Tutorial 11

January 24, 2020

1 Determine whether the relation R is a partial order on the set A.

1. $A = Z$, and aRb if and only if $a = 2b$.
   (a) $1 \neq 2\,(1)\,, 2 \neq 2\,(2) \rightarrow$ not reflexive
   (b) $1 \neq 2\,(2)\,, 2 = 2\,(1) \rightarrow$ antisymmetric. No counterexamples
   (c) $1 \neq 2\,(2)\,, 2 \neq 2\,(3)\,, 1 \neq 2\,(3) \rightarrow$ not transitive.
2. $A = R$, and aRb if and only if $a \leq b$.
   (a) $1 \leq 1, 2 \leq 2, ... \rightarrow$ reflexive. No counterexamples.
   (b) $1 \leq 2, 2 \leq 1 \rightarrow$ antisymmetric. No counterexamples.
   (c) $1 \leq 2, 2 \leq 3, 1 \leq 3 \rightarrow$ transitive. No counterexamples.
   (d) $\therefore$ partial order.
3. $M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$
   (a) Reflexive. The main diagonal is all 1's
   $$M_R = \begin{bmatrix} \mathbf{1} & 0 & 1 & 0 \\ 0 & \mathbf{1} & 1 & 0 \\ 0 & 0 & \mathbf{1} & 1 \\ 1 & 1 & 0 & \mathbf{1} \end{bmatrix}$$
   (b) Not symmetric. Not all $M_{ij} = M_{ji}$
   $$M_R = \begin{bmatrix} 1 & 0 & \mathbf{1} & 0 \\ 0 & 1 & 1 & 0 \\ \mathbf{0} & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

(c) Not transitive.

$$M_R^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

i. Since $M_{1,4} = 0$, $M_{1,4}^2 = 1$, the relation is NOT transitive.

(d) **NOT partial order**

4. $M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(a) Reflexive. The main diagonal is all 1's

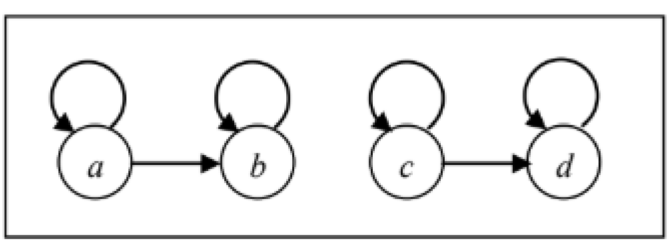
$$M_R = \begin{bmatrix} \mathbf{1} & 1 & 1 & 1 & 1 \\ 0 & \mathbf{1} & 0 & 1 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

(b) Antisymmetric . All M_{ij} which is 1, $M_{ij} = M_{ji}$

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

(c) Transitive. For ifM_{ij} is 1 in M_R, then M_{ij} is 1 in M_R^2

(d) Partial order

5. 

(a) Reflexive. All elements have loops

(b) Antisymmetric. All M_{ij} which is 1, $M_{ij} = M_{ji}$

(c) Not transitive. There exists no elements such that if $aRb \wedge bRc \to aRc$

(d) NOT a partial order.

2 Find the lexicographic ordering of the following strings of lowercase English letters:

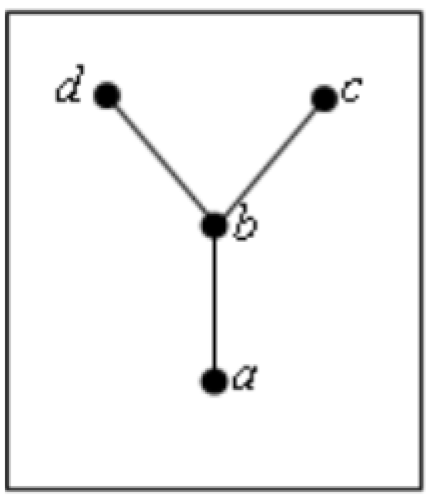
1. quack, quick, quicksilver, quicksand, quacking

   (a) ANSWER: $quack, quacking, quick, quicksand, quicksilver$

2. zoo, zero, zoom, zoology, zoological

   (a) ANSWER: $zero, zoo, zoology, zoological, zoom$

3 List all ordered pairs in the partial order whose Hasse diagram is shown as below.

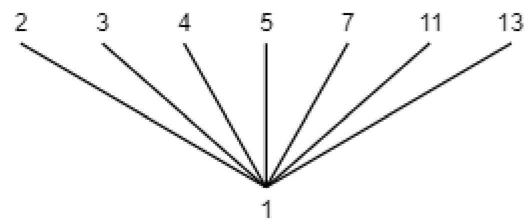
1. 

