

Calc II - C2: Differential Eq. with Modelling Apps

January 17, 2020

1 Introduction

1. Differential equations (*D.E.*): Equations containing derivatives.
2. In this chapter: Find $y = f(x)$ which satisfies (*D.E.*)

1.1 Definitions

Term	Definition
Differential equations (<i>D.E.</i>)	Equation containing derivatives
Ordinary differential equation (<i>O.D.E.</i>)	<i>D.E.</i> with only ONE independent variable
Order	Highest derivative presentation
Degree	Greatest power of highest order derivative

1.2 Example

Differential Equation	Independent Variables	Dependent Variables	Order	Degree
$\frac{d^2y}{dx^2} + x^3y = \sin x$	x	y	2(Note: $\frac{d^2y}{dx^2}$)	1(Note: $\left(\frac{d^2y}{dx^2}\right)^1$)
$\frac{d^3y}{dx^3} + 4y\left(\frac{dy}{dx}\right)^2 = x\frac{d^2y}{dx^2} + e^x$	x	y	3	1
$\frac{du}{dx} + \frac{dv}{dx} = e^x$	x	u, v	1	1
$\left(\frac{d^4s}{dt^4}\right)^2 + \left(\frac{d^2s}{dt^2}\right)^5 + \frac{ds}{dt} = 0$	t	s	4	2

1.3 Solution of *D.E.*

Definition 2.5:

1. **Solution of *DE*:** Function f (free of derivatives), satisfies identically a *D.E.*

1.4 Example

Show that the function $f(x) = 2x^3 - 5x + C$ for any real number C , is the solution of the *D.E.* $y' = 6x^2 - 5$.

1.4.1 Solution

1. Let $y = 2x^3 - 5x + C$
2. Differentiate y with respect to x , we have

$$y' = 6x^2 - 5$$

3. Substitute into *D.E.*, we have

$$6x^2 - 5 = 6x^2 - 5$$

4. Thus, $y = f(x) = 2x^3 - 5x + C$ for any real number C is the solution of $y' = 6x^2 - 5$
5. **General solution:** $y = f(x) = 2x^3 - 5x + C$.

(a) Every solution is in this form.

6. **Particular solution:** Assigning C to specific value.

1.5 Example

Show that every member of the family of function $y(t) = \frac{1+ce^t}{1-ce^t}$ is a solution of the D.E. $y'(t) = (y^2 - 1)/2$.

1. Let $y = \frac{1+ce^t}{1-ce^t}$
2. Find the derivative

$$\begin{aligned} y' &= \frac{(1 - ce^t) \frac{d}{dx} [1 + ce^t] - (1 + ce^t) \frac{d}{dx} [1 - ce^t]}{(1 - ce^t)^2} \\ &= \frac{(1 - ce^t)(ce^t) - (1 + ce^t)(-ce^t)}{(1 - ce^t)^2} \\ &= \frac{ce^t - ce^{2t} - (-ce^t - ce^{2t})}{(1 - ce^t)^2} \\ &= \frac{ce^t - ce^{2t} + ce^t + ce^{2t}}{(1 - ce^t)^2} \\ y' &= \frac{2ce^t}{(1 - ce^t)^2} \end{aligned}$$

3. Simplify the second equation, to make them equal

$$\begin{aligned} y' &= \frac{(y^2 - 1)}{2} \\ &= \frac{\left(\left(\frac{1+ce^t}{1-ce^t}\right)^2 - 1\right)}{2} \\ &= \frac{\left(\frac{1+ce^t}{1-ce^t}\right)^2 - 1}{2} \\ &= \frac{\frac{(1+ce^t)^2}{(1-ce^t)^2} - 1}{2} \\ &= \frac{\frac{(1+ce^t)^2}{(1-ce^t)^2} - 1}{2} \\ &= \frac{(1+ce^t)^2 - (1-ce^t)^2}{2(1-ce^t)^2} \\ &= \frac{1 + 2ce^t + ce^{2t} - (1 - 2ce^t + ce^{2t})}{2(1-ce^t)^2} \\ &= \frac{1 + 2ce^t + ce^{2t} - 1 + 2ce^t - ce^{2t}}{2(1-ce^t)^2} \\ &= \frac{4ce^t}{2(1-ce^t)^2} \\ y' &= \frac{2ce^t}{(1-ce^t)^2} \end{aligned}$$

4. Conclusion, since both of the equations result in the same y' , every member of the family of function $y(t) = \frac{1+ce^t}{1-ce^t} = \frac{u}{v}$ is a solution of $y'(t)$.

1.6 Example (Continue from 1.5)

Given $y(0) = 2$ (initial condition), solve the initial value problem $y' = \frac{(y^2-1)}{2}$.

$$\begin{aligned} y(0) &= \frac{1 + ce^0}{1 - ce^0} \\ &= \frac{1 + c}{1 - c} \end{aligned}$$

$$\begin{aligned}
2 &= \frac{1+c}{1-c} \\
2(1-c) &= 1+c \\
2-2c &= 1+c \\
2-1 &= c+2c \\
1 &= 3c \\
c &= \frac{1}{3}
\end{aligned}$$

1. Substitute the c into the equation

$$y(t) = \frac{1 + \frac{1}{3}e^t}{1 - \frac{1}{3}e^t}$$

2 Modelling with Differential Equations

1. **Mathematical model:** Mathematical description of a real-world phenomenon.

2.1 Models of Population Growth

1. **Assumption:** Population grows at rate proportional to size of population.

2. Variables

(a) $t = \text{time}$ (the independent variable)

(b) $P =$ the number of individuals in the population (dependent variable)

3. Rate of growth: $\frac{dP}{dt}$

$$\begin{aligned}
\frac{dP}{dt} &\propto P \\
\frac{dP}{dt} &= kP, k = \text{proportionality constant}
\end{aligned}$$

2.1.1 Example

Show that any exponential function of the form $P(t) = Ce^{kt}$ is a solution of $\frac{dP}{dt} = kP$

1. Find the derivative

$$\begin{aligned}
P(t) &= Ce^{kt} \\
P'(t) &= k(Ce^{kt}) \\
&= kP
\end{aligned}$$

2.2 A Model for the Motion of A Spring

1. Consider the motion of an object with mass m at the end of a vertical spring.
2. If the spring is stretched (or compressed) x units from its natural length, then, by Hooke's Law, it exerts a force that is proportional to x :

$$\text{Restoring force} = -kx$$

Where k is a positive constant (called the spring constant).

