Calculus 1 C3: Differentiation

July 23, 2019

1 Example

1. $10x$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(h)}{h} \\
&= \lim_{h\to 0} \frac{10(x+h) - 10x}{h} \\
&= \lim_{h\to 0} \frac{\cancel{10x} + 10h \cancel{-10x}}{h} \\
&= \lim_{h\to 0} \frac{+10h}{h} \\
f'(x) &= 10
\end{aligned}$$

2. $3x^2 + x + 1$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(h)}{h} \\
&= \lim_{h\to 0} \frac{3(x+h)^2 + x + h + 1 - 3x^2 + x + 1}{h} \\
&= \lim_{h\to 0} \frac{3\left(x^2 + 2hx + h^2\right) + x + h + 1 - 3x^2 + x + 1}{h} \\
&= \lim_{h\to 0} \frac{\cancel{3x^2} + 6hx + 3h^2 + \cancel{x} + h \cancel{+1 - 3x^2 + x + 1}}{h} \\
&= \lim_{h\to 0} \frac{6hx + 3h^2 + h}{h} \\
&= \lim_{h\to 0} 6x + 3h + 1 \\
\boldsymbol{f'(x)} &\boldsymbol{= 6x + 1}
\end{aligned}$$

3. $\sqrt{x}$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(h)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \lim_{h\to 0} \left(\frac{1}{\sqrt{x+h}+\sqrt{x}}\right) \\
f'(x) &= \frac{1}{2\sqrt{x}}
\end{aligned}
$$

2 Example

1. Evaluate the following derivatives

   (a) $y = 8x^5$

   $$
   \begin{aligned}
   y &= 8x^5 \\
   y' &= 40x^4
   \end{aligned}
   $$

   (b) $y = 3\ln(x)$

   $$y' = \frac{3}{x}$$

   (c) $y = -7e^x$

   $$
   \begin{aligned}
   y &= -7e^x \\
   \frac{dy}{dx} &= -7 * 1e^x \\
   &= -7e^x
   \end{aligned}
   $$

   (d) $y = \frac{1}{2}cos(x)$

   $$
   \begin{aligned}
   \frac{dy}{dx} &= \frac{1}{2}(-sin(x)) \\
   &= -\frac{1}{2}sin(x)
   \end{aligned}
   $$

   (e) $y = \pi x$

   $$\frac{dy}{dx} = \pi$$

(f) $y = \frac{1}{3x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{3}\frac{dy}{dx}\left(x^{-1}\right)\\ &= -\frac{1}{3}x^{-2}\\ &= -\frac{1}{3x^2}\end{aligned}$$

3 Example

1. Evaluate the derivatives

(a) $y = 2x^4 + \frac{3}{x^2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dx}\left(2x^4\right) + \frac{dy}{dx}\left(\frac{3}{x^2}\right)\\ &= 8x^3 - \frac{6}{x^3}\end{aligned}$$

(b) $y = e^x - 5sin\left(x\right)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dx}\left(e^x\right) - 5\frac{dy}{dx}\left(sin\left(x\right)\right)\\ &= e^x - 5\left(cos\left(x\right)\right)\\ &= e^x - 5cos\left(x\right)\end{aligned}$$

(c) $y = \frac{x^3+4}{x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dx}\left(x^2\right) + \frac{dy}{dx}\left(\frac{4}{x}\right)\\ &= 2x - \frac{4}{x^2}\end{aligned}$$

4 Example

1. Evaluate the derivatives

(a) $y = x\ln x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dx}\left(x\right)\cdot\left(\ln x\right) + x\cdot\frac{dy}{dx}\left(\ln x\right)\\ &= \ln\left(x\right) + x\cdot\frac{1}{x}\\ \boldsymbol{\frac{dy}{dx}} &\boldsymbol{= \ln\left(x\right) + 1}\end{aligned}$$

(b) $y = 2x^2 \sin x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dx}\left(2x^2\right) \cdot (\sin x) + 2x^2 \cdot \frac{dy}{dx}(\sin x) \\ &= 4x \cdot \sin(x) + 2x^2 \cdot (\cos(x)) \\ &= 4x \sin(x) + 2x^2 \cos(x)\end{aligned}$$

