Chapter 5: Integrals

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1 Example

1. $\int \sin^2 x \, dx$

$$\begin{aligned}\int \sin^2 x \, dx &= \int \frac{1-\cos 2x}{2} \\ &= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + c\end{aligned}$$

2. $\int \cos^2 x \, dx$

$$\begin{aligned}\int \cos^2 x \, dx &= \int \frac{\cos 2x + 1}{2} \, dx \\ &= \frac{1}{2}\int \cos 2x + 1 \, dx \\ &= \frac{1}{2}\left[\frac{1}{2}\sin 2x + x\right] + c \\ &= \frac{1}{4}\sin 2x + \frac{1}{2}x + c\end{aligned}$$

3. $\int \tan^2 x \, dx$

$$\begin{aligned}\int \tan^2 x \, dx &= \int \left(\sec^2 x - 1\right) dx \\ &= \tan x - x + c\end{aligned}$$

4. $\int \cot^2 x \, dx$

$$\int \csc^2 x + 1 \, dx = -\cot x + x + c$$

2 Example

1. $\int \cos 3x \cos x \, dx$

$$\begin{aligned}
\int \cos 3x \cos x \, dx &= \int \frac{1}{2} \left[\cos(3x + x) + \cos(3x - x)\right] \\
&= \frac{1}{2} \int \left[\cos(4x) + \cos(2x)\right] \, dx \\
&= \frac{1}{2} \left(\frac{1}{4} \sin(4x) + \frac{1}{2} \sin(2x) \right) + c \\
&= \frac{1}{8} \sin 4x + \frac{1}{4} \sin 2x + c
\end{aligned}$$

2. $\int \sin 3x \cos x \, dx$

$$\begin{aligned}
\int \sin 3x \cos x \, dx &= \int \frac{1}{2} \left[\sin(3x + x) + \sin(3x - x)\right] \, dx \\
&= \frac{1}{2} \int \left(\sin(4x) + \sin(2x)\right) dx \\
&= \frac{1}{2} \left[-\frac{1}{4} \cos 4x - \frac{\cos 2x}{2} \right] + c \\
&= -\frac{1}{8} \cos 4x - \frac{1}{4} \cos 2x + c
\end{aligned}$$

3 Example

1. $\int (2 + 3x)^{15} \, dx$

   (a) Let $u = 2 + 3x$

$$\begin{aligned}
\frac{du}{dx} &= 3 \\
dx &= \frac{du}{3}
\end{aligned}$$

   (b) Find the derivative with respect to u

$$\begin{aligned}
\int u^{15} \frac{du}{3} &= \frac{1}{3} \int u^{15} du \\
&= \frac{1}{3} \left(\frac{u^{16}}{16} \right) + c \\
&= \frac{u^{16}}{48} + c
\end{aligned}$$

   (c) Substitute $2 + 3x$ back in to replace u

$$\int (2 + 3x)^{15} \, dx = \frac{(2 + 3x)^{16}}{48} + c$$

2. $\int 2x\left(x^2+3\right)^5 dx$

   (a) Let $u = x^2 + 3$

$$\begin{aligned}\frac{du}{dx} &= 2x \\ dx &= \frac{du}{2x}\end{aligned}$$

   (b) Find the derivative with respect to u

$$\begin{aligned}\int 2x\left(x^2+3\right)^5 dx &= \int u^5 du \\ &= \frac{u^6}{6} + c\end{aligned}$$

   (c) Substitute back in

$$\int 2x\left(x^2+3\right)^5 dx = \frac{1}{6}\left(x^2+3\right)^6 + c$$

3. $\int \frac{3x^2-1}{\left(x^3-x+4\right)^4} dx$

   (a) Let $u = \left(x^3 - x + 4\right)$

$$\begin{aligned}\frac{du}{dx} &= 3x^2 - 1 \\ dx &= \frac{du}{3x^2-1}\end{aligned}$$

   (b) Substitute into question

$$\begin{aligned}\int \frac{3x^2-1}{\left(x^3-x+4\right)^4} dx &= \int \left(3x^2-1\right) u^{-4} \frac{du}{\left(3x^2-1\right)} \\ &= \int u^{-4} du \\ &= \frac{u^{-3}}{-3} + c\end{aligned}$$

