Calc II T9

December 30, 2019

1. By using geometric series $\sum_{n=0}^{8} x^n = \frac{1}{1-x}$ find the power series representations for the following functions:

   (a) $\frac{x}{4x+1}$

$$
\begin{aligned}
\frac{x}{4x+1} &= \frac{x}{4x+1} \\
&= x\left(\frac{1}{1+4x}\right) \\
&= x\left(\frac{1}{1-(-4x)}\right) \\
&= x\sum_{n=0}^{\infty}(-4x)^n \\
&= x\sum_{n=0}^{\infty}(-4)^n x^n \\
&= \sum_{n=0}^{\infty}(-4)^n x^{n+1}
\end{aligned}
$$

   i. Alternatively

$$
\begin{aligned}
&= x\left(\frac{1}{1-(-4x)}\right) \\
&= x\sum_{n=0}^{\infty}(-4x)^n
\end{aligned}
$$

   ii. Series is convergent when

$$
\begin{aligned}
|-4x| &< 1 \\
|-x| &< \frac{1}{4} \\
-\frac{1}{4} < -x &< \frac{1}{4} \\
-\frac{1}{4} < x &< \frac{1}{4}
\end{aligned}
$$

(b) $\frac{1}{1+9x^2}$

$$\begin{aligned}\frac{1}{1+9x^2} &= \frac{1}{1-(-9x^2)} \\ &= \left(-9x^2\right)^n\end{aligned}$$

i. Series is convergent when

$$\begin{aligned}-1 < -9x^2 &< 1 \\ 1 > 9x^2 &> 0 \text{Note: because it is square} \\ \frac{1}{9} &> 9x^2 > 0 \\ \sqrt{\frac{1}{9}} &> x > \frac{1}{3} \\ -\frac{1}{3} &< x < \frac{1}{3}\end{aligned}$$

(c) $\frac{1}{4+x^2}$=

$$\begin{aligned}\frac{1}{4+x^2} &= \frac{1}{4\left(1+\frac{x^2}{4}\right)} \\ &= \frac{1}{4}\left(\frac{1}{1-\left(-\frac{x^2}{4}\right)}\right) \\ &= \frac{1}{4}\left(\frac{1}{1-\left(-\frac{x^2}{4}\right)}\right) \\ &= \frac{1}{4}\left(-\frac{x^2}{4}\right)^n \\ &= \frac{1}{4}\left(-\frac{x^2}{4}\right)^n \\ &= (-1)^n\,\frac{x^{2n}}{4^{n+1}}\end{aligned}$$

i. Series is convergent when

$$\begin{aligned}\left|-\frac{x^2}{4}\right| &< 1 \\ 0 <\frac{x^2}{4} &< 1 \\ 0 <x^2 &< 4 \\ -2 <x &< 2\end{aligned}$$

2. Use the related Maclaurin series to find the first three nonzero terms in the Maclaurin series for the given functions:

(a) $f(x) = \sin\left(x^4\right)$

(b) $f(x) = x\cos 2x$

(c) $f(x) = e^{-\frac{x}{2}}$

3. Use the definition of Taylor series to find the Taylor polynomials of degree 3 for the following functions:

   (a) $f(x) = \sqrt{x}$ about $x = 4$

   (b) $f(x) = x^3 - 10x^2 + 6$ about $x = 3$

   (c) $f(x) = \frac{1}{x^2}$ about $x = -1$

4. Use series to evaluate the following limits:

   (a) $\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x}$

   (b) $\lim_{x\to 0} \frac{x-\tan^{-1} x}{x^3}$

5. Use series to approximate the following definite integral (3 decimal places):

   (a) $\int_0^{0.5} \cos\left(x^2\right) dx$

   (b) $\int_0^{0.5} \cos\left(x^2\right) dx$

6. Expand the following functions as a power series:

   (a) $\frac{1}{(1+x)^4}$

$$\begin{aligned}\frac{1}{(1+x)^4} &= (1+x)^{-4}\\ &= 1 + (-4)(x) + \frac{(-4)(-4-1)}{2!}x^2 + \frac{-4(-4-1)(-4-2)}{3!}x^3 + ...\\ &= 1 - 4x + 10x^2 - 20x^3 + ..\end{aligned}$$