(c) $y = \left(2x^3 + 1\right)(x - 5)$

$$\begin{aligned}\frac{dy}{dx} &= \left(2x^3 + 1\right)(x - 5) \\ &= \frac{dy}{dx}\left(2x^3 + 1\right)(x - 5) + \left(2x^3 + 1\right)\frac{dy}{dx}(x - 5) \\ &= 6x^2(x - 5) + \left(2x^3 + 1\right)1 \\ &= 6x^3 - 30x^2 + 2x^3 + 1 \\ \frac{dy}{dx} &= 8x^3 - 30x^2 + 1\end{aligned}$$

(d) $y = e^x\left(x^2 + 7\right)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dx}(e^x) \cdot \left(x^2 + 7\right) + e^x \cdot \frac{dy}{dx}\left(x^2 + 7\right) \\ &= e^x \cdot \left(x^2 + 7\right) + e^x \cdot 2x \\ &= e^x\left(x^2 + 7\right) + 2xe^x \\ &= e^x\left(x^2 + 2x + 7\right)\end{aligned}$$

5 Proof & Examples

5.1 Proofs

1. $\frac{d}{dx}(\tan x) = \sec^2(x)$ (REPROOF WITH SIMPLER METHOD)

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) \\ &= \frac{\cos(x)\frac{d}{dx}(\sin x) - \sin(x)\frac{d}{dx}(\cos(x))}{(cosx)^2} \\ &= \frac{\cos(x) \cdot (\cos(x)) - \sin(x) \cdot (-\sin(x))}{cos^2 x} \\ &= \frac{\cos^2 x + \sin^2(x)}{\cos^2 x}; \text{note: } sin^2(x) + cos^2(x) = 1 \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x\end{aligned}$$

2. $\frac{d}{dx}(\cot x) = -csc^2(x)$ (REPROOF WITH SIMPLER METHOD)

$$\begin{aligned}
\frac{d}{dx}(\cot x) &= \frac{d}{dx}\left(\tan^{-1}(x)\right) \\
&= \frac{d}{dx}\left(\frac{cos(x)}{sin(x)}\right) \\
&= \frac{d}{dx}\left(\frac{cos(x)}{sin(x)}\right) \\
&= \frac{\sin(x)\frac{d}{dx}(cos(x)) - \cos(x)\frac{d}{dx}(\sin(x))}{(sin(x))^2} \\
&= \frac{\sin(x)(-\sin(x)) - \cos(x)(\cos(x))}{(sin(x))^2} \\
&= \frac{-\left(\sin^2(x)\right) - cos^2(x)}{(sin(x))^2} \\
&= \frac{-\left(sin^2(x) + cos^2(x)\right)}{sin^2(x)} \\
&= -\frac{1}{sin^2(x)} \\
&= -\csc^2(x)
\end{aligned}$$

3. $\frac{d}{dx}(\sec x) = \sec x \tan x$ (REPROOF WITH SIMPLER METHOD)

$$\begin{aligned}
\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{cos\,(x)}\right) \\
&= \frac{d}{dx}\left(\frac{1}{cos\,(x)}\right) \\
&= \frac{d}{dx}\left(\frac{\tan(x)}{\sin(x)}\right) \\
&= \frac{\sin(x)\,\frac{d}{dx}(\tan(x)) - \tan(x)\,\frac{d}{dx}(\sin(x))}{sin^2(x)} \\
&= \frac{\sin(x)\left(\sec^2(x)\right) - \tan(x)(\cos(x))}{\sin^2(x)} \\
&= \frac{\sin(x)\left(\sec^2(x)\right) - \frac{\sin(x)}{\cos(x)}(\cos(x))}{\sin^2(x)} \\
&= \frac{\sin(x)\left(\sec^2(x)\right) - \sin(x)}{\sin^2(x)} \\
&= \frac{\sin(x)\left(\sec^2(x) - 1\right)}{\sin^2(x)} \\
&= \left(\sec^2(x) - 1\right)\cdot\left(\frac{1}{sin\,(x)}\right) \\
&= \left(\tan^2(x)\right)\cdot\left(\frac{1}{\sin(x)}\right) \\
&= \tan(x)\cdot\frac{\cancel{\sin(x)}}{\cos(x)}\cdot\frac{1}{\cancel{\sin(x)}} \\
&= \tan(x)\cdot\frac{1}{cos\,(x)} \\
&= \tan(x)\sec(x)
\end{aligned}$$