   (a) $\{(a,a), (a,b), (b,b), (b,d), (b,c), (c,c), (d,d)\}$

4 Draw the Hasse diagram for each of the following posets.

1. a is a divisor of b on the set $\{1, 2, 3, 5, 7, 11, 13\}$.

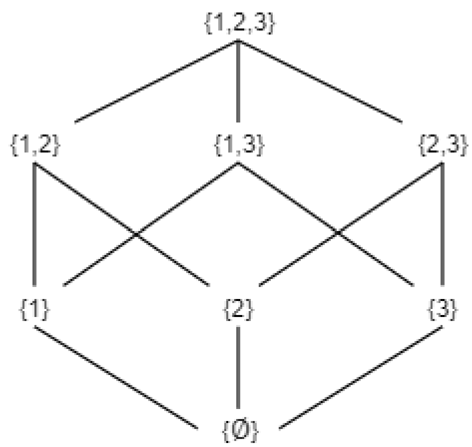
(a) Set notation: $\{(1,2), (1,3), (1,5), (1,7), (1,11), (1,13), (2,2), (3,3), (5,5), (7,7), (11,11), (13,13)\}$



(b)

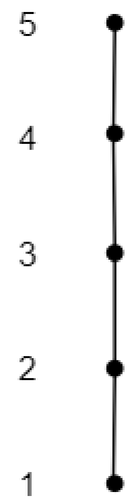
2. X is a subset of Y on the set of all subsets of $\{1,2,3\}$.

(a) Subset of $\{1,2,3\} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

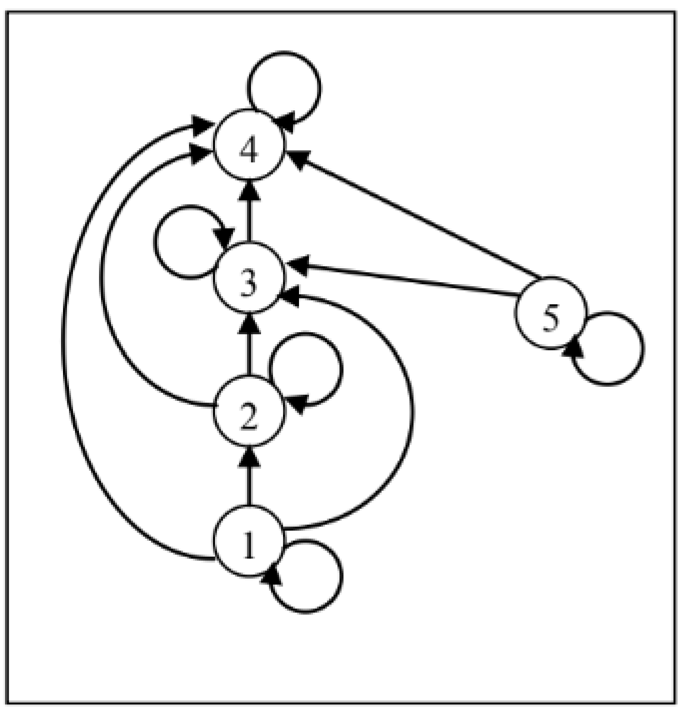


(b)