   (c) Substitute back in $u = \left(x^3 - x + 4\right)$

$$\begin{aligned}\int \frac{3x^2-1}{\left(x^3-x+4\right)^4} dx &= -\frac{1}{3\left(x^3-x+4\right)} + c \\ &= \frac{1}{-3x^2+x-4} + c\end{aligned}$$

4. $\int e^{5x} dx$

(a) Let $u = 5x$

$$\begin{aligned} u &= 5x \\ \frac{du}{dx} &= 5 \\ dx &= \frac{du}{5} \end{aligned}$$

(b) Substitute into equation

$$\begin{aligned} \int e^u \frac{du}{5} &= \frac{1}{5} \int e^u du \\ &= \frac{1}{5} [e^u] + c \end{aligned}$$

(c) Substitute back in $u = 5x$

$$\int e^{5x} dx = \frac{1}{5} e^{5x} + c$$

5. $\int 2xe^{1-x^2}\, dx$

(a) Let $u = 1 - x^2$

$$\begin{aligned} u &= 1 - x^2 \\ \frac{du}{dx} &= -2x \\ dx &= -\frac{du}{2x} \end{aligned}$$

(b) Substitute into the equation

$$\begin{aligned} \int 2xe^u \left(-\frac{du}{2x}\right) &= -\int e^u du \\ &= -e^u + c \end{aligned}$$

(c) Substitute back in $u = 1 - x^2$

$$\int 2xe^u \left(-\frac{du}{2x}\right) = -e^{1-x^2} + c$$

6. $\int \frac{x}{e^{4x^2}}\, dx$

(a) Let $u = 4x^2$

$$\begin{aligned} u &= 4x^2 \\ \frac{du}{dx} &= 8x \\ dx &= \frac{du}{8x} \end{aligned}$$

(b) Substitute into the equation

$$\begin{aligned}
\int \frac{x}{e^u}\frac{du}{8x} &= \int \frac{1}{8e^u} du \\
&= \frac{1}{8}\int e^{-u} du \\
&= \frac{1}{8}\left(-e^{-u}\right) + c \\
&= -\frac{1}{8}e^{-u} + c \\
&= -\frac{1}{8e^u} + c
\end{aligned}$$

(c) Substitute back in $u = 4x^2$

$$\int \frac{x}{e^{4x^2}}\, dx = -\frac{1}{8e^{4x^2}} + c$$

7. $\int \frac{2x}{x^2+3}\, dx$

(a) Let $u = x^2 + 3$

$$\begin{aligned}
\frac{du}{dx} &= 2x \\
dx &= \frac{du}{2x}
\end{aligned}$$

(b) Substitute inside the question

$$\begin{aligned}
\int \frac{2x}{x^2+3}\, dx &= \int \frac{2x}{u}\frac{du}{2x} \\
&= \int \frac{1}{u} du \\
&= \ln|u| + c
\end{aligned}$$

(c) Substitute back in $u = x^2 + 3$

$$\int \frac{2x}{x^2+3}\, dx = \ln\left|x^2+3\right| + c$$

8. $\int \tan x\, dx$

(a)

$$\int \frac{\sin x}{\cos x} dx$$

(b) Let $u = \cos x$

$$\begin{aligned}
\frac{du}{dx} &= -\sin x \\
dx &= \frac{du}{-\sin x}
\end{aligned}$$

(c)

i. Note

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$$

ii. Calculation

$$\begin{aligned} \int \frac{\sin x}{\cos x}\, dx &= \int \frac{\sin x}{u} \frac{du}{-\sin x} \\ &= -\int \frac{1}{u} du \\ &= -\ln |u| + c \\ &= -\ln |\cos x| + c \end{aligned}$$

9. $\int x (2-3x)^{11}\, dx; u = 2-3x$

$$\begin{aligned} \frac{du}{dx} &= -3 \\ dx &= -\frac{1}{3} du \end{aligned}$$

$$\begin{aligned} u &= 2-3x \\ x &= -\frac{u-2}{3} \\ x &= \frac{2-u}{3} \end{aligned}$$

$$\begin{aligned} \int xu^{11} \left(-\frac{1}{3}du\right) &= -\frac{1}{3} \int \left(\frac{2-u}{3}\right) \left(u^{11}\right) du \\ &= -\frac{1}{9} \int 2u^{11} - u^{12} du \\ &= -\frac{1}{9} \left(\frac{2}{12}u^{12} - \frac{u^{13}}{13}\right) + c \\ &= -\frac{1}{9} \left(\frac{1}{6}u^{12} - \frac{1}{13}u^{13}\right) + c \\ &= -\frac{1}{54} (2-3x)^{12} + \frac{1}{117} (2-3x)^{13} + c \end{aligned}$$