$$\boldsymbol{\frac{d}{dx}(\sec x) = \sec(x)\tan(x)}$$

4. $\frac{d}{dx}(\csc(x)) = -\csc x \cot x$

$$\begin{aligned}
\frac{d}{dx}(\csc(x)) &= \frac{d}{dx}\left(\frac{1}{\sin(x)}\right) \\
&= \frac{\sin(x)\frac{d}{dx}(1) - \frac{d}{dx}(\sin(x))}{\sin^2(x)} \\
&= \frac{\sin(x)(0) - \cos(x)}{\sin^2(x)} \\
&= \frac{-\cos(x)}{\sin^2(x)} \\
&= -\csc(x)\cot(x)
\end{aligned}$$

5.2 Example

1. Evaluate the following derivatives

   (a) $y = \frac{e^x}{\sin(x)}$

$$\begin{aligned}
y' &= \frac{\sin(x)\cdot\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(\sin(x))}{\sin^2(x)} \\
&= \frac{\sin(x)\cdot e^x - e^x\cos(x)}{\sin^2(x)} \\
&= e^x\frac{(\sin(x) - \cos(x))}{\sin^2(x)}
\end{aligned}$$

   (b) $y = \frac{x}{\ln x}$

$$\begin{aligned}
y' &= \frac{\ln(x)\frac{d}{dx}(x) - x\frac{d}{dx}(\ln x)}{(\ln x)^2} \\
&= \frac{\ln(x) - x\frac{1}{x}}{(\ln x)^2} \\
&= \frac{\ln(x) - 1}{(\ln x)^2}
\end{aligned}$$

(c) $y = \frac{2x}{(x+2)^2}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(x+2)^2 \frac{d}{dx}(2x) - 2x \cdot \frac{d}{dx}(x+2)^2}{(x+2)^4} \\
&= \frac{2(x+2)^2 - 2x \cdot 2(x+2)(1)}{(x+2)^4} \\
&= \frac{2(x+2)^2 - 4x(x+2)}{(x+2)^4} \\
&= \frac{2(x+2)((x+2) - 2x)}{(x+2)^4} \\
&= \frac{2(2-x)}{(x+2)^3}
\end{aligned}$$

(d) $y = \sec x$

$$\frac{dy}{dx} = \sec x \tan x$$

6 Example

1. Evaluate the derivatives.

(a) $y = (2x-3)^5$

$$\begin{aligned}
u &= 2x - 3 \\
y &= u^5 \\
\frac{dy}{dx} &= 2 \\
\frac{dv}{dx} &= 5u^4 \cdot \frac{dy}{dx}
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= 5(2x-3)^4(2) \\
\frac{dy}{dx} &= 10(2x-3)^4
\end{aligned}$$

(b) $y = \ln\left(2x - x^5\right)$

$$\begin{aligned}
u &= 2x - x^5 \\
\frac{du}{dx} &= 2 - 5x^4
\end{aligned}$$

$$\begin{aligned}
y &= \ln(u) \\
\frac{dy}{dx} &= \frac{1}{u} \cdot \frac{du}{dx} \\
&= \frac{2 - 5x^4}{2x - x^5}
\end{aligned}$$

(c) $y = \sin\left(7x^2\right)$

$$\begin{aligned} u &= 7x^2 \\ \frac{du}{dx} &= 14x \end{aligned}$$

$$\begin{aligned} y &= \sin u \\ \frac{dy}{dx} &= \frac{du}{dx} \cos u \\ \boldsymbol{\frac{dy}{dx}} &\boldsymbol{= 14x \cos\left(7x^2\right)} \end{aligned}$$

(d) $y = 2e^{4x}$

$$\begin{aligned} \frac{dy}{dx} &= 2\frac{dy}{dx}e^{4x} \\ &= 2\frac{dy}{dx}\left(e^x\right)^4 \end{aligned}$$

$$\begin{aligned} u &= e^x \\ \frac{du}{dx} &= e^x \end{aligned}$$

$$\begin{aligned} y &= 2u^4 \\ \frac{dy}{dx} &= 8u^3 \cdot \frac{du}{dx} \\ &= 8\left(e^x\right)^3 \cdot e^x \\ &= 8e^{3x} \cdot e^x \\ &= 8e^{4x} \end{aligned}$$

