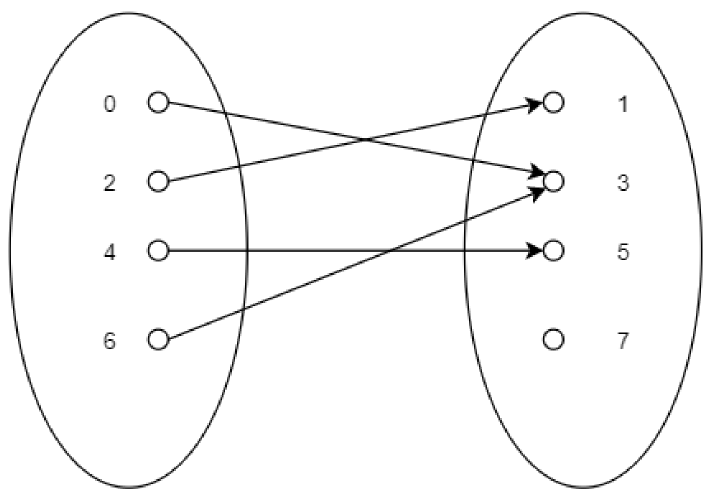
DM Tutorial 10

December 23, 2019

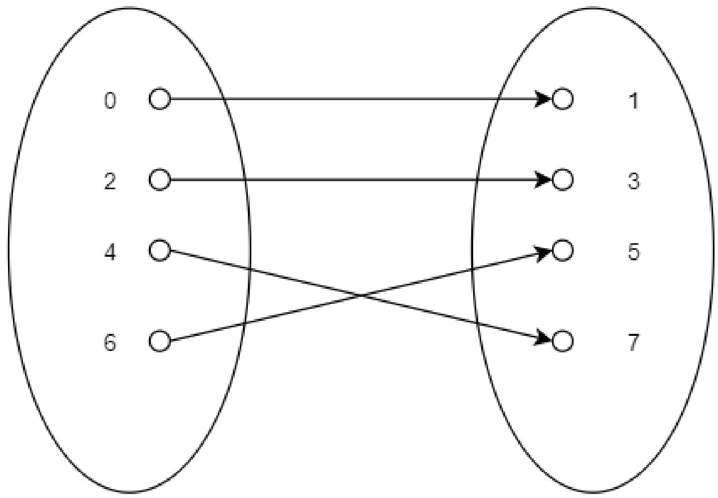
1. Let $A = \{0, 2, 4, 6\}$ and $B = \{1, 3, 5, 7\}$. Determine which of the following relations between A and B forms a function with domain A and codomains B. For those whose are functions, determine whether they are injective, surjective or bijective.

   (a) {(6, 3), (2, 1), (0, 3), (4, 5)}

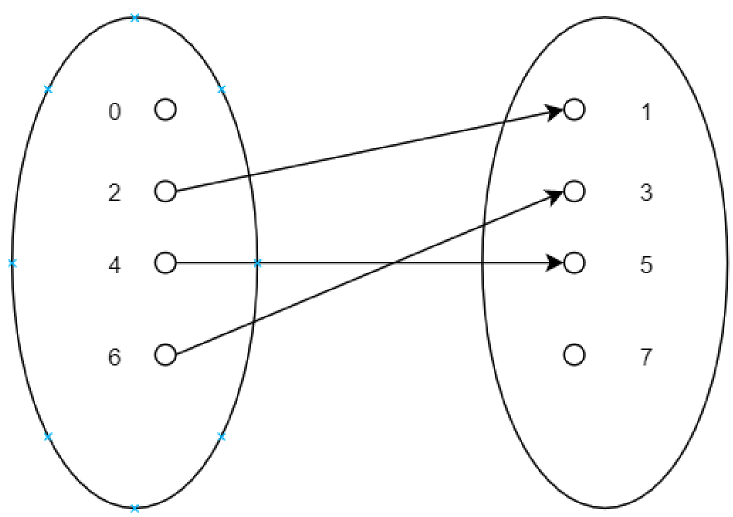
      i.
      ii. f is not a function. (No longer need to determine anything else)

