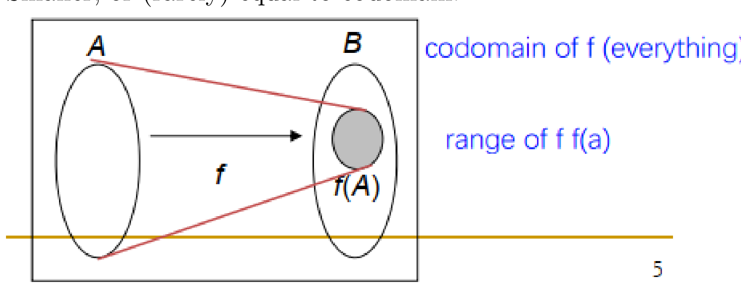
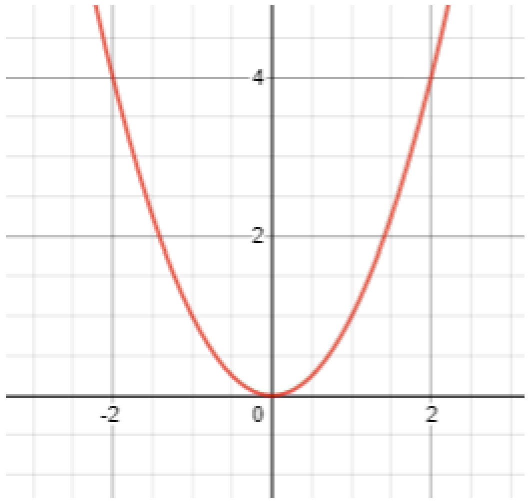
DM Chapter 5: Functions

January 16, 2020

1 Introduction

1. **Functions:** Binary relations with restrictions on occuring pairs
   (a) Every element in set A associated with only one element in set B
   (b) Also called mappings/transformations.
2. **Digraph:** Only 1 arc leaving every element
3. Let f be function from set A to set B
   (a) $y = f(x)$: For each $a \in A$, exists uniquely determined $y \in B$ such that $(x, y) \in f$.
   (b) $f(x)$ is the **image** of x under f.
   (c) A is called **domain** of f
   (d) B is called **codomain** of f
   (e) **Range of f:** Set of images of elements in A under f, $f(A)$. $f(A) = \{f(x) : x \in A\}$
      i. Smaller, or (rarely) equal to codomain.
      ii. 

4. Graph: $f : R \to R$ given by $f(x) = x^2$

(a) 

(b) **Domain** R: $x - axix$

(c) **Codomain** $R : y - axis$

(d) **Range** $R :$ all the y's on the lin

(e) Question: Find image of f under f when $x = 2$

   i. Answer: $x = 2, f(2) = 2^2 = 4$

2 Properties of Functions

1. Let $f : A \to B$

2. **Injective:** one-to-one

   (a) $\forall a_1, a_2 \in A, f(a_1) = f(a_2) \implies a_1 = a_2$

   (b) $a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$

3. **Everywhere defined:** $Dom(f) = A$

4. **Surjective** (or **onto**): Range of f is same as codomain of f

$$f(A) = B$$

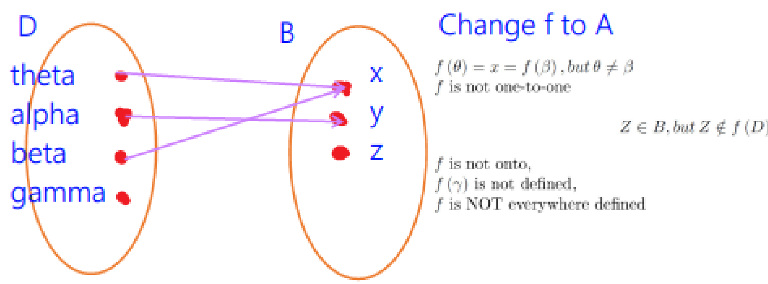
   (a) $\forall b \in B, \exists a \in A, b = f(a)$

5. **Bijective:** Injective + surjective

6. **One-to-one correspondence** between A and B: Injective + surjective + everywhere defined.

2.1 Show functions are not injective/surjective

1. **Limited/Small Domain:** Use diagrams, check the edges

   (a)

2. **Large Domain:** Use proofing

   (a) **Injective:** Show that every imagei in set B,

   (b) **Surjective:** Show that for every value in $f(x)$, or c, we can find an x to map to it.