10. $\int \sin^3 x\, dx; u = \cos x$

(a)

$$\begin{aligned} \frac{du}{dx} &= -\sin x \\ dx &= \frac{du}{-\sin x} \end{aligned}$$

(b)

$$\begin{aligned}
u &= \cos x \\
\frac{du}{dx} &= -\sin x \\
dx &= \frac{du}{-\sin x}
\end{aligned}$$

$$\begin{aligned}
\int \sin^3 x \, dx &= \int \sin^2 x \sin x \, dx \\
&= \int \left(1 - \cos^2 x\right) \sin x \, dx \\
&= \int \left(1 - u^2\right) \sin x \frac{du}{-\sin x} \\
&= -\int \left(1 - u^2\right) \sin x \frac{du}{-\sin x} \\
&= -\int \left(1 - u^2\right) du \\
&= -\left(u - \frac{u^3}{3}\right) + c \\
&= -u + \frac{1}{3}u^3 + c \\
&= -u + \frac{1}{3}u^3 + c \\
&= -\cos x + \frac{1}{3}\cos^3 x + c
\end{aligned}$$

11. $\int \sqrt{1 - x^2} dx; x = \sin\theta$

    (a) Let $x = \sin\theta$

$$\begin{aligned}
\frac{dx}{d\theta} &= \cos\theta \\
dx &= \cos\theta d\theta
\end{aligned}$$

(b) $\int \sqrt{1-x^2}dx$

$$\begin{aligned}
\int \sqrt{1-x^2}dx &= \int \sqrt{1-(\sin\theta)^2}\cos\theta d\theta \\
&= \int \sqrt{\cos^2\theta}\cos\theta d\theta \\
&= \int \cos^2\theta d\theta \\
&= \int \frac{\cos 2\theta + 1}{2} d\theta \\
&= \frac{1}{2}\left(\frac{\sin 2\theta}{2} + \theta\right) + c \\
&= \frac{1}{4}\sin 2\theta + \frac{1}{2}\theta + c \\
&= \frac{1}{4}(2\sin\theta\cos\theta) + \frac{1}{2}\theta + c \\
&= \frac{1}{2}(\sin\theta\cos\theta + \theta) + c
\end{aligned}$$

4 Example (Integration by Partial Fractions)

1. $\int \frac{1}{x(x+1)}dx$

   (a) Perform method of long division to obtain degree of numerator less than degree of denominator

   $$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

   $$A(x+1) + Bx = 1$$

      i. Let $x = 0, A = 1$
      ii. Let $x = -1, B = -1$

   (b) $\int \frac{1}{x(x+1)}dx$

   $$\begin{aligned}
   \int \frac{1}{x(x+1)}dx &= \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx \\
   &= \ln x - \ln(x+1) + c \\
   &= \ln\left(\frac{x}{x+1}\right) + c
   \end{aligned}$$

2. $\int \frac{2x^3-11x^2-2x+2}{2x^2+x-1}\,dx$

   (a) Perform long division

i.

	x	-6		
$\left(2x^2+x-1\right)$	$2x^3$	$-11x^2$	$-2x$	$+2$
	$2x^3$	$+x^2$	$-x$	
		$-12x^2$	$-x$	$+2$
		$-12x^2$	$-6x$	$+6$
			$5x$	-4

A. Therefore

$$\frac{2x^3-11x^2-2x+2}{2x^2+x-1} = x-6+\frac{5x-4}{2x^2+x-1}$$

(b) Un-polynomialize $\frac{5x-4}{2x^2+x-1}$

$$\frac{5x-4}{2x^2+x-1} = \frac{A}{(2x-1)} + \frac{B}{(x+1)}$$

$$\begin{aligned} \frac{5x-4}{(2x-1)(x+1)} &= \frac{A(x+1)+B(2x-1)}{(2x-1)(x+1)} \\ 5x-4 &= A(x+1)+B(2x-1) \\ A(x+1)+B(2x-1)-5x+4 &= 0 \end{aligned}$$

i. Find A and B

$$A(x+1)+B(2x-1)-5x+4=0$$

A. When $x=-1$

$$\begin{aligned} A(0)+B(2(-1)-1)-5(-1)+4 &= 0 \\ B(-2-1)+5+4 &= 0 \\ -3B &= -9 \\ B &= 3 \end{aligned}$$