(e) $y = \tan^7(x)$

$$\begin{aligned} y &= (\tan(x))^7 \\ u &= \tan(x) \\ \frac{du}{dx} &= \sec^2(x) \end{aligned}$$

$$\begin{aligned} y &= u^7 \\ \frac{dy}{dx} &= 7u^6 \cdot \frac{du}{dx} \\ &= 7(\tan x)^6 \cdot \sec^2(x) \\ &= 7\tan^6 x \cdot \sec^2(x) \end{aligned}$$

(f) $y = \sqrt{1+x^2}$

$$\begin{aligned}
y &= \left(1+x^2\right)^{\frac{1}{2}} \\
u &= 1+x^2 \\
\frac{du}{dx} &= 2x
\end{aligned}$$

$$\begin{aligned}
y &= u^{\frac{1}{2}} \\
\frac{dy}{dx} &= \frac{1}{2}u^{-\frac{1}{2}} \cdot \frac{du}{dx} \\
&= \frac{1}{2\sqrt{1+x^2}} \cdot 2x \\
&= \frac{x}{\sqrt{1+x^2}}
\end{aligned}$$

7 Parametric Differentiation

1. $x = t^3, y = t^2$

(a) Differentiate both parametric equations

$$\begin{aligned}
\frac{dx}{dt} &= 3t^2 \\
\frac{dy}{dt} &= 2t
\end{aligned}$$

(b) Utilize chain rule to find $\frac{dy}{dx}$ (as for why $\frac{dy}{dx}$, simple, your cartesian plane only have $x-coordinate$ and $y-coordinate$, no $t-coordinate$, so you need your gradient be a derivative of y (dy) with respect to x (dx).

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dt} * \frac{dt}{dx} \\
&= 2t * \frac{1}{\frac{dx}{dt}} \\
&= 2t * \frac{1}{3t^2} \\
\frac{dy}{dx} &= \frac{2}{3t}
\end{aligned}$$

2. $x = \cos 2\theta, y = 1 + \sin 2\theta$,find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{6}$

(a) Find the derivative

$$\begin{aligned}
\frac{dx}{d\theta} &= -\sin 2\theta * 2 \\
&= -2\sin 2\theta
\end{aligned}$$

$$\frac{dy}{d\theta} = 2\cos 2\theta$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{d\theta} * \frac{d\theta}{dx} \\ &= 2\cos 2\theta * \frac{1}{-2\sin 2\theta} \\ &= -\cot 2\theta\end{aligned}$$

(b) At $\theta = \frac{\pi}{6}$

$$\begin{aligned}\frac{dy}{dx} &= -\cot\frac{\pi}{3} \\ &= \frac{-1}{\sqrt{3}}\end{aligned}$$

3. $x = \frac{2-3t}{1+t}, y = \frac{3+2t}{1+t}$

(a) Find the derivative $\frac{dy}{dx}$

$$\begin{aligned}\frac{dx}{dt} &= \frac{(1+t)\cdot(-3) - (2-3t)\,(1)}{(1+t)^2} \\ &= \frac{-5}{(1+t)^2}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dt} &= \frac{(1+t)\cdot(2) - (3+2)\,(1)}{(1+t)^2} \\ &= -\frac{1}{(1+t)^2}\end{aligned}$$

i. $\frac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} * \frac{dt}{dx} \\ &= \frac{dy}{dt} * \frac{1}{\frac{dx}{dt}} \\ &= -\frac{1}{\cancel{(1+t)^2}} * -\frac{\cancel{(1+t)^2}}{5} \\ &= \frac{1}{5}\end{aligned}$$

8 Implicit Differentiation

8.1 Example 3.8

1. Find the $\frac{dy}{dx}$

(a) $x^2 + y^2 - 2x - 6y + 5 = 0$

$$\begin{aligned} x^2 + y^2 - 2x - 6y + 5 &= 0 \\ y^2 - 6y &= x^2 - 2x + 5 \end{aligned}$$

i. Hmm, we cannot separate completely (like $y = ...$ or $x = ...$), so lets try implicit differentation

$$\begin{aligned} \frac{d}{dx}\left(y^2 - 6y\right) &= \frac{d}{dx}\left(x^2 - 2x + 5\right) \\ \frac{dy}{dx}(2y) - 6\frac{dy}{dx} &= 2x - 2 \\ \frac{dy}{dx}(2y - 6) &= 2x - 2 \\ \frac{dy}{dx} &= \frac{2x-2}{2y-6} \\ \frac{dy}{dx} &= \frac{x-1}{y-3} \end{aligned}$$