3. By Newton's Second Law ($F = ma$), we have

$$\begin{aligned}
m \frac{d^2x}{dt^2} &= -kx \\
\frac{d^2x}{dt^2} &= -\frac{k}{m}x
\end{aligned}$$

2.2.1 Example

Find the nonzero values of k in function $x = \sin kt$ which satisfy the D.E., $\frac{d^2x}{dt^2} = -9x$

1. $x = \sin kt$

$$\begin{aligned}\frac{dx}{dt} &= k \cos kt \\ \frac{d^2x}{dt^2} &= k(-k \sin kt) \\ \frac{d^2x}{dt^2} &= -k^2 \sin kt \\ &= -k^2 x \\ -9x &= -k^2 x \\ 9 &= k^2 \\ k &= 3\end{aligned}$$

2.3 Direction fields

1. Impossible to obtain an explicit formula for most differential equation
2. Another way to learn solution through graphical approach.
3. **Example:** $y' = F(x, y)$
 - (a) $F(x, y) \equiv$ Slope of a solution curve at a point (x, y)
 - (b) **Solution curve:** Graph of solution of ODE
 - (c) **Direction/slope field:** Drawing of short line segments with slope $F(x, y)$ at several points.
 - i. Indicates direction the solution curve is heading
 - ii. Visualize general shape of solution curve.

2.3.1 Example

1. Sketch the direction field for the DE $y' = x^2 + y^2 - 1$
2. Use part (a) to sketch the solution curve that passes through the origin.
3. Solution:

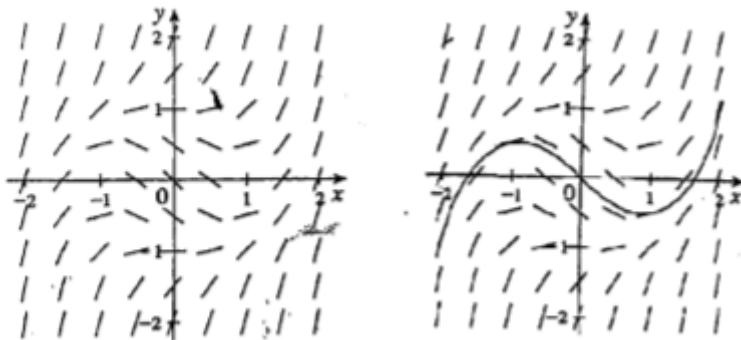
(a)

x	-2	-1	0	1	2
y	0	0	0	0	0
y'	3	0	-1	0	3

(b)

x	-2	-1	0	1	2
y	1	1	1	1	1
y'	4	1	0	1	4

- (c) Notice that we originally are required to find the other values, however, remember that $(-x)^2 = x^2$ and $(-y)^2 = y^2$. Therefore, we could omit inverse values since they are simply the same. Besides that, as you might see, all the slope are slowly increasing as the x and y increases. Therefore, we could start graphing it out.



(d)

2.3.2 Equilibrium solutions

1. An ODE of the form $y' = F(y)$ in which the independent variable is missing from the right side, is called **autonomous**.
2. For any autonomous equation $y' = F(y)$, if $F(c) = 0$, then a constant solution $y = c$ of the ODE is called an **equilibrium solution**.
3. Example: Which of the following DE are autonomous? Determine the equilibrium solution of each autonomous equation.
 - (a) $\frac{dy}{dx} = 1 - y^2$
 - i. Autonomous.

ii. Finding equilibrium solution

$$1 - y^2 = 0$$

$$y = 1, -1$$

- (b) $\frac{dx}{dt} = 1 + t^3$
 i. Not autonomous.
- (c) $\frac{dP}{dt} = P \left(1 - \frac{P}{K}\right)$, K is a constant
 i. Autonomous.

$$P \left(1 - \frac{P}{K}\right) = 0$$

$$P - \frac{P^2}{K} = 0$$

$$\frac{P^2}{K} = P$$

$$P^2 = PK$$

$$K = \frac{P^2}{P}$$

$$K = P$$

$$P = K$$

- (d) $\frac{dy}{dx} = 2xy$
 i. Not autonomous.

2.4 Euler's Method

- Numerical process to generate table of approximate values of the function that solves the initial value problem $\frac{dy}{dx} = y'$ = $F(x, y)$, and $y(x_0) = y_0$.
- Iteration formula for Euler's method
 - $x_n = x_{n-1} + h$
 - $TL; DR$: The next x value is simply the previous one + step size.
 - $y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$
 - $TL; DR$: The next y value is simply the previous one + step size * the gradient of the previous step. Remember that $\frac{dy}{dx} = F(x, y)$ as per point 1.
 - h is the horizontal distance called the step size.
- The smaller the step size, h , the better the approximation

2.4.1 Example

Use Euler's method to find approximate values for the solution of the initial value problem $F = \frac{dy}{dx} = x - y$, $y(0) = 1$ on the interval $[0, 1]$ using five steps of size $h = 0.2$.

n	X_n	$y_n = y_{n-1} + 0.2(x_{n-1} - y_{n-1})$
0	0	1
1	.2	$1 + 0.2(-1) = 1 - 0.2 = 0.8$
2	.4	$0.8 + 0.2(.2 - .8) = 0.68$
3	.6	$0.68 + 0.2(.4 - .68) = 0.624$
4	.8	$0.624 + 0.2(.6 - 0.624) = 0.6192$
5	1	$0.6192 + 0.2(.8 - 0.6192) = 0.65536$

2.4.2 Example

Use Euler's method with $n = 5$ to approximate the solution of the initial value problem $\frac{dy}{dx} = -2xy^2$, $y(0) = 1$ on the interval $[0, 0.5]$. Find the actual solution of the initial value problem. Finally, sketch the graphs of the approximate solutions and the actual solution for $0 \leq x \leq 0.5$ on the same set of axes.