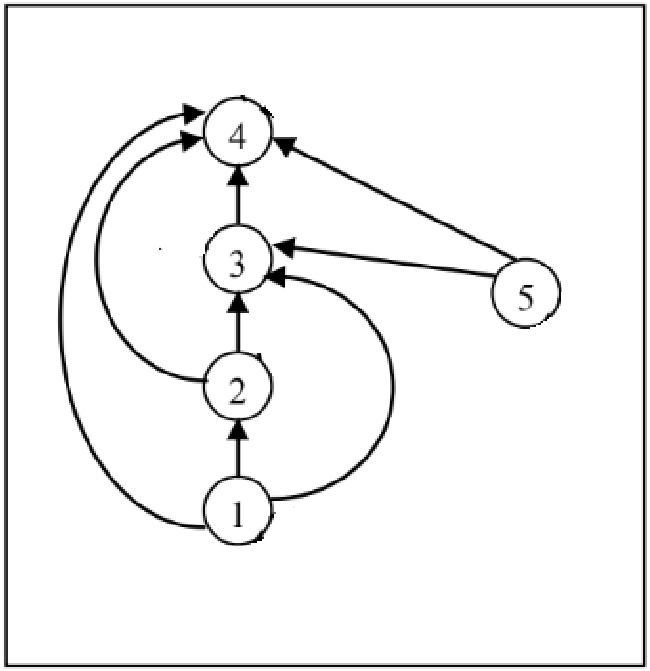
3. $A = \{1,2,3,4,5\}$, $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$



(a)

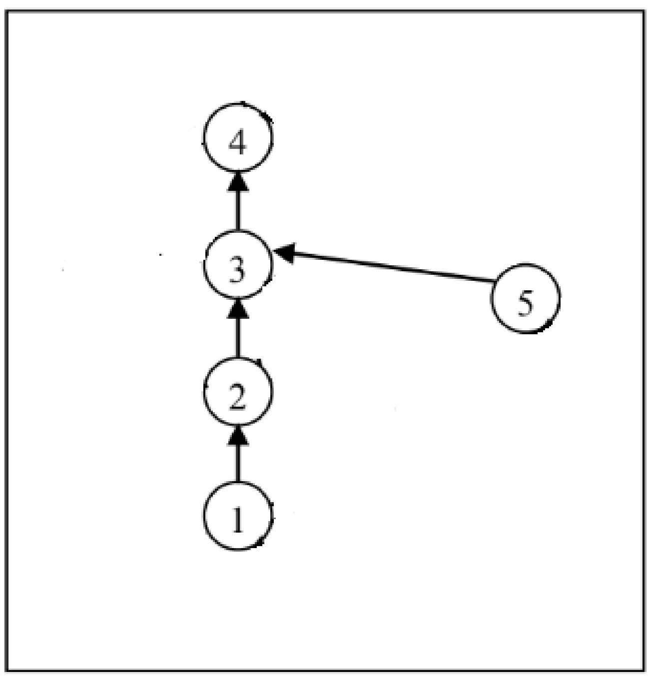
4. 

(a) Alright guys you know the drill
    i. Remove all reflexive links



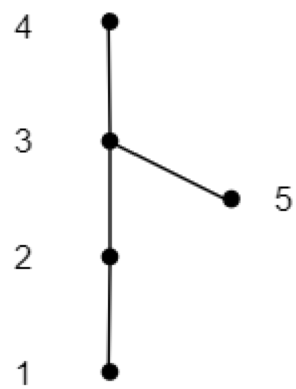
A.

ii. Remove all transitive links



A.

iii. Arrange them in proper levels and connect them with dots



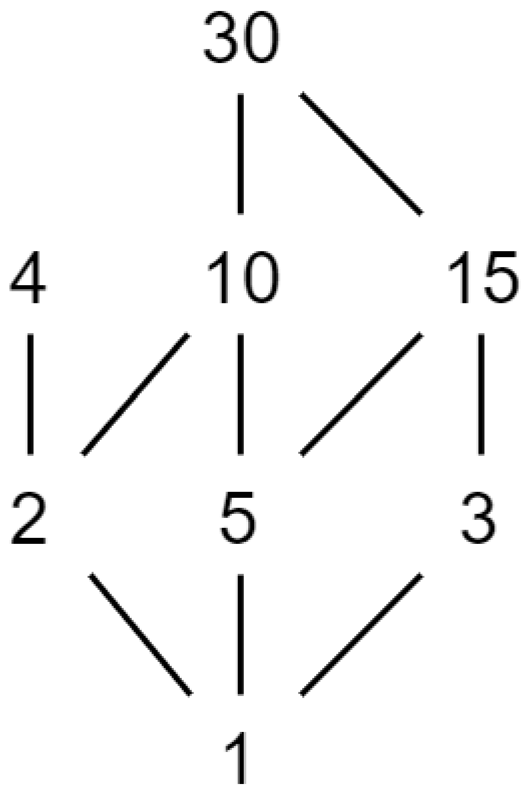
A.

