DM Chapter 6 Notes

January 18, 2020

1 Partially Ordered Sets

1. **Partial order:** Reflexive, anti-symmetric, transitive relation set.
   (a) Partial order cause elements inside are "partially" comparable (ordered). Basically, not all are comparable.
2. **Poset:** Partially ordered set.
   (a) A poset denoted as (A, R) means that it is a partially ordered set derived from set A after applying the relation R, where,
      i. A : set
      ii. R : partial order R relation
   (b) **Example**: $(A, \subseteq)$ means it is a partially ordered set (poset) derived from set A after applying the relation $\subseteq$ onto set A.
   (c) However,
      i. the usual relations with equality $(\leq, \geq)$ are posets on Z^+
      ii. Non-equality ones $(<, >)$ are not because they are NOT reflexive.

1.1 Duality

1. **Dual poset**: $\left(A, R^{-1}\right)$, if we have a poset (A, R).
2. Example: $(A, \leq)$ and $(A, \geq)$ **on** $\mathbb{Z}$ **or** $\mathbb{R}$.
   (a) Note: Unless otherwise specified (like above example), $\leq$ and $\geq$ symbols are used for relations in poset terms.
3.

Poset	Dual Poset
$(A, \leq)$	$(A, \geq)$
$(A, \leq_1)$	$(A, \geq_1)$
$(B, \leq')$	$(B, \geq')$

1.2 Comparability

1. If $(A, \leq)$ is a poset, elements a, b in set A are comparable if
   (a) $a \leq b$, (a is related to b, note we use $\leq$ to replace R, this is NOT "less than or equal" in poset terms) OR,
   (b) $b \leq a$ (b is related to a)
2. **Example**: $(A, \leq) = (Z^+, |)$. The "|" sign represents divides.
   (a) Comparable example: 2 and 6
      i. $2 \leq 6$ (2 is related to 6) or $2|6$ (2 divides 6)
   (b) Not comparable example: 2 and 7 since $2 \nmid 7$ and $7 \nmid 2$.

1.3 Linear order

1. **Linearly ordered set:** A set where every pair of elements are comparable, also known as **chain.**
2. **Linear order:** Linearly ordered partial order
3. **Example**: $(Z^+, \leq)$. Every element pair (remember this time its Z^+) is comparable (less than or equal) to another element.

1.4 Theorem 1: Cartesian product of posets

1. If $(A, \leq)$ and $(B, \leq)$ are posets, then $(A \times B, \leq)$ is a poset, with partial order defined $\leq$ defined by:
   (a) $(a, b) \leq (a', b')$ **if** $a \leq a'$ in A and $b \leq b'$ in B . (Huge if, if it does not satisfy, then it is not the valid)
   (b) The above reads as: One of the elements in (a, b) is related to one of the elements in (a', b'), if a is related to a' and b is related to b'. That is, if each element is related independently, then they must also be (still) related by a certain relation even if we mash them together.
   (c) The '$\leq$' sign is used to denote 3 distinct partial order (you can see 3 '$\leq$' there).
   (d) **Product partial order:** $(A \times B, \leq)$
   (e) Let $A = \{1, 3, 5\}$, $B = \{2, 4, 8\}$, and $\leq_A$ means "less than or equal to", | means "divides", then $(A, \leq_A)$ and $(B, |)$ are posets.
      i. $(A \times B, \leq)$ is also a poset, as example
         A. $1 \leq_A 3, 2 \mid 4$, $(1, 2) \leq (3, 4)$
         B. $3 \leq_A 5, 4 \mid 8, (3, 4) \leq (5, 8)$
      ii. Then, we also have
         A. $(1, 2) \leq (5, 8)$ because $1 \leq_A 5, 2 \mid 8$