(b) $x^2 + y^3 = xy$

i. Hate at first sight, don't even think I can break them up well, so lets try implicit differentiation again.

$$\begin{aligned} \frac{d}{dx}\left(x^2 + y^3\right) &= \frac{d}{dx}(xy) \\ 2x + 3y^2 \cdot \frac{dy}{dx} &= 1 \cdot y + x \cdot \frac{dy}{dx} \\ 3y^2 \cdot \frac{dy}{dx} - x \cdot \frac{dy}{dx} &= y - 2x \\ \frac{dy}{dx}\left(3y^2 - x\right) &= y - 2x \\ \boldsymbol{\frac{dy}{dx}} &\boldsymbol{=} \boldsymbol{\frac{y-2x}{3y^2-x}} \end{aligned}$$

(c) $\frac{x}{y} + x^2 = 2y$

i. Okay screw this again, I mean you could potentially make them have the same denominator, and then move the y to the opposite side, but you will be still stuck with x and y together. Darned lovebirds. Anyways, lets try implicit differentiation again.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x}{y}+x^2\right) &= \frac{d}{dx}\left(2y\right) \\
\frac{y-x\frac{dy}{dx}}{y^2}+2x &= 2\frac{dy}{dx} \\
\frac{1}{y}-\frac{dy}{dx}\left(\frac{x}{y^2}\right)+2x &= 2\frac{dy}{dx} \\
\frac{1}{y}+2x &= 2\frac{dy}{dx}+\frac{dy}{dx}\left(\frac{x}{y^2}\right) \\
&= \frac{dy}{dx}\left(2+\frac{x}{y^2}\right) \\
\frac{dy}{dx} &= \frac{\frac{1}{y}+2x}{2+\frac{x}{y^2}} \\
&= \frac{\frac{1}{y^2}\left(1+2xy^2\right)}{\frac{1}{y^2}\left(2y^2+x\right)} \\
\boldsymbol{\frac{dy}{dx}} &\boldsymbol{=} \boldsymbol{\frac{\left(1+2xy^2\right)}{\left(2y^2+x\right)}}
\end{aligned}$$

(d) $\sin xy = 1$

$$\begin{aligned}
\frac{d}{dx}\left(\sin xy\right) &= 1 \\
\cos xy\cdot\frac{d}{dx}\left(xy\right) &= \frac{d}{dx}\left(1\right) \\
\cos xy\cdot\left(y+x\frac{dy}{dx}\right) &= 0 \\
x\frac{dy}{dx}\cos xy+y\cos xy &= 0 \\
x\frac{dy}{dx}\cos xy &= -y\cos xy \\
\frac{dy}{dx}\left(x\cos xy\right) &= -y\cos xy \\
\frac{dy}{dx} &= -\frac{y\cos xy}{x\cos xy} \\
&= -\frac{y}{x}
\end{aligned}$$

9 Differentiation of Inverse Trigonometric Functions

9.1 Cheatsheet Prime

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\begin{aligned}\frac{d}{dx}\left(\cos^{-1} x\right) &= -\left(\frac{d}{dx}\sin^{-1} x\right)\\ &= -\frac{1}{\sqrt{1-x^2}}\end{aligned}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

9.1.1 Note: $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$ is not given in exams

9.2 Proofings

1. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$

   (a) Let $f(x) = \sin^{-1}(x)$

   $$\begin{aligned}f(\sin x) &= \sin\left(\sin^{-1} x\right)\\ &= x\\ f'(\sin x)\cos x &= 1\\ f'(\sin x) &= \frac{1}{\cos x}\end{aligned}$$

   i. Use the identity $\sin^2 x + \cos^2 x = 1$

   $$\begin{aligned}\cos^2 x &= 1\\ &= 1-\sin^2 x\\ \cos x &= \sqrt{1-\sin^2 x}\end{aligned}$$

   ii. Remember earlier in $f(x)$ where we made our $x = \sin x$?