5 Consider the partial order of divisibility on the set A. Draw the Hasse diagram of the poset and determine which posets are linearly ordered.

1. $A = \{1, 2, 3, 4, 5, 10, 15, 30\}$

(a)

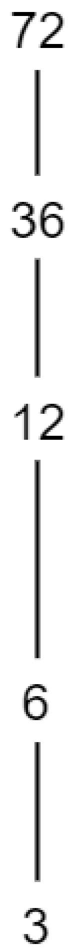
$\{$	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,10)$	$(1,15)$	$(1,30)$	
		$(2,2)$		$(2,4)$		$(2,10)$		$(2,30)$	
			$(1,3)$				$(3,15)$	$(3,30)$	
				$(4,4)$					
					$(5,5)$	$(5,10)$	$(5,15)$	$(5,30)$	
						$(10,10)$		$(10,30)$	
							$(15,15)$	$(15,30)$	
								$(30,30)$	$\}$



(b)

(c) NOT Linearly Ordered

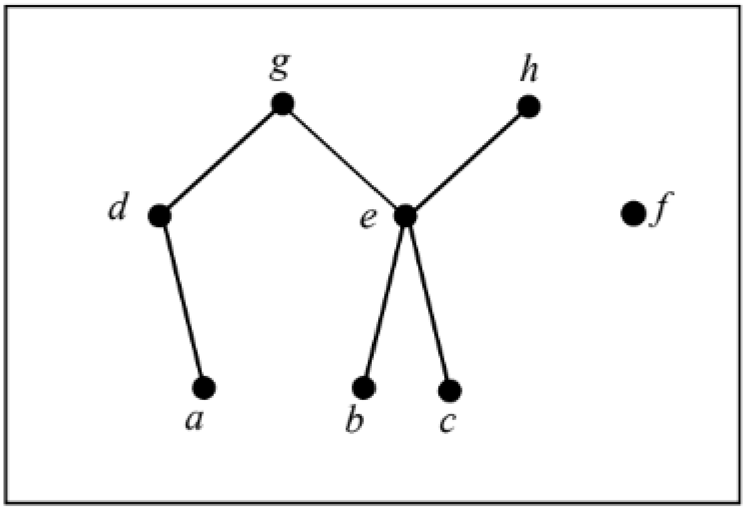
2. $A = \{3, 6, 12, 36, 72\}$



(a)

(b) Linearly ordered

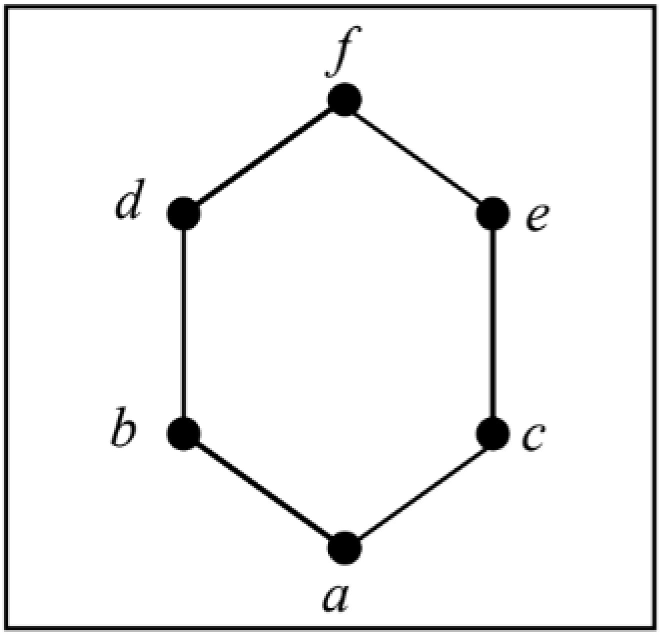
6 Given the Hasse diagram of a partial order R on $A = \{a, b, c, d, e, f, g, h\}$. List the elements of R and write down the maximal and minimal elements of A.