   (c) Example: Show that the function $k : \mathbb{R} \to \mathbb{R}$ given by $k(x) = 4x + 3$ is bijective.

      i. Injective test

      $$\begin{aligned} k(a) &= k(b) \\ 4a + 3 &= 4b + 3 \\ 4a &= 4b \\ a &\neq b \end{aligned}$$

         A. $\therefore k$ is injective

      ii. Surjective test

         A. Let $c \in \mathbb{R}, k(x) = c$

         $$\begin{aligned} 4x + 3 &= c \\ 4x &= c - 3 \\ x &= \frac{c-3}{4} \end{aligned}$$

         B. This means for every value in $\mathbb{R}$, we can find value in x.

         $$k(\mathbb{R}) = \mathbb{R}$$

3 Functions for Computer Science

3.1 From subset of universal set to Boolean Set

1. Let A be subset of universal set $U = \{u_1, u_2, ..., u_n\}$

2. Function A, function from U to $\{0, 1\}$

$$f_A(u_i) = \begin{cases} 1 & \text{if } u_i \in A \\ 0 & \text{if } u_i \notin A \end{cases}$$

3. If $A = \{4, 7, 9\}$, $U = \{1, 2, 3, ..., 10\}$, then $f_A(2) = 0$, $f_A(4) = 1$, $f_A(12)$ is undefined.

4. f_A is everywhere defined, and onto, but not one-to-one

3.2 Family of $\mod -n$ functions

1. One for each positive integer n, $f_n(m) = m \pmod n$

2. For a fixed n, any non-negative integer z can be written as $z = kn + r$ with $0 \leq r < n$

3. $f_n(z) = r$ can be written as $z \equiv r \pmod n$

4. Each member is everywhere defined, onto, but not-one-to-one $-a$ and a yields same result.

3.3 Floor & Ceiling Function

1. Floor: $f(q) = \lfloor q \rfloor$

2. Ceiling: $c(q) = \lceil q \rceil$

3.4 Common Algebraic Functions

1. Polynomial with integer coefficients

   (a) Example: $f(x) = 2x^2 + 4x + 1$

2. Exponential functions

   (a) Example, base-2 exponential: $f(x) = 2^x$

3.5 Functions without numeric domains and/or codomains

1. Length of string

2. Transposition of matrices (everywhere defined, onto, one-to-one)

3. Boolean functions, $B = \{True, False\}$

4 Permutations

1. **Permutation:** All possible combinations

   (a) A bijection (injection + surjection) from set A to itself.

   (b) Number of permutations: $n!$, where n is the number of elements

   - E.g. Let $A = \{1, 2, 3\}$. Then all the permutations of A are

   (1,2)(2,1)(3,3)

   $$I_A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

   $$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

   (c) Example:

2. **Theorem 1: Number of permutations**

   (a) If $A = \{a_1, a_2, ..., a_n\}$, then there are $n!$ permutations on A.

3. **Cyclic Permutation**

   (a) Permutations forming a cycle, $p : A \to A$ defined by:

   $$\begin{aligned} p(b_1) &= b_2 \\ p(b_2) &= b_3 \\ &\dots\dots \\ p(b_{r-1}) &= b_r \\ p(b_r) &= b_1 \end{aligned}$$

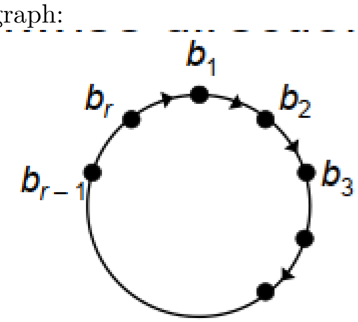
      i. $p(x) = x$, if $x \in A$, $x \notin \{b_1, b_2, ..., b_r\}$

   (b) This is calle a:

      i. cyclic permutation of length r, or,

      ii. a cycle or length r

      iii. Digraph:



iv. Let $A = \{1, 3, 5\}$. The cycle $(1, 3, 5)$ denotes permutation: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$.
   A. As you can see, the first row is $1 - 5$.
   B. The second row denotes "where the element in the same column, but row above goes to". For example $\binom{1}{3}$ denotes vertex 1 goes to vertex 3 (in the cycle).
   C. For elements which don't exist in cycle, they simply go to themselves.

v. The notation for a cycle does not indicate number of elements. We need to specify a set where the cycle applies outselves.

vi. **Identity permutation, I_A:** Length 1 cycle on set A

vii. **Disjoint cycles:** No element a exists in common in both cycles.
   A. Disjoint: $(1, 2, 3)$ and $(4, 5, 6)$
   B. Not disjoint: $(1, 2, 3)$ and $(3, 4, 5)$

(c) **Theorem 2: Products of Disjoint Cycles**
   i. All finite permutation can be written as an identity, cycle, or a product of disjoint cycles
   ii. A product of disjoiny cycles is unique except for ordering

4. Inverses & composition of cycles
   (a) Let $A = \{1, 2, 3\}$ and $p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, $p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, $p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.
   (b) Find:
      i. p_1^{-1}
         A. First, write the first row in sequence (cause we don't know their mapping yet)

$$p_1^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ ? & ? & ? \end{pmatrix}$$