- Approximate solution

n	Xn	$y_n = y_{n-1} + 0.1 \left(-2x_{n-1}y_{n-1}^2 \right)$
0	0	1
1	0.1	$1 + 0.1 \left(-2(0)(1)^2 \right) = 1$
2	0.2	$1 + 0.1 \left(-2(0.1)(1)^2 \right) = 0.98$
3	0.3	$0.98 + 0.1 \left(-2(0.2)(0.98)^2 \right) = 0.9416$
4	0.4	$0.9416 + 0.1 \left(-2(0.3)(0.9416)^2 \right) = 0.8884$
5	0.5	$0.8884 + 0.1 \left(-2(0.4)(0.8884)^2 \right) = 0.8253$

2. Actual solution (Method: Separation of variables)

$$\frac{dy}{dx} = -2xy^2$$

$$dy = -2xy^2 dx$$

$$\frac{1}{y^2} dy = -2x dx$$

$$\frac{1}{y^2} dy + 2x dx = 0$$

(a) Now integrate them

$$\int \left(\frac{1}{y^2} dy + 2x dx \right) = 0$$

$$\int y^{-2} dy + \int 2x dx = 0$$

$$-\frac{1}{y} + c + \frac{2x^2}{2} + c = 0$$

$$-\frac{1}{y} + x^2 + c = 0$$

(b) Then when $x = 0, y = 1$, so:

$$-\frac{1}{y} + x^2 + c = 0$$

$$-1 + 0^2 + c = 0$$

$$c = 1$$

(c) Finally, we arrive at our equation


$$-\frac{1}{y} + x^2 + 1 = 0$$

$$-\frac{1}{y} = -(x^2 + 1)$$

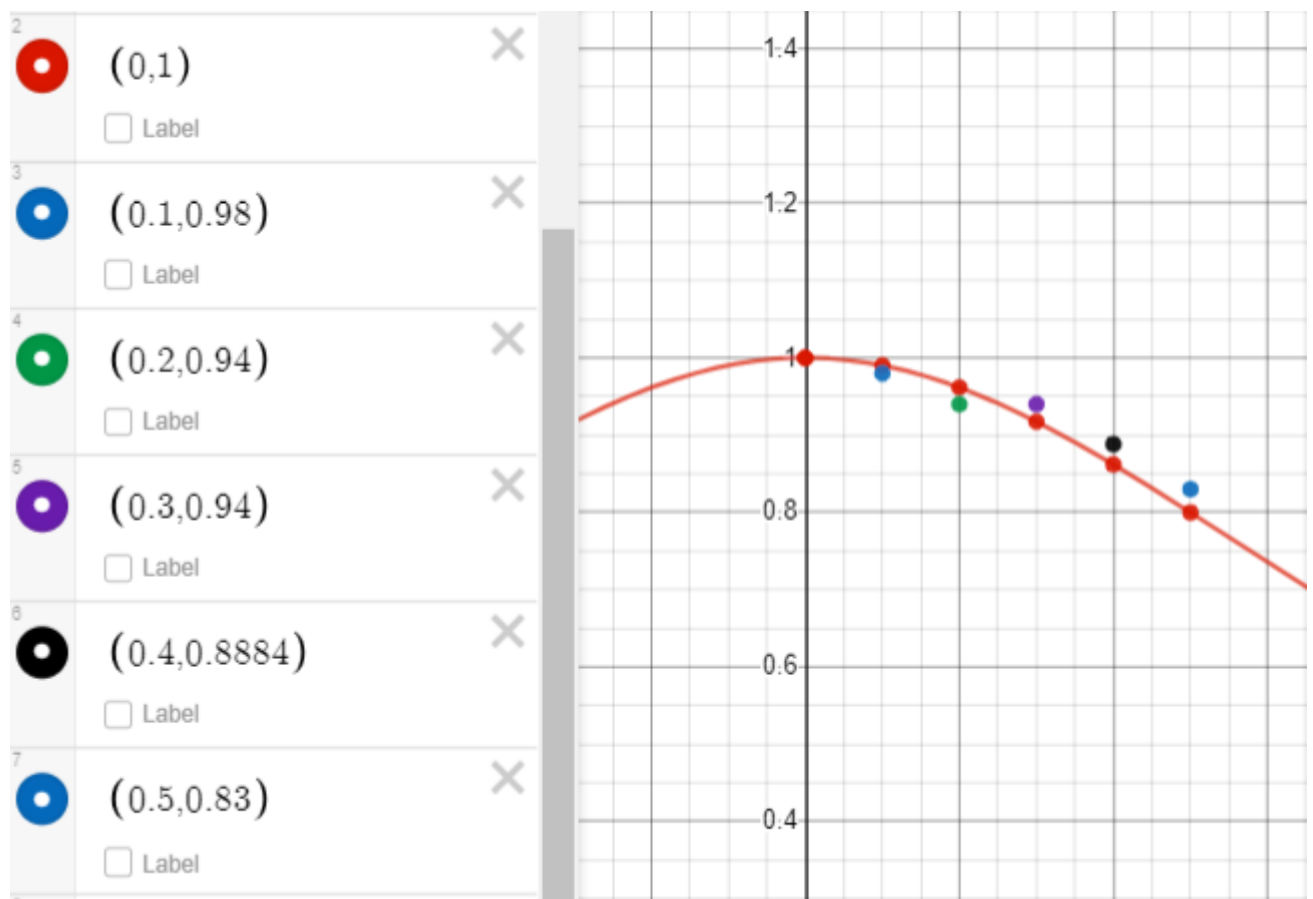
$$\frac{1}{y} = x^2 + 1$$

$$y = \frac{1}{x^2 + 1}$$

(d) Lets check our answers

x	 $\frac{1}{x^2 + 1}$
0	1
0.1	0.99009901
0.2	0.96153846
0.3	0.91743119
0.4	0.86206897
0.5	0.8

i.



ii.

(e) That's pretty close!

2.4.3 Example

Consider the initial-value problem $\frac{dy}{dx} = 0.1\sqrt{y} + 0.4x^2$, $y(2) = 4$. Use Euler's method to obtain an approximation to $y(2.5)$ using $h = 0.1$.