1.5 Lexicographic order

1. **Lexicographic order**: Dictionary order of tuples in posets
   (a) Use first element of each tuples as ordering, if “tie”, then use second elements in each tuples.
2. **Definition**: $(A \times B, \prec) = (a, b) \prec (a', b')$ if $a < a'$ or if $a = a'$ and $b \leq b'$
   (a) Note: suppose $(A, \leq)$ and $(B, \leq)$ are posets. They can have different relations, $\leq_1$ and $\leq_2$, or same.
   (b) Note 2: Notice that for partial order, w use $a \leq a'$, or the related sign. but for lexicographic order, we have $a < a'$. In lexicographic, NO 2 elements (or tuples) can be exactly the same.
      i. Another reason why we use the '$\prec$' precedes sign.
   (c) Note 3: Lexicographic ordering can be extended to more cartesian products, e.g. $A_1 \times A_2 \times A_3$. In this case, we can extend our first coordinate to $a_1 = a_1'$, $a_2 = a_2'$, and $a_3 < a_3'$ or ... (b part)
   (d) Note 4: We can also use words, for example $par\boldsymbol{k} \prec par\boldsymbol{t}$
      i. k precedes t
3. Extending lexicographic order to S^*.
   (a) If we have $x = a_1a_2a_3...a_n$, $y = b_1b_2b_3...b_k$, and $n \leq k$ in S^*, we can “extend” (extending influence, not characters) n to k.
      i. We can say that $x \prec y$ if $(a_1a_2...a_n) \prec (b_1b_2...b_n)$, even if y has k elements which is more than n.
      ii. That’s because if the front dominates, the back doesn’t matter.
      iii. Example:

$$park \prec \boldsymbol{part} \implies park \prec \boldsymbol{part}ition$$

1.6 Theorem 2: Partial order digraph cycles

1. Digraph of partial order has no cycle of length greater than 1.
2. This should make sense, because a partial order is **antisymmetric**.

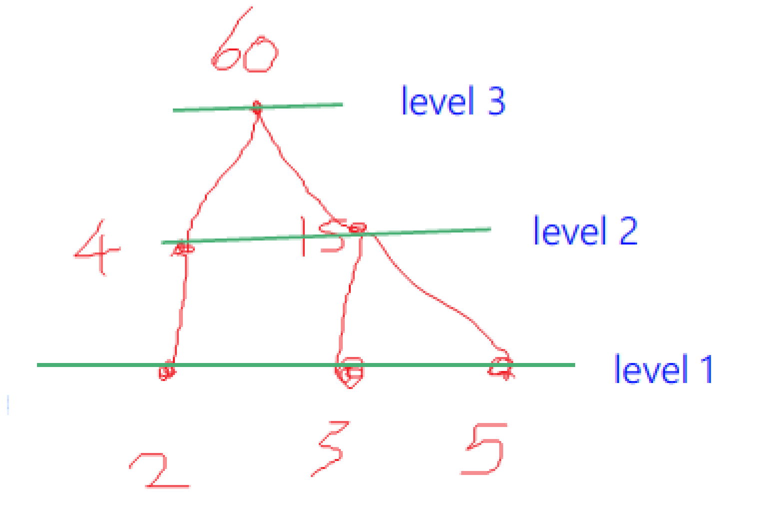
2 Hasse Diagram

1. Simplification of digraph
2. Steps:
   (a) Omitting all cycles of length 1 (or loops) (imply reflexivity)
   (b) Omitting all edges implying transitivity

(c) Drawing all edges slanting upwards to eliminate arrows, or 'stretch into lines moving upwards'

(d) Represent vertices by dots

3. Always good to break them into separate levels, for example:



4. Duality in Hasse Diagram

   (a) If $(A, \leq)$ is a poset and $(A, \geq)$ is the dual poset, then the Hasse diagram of $(A, \geq)$ is just the Hasse diagram of $(A, \leq)$ turned **upside down**.

5. Geometry in Hasse Diagram

   (a) Some Hasse Diagram can be drawn in a cube, cuboid and so on. If you can see it, then draw it as so, cause its easier to see.

2.1 Topological Sorting

1. **Topological sorting:** Process of constructing a linear order (such as $\prec$)

2. **For partial orders:** Usually involves extending partial orders to "linearize" them so that if $a \leq b$, then $a \prec b$.

