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Calculus 1 T9

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1 Question 1

Find the dimensions of a rectangle with perimeter 100m whose area is as large as possible.

$$\begin{aligned} x &= width \\ y &= length \end{aligned}$$

$$\begin{aligned} P &= perimeter \\ 2x + 2y &= 100 \\ x + y &= 50 \end{aligned}$$

Requirements: Area as large as possible.

$$A = xy$$

Try to make the equation in terms of x

$$y = 50 - x$$

Substitute back into area formula

$$\begin{aligned} A &= x\,(50 - x) \\ &= 50x - x^2 \end{aligned}$$

$$\frac{dA}{dx} = 50 - 2x$$

Find the stationary point (where maximum/mininum values occur)

$$\begin{aligned} \frac{dA}{dx} &= 0 \\ 50 - 2x &= 0 \\ x &= 25 \end{aligned}$$

Find y

$$\begin{aligned} y &= 50 - 25 \\ &= 25 \end{aligned}$$

Dimensions

$$25m * 25m$$

2 Question 2

A box with an open top is to be constructed from a square piece of cardboard, 0.9m wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

$$\begin{aligned} Sides &= s \\ &= 0.9 \end{aligned}$$

$$V = sides^2 * h$$

$$\begin{aligned} A &= area \\ &= s^2 \\ &= 0.81m \end{aligned}$$

$$Cuts = x$$

Sides after cutted

$$s = 0.9 - 2x$$

Height of box

$$x$$

Find the volume formula

$$\begin{aligned} V &= (0.9 - 2x)^2 * x \\ &= x\,(0.9 - 2x)^2 \\ &= x\left(4x^2 - 3.6x + 0.81\right) \\ &= 4x^3 - 3.6x^2 + 0.81x \end{aligned}$$

Find the derivative

$$\begin{aligned} \frac{dV}{dx} &= 4x^3 - 3.6x^2 + 0.81x \\ &= 12x^2 - 7.2x + 0.81 \end{aligned}$$

Find the stationary point

$$\frac{dV}{dx} = 0$$
$$12x^2 - 7.2x + 0.81 = 0$$
$$4x^2 - 2.4x + 0.27 = 0$$
$$x = \frac{9}{20}, x = \frac{3}{20}$$

Find the volume:
When $x = \frac{9}{20}$

$$V = 4\left(\frac{9}{20}\right)^3 - 3.6\left(\frac{9}{20}\right)^2 + 0.81\left(\frac{9}{20}\right)$$
$$= 0m^3\ (ignored, volume\, cannot = 0)$$

When $x = \frac{3}{20}$

$$V = 4\left(\frac{3}{20}\right)^3 - 3.6\left(\frac{3}{20}\right)^2 + 0.81\left(\frac{3}{20}\right)$$
$$= 0.054m^3$$

Conclusion

$$V = 0.054m^3$$

3 Question 3

Find the point on the line y = 4x + 7 that is closest to the origin
Question ask for (x, y)pair closest to $(0, 0)$

$$y = 4x + 7$$

$$distance = s = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$
$$s = \sqrt{(y - 0)^2 + (x - 0)^2}$$
$$= \left(y^2 + x^2\right)^{\frac{1}{2}}$$

Substitute y for $4x+7$

$$\begin{aligned}
s &= \left((4x+7)^2 + x^2\right)^{\frac{1}{2}} \\
&= \left(17x^2 + 28x + 28x + 49\right)^{\frac{1}{2}} \\
\frac{ds}{dt} &= \frac{1}{2}\left(17x^2 + 56x + 49\right)^{-\frac{1}{2}} \cdot (34x+56) \\
\frac{ds}{dt} &= \frac{(34x+56)}{2\sqrt{17x^2+56x+49}} \\
&= \frac{17x+28}{\sqrt{17x^2+56x+49}}
\end{aligned}$$

When $\frac{ds}{dt} = 0$

$$\begin{aligned}
\frac{17x+28}{\sqrt{17x^2+56x+49}} &= 0 \\
17x+28 &= 0 \\
x &= -\frac{28}{17} \\
&= -\frac{28}{17}
\end{aligned}$$

Check y

$$\begin{aligned}
y &= 4\left(-\frac{28}{17}\right) + 7 \\
&= \frac{7}{17}
\end{aligned}$$

4 Question 4

A piece of wire 10m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is a minimum?

