Ginwidth=

Calc 2 : Tutorial 7

December 9, 2019

1. Find a formula for the general term a_n of the following sequences:

   (a) $\{-1, 1, -1, 1, -1, ...\}$

   $$a_n = (-1)^n \, , n = 1, 2, 3...$$

   (b) $\left\{1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, ...\right\}$

   $$a_n = \frac{(-1)}{n^2}, n = 1, 2, 3, ...$$

   (c) $\left\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \frac{16}{81}, ...\right\}$

   $$\begin{aligned} r &= \frac{-\frac{2}{3}}{1} \\ &= -\frac{2}{3} \end{aligned}$$

   $$a_n = ar^{n-1}$$
   $$\boldsymbol{a_n = \left(-\frac{2}{3}\right)^{n-1}, n = 1, 2, 3...}$$

   (d) $\{2, 7, 12, 17, 22, ...\}$

   $$a = 2$$

   $$\begin{aligned} d &= 7 - 2 \\ &= 5 \end{aligned}$$

   $$\begin{aligned} a_n &= 2 + (n-1)\,5 \\ &= 2 + 5n - 5 \\ \boldsymbol{a_n} &\boldsymbol{= 5n - 3, n = 1, 2, 3...} \end{aligned}$$

   (e) $\left\{0, 1, \sqrt{2}, \sqrt{3}, 2, ...\right\}$

   $$a_n = \sqrt{n}, n = 0, 1, 2....$$
   $$a_n = \sqrt{n-1} = n = 1, 2, 3, ...$$

(f) $\left\{\frac{2}{5}, \frac{4}{25}, \frac{6}{125}, \frac{8}{625}, ...\right\}$

$$a_n = \frac{2n}{5^n}, n = 1, 2, 3, ...$$

(g) $\left\{\frac{1}{2}, e, \frac{3}{2}e^2, 2e^3, \frac{5}{2}e^4, ...\right\}$

$$a_n = \frac{n}{2}e^{n-1}, n = 1, 2, 3, ...$$

(h) $\left\{-\frac{1}{3}, \frac{8}{4}, -\frac{27}{5}, \frac{32}{3}, -\frac{125}{7}, ...\right\}$

$$a_n = \frac{(-1)^n \cdot (n)^3}{n+2}, n = 1, 2, ...$$

2. Determine whether the sequence converges or diverges. If it converges, find the limit.

   (a) $a_n = \frac{2n+4n^2}{6n^2-1}$

      i. Converges

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{2n+4n^2}{6n^2-1} \\
&= \lim_{n\to\infty} \frac{\frac{2n}{n^2} + \frac{4n^2}{n^2}}{\frac{6n^2}{n^2} - \frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{\frac{2}{n} + 4}{6 - \frac{1}{n^2}} \\
&= \frac{4}{6} \\
&= \frac{2}{3}
\end{aligned}$$

   (b) $a_n = \frac{3n^2-2n+3}{4+n^2}$

      i. Converges

      ii. Limit

$$\begin{aligned}
a_n &= \lim_{n\to\infty} \frac{\frac{3n^2}{n^2} - \frac{2n}{n^2} + \frac{3}{n^2}}{\frac{4}{n^2} + \frac{n^2}{n^2}} \\
&= \lim_{n\to\infty} \frac{3 - \frac{2}{n} + \frac{3}{n^2}}{\frac{4}{n^2} + 1} \\
&= \frac{3}{1} \\
a_n &= 3
\end{aligned}$$

   (c) $a_n = n(n-1)$

i. Proof

$$\begin{aligned}\lim_{n\to\infty} n\,(n-1) &= \infty\cdot\infty\\ &= \infty\end{aligned}$$

ii. Diverges

(d) $a_n = \frac{\ln n^2}{n}$

i. Converges

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\frac{1}{n}\ln n^2\\ &= 0\cdot\ln\infty^2\\ \lim_{n\to\infty} a_n &= 0\end{aligned}$$

