Ginwidth=

Calc 1: Tutorial 8

September 6, 2019

1. Find the critical numbers of the function.

   (a) $f(x) = x^3 + x^2 - x$

   $$f(x) = x^3 + x^2 - x$$

   i. Critical numbers, when $\frac{dy}{dx} = 0$

   $$f'(x) = 3x^2 + 2x - 1$$

   ii. When $f'(x) = 0$

   $$\begin{aligned} 3x^2 + 2x - 1 &= 0 \\ (3x-1)(x+1) &= 0 \\ x &= \frac{1}{3}, x = -1 \end{aligned}$$

   iii. Critical numbers: $\frac{1}{3}, -1$

   (b) $g(\theta) = \theta + \sin\theta$

   i. Find $g'(\theta)$

   $$\begin{aligned} g(\theta) &= \theta + \sin\theta \\ g'(\theta) &= 1 + \cos\theta \end{aligned}$$

   ii. Find $g'(\theta) = 0$

   $$\begin{aligned} 1 + \cos\theta = 0 & \\ \theta &= \cos^{-1}(1) \\ &= (\pi + 2\pi n)\text{,n is an integer} \end{aligned}$$

   iii. Critical numbers: $(\pi + 2\pi n)$

2. Find the absolute maximum and absolute minimum values of f on the given interval

   (a) $f(x) = x^3 - 3x + 1, [0, 3]$

i. Find the endpoints

A. $f(0) = 1$

B. $f(3) = 3^3 - 3(3) + 1 = 19$

ii. Find the formula for $\frac{dy}{dx}$

$$f'(x) = 3x^2 - 3$$

iii. Find all stationary points

$$\begin{aligned} 3x^2 - 3 &= 0 \\ 3\left(x^2 - 1\right) &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

iv. Find the values at the stationary points

$$\begin{aligned} f(1) &= 1^3 - 3(1) + 1 \\ &= 2 - 3 \\ &= -1 \end{aligned}$$

$$\begin{aligned} f(-1) &= (-1)^3 - 3(-1) + 1 \\ &= 3 \end{aligned}$$

v. Determine abs. max and abs.min points

A. Abs. max: 19

B. Abs. min = -1

(b) $f(x) = \frac{x}{x^2+4}, [0, 3]$

i. Find the endpoints

A. $f(0) = \frac{0}{0^2+4} = 0$

B. $f(3) = \frac{3}{3^2+4} = \frac{3}{13}$

ii. Find the formula for $\frac{dy}{dx}$

$$\begin{aligned} f'(x) &= \frac{\left(x^2+4\right)\frac{d}{dx}[x] - x\frac{d}{dx}\left[x^2+4\right]}{\left(x^2+4\right)^2} \\ &= \frac{\left(x^2+4\right) - x(2x)}{\left(x^2+4\right)^2} \\ &= \frac{x^2+4-2x^2}{\left(x^2+4\right)^2} \\ f'(x) &= \frac{-x^2+4}{\left(x^2+4\right)^2} \end{aligned}$$

iii. Find all stationary points

$$\frac{-x^2+4}{\left(x^2+4\right)^2}=0$$
$$-x^2+4=0$$
$$x^2=4$$
$$x=\pm 2$$

iv. Find the values at the stationary points

$$\begin{aligned} f(2) &= \frac{2}{2^2+4} \\ &= \frac{2}{8} \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} f(-2) &= \frac{-(-2)^2+4}{\left((-2)^2+4\right)^2} \\ &= 0 \end{aligned}$$

v. Determine abs. max and abs.min points

A. Abs. max: $\frac{1}{4}$

B. Abs. min $=0$

(c) $f(x)=x-2\cos x, [-\pi,\pi]$

i. Find the endpoints

A.