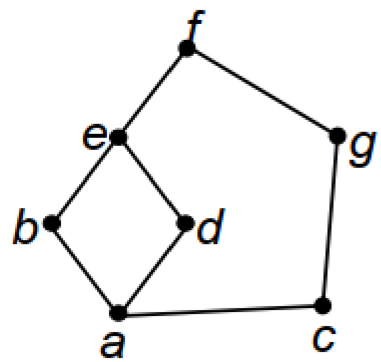
3. **Problems with poset:** When entering into computer, we must enter in order, but we want the poset to be preserved at the same time (recall that not all elements in posets are comparable)

   (a) If $a \leq b$, then a is entered **before** b

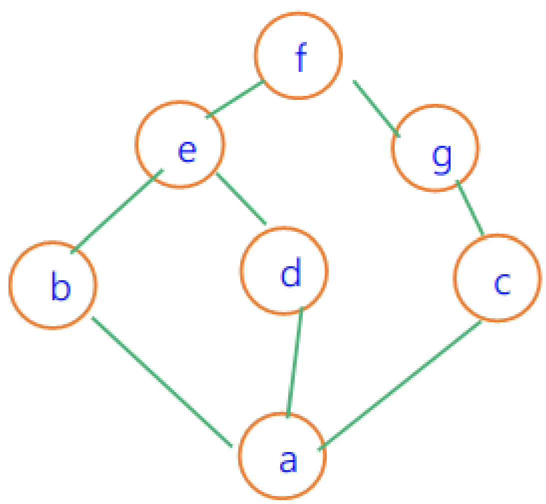
   (b) Topological sorting gives order of entry that meets this condition.

4. To topological sort posets, we go from botton up

(a) Hasse Diagram example

i. 

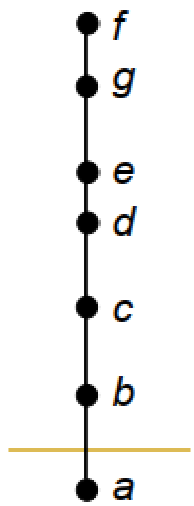
A. Before layering

ii. 

A. After layering

(b) After topological sorting, it can appear as either one of these:

i. Layer-by-layer, bottom to up, smallest element first

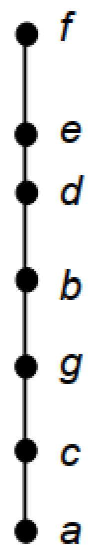


A.

ii. Chain-by-chain (until the point where both chain join), left-chain-to-right-chain, smallest element first.

A.

iii. Chain-by-chain (until the point where both chain join), right-chain-to-left-chain, smallest element first.

A. 

2.2 Isomorphism

1. **Isomorphism**: a function, that maps one poset to another poset.
   (a) In math: $a \leq b$ iff $f(a) \leq' f(b)$, assuming:
      i. $(A, \leq)$ and $(A', \leq')$ are posets
      ii. a and b are elements in A.
      iii. $f : A \to A'$ is a one-to-one corresponding function.
         A. **one-to-one corresponding:** one-to-one, onto, everywhere defined
   (b) This reads as a is related to b in A if and only if the mapping of a in A' is related to the mapping of b in A'.
2. **Isomorphic posets:** Posets that form an isomorphism.
   (a) In this case, $(A, \leq)$ and $(A', \leq')$
3. **Example:**
   (a) Information
      i. A : the set of positive integers
      ii. A' : the set of positive even integers
      iii. $\leq$: partial order on A
         A. The poset of A is $(A, \leq)$
         B. **Note**: we're on applications, so this '$\leq$' sign reads as 'less than or equal'
      iv. $\leq'$: partial order on A'

A. The poset of A is $(A, \leq')$

B. **Note**: we're on applications, so this '$\leq$' sign reads as 'less than or equal'

v. $f : A \to A'$: $f(a) = 2a$

(b) **Question**: Show that f is an isomorphism

(c) **Answer:**

i. First, check if the function is one-to-one corresponding. If fail, then conclude f is NOT an isomorphism.