         B. Then, go the opposite way, from the $p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, we see that
            1. 1 goes to 3
            2. 2 goes to 1
            3. 3 goes to 2
         C. Therefore, we inverse it, or flip every single value so it goes backwards, this means:
            1. 3 goes to 1
            2. 1 goes to 2
            3. 2 goes to 3

D. Hence arriving at our answer:

$$p_1^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

ii. $p_3 \circ p_2$

A. Remember it's in reverse order, if you go forward order then 0 marks for you (why can't we just go forward).

$$p_3 \circ p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

B. So, again, start with listing the top row

$$\begin{pmatrix} 1 & 2 & 3 \\ ? & ? & ? \end{pmatrix}$$

C. Then, you want to find the "end-result" of each edge after traversing into the second matrix, so starting from the first one $(1,2) \cdot (2,3) = (1,3)$

D. Now, repeat this for the rest, and you get

$$p_3 \circ p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

E. Since this forms a perfect cycle, we can simplify it very neatly into cycle form, so why not? (This is optional tho)

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1,3)$$
$$p_3 \circ p_2 = (1,3)$$

5. Writing permutations as disjoint cycles

(a) **Question**: Write the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 6 & 5 & 2 & 1 & 8 & 7 & 9 \end{pmatrix}$

i. First, lets start with the first cycle

A. $\begin{pmatrix} \mathbf{1} & 2 & \mathbf{3} & 4 & 5 & \mathbf{6} & 7 & 8 & 9 \\ \mathbf{3} & 4 & \mathbf{6} & 5 & 2 & \mathbf{1} & 8 & 7 & 9 \end{pmatrix}$

B. So, we get $(1,3,6)$. Note we don't need to write 1 because its a cycle.

ii. Second, lets move on to the second cycle

A. $\begin{pmatrix} X & \mathbf{2} & X & \mathbf{4} & \mathbf{5} & X & 7 & 8 & 9 \\ X & \mathbf{4} & X & \mathbf{5} & \mathbf{2} & X & 8 & 7 & 9 \end{pmatrix}$

B. So, we get $(2,4,5)$

iii. Lets move on to the third

A. $\begin{pmatrix} X & X & X & X & X & X & \mathbf{7} & \mathbf{8} & 9 \\ X & X & X & X & X & X & \mathbf{8} & \mathbf{7} & 9 \end{pmatrix}$

B. So, we get $(7, 8)$

iv. Finally, lets move on to...hold on. This is an identity permutation, so we ignore that.

A. $\begin{pmatrix} X & X & X & X & X & X & X & X & \mathbf{9} \\ X & X & X & X & X & X & X & X & \mathbf{9} \end{pmatrix}$

v. Once we're done, we form a product of disjoint cycles, in order

$$(1, 3, 6) \circ (2, 4, 5) \circ (7, 8)$$

4.1 Even and Odd Permutations

1. **Transposition:** Cycle of length 2

2. **Identity permutation:** Cycle of length 1. However, if $p = (a_i, a_j)$ is a transposition of A, then $p \circ p = 1_A$ (p transposition p, remember the "find where it ends, yes, this will make it $p \circ p = (a_i, a_i)$)

3. Every cycle can be written as product of transpositions

$$(b_1, b_2, ..., b_{k+1}) = (b_1, b_{k+1}) \circ ... \circ (b_1, b_3) \circ (b_1, b_2)$$

   (a) Where each permutation is simply a cycle of first element and the nth next element. But we must go from **last to first**, because remember for some odd reason the $\circ$ sign composition goes from last to first.

4.2 Corralary 1: Permutations of finite set

1. Every permutation of finite set with at least 2 elements can be written as a product of transpositions which need not be disjoint

2. Every cycle can be written as a product of transpositions in many different ways.

4.3 Theorem 3: Evenness & oddness of an permutation

1. A permutation of a finite set can be written as a product of an even number of transpositions.

2. If so, it can never be written as an odd number of transpositions, and conversely.

3. **Even permutations:** Can be written as even number of transpositions

4. **Odd permutation:** Can be written as odd number of transpositions

5. Combinations:

   (a) $even \circ even = even$

   (b) $odd \circ odd = even$

(c) $even \circ odd = odd$

6. Theorem 4: Number of permutations

   (a) If set A is a finite set with n elements, there are:

      i. $\frac{n!}{2}$ even permutations

      ii. $\frac{n!}{2}$ odd permutations

      iii. Recall: a set can have $n!$ permutations, so it makes sense that each of them contribute to half.

7. Determining whether permutations are even or odd:

   (a) Write them out as product of disjoint cycles

   (b) Count them

   (c) If even, then even

   (d) If odd, then odd