B.

$$\begin{aligned} f(-\pi) &= -\pi-(-\pi)\cos(-\pi) \\ &= -\pi-(-\pi)\cos(-\pi) \\ &= -\pi-(-\pi)(-1) \\ &= -2\pi \end{aligned}$$

C.

$$\begin{aligned} f(\pi) &= \pi-(\pi)\cos(\pi) \\ &= \pi-(\pi)\cos(\pi) \\ &= \pi-(\pi)(-1) \\ &= 2\pi \end{aligned}$$

ii. Find the formula for $\frac{dy}{dx}$

$$\begin{aligned} f'(x) &= 1-2(-\sin x) \\ &= 1+2\sin x \end{aligned}$$

iii. Find all stationary points

$$(\pi 2 + 2\pi n, 1), (3\pi 2 + 2\pi n, 1), (0 + 2\pi n, 0), (\pi + 2\pi n, 0)$$

, n is an integer

iv. Find the values at the stationary points

$$\begin{aligned} f(2) &= \frac{2}{2^2+4} \\ &= \frac{2}{8} \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} f(-2) &= \frac{-(-2)^2+4}{\left((-2)^2+4\right)^2} \\ &= 0 \end{aligned}$$

v. Determine abs. max and abs.min points

A. Abs. max: 3

B. Abs. min $= -3$

3. For the following functions

(a)

$$g(x) = 200 + 8x^3 + x^4$$

i. Find local max and local mins

$$\begin{aligned} g'(x) &= 24x^2 + 4x^3 \\ &= 4x^2(6+x) \end{aligned}$$

$$\begin{aligned} g'(x) &= 0 \\ 4x^2(6+x) &= 0 \\ x &= 0, x = -6 \end{aligned}$$

A. Find the coordinates

$$\begin{aligned} g(0) &= 200 + 8(0)^3 + (0)^4 \\ &= 200 \end{aligned}$$

$$\begin{aligned} g(-6) &= 200 + 8(-6)^3 + (-6)^4 \\ &= -232 \end{aligned}$$

B. Local max. and local min.

$$(0, 200), (-6, -232)$$

ii. Determine the interval(s) where the curve is concave upward and the interval(s) where it is concave downward. Find the coordinates of the inflection point.

A. Find the second derivative

$$\begin{aligned} g''(x) &= \frac{d}{dx}\left[24x^2 + 4x^3\right] \\ &= 48x + 12x^2 \end{aligned}$$

B. Find the inflection points

$$\begin{aligned} 0 &= 12x(4 + x) \\ x &= 0, x = -4 \end{aligned}$$

C. Find the coordinates

$$(0, 200)$$

$$\begin{aligned} g(-4) &= 200 + 8(-4)^3 + (-4)^4 \\ &= -56 \end{aligned}$$

$$(-4, -56)$$

iii.

<table>
<tr><td>$f'(x)$</td><td>dec.</td><td>0</td><td colspan="2">inc.</td><td>0</td><td>inc.</td></tr>
<tr><td>x</td><td>-7</td><td>-6</td><td>-4</td><td>-1</td><td>0</td><td>1</td></tr>
<tr><td>$f''(x)$</td><td colspan="2">c.up</td><td>0</td><td>c.down</td><td>0</td><td>c.up</td></tr>
</table>

(b) $f(x) = 2x^3 - 7x^2 + 4x + 5$

i. Find the local maximum and minimum points of the curve.

A. When $f'(x) = 0$

$$\begin{aligned} f'(x) &= 6x^2 - 14x + 4 \\ &= 2\left(3x^2 - 7x + 2\right) \\ &= 2(3x - 1)(x - 2) \\ 0 &= (3x - 1)(x - 2) \\ x &= \frac{1}{3}, x = 2 \end{aligned}$$

B. Coordinates

$$\begin{aligned} f\left(\frac{1}{3}\right) &= 2\left(\frac{1}{3}\right)^3 - 7\left(\frac{1}{3}\right)^2 + 4\left(\frac{1}{3}\right) + 5 \\ &= \frac{152}{27} \end{aligned}$$

$$\left(\frac{1}{3}, \frac{152}{27}\right)$$

$$\begin{aligned} f(2) &= 2(2)^3 - 7(2)^2 + 4(2) + 5 \\ &= 1 \end{aligned}$$

$$(2, 1)$$

ii. Determine the interval(s) where the curve is concave upward and the interval(s) where it is concave downward. Find the coordinates of the inflection point.