1. Let the amount cut for the square be x. The amount left for the equilateral triangle will be $10 - x$
2. Side length of square is $\frac{x}{4}$

$$\begin{aligned}
A_s &= \left(\frac{x}{4}\right)^2 \\
&= \frac{x^2}{16}
\end{aligned}$$

3. Area of equilateral triangle with side length x is $\frac{\sqrt{3}}{4}x^2$

(a) Side length of equilateral triangle is $\frac{10-x}{3}$

(b) Area of equilateral triangle

$$\frac{\sqrt{3}}{4}\left(\frac{10-x}{3}\right)^2 = \frac{\sqrt{3}\left(10-x\right)^2}{36}$$

4. Total area $= A_s + A_t$

$$A\left(x\right) = \frac{x^2}{16} + \frac{\sqrt{3}\left(10-x\right)^2}{36}$$

(a) Differentiate

$$A'\left(x\right) = \frac{x}{8} - \frac{\sqrt{3}\left(10-x\right)}{18}$$

5. Solve $A'\left(x\right) = 0$

$$\frac{x}{8} - \frac{\sqrt{3}\left(10-x\right)}{18} = 0$$

$$\frac{x}{8} = \frac{\sqrt{3}\left(10-x\right)}{18}$$

(a) Cross Multiply

$$18x = 8\sqrt{3}\left(10-x\right)$$

$$18x = 80\sqrt{3} - 8\sqrt{3}x$$

$$\left(18+8\sqrt{3}\right)x = 80\sqrt{3}$$

$$x = \frac{80\sqrt{3}}{\left(18+8\sqrt{3}\right)}$$

$$\approx 4.35$$

6. Evaluate the endpoints

$$A\left(0\right) = \frac{0}{16} - \frac{\sqrt{3}\left(10-0\right)^2}{36}$$

$$\approx 4.81$$

$$A\left(4.35\right) = \frac{4.35}{16} - \frac{\sqrt{3}\left(10-4.35\right)^2}{36}$$

$$\approx 2.72$$

$$A\left(10\right) = \frac{10}{16} - \frac{\sqrt{3}\left(10-10\right)^2}{36}$$

$$= 6.25$$

7. For minimum area, $x = 4.35m$

5 Question 5

For the given cost and price functions, find the production level that will maximize profit.

$$C(x) = 16000 + 500x - 1.6x^2 + 0.004x^3$$

$$p(x) = 1700 - 7x$$

$$\begin{aligned} C'(x) &= 500 - 3.2x + 0.012x^2 \\ R(x) &= x \cdot p(x) \\ &= 1700x - 7x^2 \\ R'(x) &= 1700 - 14x \end{aligned}$$

To maximize the profit, marginal revenue must equal marginal cost

$$\begin{aligned} C'(x) &= R'(x) \\ 500 - 3.2x + 0.012x^2 &= 1700 - 14x \\ 0.012x^2 + 10.8x - 1200 = 0 \\ x &= 100, -1000 \end{aligned}$$

Therefore

$$x = 100$$

6 Question 6

Suppose $p(x) = 100 - 0.04x$ and $c(x) = 500 + 40x + 0.04x^2$ are the price function (in dollars per unit) and cost function (in dollars) and x is the number of units produced. Find the production level when profit is maximized. What is the price and the average cost at this optimum level?

1. Let P be the profit. For the profit to be maximized, we need the marginal revenue, R to be at least equal to the marginal cost.

$$\begin{aligned} R(x) &= x \cdot p(x) \\ &= x(100 - 0.04x) \\ &= 100x - 0.04x^2 \end{aligned}$$

$$R'(x) = 100 - 0.08x$$

$$\begin{aligned} c'(x) &= 500 + 40x + 0.04x^2 \\ &= 40 + 0.08x \end{aligned}$$

2. Let the marginal cost be equal to the marginal revenue

$$
\begin{aligned}
R'(x) &= c'(x) \\
100 - 0.08x &= 40 + 0.08x \\
60 &= 0.16x \\
x &= 375
\end{aligned}
$$

3. Fid the price and average cost at this optimum level

   (a) Price

$$
\begin{aligned}
p(375) &= 100 - 0.04(375) \\
&= 85
\end{aligned}
$$

   (b) Average cost

      i. Total cost

$$
\begin{aligned}
c(375) &= 500 + 40(375) + 0.04(375)^2 \\
&= 21125
\end{aligned}
$$

      ii. Average cost

$$
\begin{aligned}
c &= \frac{21125}{375} \\
&= 56.33
\end{aligned}
$$

7 Question 7

Use Newton's method with initial approximation $x_1 = 1$ to find x_3, the third approximation to the root of the equation $x^3 - x^2 - 1 = 0$. (Give your answer to three decimal places.)