B. When $x=\frac{1}{2}$

$$\begin{aligned} A\left(\frac{1}{2}+1\right)+B(0)-5\left(\frac{1}{2}\right)+4 &= 0 \\ \frac{3}{2}A-\frac{5}{2}+4 &= 0 \\ 3A-5+8 &= 0 \\ 3A+3 &= 0 \\ A &= \frac{-3}{3} \\ A &= -1 \end{aligned}$$

ii. Substitute back into the equation

$$\frac{5x-4}{2x^2+x-1} = \frac{A}{(2x-1)} + \frac{B}{(x+1)}$$
$$\boldsymbol{\frac{5x-4}{2x^2+x-1} = -\frac{1}{(2x-1)} + \frac{3}{(x+1)}}$$

(c) Solve the integration problem

$$\int \frac{2x^3-11x^2-2x+2}{2x^2+x-1}\,dx = \int x-6+\frac{5x-4}{2x^2+x-1}\,dx$$
$$= \int x-6+\left(-\frac{1}{(2x-1)}+\frac{3}{(x+1)}\right)dx$$
$$= \int x-6-\frac{1}{(2x-1)}+\frac{3}{(x+1)}dx$$
$$\boldsymbol{\int \frac{2x^3-11x^2-2x+2}{2x^2+x-1}\,dx = \frac{x^2}{2}-6x-\frac{1}{2}\ln(2x-1)+3\ln(x+1)+c}$$

5 Example (Integration by Parts, with $\ln x$)

Note: If integrate $\ln x$ with other functions, then let $\ln x = u$

1. $\int \ln x\, dx$

   (a) Let $u = \ln x$,

   $$\frac{du}{dx} = \frac{1}{x}$$
   $$du = \frac{1}{x}dx$$

   (b) $dv = dx$

   $$\int 1\cdot dv = \int 1\cdot dx$$
   $$v = x$$

   (c) Integrate it by parts

   $$\int \ln x\, dx = uv - \int vdu$$
   $$= x\ln x - \int x\frac{1}{x}dx$$
   $$= x\ln x - x + c$$

2. $\int x\ln x\, dx$

(a) Let $u = \ln x$, $dv = xdx$

$$du = \frac{1}{x}dx$$

$$\int dv = \int xdx$$
$$v = \frac{x^2}{2}$$

$$\begin{aligned}
\int x \ln x \, dx &= uv - \int vdu \\
&= \ln x \left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right) dx \\
&= \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx \\
&= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx \\
&= \frac{x^2}{2} \ln x - \frac{1}{2} \left(\frac{x^2}{2} + c\right) \\
&= \frac{x^2}{2} \ln x - \frac{x^2}{4} + c
\end{aligned}$$

6 Example (Integration by Parts, without $\ln x$, but have x^n)

Note: If integrate x^n with other functions (except $\ln x$), then let $u = x^n$

1. $\int xe^x \, dx$

   (a) Find u and dv, and from them, find du and v

      i. Let $u = x$

$$\begin{aligned}
u &= x \\
\frac{d}{dx}[u] &= \frac{d}{dx}[x] \\
\frac{du}{dx} &= 1 \\
du &= dx
\end{aligned}$$

      ii. Let $dv = e^x dx$ (Note $\int dv = \int 1dv$)

$$\begin{aligned}
\int dv &= \int e^x dx \\
v &= e^x
\end{aligned}$$

iii. Utilize the formula

$$\begin{aligned}
\int xe^x dx &= uv - \int vdu \\
&= x\left(e^x\right) - \int e^x dx \\
&= xe^x - e^x + c \\
\int xe^x dx &= e^x\left(x-1\right) + c
\end{aligned}$$