N	x_n	$y_n = y_{n-1} + 0.1(0.1\sqrt{y_{n-1}} + 0.4x_{n-1}^2)$
0	2	4
1	2.1	$y_n = 4 + 0.1(0.1\sqrt{4} + 0.4(2)^2) = 4.18$
2	2.2	$y_n = 4.18 + 0.1(0.1\sqrt{4.18} + 0.4(2.1)^2) = 4.3768$
3	2.3	$y_n = 4.3768 + 0.1(0.1\sqrt{4.3768} + 0.4(2.2)^2) = 4.5913$
4	2.4	$y_n = 4.5913 + 0.1(0.1\sqrt{4.5913} + 0.4(2.3)^2) = 4.8243$
5	2.5	$y_n = 4.8243 + 0.1(0.1\sqrt{4.8243} + 0.4(2.4)^2) = 5.0767$

2.5 First Order Linear Differential Equations

2.5.1 Definition

An n-th order linear Ordinary Differential Equation (O.D.E.) has the form of:

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x) \frac{dy}{dx} + a_0(x) y = f(x)$$

1. A linear O,D,E, is said to be homogenous (or alike) if $f(x) = 0$, and vice versa.

Important properties of linear O.D.E.

1. Dependent variables & derivatives are ONLY **power of one**.
2. Non-linear O.D.E. =O.D.E. that is not linear
3. Coefficient of derivatives and y (A.K.A the multiplier) must be function of independent variable x ONLY.
4. Function on RHS must all be functions of independent variable x .
5. Constant can be considered function of x , e.g. $a(x) = 4$

Example

D.E.	Order	Linear/Non-linear	Homogeneous/Non-homogenous (only for linear D.E.)
$x^2 \frac{dy}{dx} + (\sin x) y = 0$	1	Linear	Homogeneous
$\frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} + 9y = 0$	2	Non-linear	N/A
$\frac{d^4 y}{dx^4} + x^2 \frac{d^3 y}{dx^3} - x^3 \frac{dy}{dx} = x e^x$	4	Linear	Non-homogenous
$\frac{d^2 y}{dx^2} + \left(\frac{d^2}{dx^2}\right)^2 + 6y = x$	2	Non-linear	N/A

2.5.2 Definition

A first order linear D.E. is in the form:

$$a_1(x) \frac{dy}{dx} + a_0(x) y = f(x), \text{ where } a_1(x) \neq 0$$

1. Dividing by $a_1(x)$, we have

$$\frac{dy}{dx} + p(x) y = q(x)$$

- (a) Where $p(x)$ and $q(x)$ are continuous functions.

2. **Standard form of F.O.D.E.:** $\frac{dy}{dx} + p(x) y = q(x)$

Solution Process

1. Put *D.E.* in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.
2. Find the integrating factor $\mu(x) = e^{\int p(x) dx}$.
 - (a) Note: For this step, the $+c$ is omitted when integrating, else in the end you'll get an extra c somewhere as a constant. That would be a waste of time.
3. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx} [\mu(x) y]$ and write it as such.
4. Integrate both sides, make sure you properly deal with the constant of integration.
5. Solve for the solution $y(t)$

2.5.3 Example 1

Solve $\frac{dy}{dx} + 3y = e^{2x}$.

1. Put *D.E.* in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.
2. Find the integrating factor $\mu(x) = e^{\int p(x) dx}$.

$$\begin{aligned} \frac{dy}{dx} + 3y &= e^{2x}, p(x) = 3 \\ \mu(x) &= e^{\int 3 dx} \\ \mu(x) &= e^{3x} \end{aligned}$$

3. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx} [\mu(x) y]$ and write it as such.

$$\begin{aligned} e^{3x} \left(\frac{dy}{dx} + 3y \right) &= e^{3x} (e^{2x}) \\ e^{3x} \frac{dy}{dx} + 3e^{3x} y &= e^{3x} (e^{2x}) \\ \frac{d}{dx} (e^{3x} y) &= e^{5x} \quad \left(\text{if you didn't notice, } \frac{d}{dx} (ab) = a'b + ab' \right) \end{aligned}$$

4. Integrate both sides, make sure you properly deal with the constant of integration.

$$\begin{aligned} \int \frac{d}{dx} (e^{3x} y) dx &= \int e^{5x} dx \\ e^{3x} y &= \frac{1}{5} e^{5x} + c \end{aligned}$$

5. Solve for the solution $y(t)$

$$y = \frac{1}{5} e^{2x} + c e^{-3x}$$

2.5.4 Example 2

Solve $x \frac{dy}{dx} + y = x^3$

1. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned} x \frac{dy}{dx} + y &= x^3 \\ \frac{x \frac{dy}{dx} + y}{x} &= \frac{x^3}{x} \\ \frac{dy}{dx} + \frac{y}{x} &= x^2, p(x) = \frac{1}{x} \end{aligned}$$

2. Find the integrating factor $\mu(x) = e^{\int p(x) dx}$.

$$\begin{aligned} \mu(x) &= e^{\int \frac{1}{x} dx} \\ \mu(x) &= e^{\ln x} \\ &= x \end{aligned}$$

3. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx} [\mu(x) y]$ and write it as such.

$$\begin{aligned} x \left(\frac{dy}{dx} + \frac{y}{x} \right) &= x(x^2) \\ x \frac{dy}{dx} + y &= x^3 \\ \frac{d}{dx} [xy] &= x^3 \end{aligned}$$

4. Integrate both sides, make sure you properly deal with the constant of integration.

$$\begin{aligned} \int \frac{d}{dx} [xy] dx &= \int x^3 dx \\ xy &= \frac{x^4}{4} + c \end{aligned}$$

5. Solve for the solution $y(t)$

$$y = \frac{1}{4}x^3 + \frac{c}{x}$$

2.5.5 Example 3

Solve $x \frac{dy}{dx} + 2y = x^2 - x + 1$, $y(1) = \frac{1}{2}$.

1. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned} \frac{1}{x} \left(x \frac{dy}{dx} + 2y \right) &= \frac{1}{x} (x^2 - x + 1) \\ \frac{dy}{dx} + \frac{2}{x} y &= x - 1 + \frac{1}{x}, p(x) = \frac{2}{x} \end{aligned}$$

2. Find the integrating factor $\mu(x) = e^{\int p(x) dx}$.

$$\begin{aligned} \mu(x) &= e^{\int \frac{2}{x} dx} \\ \mu(x) &= e^{\ln x^2} \\ \mu(x) &= x^2 \end{aligned}$$

3. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx} [\mu(x) y]$ and write it as such.

$$\begin{aligned} x^2 \left(\frac{dy}{dx} + \frac{2}{x} y \right) &= x^2 \left(x - 1 + \frac{1}{x} \right) \\ x^2 \frac{dy}{dx} + x^2 \frac{2}{x} y &= x^3 - x^2 + x \\ x^2 \frac{dy}{dx} + 2xy &= x^3 - x^2 + x \\ \frac{d}{dx} (x^2 y) &= x^3 - x^2 + x \end{aligned}$$

4. Integrate both sides, make sure you properly deal with the constant of integration.

$$\int \frac{d}{dx} (x^2 y) dx = \int x^3 - x^2 + x dx$$

$$x^2 y = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + c$$

5. Solve for the solution $y(t)$

$$y = \frac{1}{x^2} \left(\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + c \right)$$

$$y = \frac{x^2}{4} - \frac{x}{3} + \frac{1}{2} + \frac{c}{x^2}$$

- (a) When $y(1) = \frac{1}{2}$

$$\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c$$

$$c = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{2}$$

$$c = \frac{1}{12}$$

- (b) The solution

$$y = \frac{x^2}{4} - \frac{x}{3} + \frac{1}{2} + \frac{1}{12x^2}$$

2.5.6 Example

Solve the equation $(x-2) \frac{dy}{dx} - y = (x-2)^3$

$$\frac{dy}{dx} - \frac{y}{(x-2)} = \frac{(x-2)^3}{(x-2)}$$

$$\frac{dy}{dx} - \frac{y}{(x-2)} = (x-2)^2, p(x) = -\frac{1}{(x-2)}$$

1. Find the integrating factor

$$\mu(x) = e^{\int -(x-2)^{-1} dx}$$

$$= e^{-\ln(x-2)}$$

$$= e^{\ln(x-2)^{-1}}$$

$$\mu(x) = \frac{1}{x-2}$$

2. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx} [\mu(x)y]$ and write it as such.

$$\frac{dy}{dx} - \frac{y}{(x-2)} = (x-2)^2$$

$$\frac{1}{x-2} \left(\frac{dy}{dx} - \frac{y}{(x-2)} \right) = \frac{1}{x-2} (x-2)^2$$

$$\frac{1}{x-2} \frac{dy}{dx} - \frac{1}{(x-2)} \frac{y}{(x-2)} = x-2$$

$$(x-2)^{-1} \frac{dy}{dx} - y (x-2)^{-2} = x-2$$

$$\int \frac{d}{dx} [(x-2)^{-1} y] dx = \int x-2 dx$$

$$\frac{y}{x-2} = \frac{x^2}{2} - 2x + c$$

$$y = \left(\frac{x^2}{2} - 2x + c \right) (x-2)$$

2.6 Seperable Equations

Definition 2.8 A separable D.E. is any D.E. that we can write in the form of

$$g(y) \frac{dy}{dx} = f(x)$$

Solution Process

1. Write D.E. as $g(y) dy = f(x) dx$
2. Integrate both sides

$$\int g(y) dy = \int f(x) dx + C$$

3. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)
 - (a) DO NOT forget to include C , constant of integration

2.6.1 Example

Solve $\frac{dy}{dx} = \frac{2x}{y+1}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x}{y+1} \\ (y+1) dy &= 2x dx \\ \int (y+1) dy &= \int 2x dx \\ \frac{y^2}{2} + y &= \frac{2x^2}{2} + c \\ y &= x^2 - \frac{1}{2}y^2 + c\end{aligned}$$

2.6.2 Example

Solve $\frac{dy}{dx} = \frac{3y-1}{x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{3y-1}{x} \\ \int \frac{1}{3y-1} dy &= \int \frac{1}{x} dx \\ \frac{1}{3} \ln(3y-1) &= \ln x + c \\ \ln(3y-1) &= 3(\ln x + c) \\ \ln(3y-1) &= \ln x^3 + \ln A, A = e^{3c} \\ 3y-1 &= Ax^3 \\ y &= \frac{A}{3}x^3 + \frac{1}{3}\end{aligned}$$

2.6.3 Example

Solve the differential equation $y^2 \frac{dy}{dx} = x^2 + 1$ given that $y = 1$ when $x = 2$

$$\begin{aligned}y^2 \frac{dy}{dx} &= x^2 + 1 \\ y^2 dy &= (x^2 + 1) dx \\ \int y^2 dy &= \int x^2 + 1 dx \\ \frac{y^3}{3} &= \frac{x^3}{3} + x + c \\ y^3 &= x^3 + 3x + c \\ y &= \sqrt[3]{x^3 + 3x + c}\end{aligned}$$

2.6.4 Example

Solve $\frac{dy}{dx} = e^{3x-5y}$

$$\begin{aligned}\frac{dy}{dx} &= e^{3x-5y} \\ &= \frac{e^{3x}}{e^{5y}} \\ e^{5y} dy &= e^{3x} dx \\ \int e^{5y} dy &= \int e^{3x} dx \\ \frac{1}{5} e^{5y} &= \frac{1}{3} e^{3x} + c \\ e^{5y} &= \frac{5}{3} e^{3x} + c\end{aligned}$$

2.6.5 Example

Solve $4ydx - (1+x)dy = 0, y(0) = 1$

$$\begin{aligned}4ydx &= (1+x)dy \\ \int \frac{1}{1+x} dx &= \int \frac{1}{4y} dy \\ \ln(1+x) &= \frac{1}{4} \ln y + c\end{aligned}$$

1. $y(0) = 1$

$$\begin{aligned}\ln(1) &= \frac{1}{4} \ln 1 + c \\ c &= 0\end{aligned}$$

2. Solve the rest of the equation

$$\begin{aligned}\ln(1+x) &= \frac{1}{4} \ln y \\ 1+x &= y^{\frac{1}{4}} \\ y &= (1+x)^4\end{aligned}$$

2.7 Second Order Differential Equation

A second order O.D.E. has the form

$$a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y = f(x)$$

1. If $a_2(x), a_1(x)$ and $a_0(x)$ are constants, then we have a second order D.E. with constant coefficients.

Definition 2.9

1. Second Order Linear D.E.

- (a) $ay'' + by' + cy = f(x)$ where a, b, c are constants and $a \neq 0$.
 (b) **Homogeneous:** $f(x) = 0$, and vice versa

2.7.1 Second Order Homogeneous Linear D.E.

1. **Form:** $ay'' + by' + cy = 0$

2. **General solution:**

- (a) $y = C_1 y_1 + C_2 y_2$, where $y_1 \neq k y_2$ for any constant k , if y_1 and y_2 are the solutions.

