Ginwidth=

Calc II Tutorial 2

October 15, 2019

1. Find the length of arc of the curve:

   (a) $x = e^t cost,\ y = e^t \sin t, 0 \leq t \leq \pi$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\begin{aligned} \frac{dx}{dt} &= e^t \cos t + e^t\left(-\sin t\right) \\ &= e^t \cos t - e^t \sin t \\ &= e^t\left(\cos t - \sin t\right) \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt} &= e^t \sin t + e^t\left(\cos t\right) \\ \frac{dy}{dt} &= e^t\left(\sin t + \cos t\right) \end{aligned}$$

$$\begin{aligned} L &= \int_0^\pi \sqrt{\left(e^t\left(\cos t - \sin t\right)\right)^2 + \left(e^t\left(\sin t + \cos t\right)\right)^2} \\ &= \int_0^\pi \sqrt{\left(e^t\left(\cos t - \sin t\right)\right)^2 + \left(e^t\left(\sin t + \cos t\right)\right)^2} \\ &= \int_0^\pi \sqrt{e^{2t}\left(\cos t - \sin t\right)^2 + e^{2t}\left(\sin t + \cos t\right)^2} \\ &= \int_0^\pi \sqrt{e^{2t}\left[\left(\cos t - \sin t\right)^2 + \left(\sin t + \cos t\right)^2\right]} \\ &= \int_0^\pi e^t \sqrt{1 - \sin\left(2t\right) + 1 + \sin\left(2t\right)} \\ &= \sqrt{2}\int_0^\pi e^t \\ &= \sqrt{2}\left(e^\pi - e^0\right) \\ &= \sqrt{2}\left(e^\pi - 1\right) \end{aligned}$$

(b) $x = e^t + e^{-t}, y = 5 - 2t, 0 \leq t \leq 3$

$$\frac{dx}{dt} = e^t - e^{-t}$$

$$\frac{dy}{dt} = -2$$

$$\begin{aligned} L &= \int_0^3 \sqrt{\left(e^t - e^{-t}\right)^2 + \left(-2\right)^2} dt \\ &= \int_0^3 \sqrt{e^{2t} + e^{-2t} - 2 + 4} dt \\ &= \int_0^3 \sqrt{e^{2t} + e^{-2t} + 2} dt \end{aligned}$$

i. Let $u = 2t$

$$\begin{aligned} \frac{du}{dt} &= 2 \\ du &= 2dt \end{aligned}$$

ii. Find the bounds with respect to u

$$\begin{aligned} u|_{x=0} &= 0 \\ u|_{x=3} &= 2\left(3\right) \\ &= 6 \end{aligned}$$

iii. Find the final equation

$$\begin{aligned} L &= \frac{1}{2} \int_0^6 \sqrt{e^u + e^{-u} + 2} du \\ &= \frac{1}{2} \int_0^6 \sqrt{e^u + e^{-u} + 2} du \\ &= \frac{1}{2} \left(-\frac{-2e^6 + 2}{e^3} \right) \\ &= -\frac{1 + e^6}{e^3} \\ &= e^3 - \frac{1}{e^3} \end{aligned}$$

(c) $y = \frac{x^3}{6} + \frac{1}{2x}, \frac{1}{2} \leq x \leq 1$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3x^2}{6} + \frac{1}{2}\left(-1\right)x^{-2} \\ &= \frac{x^2}{2} - \frac{1}{2x^2} \end{aligned}$$

$$
\begin{aligned}
L &= \int_a^b ds \\
&= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
&= \int_{\frac{1}{2}}^1 \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2} dx \\
&= \int_{\frac{1}{2}}^1 \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}} dx \\
&= \int_{\frac{1}{2}}^1 \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}} dx \\
&= \int_{\frac{1}{2}}^1 \sqrt{\frac{1}{2} + \frac{x^4}{4} + \frac{1}{4x^4}} dx \\
&= \frac{31}{48}
\end{aligned}
$$

(d) $x = \frac{y^5}{20} + \frac{1}{3y^3}$, from $y = 1$ to $y = 2$

i. Find the differentiation

$$
\begin{aligned}
\frac{dx}{dy} &= \frac{5y^4}{20} + \frac{1}{3} \cdot (-3) \cdot \left(y^{-4}\right) \\
\frac{dx}{dy} &= \frac{5y^4}{20} - y^{-4}
\end{aligned}
$$

ii. Integrate

$$
\begin{aligned}
L &= \int_a^b ds \\
&= \int_1^2 \sqrt{\left(\frac{dx}{dy}\right)^2 + 1 dy} \\
&= \int_1^2 \sqrt{\left(\frac{5y^4}{20} - y^{-4}\right)^2 + 1 dy} \\
L &= \frac{221}{120}
\end{aligned}
$$

2. Find the area of the surface generated when the arc of the curve $y = 6x$

between $x=0$ and $x=1$ rotates about the x-axis

$$\begin{aligned}
S &= 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \\
&= 2\pi \int_0^1 6x\sqrt{1+(6)^2}dx \\
&= 12\sqrt{37}\pi \int_0^1 x \\
&= 12\sqrt{37}\pi \int_0^1 x \\
&= 12\sqrt{37}\pi \cdot \frac{1}{2} \\
&= \sqrt{1332}\pi
\end{aligned}$$

3. Find the area of the surface generated when the arc of the curve $y = \sqrt{25-x^2}$, $-2 \leq x \leq 3$ rotates about the x-axis.