(b) $\int x \sin x \, dx$

i. Find u and dv, and use them to find du and v

$$\begin{aligned}
u &= x \\
\frac{du}{dx} &= 1 \\
du &= 1dx \\
du &= dx
\end{aligned}$$

$$\begin{aligned}
dv &= \sin x \, dx \\
\int dv &= \int \sin x \, dx \\
\int 1dv &= \int \sin x \, dx \\
v &= -\cos x
\end{aligned}$$

ii. Utilize the formula

$$\begin{aligned}
\int x \sin x \, dx &= uv - \int vdu \\
&= x\left(-\cos x\right) - \int \left(-\cos x\right)\left(dx\right) \\
&= -x\cos x + \sin x + c
\end{aligned}$$

2. $\int x^2 e^x dx$

(a) Find u and dv, then use it to find du and v.

$$\begin{aligned}
u &= x^2 \\
\frac{du}{dx} &= 2x \\
du &= 2x \, dx
\end{aligned}$$

$$\frac{dv}{dx} = e^x$$
$$dv = e^x dx$$
$$\int dv = \int e^x dx$$
$$v = e^x \text{ Note: } \int dv = \int 1 dv$$

(b) Utilize the formula.

$$\int x^2 e^x dx = uv - \int v du$$
$$= \left(x^2\right)\left(e^x\right) - \int \left(e^x\right) 2x\, dx$$
$$= x^2 e^x - 2\int xe^x\, dx$$
$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x\, dx$$

(c) At this point, as you can see (the bolded part), we still have one integration to work with, therefore, we need to integrate it by parts again. Its up to personal preference whether to take the entire term or just the integration part. But regardless, do not be careless.

$$\int \boldsymbol{xe^x\, dx}$$

(d) Find u and dv, and use them to find du and v

$$u = x$$
$$\boldsymbol{du = dx}$$

$$dv = e^x dx$$
$$\int dv = \int e^x dx$$
$$\boldsymbol{v = e^x}$$

(e) Utilize the formula

$$\int xe^x\, dx = uv - \int v du$$
$$= x\left(e^x\right) - \int e^x dx$$
$$\int \boldsymbol{xe^x\, dx = xe^x - e^x + c}$$

(f) Now we can combine the results. Note: $2c = c$, because c is just an arbitrary constant and it doesn't affect our results.

$$\begin{aligned}\int x^2 e^x dx &= x^2 e^x - 2\int \boldsymbol{xe^x\, dx} \\ &= x^2 e^x - 2\left(xe^x + e^x + c\right) \\ &= x^2 e^x - 2xe^x + 2e^x + 2c \\ &= e^x\left(x^2 - 2x + 2\right) + c\end{aligned}$$

7 Example (Integration by parts, without $\ln x$ nor x^n)

Note: If we are going to integrate with other functions, and none of the functions have $\ln x$ or x^n, then we are free to choose any one function to be used. The reason why we prefer to choose those two functions first is because they are more complicated than functions without ln or power to be integrated.

1. $\int e^x \sin x\, dx$

   (a) Find terms to be substituted for u and dv, and use them to find du and v

$$\begin{aligned}u &= \sin x \\ du &= \cos x\, dx\end{aligned}$$

$$\begin{aligned}dv &= e^x dx \\ \int dv &= \int e^x dx \\ v &= e^x\end{aligned}$$

   (b) Utilize the formula

$$\begin{aligned}\int e^x \sin x\, dx &= uv - \int v du \\ &= \sin x \cdot e^x - \int e^x\left(\cos x\, dx\right) \\ &= e^x \sin x - \int e^x \cos x\, dx\end{aligned}$$

   (c) We are now stuck again, but notice something, if we keep going on, we will be integrating by parts forever, then we have no choice but to do some fancy math.

$$
\begin{aligned}
\int e^x \sin x \, dx &= e^x \sin x - \int e^x \cos x \, dx \\
&= e^x \sin x - \left(e^x \cos x - \int -e^x \sin x \, dx \right) \\
&= e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx \right) \\
&= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \\
\int e^x \sin x \, dx + \int e^x \sin x \, dx &= e^x \sin x - e^x \cos x \\
2 \int e^x \sin x \, dx &= e^x \sin x - e^x \cos x \\
\frac{1}{2} \left(2 \int e^x \sin x \, dx \right) &= \frac{1}{2} \left(e^x \sin x - e^x \cos x \right) \\
\boldsymbol{\int e^x \sin x \, dx} &\boldsymbol{= \frac{1}{2} e^x \left(\sin x - \cos x \right)}
\end{aligned}
$$