$$R = \{(a,a), (a,d), (a,g), (d,d), (d,g), (b,b), (b,e), (b,g), (b,h), (e,e), (e,g), (e,h), (h,h), (f,f)\}$$

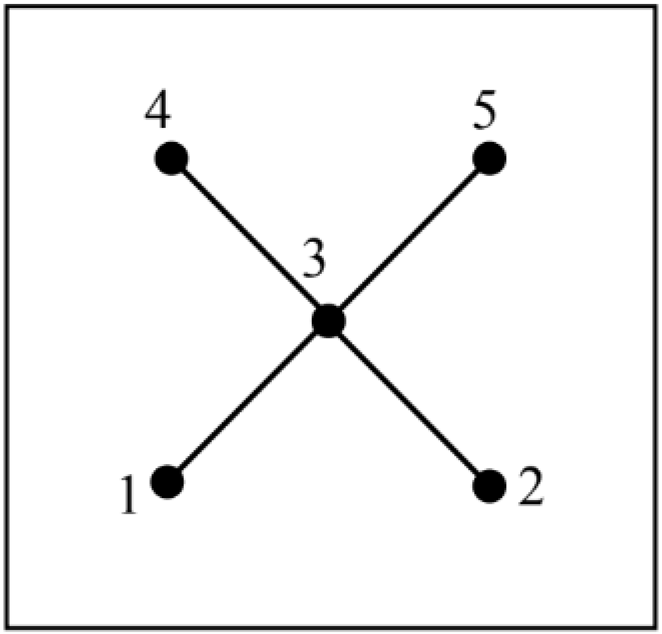
1. Maximal elements
   (a) $\{g, h, f\}$
2. Minimal elements
   (a) $\{a, b, c, f\}$

7 Determine the greatest and least element, if exist, of the following posets.

1. 

   (a) Greatest element: f

   (b) Least element: a

2. 

   (a) Greatest elements: DNE (more than 1)
   (b) Least elements: DNE (more than 1)

8 Consider the following posets whose Hasse diagrams are shown. Find, if they exist,

maximal and minimal elements;

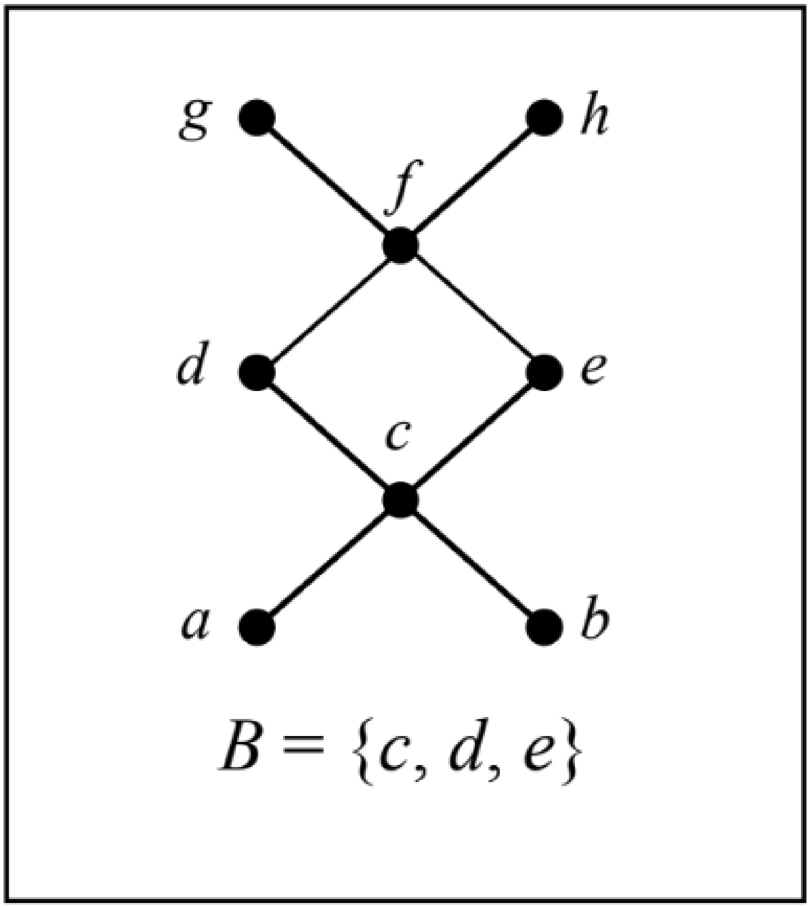
all upper bounds of B;