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left[6x^2 - 14x + 4\right] \\ &= 12x - 14 \end{aligned}$$

$$\begin{aligned} f''(x) &= 0 \\ 2(6x - 7) &= 0 \\ x &= \frac{7}{6} \end{aligned}$$

A. Coordinates

$$\begin{aligned} f\left(\frac{7}{6}\right) &= 6\left(\frac{7}{6}\right)^2 - 14\left(\frac{7}{6}\right) + 4 \\ &= -\frac{25}{6} \end{aligned}$$

$$\left(\frac{7}{6}, -\frac{25}{6}\right)$$

iii. Sketch the curve (use the table)

A.

$f'(x)$	increasing	0	decreasing	0	increasing
x	0	$\frac{1}{3}$	$1\frac{1}{6}$	2	3
$f''(x)$	-ve		0		+ve

(c) $f(x) = \ln\left(1 + x^2\right)$

i. Find the local maximum and minimum points of the curve.

$$\begin{aligned} f'(x) &= \frac{1}{1 + x^2} \cdot (2x) \\ &= \frac{2x}{1 + x^2} \end{aligned}$$

$$\begin{aligned} f'(x) &= 0 \\ \frac{2x}{1 + x^2} &= 0 \\ 2x &= 0 \\ x &= 0 \end{aligned}$$

A. Coordinates

$$\begin{aligned} f(0) &= \ln 1 \\ &= 0 \end{aligned}$$

$$(0, 0)$$

ii. Determine the interval(s) where the curve is concave upward and the interval(s) where it is concave downward. Find the coordinates of the inflection point.

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left[\frac{2x}{1+x^2}\right] \\ &= \frac{\left(1+x^2\right)(2) - (2x)(2x)}{(1+x)^2} \\ &= \frac{2\left(1+x^2\right) - 4x^2}{(1+x)^2} \\ &= \frac{2\left(\left(1+x^2\right) - 2x^2\right)}{(1+x)^2} \\ &= \frac{2\left(1-x^2\right)}{(1+x)^2} \end{aligned}$$

$$\begin{aligned} f''(x) &= 0 \\ \frac{2\left(1-x^2\right)}{(1+x)^2} &= 0 \\ \left(1-x^2\right) &= 0 \\ -(x-1)(x+1) &= 0 \\ x &= \pm 1 \end{aligned}$$

A. Coordinates

$$\begin{aligned} f(1) &= 2(1)^3 - 7(1)^2 + 4(1) + 5 \\ &= 4 \end{aligned}$$

$$\begin{aligned} f(-1) &= 2(-1)^3 - 7(-1)^2 + 4(-1) + 5 \\ &= -8 \end{aligned}$$

$$(1, 4), (-1, -8)$$

iii. Sketch the curve

A.

$f'(x)$	decr.	0	incr.

x	-2	-1	0	1	2
$f''(x)$	c.down	0	c.up	0	c.down

4. $f(x) = \frac{x}{(x-1)^3}$

(a) Find the vertical and horizontal asymptotes.

i. Vertical asymp.

$$\begin{aligned}(x-1)^3 &= 0\\ x-1 &= 0\\ x &= 1\end{aligned}$$

ii. Horizontal asymp.

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{(x-1)^3} &= \frac{\infty}{(\infty-1)^3}\\ &= \frac{\infty}{\infty^3}\\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{x\to-\infty}\frac{x}{(x-1)^3} &= \frac{-\infty}{(-\infty-1)^3}\\ &= 0\end{aligned}$$

(b) Find the intervals of increase or decrease.