7.1 Integration by reduction formula

7.1.1 Proof the reduction formula

1. Split into two terms

$$\int \sin^n x \, dx = \int \sin^{n-1} x \cdot \sin x \, dx$$

2. Find u and dv, and use them to find du and v

$$
\begin{aligned}
u &= \sin^{n-1} x \\
\frac{du}{dx} &= (n-1) \sin^{n-2} x \cos x \\
du &= (n-1) \sin^{n-2} x \cos x \, dx
\end{aligned}
$$

$$
\begin{aligned}
dv &= \sin x \, dx \\
v &= -\cos x
\end{aligned}
$$

3. Utilize the formula **(Note to self: Look this up later on)**

$$
\begin{aligned}
\int \sin^n x \, dx &= \int \sin^{n-1} x \cdot \sin x \, dx \\
&= uv - \int v \, du \\
&= \left(\sin^{n-1} x\right)(-\cos x) - \int (-\cos x)(n-1)\sin^{n-2} x \cos x \, dx \\
&= \left(\sin^{n-1} x\right)(-\cos x) - \int (-\cos x)(n-1)\sin^{n-2} x \cos x \, dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos x \cos x \, dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x \, dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \left(1 - \sin^2 x\right) \, dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x - \sin^n x \, dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x - (n-1)\int \sin^n x \, dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x - (n-1)\int \sin^n x \, dx \\
&= -\cos x \sin^{n-1} x - (n-1)\int \sin^n x \, dx + (n-1)\int \sin^{n-2} x \, dx \\
[1 + (n-1)]\int \sin^n x \, dx &= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \, dx \\
n\int \sin^n x dx &= -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin x \, dx \\
&= -\frac{1}{3}\cos x \sin^2 x + \frac{2}{3}(-\cos x) + c \\
&= -\frac{1}{3}\cos x \sin^2 x - \frac{2}{3}\cos x + c
\end{aligned}
$$

8 Example (Riemann Sum)

1. Right-hand endpoints

$$
n = 4, \Delta x = \frac{1-0}{4} = 0.25
$$

$$
\begin{aligned}
R_4 &= 0.25\,(0.25)^2 + 0.25\,(0.5)^2 + 0.25\,(0.75)^2 + 0.25\,(1)^2 \\
&= 0.46875
\end{aligned}
$$

2. Left-hand endpoints

$$n = 4, \Delta x = \frac{1-0}{4} = 0.25$$

$$\begin{aligned} L_4 &= 0.25\left(0\right)^2 + 0.25\left(0.25\right)^2 + 0.25\left(0.5\right)^2 + 0.25\left(0.75\right)^2 \\ &= 0.21875 \end{aligned}$$

3. Range (not required)

$$0.21875 < A < 0.46875$$

9 Example

$$\lim_{x\to\infty} \sum_{i=1}^{n} \left[x_i^3 + x_i \sin x_i\right] \Delta x, interval = [0, \pi]$$

$$\lim_{x\to\infty} \sum_{i=1}^{n} \left[x_i^3 + x_i \sin x_i\right] \Delta x = \int_0^{\pi} \left(x^3 + x \sin x\right) dx$$

10 Example

1. $\int_0^1 \sqrt{1-x^2}dx$

   (a) $n = 4, \Delta x = \frac{2-0}{4} = 0.5$ (use midpoint)

   (b)

Interval	0		0.5		1		1.5		2
Midpoint		0.25		0.75		1.25		1.75	

   (c)

$$f(x) = e^{-x}$$

$$\begin{aligned} M_4 &= 0.5 * e^{-0.25} + 0.5 * e^{-0.75} + 0.5 * e^{-1.25} + 0.5 * e^{-1.75} \\ A &\approx 0.8857 \end{aligned}$$

   (d) $\int_0^1 \sqrt{1-x^2}dx$ (midpoints)

$$n = 4, \Delta x = \frac{1-0}{4} = 0.25$$

$$\begin{aligned} M_4 &= 0.25 * \sqrt{1-\left(0.125\right)^2} + 0.25 * \sqrt{1-\left(0.375\right)^2} + 0.25 * \sqrt{1-\left(0.625\right)^2} + 0.25 * \sqrt{1-\left(0.875\right) \\ &\approx 0.79598 \end{aligned}$$

