} \left(\frac{2x}{x-3} \right) &= \frac{2(3)}{3^+ - 3}, 3^+ - 3 \rightarrow 0 \\
&= \infty \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\lim_{x \rightarrow 3^+} \left(\frac{2x}{x-3} \right) &= \frac{2(3)}{3^- - 3}, 3^- - 3 \rightarrow 0^- \\
&= -\infty
\end{aligned}$$

13 Example 2.13

Find $\lim_{x \rightarrow -3} \left(\frac{1}{(x+3)^2} \right)$

$$\begin{aligned}
\lim_{x \rightarrow -3^+} (x+3)^2 &\rightarrow (0^-)^2 \\
&\rightarrow 0^+
\end{aligned}$$

$$\lim_{x \rightarrow -3^-} (x+3)^2 \rightarrow (0^+)^2 \rightarrow 0^+$$

$$\begin{aligned}
\lim_{x \rightarrow -3^+} \left(\frac{1}{(x+3)^2} \right) &= \frac{1}{0^+} \\
&= \infty
\end{aligned}$$

$$\lim_{x \rightarrow -3^-} \left(\frac{1}{(x+3)^2} \right) = \frac{1}{0^+} = \infty$$

$$\lim_{x \rightarrow -3} \left(\frac{1}{(x+3)^2} = \infty \right)$$

14 Example 2.14

1. $x = 3$
2. $x = \frac{\pi}{2}$
3. $x = 0$

15 Example 2.15

1.

(a) $\lim_{x \rightarrow \infty} \left(\frac{1}{x^8} \right)$

$$\lim_{x \rightarrow \infty} \left(\frac{1}{x^8} \right) = \frac{1}{\infty^8} = 0$$

(b) $\lim_{x \rightarrow -\infty} \left(\frac{500}{x^3} \right)$

$$\frac{500}{(-\infty)^3} = 0$$

(c) $\lim_{x \rightarrow \infty} ((2x^2 + 3x - 5))$

$$2(\infty)^2 + 3(\infty) - 5 = \infty$$

(d) $\lim_{x \rightarrow \infty} \left(\frac{3x+2}{4x+9} \right)$

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\frac{3x+2}{4x+9} \right) &= \frac{\infty}{\infty} (\text{undefined}) \\ &= \lim_{x \rightarrow \infty} \left(\frac{3 + \frac{2}{x}}{4 + \frac{9}{x}} \right) (\text{divide all by } x) \\ &= \frac{3}{4} \end{aligned}$$

$$(e) \lim_{x \rightarrow \infty} \left(\frac{x}{x^2 - 1} \right)$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\frac{x}{x^2 - 1} \right) &= \lim_{x \rightarrow \infty} \left(\frac{1}{x - \frac{1}{x}} \right) \text{ (divide all by } x) \\ &= \frac{1}{\infty} \\ &= 0 \end{aligned}$$

$$(f) \lim_{x \rightarrow \infty} \left(\frac{x^3 - 8x + 1}{5x^3 - 1} \right)$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\frac{x^3 - 8x + 1}{5x^3 - 1} \right) &= \lim_{x \rightarrow \infty} \left(\frac{1 - \frac{8}{x^2} + \frac{1}{x^3}}{5 - \frac{1}{x^3}} \right) \text{ (divide all by } x^3) \\ &= \frac{1}{5} \end{aligned}$$

$$(g) \lim_{x \rightarrow \infty} \left(\frac{x^2 - 6}{x - 3} \right)$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\frac{x^2 - 6}{x - 3} \right) &= \lim_{x \rightarrow \infty} \left(\frac{x - \frac{6}{x}}{1 - \frac{3}{x}} \right) \text{ (divide all by } x) \\ &= \infty \end{aligned}$$

$$(h) \lim_{x \rightarrow \infty} \left(\frac{1 + x^2}{1 - x^2} \right)$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\frac{1 + x^2}{1 - x^2} \right) &= \lim_{x \rightarrow \infty} \left(\frac{\frac{1}{x^2} + 1}{\frac{1}{x^2} - 1} \right) \text{ (divide all by } x^2) \\ &= 1 \end{aligned}$$

16 Notes & Exercise

16.1 Note

x	$\left(1 + \frac{1}{x}\right)^x$
1	2
10	2.5937
100	2.704814
1000	2.7169
.	.
.	.
100,000,000	2.718282...
100,000,000,000	2.718282...
100,000,000,000,000	2.718282...

$$\lim_{x \rightarrow \infty} \left(\left(1 + \frac{1}{x} \right)^x \right) = e \text{ (Euler's number)}$$

To remember:

1. Count the number of characters in each word
 - (a) To
 - (b) express
 - (c) e
 - (d) remember
 - (e) to
 - (f) memorize
 - (g) a
 - (h) sentence
 - (i) to
 - (j) memorize
 - (k) this
2. Or, you can remember that after 2.7, 1828 appears twice:
 - (a) 2.7 1828 1828
 - (b) After that are just the digits of the angles $45^\circ, 90^\circ, 45^\circ$ (no real reason)
 - i. 2.7 1828 1828 45 90 45
3. e 's slope is its value.
4. e is defined as the value at **Maximum, Continuous compounding, 100% growth, one time period** ($e \cdot 1$).

16.2 Notes

1.
$$\lim_{x \rightarrow \infty} \left(\frac{x^2 - 1}{x^2 + 1} = 1 \right)$$

$y = 1$ is a horizontal asymptote.
2. $\lim_{x \rightarrow \infty} \left(\left(\frac{1}{x} - 2 \right) = -2 \right)$
 - (a) $y = -2$ is a horizontal asymptote.

17 Example

Since $f(x)$ is continuous at $[1, 2]$

$$\begin{aligned} f(1) &= 4 - 6 + 3 - 2 \\ &= -1 < 0 \end{aligned}$$

$$\begin{aligned} f(2) &= 4(2)^3 - 6(2)^2 + 3(2) - 2 \\ &= 12 > 0 \end{aligned}$$

According to the intermediate value theorem, since $f(x)$ is continuous between $[1, 2]$, and $f(1) < 0 < f(2)$. There exists a number c between 1 and 2 such that $f(c) = 0$