   (b) $\sqrt{9-x}$

$$\begin{aligned}\sqrt{9-x} &= \sqrt{9\left(1-\frac{x}{9}\right)}\\ &= \sqrt{9}\left(1-\frac{x}{9}\right)^{\frac{1}{2}}\\ &= \sqrt{9}\left[1+\left(-\frac{x}{9}\right)\right]^{\frac{1}{2}}\\ &= 3\left(1+\frac{1}{2}\left(\frac{x}{9}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{x}{9}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(-\frac{x}{9}\right)^3+...\right)\end{aligned}$$

7. Use the Binomial Theorem to expand (for natural numbers use theorem, anything else use series):

(a) $(2x-3)^4$

$$\begin{aligned}\sum_{k=0}^{4}\binom{4}{k}(2x)^{4-k}(-3)^k &= \binom{4}{0}(2x)^4(-3)^0+\binom{4}{1}(2x)^3(-3)^1+\binom{4}{2}(2x)^2(-3)^2\\ &+\binom{4}{3}(2x)^1(-3)^3+\binom{4}{4}(2x)^0(-3)^4\\ &= 16x^4-96x^3+216x^2+81\end{aligned}$$

(b) $\left(x+\frac{1}{2x}\right)^6$

$$\begin{aligned}\sum_{k=0}^{6}\binom{6}{k}x^{6-k}\left(\frac{1}{2x}\right)^k &= \binom{6}{0}x^6\left(\frac{1}{2x}\right)^0+\binom{6}{1}x^5\left(\frac{1}{2x}\right)^1+\binom{6}{2}x^4\left(\frac{1}{2x}\right)^2\\ &+\binom{6}{3}x^3\left(\frac{1}{2x}\right)^3+\binom{6}{4}x^2\left(\frac{1}{2x}\right)^4+\binom{6}{5}x^1\left(\frac{1}{2x}\right)^5\\ &+\binom{6}{6}x^0\left(\frac{1}{2x}\right)^6\\ &= x^6+3x^4+\frac{15}{4}x^2+\frac{5}{2}+\frac{15}{16x^2}+\frac{3}{16x^4}+\frac{1}{64x^6}\end{aligned}$$

(c) $\left(2+\frac{x}{4}\right)^4$

$$\begin{aligned}\sum_{k=0}^{4}\binom{4}{k}2^{4-k}\left(\frac{x}{4}\right)^k &= \binom{4}{0}2^4\left(\frac{x}{4}\right)^0+\binom{4}{1}2^3\left(\frac{x}{4}\right)^1+\binom{4}{2}2^2\left(\frac{x}{4}\right)^2\\ &+\binom{4}{3}2^1\left(\frac{x}{4}\right)^3+\left(\frac{x}{4}\right)^4\\ &= 16+12x+\frac{3}{2}x^2+\frac{x^3}{8}+\frac{x^4}{256}\end{aligned}$$

8. Expand $\left(5+\sqrt{3}\right)^4$ and $\left(5-\sqrt{3}\right)^4$, then find the value of $\left(5+\sqrt{3}\right)^4-\left(5-\sqrt{3}\right)^4$

$$\begin{aligned}\left(5+\sqrt{3}\right)^4 &= 1\,(5)^4\left(\sqrt{3}\right)^0+4\,(5)^3\left(\sqrt{3}\right)^1+1\,(5)^0\left(\sqrt{3}\right)^4\\ &= 625+500\sqrt{3}+450+60\sqrt{3}+45\\ &= 1120+560\sqrt{3}\end{aligned}$$

$$\begin{aligned}\left(5-\sqrt{3}\right)^4 &= 625-500\sqrt{3}+450-60\sqrt{3}+45\\ &= 1120-560\sqrt{3}\end{aligned}$$

$$\begin{aligned}\left(5+\sqrt{3}\right)^4-\left(5-\sqrt{3}\right)^4 &= \left(1120+560\sqrt{3}\right)-\left(1120-560\sqrt{3}\right)\\ &= 1120\sqrt{3}\end{aligned}$$