2. $n = 4, \Delta x = \frac{3-0}{4} = 0.75$ (use left-hand endpoints)

$$n = 4, \Delta x = \frac{1-0}{4} = 0.75$$

$$\begin{aligned} L_4 &= 0.75\left(f\left(0\right) + ... + f\left(2.25\right)\right) \\ &= 0.75\left(-1 - 0.25 + 0.5 + 1.25\right) \\ &= 0.375 \end{aligned}$$

11 Example (Fundamental theorem of calculus)

$$\begin{aligned} \int_0^{10} f\left(x\right)\, dx &= \int_0^{8} f\left(x\right)\, dx + \int_8^{10} f\left(x\right)\, dx \\ 17 &= 12 + \int_8^{10} f\left(x\right)\, dx \\ \int_8^{10} f\left(x\right)\, dx &= 17 - 12 \\ &= 5 \end{aligned}$$

12 Example $\left(\int_a^b \left[f\left(x\right) \pm g\left(x\right)\right]\, dx = \int_a^b f\left(x\right)\, dx \pm \int_a^b g\left(x\right)\, dx\right)$

1. $\int_1^2 \left(x+3\right)\, dx$

$$\begin{aligned} \int_1^2 \left(x+3\right)\, dx &= \int_1^2 x\, dx + \int_1^2 3\, dx \\ &= \left[\frac{x^2}{2}\right]_1^2 + \left[3x\right]_1^2 \\ &= \left(2 - \frac{1}{2}\right) + \left(3\left(2\right) - 3\left(1\right)\right) \\ &= 4.5 \end{aligned}$$

2. $\int_7^{10} \frac{7}{x^2-5x-6}\, dx$

   (a) Solve for partial fractions to split the polynomial into individual, integratable terms

$$\begin{aligned}
\frac{7}{x^2-5x-6} &= \frac{7}{(x-6)(x+1)} \\
&= \frac{A}{(x-6)} + \frac{B}{(x+1)} \\
\frac{7}{(x-6)(x+1)} &= \frac{A(x+1)+B(x-6)}{(x-6)(x+1)} \\
7 &= A(x+1)+B(x-6)
\end{aligned}$$

(b) Solve for A and B

i. Solve for A, when $x = 6$

$$\begin{aligned}
7 &= A(6+1)+B(6-6) \\
&= 7A \\
A &= 1
\end{aligned}$$

ii. Solve for B, when $x = -1$

$$\begin{aligned}
7 &= A(-1+1)+B(-1-6) \\
&= -7B \\
B &= -1
\end{aligned}$$

(c) Obtain final equation

$$\frac{7}{x^2-5x-6} = \frac{1}{(x-6)} - \frac{1}{(x+1)}$$

$$\begin{aligned}
\int_7^{10} \frac{7}{x^2-5x-6}\,dx &= \int_7^{10} \frac{A}{(x-6)} + \frac{B}{(x+1)}\,dx \\
&= \int_7^{10} \frac{1}{(x-6)} - \frac{1}{(x+1)}\,dx \\
&= \left[\ln(x-6) - \ln(x+1)\right]_7^{10} \\
&= (\ln(10-6) - \ln(10+1)) - (\ln(7-6) - \ln(7+1)) \\
&= (\ln 4 - \ln 11) - (\ln 1 - \ln 8) \\
&= \ln 4 - \ln 11 - \ln 1 + \ln 8 \\
&= \ln \frac{4*8}{11*1} \\
&= \ln \frac{32}{11}
\end{aligned}$$

3. $\int_0^3 \sqrt{9-x^2}dx; x = 3\sin\theta$

(a) Find dx

$$\begin{aligned} x &= 3\sin\theta \\ \frac{dx}{d\theta} &= 3\cos\theta \\ dx &= 3\cos\theta \cdot d\theta \end{aligned}$$

(b) Find θ when $x = 3$ and $x = 0$

i. When $x = 3$

$$\begin{aligned} 3 &= 3\sin\theta \\ \sin\theta &= 1 \\ \theta &= \sin^{-1}(1) \\ &= \frac{\pi}{2} \end{aligned}$$

ii. When $x = 0$

$$\begin{aligned} 0 &= 3\sin\theta \\ \sin\theta &= 0 \\ \theta &= 0 \end{aligned}$$