1. Let $f(x) = x^3 - x^2 - 1$, $f'(x) = 3x^2 - 2x$
2. $x_1 = 1$
3. x_2

$$
\begin{aligned}
x_2 &= 1 - \frac{1^3 - 1^2 - 1}{3(1)^2 - 2(1)} \\
&= 2
\end{aligned}
$$

4. x_3

$$
\begin{aligned}
x_3 &= 2 - \frac{2^3 - 2^2 - 1}{3(2)^2 - 2(2)} \\
&= 1.625
\end{aligned}
$$

8 Question 8

Use Newton's method to approximate $\sqrt[7]{1000}$ correct to eight decimal places. $[x_0 = 3]$.

1. Let $x = \sqrt[7]{1000}$

$$\begin{aligned} x &= \sqrt[7]{1000} \\ x^7 &= 1000 \\ x^7 - 1000 &= 0 \end{aligned}$$

$$\begin{aligned} f(x) &= x^7 - 1000 \\ f'(x) &= 7x^6 \end{aligned}$$

2. Find the approximation to eight decimal places

$$x_0 = 3$$

$$\begin{aligned} x_1 &= x_0 - \frac{(x_0)^7 - 1000}{7(x_0)^6} \\ &= 2.76739173 \end{aligned}$$

$$\begin{aligned} x_2 &= x_1 - \frac{(x_1)^7 - 1000}{7(x_1)^6} \\ &= 2.690087405 \end{aligned}$$

$$\begin{aligned} x_3 &= x_2 - \frac{(x_2)^7 - 1000}{7(x_2)^6} \\ &= 2.682756447 \end{aligned}$$

$$\begin{aligned} x_4 &= x_3 - \frac{(x_3)^7 - 1000}{7(x_3)^6} \\ &= 2.682695799 \end{aligned}$$

$$\begin{aligned} x_5 &= x_4 - \frac{(x_4)^7 - 1000}{7(x_4)^6} \\ &= 2.682695795 \end{aligned}$$

$$\begin{aligned} x_6 &= x_5 - \frac{(x_5)^7 - 1000}{7(x_5)^6} \\ &= 2.682695795 \end{aligned}$$

3. Since $x_5 = x_6$ at 8d.p., we conclude that $\sqrt[7]{1000} = 2.682695795$ to 8 d.p.

9 Question 9

Use Newton's method to find the root of the equation x4 + x − 4 = 0 in the interval [1, 2] correct to six decimal places.

$Let\, x = \sqrt[7]{1000} x^7 \quad = 1000x^7 - 1000 = 0$

$$
\begin{aligned}
Let\, f(x) &= x^7 - 1000 \\
f'(x) &= 7x^6
\end{aligned}
$$

$$x_0 = 3$$

$$
\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
&= 3 - \frac{f(3)}{f'(3)} \\
&= 3 - \frac{3^7 - 1000}{7(3)^6} \\
x_1 &= 2.76739173
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\
&= 2.76739173 - \frac{f(2.76739173)}{f'(2.76739173)} \\
&= 2.76739173 - \frac{(2.76739173)^7 - 1000}{7(2.76739173)^6} \\
x_2 &= 2.690087405
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\
&= 2.690087405 - \frac{(2.690087405)^7 - 1000}{7(2.690087405)^6} \\
x_3 &= 2.682756447
\end{aligned}
$$

$$
\begin{aligned}
x_4 &= 2.682756447 - \frac{(2.682756447)^7 - 1000}{7(2.682756447)^6} \\
&= 2.682695799
\end{aligned}
$$

$$
\begin{aligned}
x_5 &= 2.682695799 - \frac{(2.682695799)^7 - 1000}{7\,(2.682695799)^6} \\
&= 2.682695795
\end{aligned}
$$

$$
\begin{aligned}
x_6 &= 2.682695795 - \frac{(2.682695795)^7 - 1000}{7\,(2.682695795)^6} \\
&= 2.682695795
\end{aligned}
$$

Since x_5 and $x_6$agree to 8 d.p. , $\sqrt[7]{1000} \approx 2.68269580$ (8d.p.)