3. We assume all solutions to the D.E. will be of the form $y = e^{rx}$, r is a constant.

$$\begin{aligned}y &= e^{rx} \\ y' &= r e^{rx} \\ y'' &= r^2 e^{rx}\end{aligned}$$

4. Plug these into the D.E.

$$\begin{aligned}ar^2 e^{rx} + br e^{rx} + ce^{rx} &= 0 \\ e^{rx} (ar^2 + br + c) &= 0\end{aligned}$$

(a) Since $e^{rx} \neq 0$,

$$ar^2 + br + c = 0$$

i. Like, if $AB = 0$, either A or B must be 0.

5. **Characteristic/auxiliary equation:** $ar^2 + br + c = 0$ for any D.E. $ay'' + by' + c = 0$

(a) Since it is quadratic, we will have two roots, r_1 and r_2 .

(b) Can be obtained with $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, roots can be either

- i. Real, distinct roots $r_1 \neq r_2$ ($b^2 - 4ac > 0$)
- ii. Repeated real roots, $r_1 = r_2$ ($b^2 - 4ac = 0$)
- iii. Complex roots, $r_{1,2} = \alpha \pm \beta i$ ($b^2 - 4ac < 0$)

6. Note: In the following, C_1 and C_2 are arbitrary constants.

Real, distinct roots, $r_1 \neq r_2$ We have $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$,

1. **General solution:** $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ (<< MUST REMEMBER, NOT GIVEN IN EXAMS)

Example

1. Find the general solution of

(a) $y'' - 3y' - 10y = 0$

$$\begin{aligned} r^2 - 3r - 10 &= 0 \\ (r - 5)(r + 2) &= 0 \\ r_1 = 5, r_2 &= -2 \\ y &= C_1 e^{5x} + C_2 e^{-2x} \end{aligned}$$

(b) $2y'' - 5y' - 3y = 0$

$$\begin{aligned} 2r^2 - 5r - 3 &= 0 \\ (2r + 1)(r - 3) &= 0 \\ r &= -\frac{1}{2}, r = 3 \\ y &= C_1 e^{-\frac{1}{2}x} + C_2 e^{3x} \end{aligned}$$

2. Solve the following I.V.P. $y'' + 11y' + 24 = 0$, given $y(0) = 0$, $y'(0) = -7$.

$$\begin{aligned} r^2 + 11r + 24 &= 0 \\ (r + 8)(r + 3) &= 0 \\ r_1 &= -8, r_2 = -3 \\ y &= C_1 e^{-8x} + C_2 e^{-3x} \\ y(0) = 0 &\rightarrow C_1 e^0 + C_2 e^0 = 0 \rightarrow C_1 + C_2 = 0 \quad \textcircled{R} \\ y'(0) = -7 &\rightarrow y' = -8C_1 - 3C_2 = -7 \quad \textcircled{S} \end{aligned}$$

(a) Solve \textcircled{R} and \textcircled{S}

$$C_1 = -C_2$$

$$\begin{aligned} -8C_1 + 3C_1 &= -7 \\ -5C_1 &= -7 \\ C_1 &= \frac{7}{5} \end{aligned}$$

$$C_2 = -\frac{7}{5}$$

(b) Find the final equation

$$y = \frac{7}{5}e^{-8x} - \frac{7}{5}e^{-3x}$$

2.7.1.2 Repeated real root, $r_1 = r_2$

1. General solution: $y = C_1 e^{rx} + C_2 x e^{rx}$ OR $y = e^{rx} (C_1 + C_2 x)$

Example 1:

- Find the general solution for

(a) $y'' - 6y' + 9y = 0$

(b) $y'' - 10y' + 25y = 0$

- Answer

(a)

$$r^2 - 6r + 9 = 0$$

$$(r - 3)^2 = 0$$

$$r = 3$$

$$y = e^{3x} (C_1 + C_2 x)$$

(b)

$$r^2 - 10r + 25 = 0$$

$$(r - 5)(r - 5) = 0$$

$$(r - 5)^2 = 0$$

$$r = 5$$

$$y = e^{5x} (C_1 + C_2 x)$$

Example 2:

- Question: Solve the following I.V.P. $y'' - 5y' + 4 = 0$, given $y(0) = 12, y'(0) = -3$

$$r^2 - 4r + 4 = 0$$

$$r = 2$$

$$y = e^{2x} (C_1 + C_2 x)$$

(a) $y(0) = 12$

$$y(0) = 12$$

$$e^0 (C_1 + C_2 0) = 12$$

$$C_1 = 12$$

(b) Substitute into the equation, then solve using $y'(0) = -3$

$$y = e^{2x} (12 + C_2 x)$$

$$y'(0) = 2e^{2x} \cdot (12 + C_2 x) + e^{2x} (C_2)$$

$$-3 = 2 \cdot (12) + C_2$$

$$-3 = 24 + C_2$$

$$C_2 = -27$$

(c) Find the final equation

$$y = e^{2x} (12 - 27x)$$

2.7.1.3 Complex roots, $r_{1,2} = \alpha \pm \beta i$

- $y_1 = e^{(\alpha + \beta i)x}, y_2 = e^{(\alpha - \beta i)x}$

- General solution:

$$y = C_1 e^{\alpha x} \sin \beta x + C_2 e^{\alpha x} \cos \beta x$$

$$y = e^{\alpha x} (C_1 \sin \beta x + C_2 \cos \beta x)$$

Example 1

- Question

(a) Find general solution of:

i. $y'' - 10y' + 41y = 0$

ii. $y'' + 4y' + 13y = 0$

- Answer

(a) $r^2 - 10r + 41 = 0$

$$\begin{aligned} r &= \frac{10 \pm \sqrt{10^2 - 4(1)(41)}}{2(1)} \\ &= \frac{10 \pm \sqrt{-64}}{2} \\ &= \frac{10 \pm \sqrt{64}\sqrt{-1}}{2} \\ &= \frac{10 \pm 8i}{2} \\ &= 5 \pm 4i \end{aligned}$$

(b) $r^2 + 4r + 13 = 0$

$$\begin{aligned} r &= \frac{-4 \pm \sqrt{4^2 - 4(1)(13)}}{2(1)} \\ &= \frac{-4 \pm 6i}{2(1)} \\ &= \frac{-4 \pm 6i}{2} \\ &= -2 \pm 3i \end{aligned}$$