A. Since f is one-to-one, onto, everywhere defined, f is **one-to-one corresponding**

ii. Second, check relations. Check if they satisfy: $a \leq b \iff f(a) \leq' f(b)$

A. $f(a) = 2a$, $f(b) = 2b$. Both of them are even integers.

B. If $a \leq b$ then $2a \leq 2b$, if $2a \leq 2b$, then $a \leq b$. Therefore, $a \leq b \iff f(a) \leq' f(b)$

C. Thus, f is an **isomorphism**.

2.3 Theorem 1: Principle of Correspondence (Check with teacher about B' sign)

1. **Conditions**:

(a) Elements of set B has a property related to elements in set A.

i. So $(B, \leq)$ is a poset, and,

ii. $(A, \leq')$ is another poset

(b) The property, can be defined ENTIRELY in terms of a relation, $\leq$

i. note: we're on definitions again, so this sign, $\leq$ reads as 'relation'

2. **Result:**

(a) Elements of B' must possess same property defined in terms of $\leq'$

i. This reads as: the elements of B when mapped to set A (essentially, $f(b)$, where b is an element of B) must possess the same property define in terms of $\leq'$.

ii. The reason is because the elements in B behave similarly to elements in A.

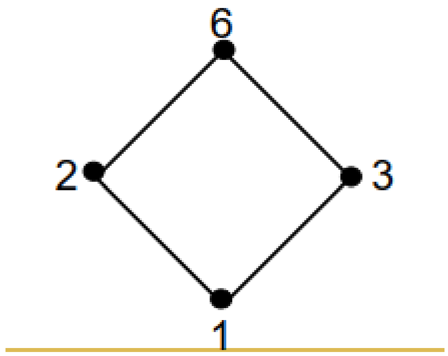
iii. So, when they map to A, since they behave similarly, they must also satisfy the relation R'.

(b) **Hasse Diagram:** In this case, two finite isomorphic posets must have the same Hasse diagram.Or in other words, they got 'relabeled'.

i. Let $A = \{1, 3, 4, 6\}$, let $\leq$ be the relation $|$ (divides)

A. The poset is $(A, |)$

B. Hasse diagram:



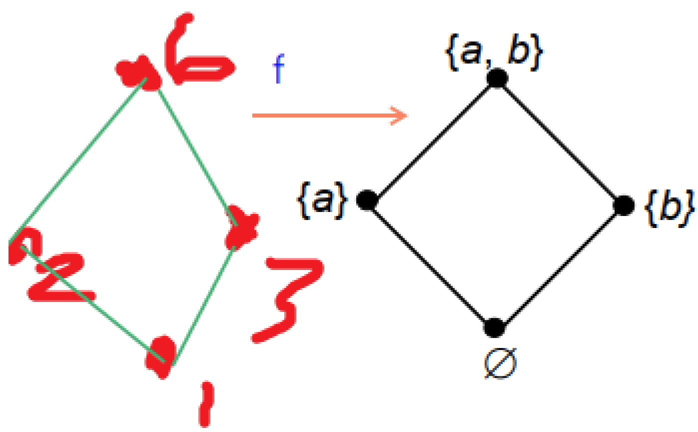
ii. Let $A' = \wp(\{a,b\}) = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$, let $\leq'$ be set containment (or subset), $\subseteq$

iii. If $f : A \to A'$ is defined by $f(1) = \varnothing, f(2) = \{a\}, f(3) = \{b\}, f(6) = \{a,b\}$

A. Then, f is one-to-one corresponding.

B. f is **order preserving**: $x|y \iff f(x) \subseteq f(y)$

iv. Therefore, the 'relabeled' hasse diagram of $(A', \leq')$ would be



v. Therefore, the function f is isomorphism.