   $$\begin{aligned}x &= \sin x\\ x^2 &= \sin^2 x\end{aligned}$$

   iii. Substitute back in

   $$\cos x = \sqrt{1-x^2}$$

iv. Substitute back into the original function, with $\sin x$ substituted by x

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

v. Remember we made $f(x) = \sin^{-1}(x)$? Now lets us turn it back, but now we have $f'(x)$, therefore, its more like after $\frac{dy}{dx}$

$$\frac{dy}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

2. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

   (a) Let $f(x) = \tan^{-1}(x)$

$$\begin{aligned} f(\tan\theta) &= \theta; f^{-1}f(x) = x \\ f'(\tan\theta)\cdot\sec^2\theta &= 1 \\ f'(\tan\theta) &= \frac{1}{\sec^2\theta} \end{aligned}$$

   (b) We will apply the following rules:

      i. $\sec^2\theta = \tan^2\theta + 1$

      ii. Since our $\tan\theta$ is made as our x inside $f(x)$,

$$\begin{aligned} x &= \tan\theta \\ x^2 &= (\tan\theta)^2 \\ x^2 &= \tan^2\theta \end{aligned}$$

      iii. Substitute it back in to i) and we will get

$$\sec^2\theta = x^2 + 1$$

   (c) Subsitute back in and we will get:

$$f'(x) = \frac{1}{1+x^2}$$

   (d) Therefore, $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

9.3 Examples 3.9

1. Find $\frac{dy}{dx}$ of the following functions

(a) $y = \sin^{-1}\sqrt{x-1}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\sqrt{1-\left(\sqrt{x-1}\right)^2}} * \frac{d}{dx}\left(\sqrt{x-1}\right) \\
&= \frac{\frac{1}{2}\left(x-1\right)^{-\frac{1}{2}} \cdot 1}{\sqrt{1-\left(\sqrt{x-1}\right)^2}} \\
&= \frac{\frac{1}{2}\left(x-1\right)^{-\frac{1}{2}}}{\sqrt{1-x+1}} \\
\frac{dy}{dx} &= \frac{1}{2\sqrt{2-x}\sqrt{x-1}}
\end{aligned}$$

(b) $y = \cos^{-1}\left(\frac{1-x}{1+x}\right)$

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{1}{\sqrt{1-\left(\frac{1-x}{1+x}\right)^2}} * \frac{d}{dx}\left(\frac{1-x}{1+x}\right) \\
&= -\frac{1}{\sqrt{1-\frac{(1-x)^2}{(1+x)^2}}} * \frac{(1+x)\cdot(-1)-(1-x)\cdot 1}{\left(1+x\right)^2} \\
&= -\frac{1}{\sqrt{\frac{(1+x)^2-(1-x)^2}{(1+x)^2}}} * \frac{-\left(1+x\right)-\left(1-x\right)}{\left(1+x\right)^2} \\
&= -\frac{1}{\sqrt{\frac{(1+x)^2-(1-x)^2}{(1+x)^2}}} * \frac{-2}{\left(1+x\right)^2} \\
&= -\frac{1}{\sqrt{\frac{(1+2x+x^2)-(1-2x+x^2)}{(1+x)^2}}} * \frac{-2}{\left(1+x\right)^2} \\
&= -\frac{1}{\sqrt{\frac{\cancel{1}+2x\cancel{+x^2}\cancel{-1}+2x\cancel{-x^2}}{(1+x)^2}}} * \frac{-2}{\left(1+x\right)^2} \\
&= -\frac{1}{\frac{\sqrt{4x}}{\sqrt{(1+x)^2}}} * \frac{-2}{\left(1+x\right)^2} \\
&= \cancel{-}\frac{(1+x)}{\cancel{2}\sqrt{x}} * \frac{\cancel{-2}}{\left(1+x\right)^2} \\
&= \frac{(1+x)}{\sqrt{x}\left(1+x\right)^2} \\
&= \frac{1}{\sqrt{x}\left(1+x\right)}
\end{aligned}$$

(c) $y = 2\tan^{-1}\sqrt{x}$

$$\begin{aligned}\frac{dy}{dx} &= 2\left(\frac{1}{1+\left(\sqrt{x}\right)^2}\right)\cdot\frac{dy}{dx}\sqrt{x}\\ &= \frac{2}{1+x}\cdot\frac{1}{2\sqrt{x}}\\ &= \frac{1}{\sqrt{x}\,(1+x)}\end{aligned}$$