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left[\frac{x}{(x-1)^3}\right]\\ &= \frac{(x-1)^3 - x\cdot 3(x-1)^2}{(x-1)^6}\\ &= \frac{(x-1)^2(x-1-3x)}{(x-1)^6}\\ &= -\frac{(1+2x)}{(x-1)^4}\end{aligned}$$

$$\begin{aligned}-\frac{(1+2x)}{(x-1)^4} &= 0\\ -1-2x &= 0\\ x &= -\frac{1}{2}\end{aligned}$$

(c) Find the local maximum and minimum values

i. Local maximum: $(-0.5, -0.148)$

ii. Local minimum: None

(d) Find the intervals of concavity and the inflection points.

$$\begin{aligned}
f''(x) &= \frac{d}{dx}\left[-\frac{(1+2x)}{(x-1)^4}\right] \\
&= -\frac{(x-1)^4(2)-(1+2x)\,4\,(x-1)^3}{(x-1)^8} \\
&= \frac{2(x-1)^4-4(1+2x)(x-1)^3}{(x-1)^8} \\
&= \frac{2(x-1)-4(1+2x)}{(x-1)^5} \\
&= \frac{2x-2-4-8x}{(x-1)^5} \\
&= \frac{-6-6x}{(x-1)^5} \\
f''(x) &= \frac{-6(1+x)}{(x-1)^5}
\end{aligned}$$

$$\begin{aligned}
\frac{-6(1+x)}{(x-1)^5} &= 0 \\
-6(1+x) &= 0 \\
x &= -1
\end{aligned}$$

$$\begin{aligned}
f(-1) &= \frac{-1}{(-1-1)^3} \\
&= \frac{-1}{(-2)^3} \\
&= \frac{1}{8}
\end{aligned}$$

$$\left(-1, \frac{1}{8}\right)$$

(e) Use the information from parts (a)-(d) to sketch the graph

i.

$f'(x)$	incr.		0	dec.
x	-2	-1	$-\frac{1}{2}$	0
$f(x)$		$\frac{1}{8}$		
$f''(x)$	c.down	0	c.up	

5.

(a)

$$\begin{aligned}\lim_{x\to 1}\frac{x^a-1}{x^b-1} &= \lim_{x\to 1}\frac{ax^{a-1}}{bx^{b-1}}\\ &= \frac{a\left(1\right)^{a-1}}{b\left(1\right)^{b-1}}\\ &= \frac{a}{b}\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 0}\frac{x+\tan x}{\sin x} &= \lim_{x\to 0}\frac{1+\sec^2 x}{\cos x}\\ &= \lim_{x\to 0}\frac{1+\sec^2\left(0\right)}{\cos\left(0\right)}\\ &= \lim_{x\to 0}\frac{1+1}{1}\\ &= 2\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln\ln x}{x} &= \lim_{x\to\infty}\frac{\frac{1}{\ln x}\cdot\left(\frac{1}{x}\right)}{1}\\ &= \lim_{x\to\infty}\frac{\frac{1}{x\ln x}}{1}\\ &= \lim_{x\to\infty}\frac{1}{x\ln x}\\ &= \frac{1}{\infty\ln\infty}\\ &= 0\end{aligned}$$

(d)

$$\begin{aligned}\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2} &= \lim_{x\to 0}\frac{-m\sin\left(mx\right)+n\sin\left(nx\right)}{2x}\\ &= \lim_{x\to 0}\frac{-m^2\cos\left(mx\right)+n^2\cos\left(nx\right)}{2}\\ &= \frac{-m^2\cos 0+n^2\cos 0}{2}\\ &= \frac{n^2-m^2}{2}\end{aligned}$$

(e)

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\tan^{-1}x} &= \lim_{x\to 0}\frac{1}{x^2+1}\\ &= \frac{1}{1}\\ &= 1\end{aligned}$$

(f)

$$\begin{aligned}\lim_{x\to -\infty} x^2e^x &= \lim_{x\to -\infty} x^2e^x\\ &= \lim_{x\to -\infty}\frac{x^2}{\frac{1}{e^x}}\\ &= \lim_{x\to -\infty}\frac{2x}{-e^{-x}}\\ &= \lim_{x\to -\infty}\frac{2}{e^x}\\ &= \frac{2}{e^{-\infty}}\\ &= 0\end{aligned}$$