(e) $a_n = \frac{2^n}{3^{n+1}}$

i. Converges

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\frac{2^n}{3^{n+1}}\\ &= \lim_{n\to\infty}\frac{2^n}{3^{n+1}}\\ &= \frac{2^\infty}{3^{\infty+1}}\\ &= 0\end{aligned}$$

(f) $a_n = \frac{(-1)^{n-1}n}{n^2+1}$

i. Diverges

(g) $a_n = \frac{\sin^2 n}{2^n}$

i. Proof (Squeeze theorem)

$$\begin{aligned}-1 \le \sin n &\le 1\\ 0 \le \sin^2 n &\le 1\\ 0 \le \frac{\sin^2 n}{2^n} &\le \frac{1}{2^n}\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} 0 \le \lim_{n\to\infty}\frac{\sin^2 n}{2^n} &\le \lim_{n\to\infty}\frac{1}{2^n}\\ 0 \le \lim_{n\to\infty}\frac{\sin^2 n}{2^n} &\le 0\end{aligned}$$

$$\lim_{n\to\infty}\frac{\sin^2 n}{2^n} = 0$$

ii. Convergent

(h) $a_n = 2 + \cos(n\pi)$

i. Proof

$$\begin{aligned} a_n &= 2 + \cos(n\pi) \\ &= 2 + \{\cos\pi, \cos 2\pi, \cos 3\pi, ...\} \\ &= \{1, 3, 1, 3, 1, 3, ...\} \\ &= diverges \end{aligned}$$

(i) $a_n = \frac{e^n + e^{-n}}{e^{2n} - 1}$

i. Converges

$$\begin{aligned} a_n &= \frac{e^n + e^{-n}}{e^{2n} - 1} \\ &= \lim_{n\to\infty} \frac{\frac{e^n}{e^{2n}} + \frac{e^{-n}}{e^{2n}}}{\frac{e^{2n}}{e^{2n}} - \frac{1}{e^{2n}}} \\ &= \lim_{n\to\infty} \frac{\frac{1}{e^n} + \frac{1}{e^{3n}}}{1 - \frac{1}{e^{2n}}} \\ a_n &= 0 \end{aligned}$$

(j) $a_n = \frac{(n+2)!}{n!}$

i. Proof

$$\begin{aligned} a_n &= \frac{(n+2)(n+1)\cancel{(n)(n-1)!}}{\cancel{n(n-1)!}} \\ &= (n+2)(n+1) \\ &= \infty \end{aligned}$$

ii. Diverges

3. Determine if the following sequences are monotonic and/or bounded.

(a) $a = \frac{1}{2n+3}$

i. Monotonic (decreasing)

ii. Bounded above

$$UB = \frac{1}{5}$$

iii. Bounded below

$$LB = 0$$

(b) $a_n = \frac{n}{n+1}$

i. Not monotonic

ii. Not bounded

(c) $a_n = \cos\left(\frac{n\pi}{2}\right)$

   i. Proof

$$\left\{\cos\frac{\pi}{2}, \cos\pi, \cos\frac{3\pi}{2}, ...\right\}$$

   ii. Not monotonic

   iii. Bounded

      A. $UB = 1$

      B. $LB = -1$

(d) $a_n = -n^2$

   i. Monotonic

$$\{-1, -4, -9, -16, ...\}$$

   ii. Bounded

      A. $UB = -1$

(e) $a_n = (-1)^{n-1}$

$$\{1, -1, 1, -1, ...\}$$

   i. Not monotonic

   ii. Bounded

      A. $UB = 1$

      B. $LB = -1$

(f) $a_n = \frac{2}{n^2}$

$$\left\{2, \frac{2}{4}, \frac{2}{9}, ...\right\}$$

   i. Monotonic

   ii. Bounded

      A. $UB = 2$

      B. $LB = 0$