(c) Find the final general solution

$$y = e^{-2x} (C_1 \sin 3x + C_2 \cos 3x)$$

Example 2

1. Solve the following I.V.P.

(a) $y'' - 8y' + 17 = 0$, given $y(0) = -4, y'(0) = -1$

2. Answer

(a) Find r

$$\begin{aligned} r^2 - 8r + 17 &= 0 \\ r &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(17)}}{2} \\ &= \frac{8 \pm \sqrt{64 - 4(68)}}{2} \\ &= 4 \pm \frac{\sqrt{-208}}{2} \\ &= 4 \pm \frac{\sqrt{-1}\sqrt{208}}{2} \\ &= 4 \pm \frac{\sqrt{208}i}{2} \\ &= 4 \pm ci \end{aligned}$$

(b) Substitute into the equation

$$\begin{aligned} y &= e^{4x} (C_1 \sin x + C_2 \cos x) \\ y(0) &= -4 \\ -4 &= e^0 (C_1 \sin 0 + C_2 \cos 0) \\ -4 &= C_2 \\ C_2 &= -4 \end{aligned}$$

$$\begin{aligned} y &= e^{4x} (C_1 \sin x - 4 \cos x) \\ y' &= 4e^{4x} (C_1 \sin x - 4 \cos x) + e^{4x} (C_1 \cos x + 4 \sin x) \\ -1 &= 4e^0 (C_1 \sin 0 - 4 \cos 0) + e^0 (C_1 \cos 0 + 4 \sin 0) \\ -1 &= -16 + (C_1) \\ C_1 &= 15 \end{aligned}$$

(c) Answer

$$y = e^{4x} (15 \sin x - 4 \cos x)$$

2.7.2 Second Order Non-Homogeneous Linear D.E.

1. Form: $ay'' + by' + cy = f(x)$
2. Solution process (the method of undetermined coefficients)
 - (a) Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)
 - (b) Find the complementary function, y_h
 - (c) Find the particular solution y_p
 - (d) Find the non-homogeneous D.E. general solution, $y = y_h + y_p$

2.7.2.1 The Method of Undetermined Coefficients

1. Determine form of y_p
2. Substitute into $ay'' + by' + cy = f(x)$
3. Notes (ALL THESE NOT GIVEN, HAVE TO MEMORIZE):
 - (a) This method is limited to non-homogeneous, linear, D.E. when $f(x)$ is a:

- i. constant (like below, e^{2x})

$f(x)$	y_p guess
a	A

- ii. exponential, polynomial, or sin/cosine function

$f(x)$	y_p guess
ae^{ax}	Ae^{ax}
$a \sin(\beta x)$	$A \sin(\beta x) + B \cos(\beta x)$
$a \cos(\beta x)$	$A \sin(\beta x) + B \cos(\beta x)$
$a \sin(\beta x) + a \cos(\beta x)$	$A \sin(\beta x) + B \cos(\beta x)$
$a \sin(\beta x) + b \cos(\beta x)$	$A \sin(\beta x) + B \cos(\beta x)$

- iii. Finite sums

$f(x)$	y_p guess
$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$	$Ax^n + Bx^{n-1} + \dots + Nx + P$

- iv. Product of above

$f(x)$	y_p guess
$e^{ax} \sin(\beta x)$ or $e^{ax} \cos(\beta x)$	$e^{ax} [A \sin(\beta x) + B \cos(\beta x)]$
$e^{ax} [a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0]$	$e^{ax} [Ax^n + Bx^{n-1} + \dots + Nx + P]$
$[a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0] \sin(\beta x)$	$[Ax^n + Bx^{n-1} + \dots + Nx + P] \sin(\beta x) + [Ax^n + Bx^{n-1} + \dots + Nx + P] \cos(\beta x)$
$[a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0] \cos(\beta x)$	$[Ax^n + Bx^{n-1} + \dots + Nx + P] \sin(\beta x) + [Ax^n + Bx^{n-1} + \dots + Nx + P] \cos(\beta x)$

- (b) y_p can be the sum of the forms inside the table.

- i. E.g. : $f(x) = 2x + e^{3x} \rightarrow y_p = Ax + B + Ce^{3x}$

- (c) If y_p have any terms duplicated in y_h , we have to multiply y_p by x until no duplicate term exists.

- i. Say $f(x) = \sin 2x$, $y_h = C_1 \sin 2x + C_2 \cos 2x$.
- ii. If we suggest $y_p = A \sin 2x + B \cos 2x$, this is duplicated in y_h (**barring the constant aside**). So, we will have to multiply by x .
- iii. This makes $y_p = x[A \sin 2x + B \cos 2x]$, which is not duplicated, and valid.
- iv. Reason we want to do this is to prevent important constants (like A) from cancelling each other. So, we need to **guess** something else. (Remember we are guessing according to the table)

Example Find the general solution of $y'' + 6y' + 9y = e^{2x}$

1. Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)

$$\begin{aligned}
 r^2 + 6r + 9 &= 0 \\
 (r + 3)^2 &= 0 \\
 r &= -3
 \end{aligned}$$

2. Find the complementary function, y_h

$$y_h = e^{-3x} (C_1 + C_2 x)$$

3. Find the particular solution y_p

- (a) Find the derivatives

$$\begin{aligned}y_p &= Ae^{2x} \\y'_p &= 2Ae^{2x} \\y''_p &= 4Ae^{2x}\end{aligned}$$

- (b) Plug into the function

$$\begin{aligned}4Ae^{2x} + 6(2Ae^{2x}) + 9(Ae^{2x}) &= e^{2x} \\25Ae^{2x} &= e^{2x}\end{aligned}$$