all lower bounds of B;

the least upper bound of B;

the greatest lower bound of B.

1.



(a) Find the maximal and minimal elements

i. Maximal: $\{g, h\}$

ii. Minimal: $\{a, b\}$

(b) all upper bounds of B;

i. $\{f, g, h\}$, because every element of B is $\leq \{f, g, h\}$.

(c) all lower bounds of B;

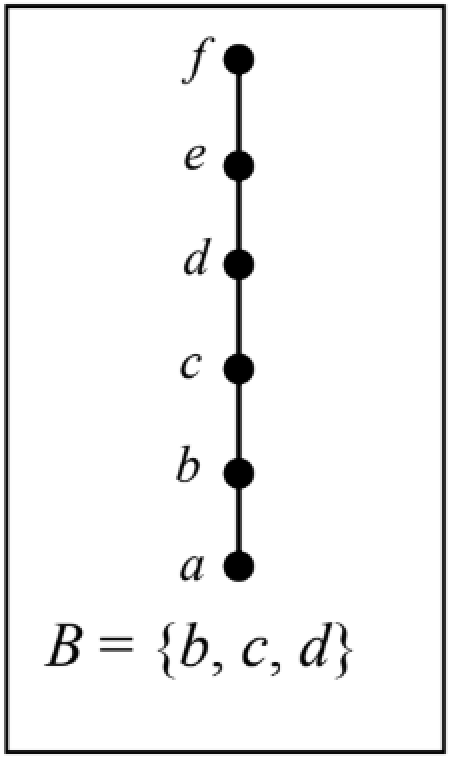
i. $\{a, b, c\}$

(d) the least upper bound of B;

i. f

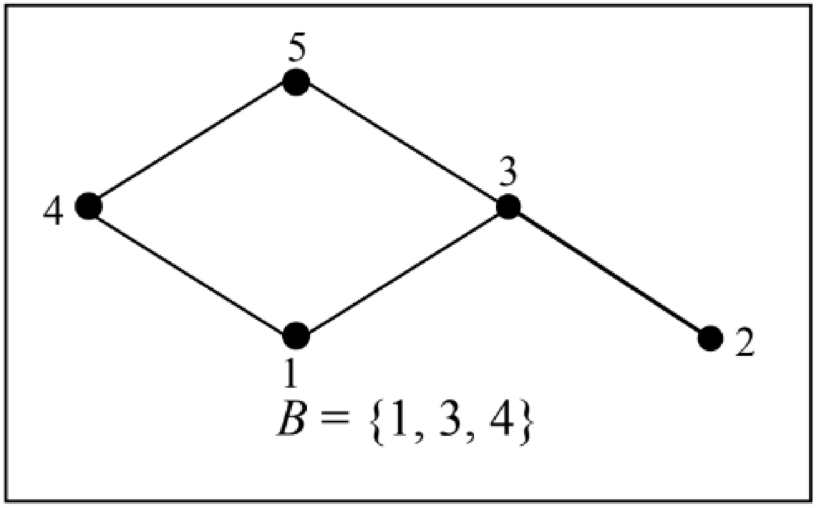
(e) the greatest lower bound of B.

i. c



2.

   (a) Find the maximal and minimal elements
      i. Maximal:$\{f\}$
      ii. Minimal: $\{a\}$
   (b) all upper bounds of B;
      i. $\{d, e, f\}$
   (c) all lower bounds of B;
      i. $\{a, b\}$
   (d) the least upper bound of B;
      i. $\{d\}$
   (e) the greatest lower bound of B.
      i. $\{b\}$

3. 