(d) $y = \left(x^2+2\right)\tan^{-1}(2x)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^2+2\right)\cdot\tan^{-1}(2x) + \left(x^2+2\right)\cdot\frac{d}{dx}\left(\tan^{-1}(2x)\right)\\ &= \frac{2x}{\tan(2x)} + \left(x^2+2\right)\cdot\frac{1}{1+(2x)^2}\cdot\frac{d}{dx}(2x)\\ &= \frac{2x}{\tan(2x)} + \frac{\left(x^2+2\right)}{1+(2x)^2}\cdot 2\\ &= \frac{2x}{\tan(2x)} + \frac{2\left(x^2+2\right)}{1+4x^2}\\ \frac{dy}{dx} &= 2x\tan^{-1}(2x) + \frac{2\left(x^2+2\right)}{1+4x^2}\end{aligned}$$

10 Higher Derivatives

10.1 Examples 3.10

1. Find the second derivatives of the following functions.

(a) $y = 2x^4 - 5x^3 + 3x^2 - 2x + 4$

$$\begin{aligned}\frac{dy}{dx} &= 8x^3 - 15x^2 + 6x - 2\\ \frac{d^2y}{dx^2} &= 24x^2 - 30x + 6\end{aligned}$$

(b) $y = e^{-x}\sin 2x$

$$\begin{aligned}y' &= \frac{d}{dx}\left(e^{-x}\right)\cdot\sin 2x + e^{-x}\cdot\frac{d}{dx}(\sin 2x)\\ &= \frac{d}{dx}\left(e^{-x}\right)\cdot\sin 2x + e^{-x}\cdot\frac{d}{dx}(\sin 2x)\\ &= -e^{-x}\cdot\sin 2x + e^{-x}\cdot\cos 2x\cdot 2\\ &= -e^{-x}\sin 2x + 2e^{-x}\cos 2x\\ &= 2e^{-x}\cos 2x - e^{-x}\sin 2x\\ &= e^{-x}\left(2\cos 2x - \sin 2x\right)\end{aligned}$$

$$
\begin{aligned}
y'' &= \frac{d}{dx}\left[e^{-x}\left(2\cos 2x - \sin 2x\right)\right] \\
&= \frac{d}{dx}\left[e^{-x}\right] * \left(2\cos 2x - \sin 2x\right) + e^{-x}\cdot\frac{d}{dx}\left[\left(2\cos 2x - \sin 2x\right)\right] \\
&= -e^{-x}\left(2\cos 2x - \sin 2x\right) + e^{-x}\left(2\left(\left(-\sin 2x\right)\cdot 2\right) - \left(\cos 2x\right)\cdot 2\right) \\
&= -e^{-x}\left(2\cos 2x - \sin 2x\right) + e^{-x}\left(-4\sin 2x - 2\cos 2x\right) \\
&= -e^{-x}\left(2\cos 2x - \sin 2x\right) - 2e^{-x}\left(2\sin 2x + \cos 2x\right) \\
&= -e^{-x}\left(2\cos 2x - \sin 2x + 2\left(2\sin 2x + \cos 2x\right)\right) \\
&= -e^{-x}\left(2\cos 2x - \sin 2x + 4\sin 2x + 2\cos 2x\right) \\
&= -e^{-x}\left(4\cos 2x + 3\sin 2x\right)
\end{aligned}
$$

(c) $y = \frac{\ln x}{x}$

$$
\begin{aligned}
y' &= \frac{x\frac{d}{dx}\left(\ln x\right) - \ln x\frac{d}{dx}x}{x^2} \\
&= \frac{x\frac{1}{x} - \ln x\cdot 1}{x^2} \\
y' &= \frac{1-\ln x}{x^2}
\end{aligned}
$$

$$
\begin{aligned}
y'' &= \frac{x^2\frac{d}{dx}\left(1-\ln x\right) - \left(1-\ln x\right)\frac{d}{dx}\left(x^2\right)}{\left(x^2\right)^2} \\
&= \frac{x^2\left(-\frac{1}{x}\right) - \left(1-\ln x\right)2x}{x^4} \\
&= \frac{-x - \left(2x - 2x\ln x\right)}{x^4} \\
&= \frac{-3x + 2x\ln x}{x^4} \\
&= \frac{x\left(2\ln x - 3\right)}{x^4} \\
&= \frac{\left(2\ln x - 3\right)}{x^3}
\end{aligned}
$$