(g)

$$\begin{aligned}\lim_{x\to\left(\frac{\pi}{2}\right)^-}\sec 7x\cos 3x &= \lim_{x\to\left(\frac{\pi}{2}\right)^-}\sec 7x\cos 3x\\ &= \lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{\cos 3x}{\cos 7x}\\ &= \lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{-3\sin 3x}{-7\sin 7x}\\ &= \frac{-3\cos\left(\frac{\pi}{2}-\frac{3\pi}{2}\right)}{-7\cos\left(\frac{\pi}{2}-\frac{7\pi}{2}\right)}\\ &= \frac{-3\cos(-\pi)}{-7\cos(-3\pi)}\\ &= \frac{-3\cos(\pi)}{-7\cos(3\pi)}\\ &= \frac{-3\,(1)}{-7\,(1)}\\ &= \frac{3}{7}\end{aligned}$$

(h) $\lim_{x\to 0} (\csc x - \cot x)$

$$\begin{aligned}
\lim_{x\to 0} (\csc x - \cot x) &= \lim_{x\to 0} (\csc x - \cot x) \\
&= \lim_{x\to 0} (\csc x) - \lim_{x\to 0} (\cot x) \\
&= \lim_{x\to 0} \left(\frac{1}{\sin x}\right) - \lim_{x\to 0} \left(\frac{1}{\tan x}\right) \\
&= \lim_{x\to 0} \left(\sin^{-1} x\right) - \lim_{x\to 0} \left(\tan^{-1} x\right) \\
&= \lim_{x\to 0} \left(\sin^{-1} x\right) - \lim_{x\to 0} \left(\tan^{-1} x\right) \\
&= \lim_{x\to 0} \left(\frac{1}{\sqrt{1-x^2}}\right) - \lim_{x\to 0} \left(\frac{1}{1+x^2}\right) \\
&= 1 - 1 \\
&= 0
\end{aligned}$$

(i) $\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

$$\begin{aligned}
\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right) &= \lim_{x\to 1} \left(\frac{x-1-\ln x}{(x-1)\ln x}\right) \\
&= \lim_{x\to 1} \left(\frac{x-1-\ln x}{x\ln x - \ln x}\right) \\
&= \lim_{x\to 1} \left(\frac{1-\frac{1}{x}}{\ln x + \frac{x}{x} - \frac{1}{x}}\right) \\
&= \lim_{x\to 1} \left(\frac{1-\frac{1}{x}}{\ln x + 1 - \frac{1}{x}}\right) \\
&= \lim_{x\to 1} \left(\frac{1-x^{-1}}{\ln x + 1 - x^{-1}}\right) \\
&= \lim_{x\to 1} \left(\frac{x^{-2}}{\frac{1}{x} + x^{-2}}\right) \\
&= \left(\frac{1}{1+1}\right) \\
&= \frac{1}{2}
\end{aligned}$$

(j) $\lim_{x\to 0} (1-2x)^{\frac{1}{x}}$

i. Let $y = (1-2x)^{\frac{1}{x}}$

$$\begin{aligned}
\ln y &= \frac{1}{x}\ln(1-2x) \\
&= \frac{\ln(1-2x)}{x} \\
\lim_{x\to 0}\ln(y) &= \lim_{x\to 0}\frac{\ln(1-2x)}{x} \\
&= \lim_{x\to 0}\frac{-\frac{2}{1-2x}}{1} \\
&= \lim_{x\to 0}-\frac{2}{1-2x} \\
&= -\frac{2}{1-2(0)} \\
\ln y &= -2
\end{aligned}$$

ii.

$$\begin{aligned}
\lim_{x\to 0}(1-2x)^{\frac{1}{x}} &= \lim_{x\to\infty} y \\
&= \lim_{x\to\infty} e^{\ln(y)} \\
&= e^{\lim_{x\to\infty}\ln(y)} \\
&= e^{-2}
\end{aligned}$$