3 Extremal Elements of Partial Ordered Sets

1. Let's say we have a random poset $(A, \leq)$

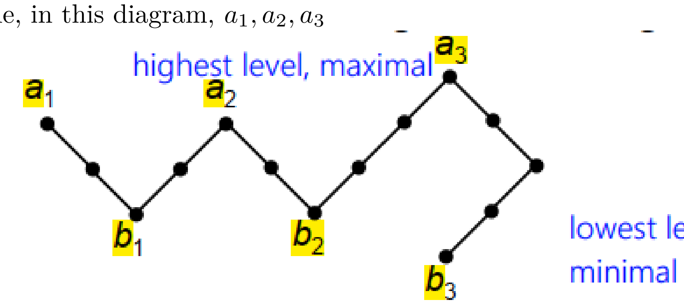
   (a) **Maximal element**: The maximum element in A

      i. An element $a \in A$ where no element $c \in A$ is $> a$.

   (b) **Minimal element:** The minimal element in A

      i. An element $b \in A$ where no element $c \in A$ such that $c < b$.

(c) There can be multiple max/minimal element if they are not "connected" in a Hasse diagram. For example, in this diagram, a_1, a_2, a_3



are maximals, and $b_1, b_2, b_3$are minimals

(d) A poset can also have no max/min elements, e.g. the $\mathbb{Z}$

3.1 Finding Topological Sorting of a Finite Poset $(A, \leq)$

1. **Theorem:** A finite, nonempty poset with partial order $\leq$ has at least 1 max & 1 min element.

2. We can use algorithm to find topological sorting of finite poset $(A, \leq)$

   (a) If $a \in A$ and $B = A - \{a\}$, then B is also a poset under the restriction of $\leq$ to $B \times B$ (tuple of values of B).

   (b) We can produce a linear array SORT by order of increasing index, such that $SORT[1] \prec SORT[2] \prec ...$, AKA topological sorting of $(A, \leq)$

   (c) Algorithm:

      i. Step 1: Choose a minimal element of A.

      ii. Step 2: Make a next entry of SORT and replace A with $A - \{a\}$.

      iii. Step 3: Repeat steps 1 and 2 until $A = \varnothing$.

      iv. End

3.2 Greatest & Least Element

1. **Greatest element**: Largest element in A

   (a) If $x \leq a$ for all $x \in A$

   (b) Denoted by 1, unit element

2. **Least element:** Smallest element in A

   (a) If $a \leq x$ for all $x \in A$

   (b) Denoted by 0, zero element

3. **Theorem 2:** Can at most have one greatest & one least for each poset,

   (a) Note: in a power set, the $\varnothing$ set is the least element.

3.3 Upper Bound & Lower Bound

1. **Consider a poset A and B, a subset of A**

   (a) **Upper bound**: The values bounding the top of subset B.

      i. An element $a \in A$ in B such that $b \leq a$ for all $b \in B$.

   (b) **Lower bound:** The values bounding the bottom of the subset B

      i. An element $a \in A$ in B such that $b \geq a$ for all $b \in B$.

   (c) Subset B may not have upper/lower bounds in A, and may not belong to part of the subset.

4 Least Upper Bound and Greatest Lower Bound

1. **Least upper bound:** The upper bound element closest to the subset.

   (a) AKA LUB(B): An element $a \in A$, such that if a is an upper bound of B and $a \leq a'$, whenever a' is an upper bound of B.

   (b) **Hasse Diagram:** First vertex reachable upwards from B elements

2. **Greatest lower bound:** The lower bound element closest to the subet.

   (a) AKA GLB(B): An element $a \in A$, such that if a is an lower bound of B and $a \geq a'$, whenever a' is an upper bound of B.

   (b) **Hasse Diagram:** First vertex reachable downward from B elements.

3. **Duality:**

   (a) **Upper bounds** in $(A, \leq)$ = lower bounds in $(A, \geq)$.

   (b) **Lower bounds** in $(A, \leq)$ = upper bounds in $(A, \geq)$.

   (c) Same for GLB and LUB

4. **Theorem 3: Uniqueness of LUB & GLB**

   (a) A subset B of A (where A is a poset) has at most 1 LUB and 1 GLB.