(d) $x = t^3, y = t^2$

i. First derivative

$$
\begin{aligned}
\frac{dx}{dt} &= 3t^2 \\
\frac{dy}{dt} &= 2t
\end{aligned}
$$

A. Use chain rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} * \frac{dt}{dx}\\ &= \frac{dy}{dt} * \frac{1}{\frac{dx}{dt}}\\ &= \frac{2t}{3t^2}\\ \frac{dy}{dx} &= \frac{2}{3t}\end{aligned}$$

ii. Second derivative

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left[\frac{2}{3t}\right]\\ &= \frac{2}{3}\frac{d}{dx}\left[t^{-1}\right]\\ &= \frac{2}{3} \cdot -t^{-2} \cdot \frac{dt}{dx}\\ &= -\frac{2}{3}t^{-2} \cdot \left(\frac{1}{\frac{dx}{dt}}\right)\\ &= -\frac{2}{3t^2} \cdot \left(\frac{1}{3t^2}\right)\\ &= -\frac{2}{9t^4}\end{aligned}$$

(e) $x^2 + y^2 - 2x + 2y = 23$at point $x = -2, y = 3$

$$\begin{aligned}\frac{d}{dx}\left[x^2 + y^2 - 2x + 2y\right] &= \frac{d}{dx}[23]\\ 2x + 2y \cdot \frac{dy}{dx} - 2 + \frac{dy}{dx} \cdot 2 &= 0\\ 2y\frac{dy}{dx} + 2\frac{dy}{dx} &= -(2x-2)\\ \frac{dy}{dx}(2y+2) &= 2 - 2x\\ \frac{dy}{dx} &= \frac{2-2x}{(2y+2)}\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dx}\left[\frac{dy}{dx}\right] &= \frac{d}{dx}\left[\frac{2-2x}{(2y+2)}\right] \\
\frac{d^2y}{dx^2} &= \frac{(2y+2)\frac{d}{dx}(2-2x)-(2-2x)\frac{d}{dx}(2y+2)}{(2y+2)^2} \\
&= \frac{(2y+2)(-2)-(2-2x)\left(2\frac{dy}{dx}\right)}{(2y+2)^2} \\
&= \frac{-4y-4+(4x-4)\left(\frac{dy}{dx}\right)}{(2y+2)^2} \\
&= \frac{-4y-4+(4x-4)\left(\frac{2-2x}{(2y+2)}\right)}{(2y+2)^2} \\
&= \frac{-4y-4+\left(\frac{(2-2x)(4x-4)}{(2y+2)}\right)}{(2y+2)^2} \\
&= \frac{-4y-4+\left(\frac{2(1-x)(4x-4)}{2(y+1)}\right)}{(2y+2)^2} \\
&= \frac{-4y-4+\left(\frac{(1-x)(4x-4)}{(y+1)}\right)}{(2y+2)^2}
\end{aligned}
$$

i. Substitute in the point $(-2,3)$where $x=-2$, $y=3$

$$
\begin{aligned}
\frac{d^2y}{dx^2}|_{x=-2,y=3} &= \frac{-4(3)-4+\left(\frac{(1-(-2))(4(-2)-4)}{(3+1)}\right)}{(2(3)+2)^2} \\
&= \frac{-12-4+\left(\frac{3(-12)}{4}\right)}{(6+2)^2} \\
&= \frac{-16+\left(\frac{-36}{4}\right)}{(8)^2} \\
&= \frac{-16-9}{(8)^2} \\
&= \frac{-25}{64} \\
&= -\frac{25}{64}
\end{aligned}
$$

11 Logarithmic Differentiation

11.1 Example

TODO

12 Exam Tips

1. If the question ask you for simplest form of a trigonometric value, be careful of common angles. Always give common angles in fraction instead of decimals. Otherwise might minus one mark.

$$Example : \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
$$instead\,of : \sin\left(\frac{\pi}{3}\right) = 0.866...$$