4.

$$\begin{aligned}
\int_0^3 \sqrt{9-x^2}dx &= \int_0^{\frac{\pi}{2}} \sqrt{9-(3\sin\theta)^2}3\cos\theta\cdot d\theta \\
&= \int_0^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta}3\cos\theta\cdot d\theta \\
&= \int_0^{\frac{\pi}{2}} 3\sqrt{1-\sin^2\theta}3\cos\theta\cdot d\theta \\
&= 9\int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\cos\theta\cdot d\theta \\
&= 9\int_0^{\frac{\pi}{2}} \cos\theta\cos\theta\cdot d\theta \\
&= 9\int_0^{\frac{\pi}{2}} \cos^2\theta\cdot d\theta \\
&= 9\int_0^{\frac{\pi}{2}} \frac{\cos 2\theta+1}{2}\cdot d\theta \\
&= 9\left[\frac{\sin 2\theta}{2*2}+\frac{1}{2}\theta\right]_0^{\frac{\pi}{2}} \\
&= \frac{9}{2}\left[\frac{\sin 2\theta}{2}+\theta\right]_0^{\frac{\pi}{2}} \\
&= \frac{9}{2}\left(\frac{\sin\pi}{2}+\left(\frac{\pi}{2}\right)-\left(\frac{\sin 0}{2}\right)-0\right) \\
&= \frac{9}{2}\left(\cancel{\frac{\sin\pi}{2}}+\left(\frac{\pi}{2}\right)\right) \\
&= \frac{9}{4}\pi
\end{aligned}$$

13 Example (Trapezoidal Rule)

1. $\int_0^3 \frac{1}{1+x^3}dx, n=6$

   (a) Find the width of each strips

   $$\begin{aligned} h &= \frac{b-a}{n} = \frac{3-0}{6} \\ &= 0.5 \end{aligned}$$

   (b) Find the values of y for each strip

x	$y = \frac{1}{1+x^3}$
0	1
0.5	0.88889
1	0.5
1.5	0.22857
2	0.11111
2.5	0.06015
3	0.03571

(c) Estimate the integration value using Trapezoidal rule

$$\begin{aligned}\int_0^3 \frac{1}{1+x^3} dx &= \frac{0.5}{2}\left[1 + 0.03571 + 2\left(0.88889 + .. + 0.06015\right)\right] \\ &= 1.15\,(2d.p.)\end{aligned}$$

14 Example (Simpson's Rule)

1. $\int_0^3 \frac{1}{1+x^3} dx, n = 6$

   (a) Find the width of each strips

   $$\begin{aligned}h &= \frac{b-a}{n} = \frac{3-0}{6} \\ &= 0.5\end{aligned}$$

   (b) Find the values of y for each strip

x	$y = \frac{1}{1+x^3}$	Odd terms	Even terms
0	1		
0.5		0.88889	
1			0.5
1.5		0.22857	
2			0.11111
2.5		0.06015	
3	0.03571		
SUM		0.17761	0.61111

   (c) Approximate the area using Simpson's rule

   $$\begin{aligned}\int_0^3 \frac{1}{1+x^3} dx &= \frac{0.5}{3}\left[1 + 0.3571 + 4\left(0.17761\right) + 2\left(0.61111\right)\right] \\ &= 1.161\,(3d.p.)\end{aligned}$$

15 Example (Simpson's Rule)

1. $\int_0^{0.8} e^{-x}\, dx,\ h = 0.1$

2. Find the $y - values$

x	$y = e^{-x}$	Odd terms	Even terms
0	1		
0.1		0.90483	
0.2			0.81873
0.3		0.74018	
0.4			0.67032
0.5		0.60653	
0.6			0.54881
0.7		0.49659	
0.8	0.44933		
Total		2.74877	2.0379

3. Approximate the definite integral using Simpson's rule

$$\begin{aligned}\int_0^{0.8} e^{-x}\,dx &\approx \frac{1}{3}(0.1)\left((1+0.44933)+4(2.74877)+2(2.0379)\right)\\ &= 0.55067\end{aligned}$$