   (a) Find the derivative

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[\left(5^2-x^2\right)^{\frac{1}{2}}\right] \\
&= \frac{1}{2}\left(5^2-x^2\right)^{-\frac{1}{2}} \cdot (-2x) \\
&= -\frac{x}{\sqrt{25-x^2}}
\end{aligned}$$

$$\begin{aligned}
S &= 2\pi \int_{-2}^{3} \sqrt{25-x^2} \cdot \sqrt{1+\left(-\frac{x}{\sqrt{25-x^2}}\right)^2}dx \\
&= 2\pi \int_{-2}^{3} \sqrt{25-x^2} \cdot \sqrt{1+\frac{x^2}{25-x^2}}dx \\
S &= 50\pi
\end{aligned}$$

4. Find the area of the surface generated when the arc of the curve $x=t$, $y=t^3$, $0 \leq t \leq 1$, rotates about the $x-axis$ through a complete revolution.

   (a) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$

$$\begin{aligned}
\frac{dx}{dt} &= 1 \\
\frac{dy}{dt} &= 3t^2
\end{aligned}$$

(b) Find the integration

$$\begin{aligned}
S &= 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= 2\pi \int_0^1 t^3\sqrt{1+(3t^2)^2}\,dt \\
&= 2\pi \int_0^1 t^3\sqrt{1+9t^4}\,dt \\
S &= \frac{\pi\left(10\sqrt{10}-1\right)}{27}
\end{aligned}$$

5. Find the surface area generated when the arc of the curve $y = \frac{x^3}{12} + \frac{1}{x}, 1 \leq x \leq 4$ rotates about the x-axis.

(a) Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{x^2}{4} - \frac{1}{x^2}$$

(b) Find the integral

$$\begin{aligned}
S &= 2\pi \int_1^4 \left(\frac{x^3}{12} + \frac{1}{x}\right) \cdot \sqrt{1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2}\,dx \\
&= 2\pi \int_1^4 \left(\frac{x^3}{12} + \frac{1}{x}\right) \cdot \sqrt{1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2}\,dx \\
&= 34\frac{3}{8}\pi \\
&= \frac{275}{8}\pi
\end{aligned}$$

6. Find the area of the surface obtained by rotating the curve $x = \sqrt{a^2 - y^2}, 0 \leq y \leq \frac{a}{2}$ about the y-axis.

(a) Find $\frac{dx}{dy}$

$$\begin{aligned}
x &= \left(a^2 - y^2\right)^{\frac{1}{2}} \\
\frac{dx}{dy} &= -\frac{y}{(a^2 - y^2)^{\frac{1}{2}}}
\end{aligned}$$

(b) Find the integral

$$
\begin{aligned}
S &= 2\pi \int_0^{\frac{a}{2}} \sqrt{a^2 - y^2} \cdot \sqrt{1 + \left(- \frac{y}{(a^2 - y^2)^{\frac{1}{2}}} \right)^2} dy \\
&= 2\pi \int_0^{\frac{a}{2}} \sqrt{a^2 - y^2} \cdot \sqrt{1 + \frac{y^2}{a^2 - y^2}} dy \\
&= 2\pi \int_0^{\frac{a}{2}} \sqrt{a^2 - y^2} \cdot \sqrt{\frac{a^2 - y^2 + y^2}{a^2 - y^2}} dy \\
&= 2\pi \int_0^{\frac{a}{2}} \sqrt{a^2 - y^2} \cdot \sqrt{\frac{a^2}{a^2 - y^2}} dy \\
&= 2\pi \int_0^{\frac{a}{2}} \cancel{\sqrt{a^2 - y^2}} \cdot \frac{\sqrt{a^2}}{\cancel{\sqrt{a^2 - y^2}}} dy \\
&= 2\pi \int_0^{\frac{a}{2}} \sqrt{a^2} dy \\
&= 2\pi \int_0^{\frac{a}{2}} a dy \\
&= 2a\pi \int_0^{\frac{a}{2}} 1 \, dy \\
&= 2a\pi \, [y]_0^{\frac{a}{2}} \\
&= 2a\pi \left(\frac{a}{2} \right) \\
&= \pi a^2
\end{aligned}
$$

7. Find the area of the surface generated when the arc of the curve $x = 1 + 2y^2, 1 \leq y \leq 2$,rotates about the $x - axis$.

   (a) Find y

$$
\begin{aligned}
2y^2 &= x - 1 \\
y^2 &= \frac{x - 1}{2} \\
y &= \sqrt{\frac{x - 1}{2}}
\end{aligned}
$$

(b) Find $\frac{dy}{dx}$

$$\begin{aligned}
y &= \sqrt{\frac{x-1}{2}} \\
y &= \sqrt{\frac{1}{2}} \cdot \sqrt{x-1} \\
\frac{dy}{dx} &= \sqrt{\frac{1}{2}} \cdot \frac{1}{2\sqrt{x-1}} \\
&= \frac{\sqrt{1}}{2\sqrt{2}\sqrt{x-1}}
\end{aligned}$$

(c) Find the boundaries in terms of x

$$\begin{aligned}
x|_{y=1} &= 1+2 \\
&= 3
\end{aligned}$$

$$\begin{aligned}
x|_{y=2} &= 1+2\,(2)^2 \\
&= 1+8 \\
&= 9
\end{aligned}$$

(d) Find integral

$$\begin{aligned}
S &= 2\pi \int_3^9 \left(\sqrt{\frac{x-1}{2}}\right) \cdot \sqrt{1+\left(\frac{\sqrt{1}}{\sqrt{8}\sqrt{x-1}}\right)^2}\,dx \\
&= 2\pi\left(\frac{65\sqrt{65}-17\sqrt{17}}{48}\right) \\
&= \pi\left(\frac{65\sqrt{65}-17\sqrt{17}}{24}\right) \\
&= 18.9147\pi \\
&\approx 18.91\pi
\end{aligned}$$