   (a) Find the maximal and minimal elements
      i. Maximal:$\{5\}$
      ii. Minimal: $\{1, 2\}$
   (b) all upper bounds of B;
      i. $\{5\}$
   (c) all lower bounds of B;
      i. $\{1\}$
   (d) the least upper bound of B;
      i. $\{5\}$
   (e) the greatest lower bound of B.
      i. $\{1\}$

9 Answer the following questions concerning the poset ({3, 5, 9, 15, 24, 45}, |).

1. Find the maximal and minimal elements.
2. Determine the greatest element and least element, if exist.
3. Find all upper bounds and least upper bounds of {3, 5}, if exist.
4. Find all lower bounds of {15, 45}. Hence determine the greatest lower bound of {15, 45}, if exist.

9.1 Answer

1. Find the poset mapping

$$
\begin{aligned}
\{\,&(3,3)\,,(3,9)\,,(3,15)\,,(3,24)\,,(3,45)\,,(5,5)\,,\\
&(5,15)\,,(5,45)\,,(9,45)\,,(15,15)\,,(15,45)\,,\\
&(24,24)\,,(45,45)\}
\end{aligned}
$$

2. Find the maximal and minimal elements.

   (a) Maximal: $\{24, 45\}$

   (b) Minimal: $\{3, 5\}$

3. Determine the greatest element and least element, if exist.

   (a) Greatest element

      i. **Does not exist** (Explanation: split off at end)

   (b) Least element

      i. **Does not exist** (Explanation: Split off at end)

4. Find all upper bounds and least upper bounds of {3, 5}, if exist.

   (a) All upper bounds: $\{15, 45\}$

   (b) Least upper bound: $\{15\}$

5. Find all lower bounds of {15, 45}. Hence determine the greatest lower bound of {15, 45}, if exist.

   (a) Lower bounds

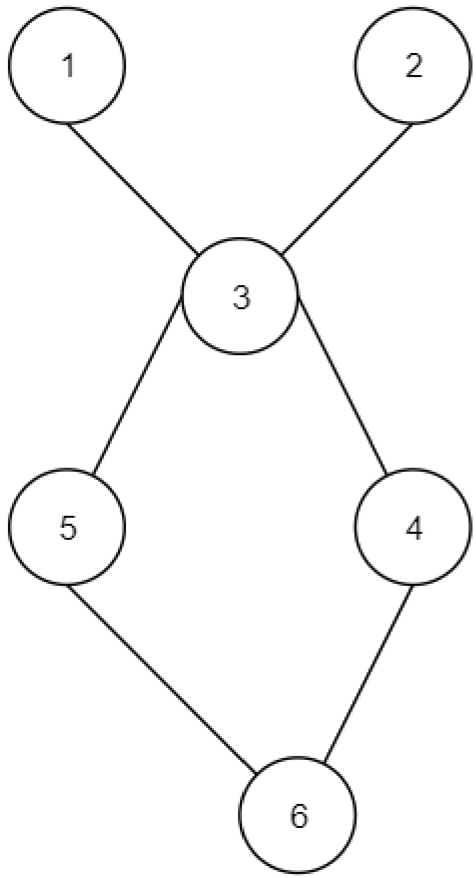
$$\{15, 45\}$$

   (b) GLB

$$\{15\}$$

10 Let $A = \{1, 2, 3, 4, 5, 6\}$ and consider the partial order R on A as R = {(6, 6), (6, 5), (6, 4), (6, 3), (6, 2), (6, 1), (5, 5), (5, 3), (5, 2), (5, 1), (4, 4), (4, 3), (4, 2), (4, 1), (3, 3), (3, 2), (3, 1), (2, 2), (1, 1)}.

1. Draw a Hasse diagram of the poset $[A, R]$



(a)

2. Find the minimal and maximal elements of the poset $[A, R]$
   (a) Minimal elements: $\{6\}$
   (b) Maximal elements: $\{1, 2\}$
3. Find the least upper bound of $2, 5$, if it exists.
   (a) $\{2\}$
4. Find the greatest lower bound of $5, 4$, if exists.
   (a) $\{6\}$