- (c) Find A

$$\begin{aligned}25Ae^{2x} &= e^{2x} \\A &= \frac{1}{25}\end{aligned}$$

- (d) Find y_p

$$y_p = \frac{1}{25}e^{2x}$$

4. Find the non-homogeneous D.E. general solution, $y = y_h + y_p$

$$y = e^{-3x}(c_1 + c_2x) + \frac{1}{25}e^{2x}$$

Example

1. Question: Suggest the form of y_p of $ay'' + by' + cy = f(x)$ if

- (a) $f(x) = 2x^3$
 - i. $y_p = Ax^3 + Bx^2 + Cx + D$
- (b) $f(x) = 4e^{-3x}$
 - i. $y_p = Ae^{-3x}$
- (c) $f(x) = 2(\sin 2x + \cos 2x)$
 - i. $A \sin 2x + B \cos 2x$
- (d) $f(x) = (9x^2 - 3)e^{4x}$
 - i. $Y_p = e^{4x}(Ax^2 + Bx + C)$
- (e) $f(x) = e^{4x} \sin 2x$
 - i. $y_p = e^{4x}(A \sin 2x + B \cos 2x)$
- (f) $f(x) = \sin x + \cos 2x$
 - i. $y_p = A \sin x + B \cos x + C \sin 2x + D \cos 2x$
- (g) $f(x) = e^{2x} - e^{4x}$
 - i. $y_p = Ae^{2x} + Be^{4x}$

Example 1

1. Question: Find the general solution of $y'' + 4y = \sin x$

2. Answer

- (a) Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)

$$\begin{aligned}y'' + 4y &= 0 \\r^2 + 4 &= 0 \\r &= 0 \pm 2i \leftarrow (\text{remember : } \alpha \pm \beta i)\end{aligned}$$

- (b) Find the complementary function, y_h . Note: repeated real roots

$$y_h = e^{0x}(C_1 \cos 2x + C_2 \sin 2x)$$

- (c) Find the particular solution y_p

$$\begin{aligned}f(x) &= \sin x \\y_p &= A \sin x + B \cos x \\y'_p &= A \cos x - B \sin x \\y''_p &= -A \sin x - B \cos x \\-A \sin x - B \cos x + 4(A \sin x + B \cos x) &= \sin x \\\sin x(-A + 4A) + \cos x(-B + 4B) &= \sin x \\\sin x(3A) + \cos x(3B) &= \sin x\end{aligned}$$

- i. Compare $\sin x (3A)$ with $\sin x$

$$\begin{aligned}\sin x (3A) &= \sin x \\ 3A &= 1 \\ A &= \frac{1}{3}\end{aligned}$$

- ii. Compare $\cos x (3B)$ with 0

$$\begin{aligned}\cos x (3B) &= 0 \\ B &= 0\end{aligned}$$

- iii. Find y_p

$$\begin{aligned}y_p &= \frac{1}{3} \sin x + 0 \cos x \\ y_p &= \frac{1}{3} \sin x\end{aligned}$$

- (d) Find the non-homogeneous D.E. general solution, $y = y_h + y_p$

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{3} \sin x$$

Example 2

1. Question

- (a) Find the general solution of $y'' - 4y' - 12y = 2x^3 - x + 3$

2. Answer

- (a) Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)

$$\begin{aligned}r^2 - 4r - 12 &= 0 \\ r &= 6, -2\end{aligned}$$

- (b) Find the complementary function, y_h (Note, non-repeated real roots)

$$y_h = C_1 e^{6x} + C_2 e^{-2x}$$

- (c) Find the particular solution y_p

$$\begin{aligned}y_p &= Ax^3 + Bx^2 + Cx + D \\ y_p' &= 3Ax^2 + 2Bx + C \\ y_p'' &= 6Ax + 2B\end{aligned}$$

$$\begin{aligned}(6Ax + 2B) - 4(3Ax^2 + 2Bx + C) - 12(Ax^3 + Bx^2 + Cx + D) &= 2x^3 - x + 3 \\ x^3(-12A) + x^2(-12A - 12B) + x(6A - 8B - 12C) + 2B - 4C - 12D &= 2x^3 - x + 3\end{aligned}$$

- i. $-12A = 2$

$$A = -\frac{1}{6}$$

- ii. $-12\left(-\frac{1}{6}\right) - 12B = 0$

$$B = \frac{1}{6}$$

- iii. $6\left(-\frac{1}{6}\right) - 8\left(\frac{1}{6}\right) - 12C = -1$

$$C = -\frac{1}{9}$$

- iv. $2\left(\frac{1}{6}\right) - 4\left(-\frac{1}{9}\right) - 12D = 3$

$$D = -\frac{5}{27}$$

- (d) Find the non-homogeneous D.E. general solution, $y = y_h + y_p$ (REMEMBER THIS PART)

$$y = C_1 e^{6x} + C_2 e^{-2x} - \frac{1}{6}x^3 + \frac{1}{6}x^2 - \frac{1}{9}x - \frac{5}{27}$$

Example 3

1. Solve the following I.V.P.

2. $y'' + y = 4x + 10 \sin x$, given $y(\pi) = 0, y'(\pi) = 2$

(a) Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)

$$r^2 + 1 = 0 \rightarrow r = 0 \pm i$$

$$y_h = C_1 \sin x + C_2 \cos x$$

(b) Find the complementary function, y_h (Note, irrational roots)

$$y_h = C_1 \sin x + C_2 \cos x$$

(c) Find the particular solution y_p

$$y_p = (A_x + B) + (Cx \sin x + Dx \cos x)$$

$$y'_p = A + (C \sin x + Cx \cos x + D \cos x - Dx \sin x)$$

$$= A + (C \sin x + D \cos x) + (Cx \cos x - Dx \sin x)$$

$$y''_p = C \cos x - D \sin x + (C \cos x - Cx \sin x - D \sin x - Dx \cos x)$$

$$= 2C \cos x - 2D \sin x - Cx \sin x - Dx \cos x$$

i. $2C \cos x - 2D \sin x - Cx \sin x - Dx \cos x = 4x + 10 \sin x$

$$A = 4, B = 0, C = 0, D = -5$$

$$y_p = 4x - 5x \cos x$$

(d) Find the non-homogeneous D.E. general solution, $y = y_h + y_p$

$$y = C_1 \sin x + C_2 \cos x + 4x - 5x \cos x$$

(e) Find C_1 and C_2

$$y(\pi) = 0$$

$$-C_2 + 4\pi + 5\pi = 0$$

$$C_2 = 9\pi$$

$$y = C_1 \sin x + 9\pi \cos x + 4x - 5x \cos x$$

$$y' = C_1 \cos x - 9\pi \sin x + 4 - 5 \cos x + 5x \sin x$$

$$y'(\pi) = -C_1 + 4 + 5 = 2$$

$$C_1 = 7$$

(f) Find the solution

$$y = 7 \sin x + 9\pi \cos x + 4x - 5x \cos x$$