   (b) {(2, 3), (4, 7), (0, 1), (6, 5)}



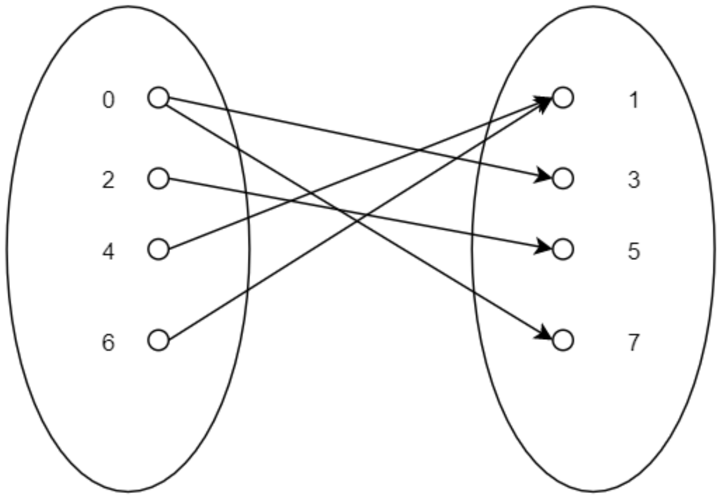
i.
ii. Injective
iii. Surjective
iv. Bijective

(c) {(2, 1), (4, 5), (6, 3)}



i.
ii. Injective
iii. Not surjective
iv. Not bijective

(d) $\{(6, 1), (0, 3), (4, 1), (0, 7), (2, 5)\}$



i.
ii. Not injective
iii. Surjective
iv. Not bijective

2. Let $A = \{-1, 0, 1, 2\}$ and $f : A \to \mathbb{Z}$ be given by $f(x) = \lfloor \frac{x^2+1}{3} \rfloor$

(a) Find the range of f.

i. $f(-1) = \lfloor \frac{(-1)^2+1}{3} \rfloor$

$$\lfloor \frac{2}{3} \rfloor = 0$$

ii. $f(0) = \lfloor \frac{(0)^2+1}{3} \rfloor$

$$\lfloor \frac{(0)^2+1}{3} \rfloor = 0$$

iii. $f(1) = \lfloor \frac{(1)^2+1}{3} \rfloor$

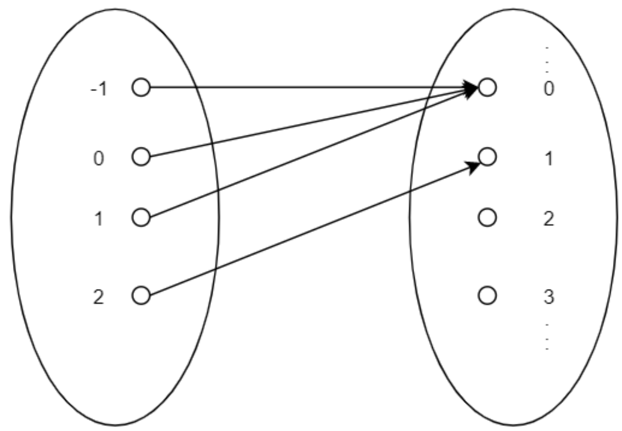
$$\lfloor \frac{(1)^2+1}{3} \rfloor = 0$$

iv. $f(2) = \lfloor \frac{(2)^2+1}{3} \rfloor$

$$\begin{aligned} \lfloor \frac{(2)^2+1}{3} \rfloor &= \lfloor \frac{5}{3} \rfloor \\ &= 1 \end{aligned}$$

v. $R_f = \{0, 5\}$

(b) Determine whether the function f is injective, surjective or bijective. Justify your answer.



i.
ii. Not injective
iii. Not Surjective
iv. Not bijective

3. Given $f(x) = 2x - 1$, a function from $X = \{1, 2, 3\}$ to $Y = \{1, 2, 3, 4, 5\}$. Find the domain and range of the function f. Hence determine whether the function is a bijective function and explain your answer.

$$D_f = \{1, 2, 3\}$$
$$R_f = \{1, 3, 5\}$$

(a) Injective

(b) Not surjective, because $\{2, 4\} \in Y$ but $\{2, 4\} \notin R_f$.

(c) Therefore, the function is NOT bijective.

4. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix}$, $p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 4 & 6 \end{pmatrix}$, $p_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 5 & 4 & 1 \end{pmatrix}$. Compute the following.

(a) p_1^{-1}

$$\begin{aligned} p_1^{-1} &= \begin{pmatrix} 3 & 4 & 1 & 2 & 6 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix} \end{aligned}$$

(b) $p_3 \circ p_1$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 5 & 4 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 6 & 3 & 1 & 4 \end{pmatrix}$$
$$= (1,2,5) \circ (3,6,4)$$

(c) p_3^{-1}

$$\begin{aligned} p_3^{-1} &= \begin{pmatrix} 6 & 3 & 2 & 5 & 4 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 5 & 4 & 1 \end{pmatrix} \end{aligned}$$

(d) $p_1^{-1} \circ p_2^{-1}$

$$\begin{aligned} p_2^{-1} &= \begin{pmatrix} 2 & 3 & 1 & 5 & 4 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 4 & 6 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} p_1^{-1} \circ p_2^{-1} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 4 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 6 & 2 & 5 \end{pmatrix} \end{aligned}$$

5. Let $A = \{1,2,3,4,5,6,7,8\}$. Compute the following products.

(a) $(3,5,7,8) \circ (1,3,2)$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 5 & 4 & 7 & 6 & 8 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 5 & 4 & 7 & 6 & 8 & 3 \end{pmatrix}$$

(b) $(2,6) \circ (3,5,7,8) \circ (2,5,3,4)$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 5 & 4 & 7 & 6 & 8 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 5 & 4 & 7 & 6 & 8 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 5 & 4 & 7 & 6 & 8 & 3 \end{pmatrix}$$

6. Let $A = \{1,2,3,4,5,6,7,8\}$. Write each permutation as a product of transpositions.

(a) $(2,1,4,5,8,6)$

$$(2,6) \circ (2,8) \circ (2,5) \circ (2,4) \circ (2,1)$$

(b) $(3,1,6) \circ (4,8,2,5)$

$$(3,6) \circ (3,1) \circ (4,5) \circ (4,2) \circ (4,8)$$

7. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Determine the given permutation is even or odd.

   (a) $(6, 4, 2, 1, 5)$

   $$(6, 5) \circ (6, 1) \circ (6, 2) \circ (6, 4)$$

      i. 4 transpositions, **Even**

   (b) $(4, 8) \circ (3, 5, 2, 1) \circ (2, 4, 7, 1)$

   $$(4, 8) \circ (3, 1) \circ (3, 2) \circ (3, 5) \circ (2, 1) \circ (2, 7) \circ (2, 4)$$

      i. 7 transpositions, **odd**

8. Let $A = \{1, 2, 3, 4, 5\}$. Let $f = (5, 3, 2)$ and $g = (3, 4, 1)$ be permutations of A. Compute each of the following and write the results as the product of disjoint cycles.

   (a) $f \circ g$

   $$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 2 & 4 & 3 \end{pmatrix}$$
   $$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix}$$
   $$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 \end{pmatrix}$$

   (b) $f^{-1} \circ g^{-1}$

   $$f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix}$$
   $$g^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 3 & 5 \end{pmatrix}$$

   $$\begin{aligned} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 3 & 5 \end{pmatrix} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 3 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \end{pmatrix} \end{aligned}$$

9. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 1 & 5 & 6 \end{pmatrix}$ be a permutation of A.

   (a) Write p as a product of disjoint cycles.

   $$\begin{aligned} p &= (1, 2, 4) \circ (3) \circ (5) \circ (6) \\ &= (1, 2, 4) \end{aligned}$$

(b) Compute p^{-1}

$$
\begin{aligned}
p^{-1} &= \begin{pmatrix} 2 & 4 & 3 & 1 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 2 & 5 & 6 \end{pmatrix}
\end{aligned}
$$

$$p^{-1} = (1,4) \circ (3) \circ (5) \circ (6)$$

(c) Compute p^2 (basically, hop two times)

$$
\begin{aligned}
p^2 &= p \circ p \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 1 & 5 & 6 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 1 & 5 & 6 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 2 & 5 & 6 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
p^2 &= (1,4,2) \circ (3) \circ (5) \circ (6) \\
&= (1,4,2)
\end{aligned}
$$