Chapter 3: Continuous Probability Distribution

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1 Continuous Uniform Distribution

A continuous random variable defined over the interval (a, b) is said to follow a continuous uniform distribution if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & otherwise \end{cases}$$

1.1 Example 1

Given $X \sim U(1, 5)$. Find:
(a) $P(2 < X < 4.6)$

$$\begin{aligned} c &= \frac{1}{b-a} \\ &= \frac{1}{5-1} \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} P(2 < X < 4.6) &= \int_2^{4.6} \frac{1}{4} dx \\ &= \frac{1}{4} x]_2^{4.6} \\ &= \frac{4.6}{4} - \frac{1}{2} \\ \boldsymbol{P(2 < X < 4.6)} &\boldsymbol{= 0.65} \end{aligned}$$

(b)$P(X < 3.8)$
!!!!
(c) $P(X > 4.3)$

$$c = \frac{1}{4}\text{like above}$$

$$
\begin{aligned}
P(X > 4.3) &= \int_{4.3}^{5} \frac{1}{4} dx \\
&= \frac{1}{4} x]_{4.3}^{5} \\
&= \frac{5}{4} - \frac{4.3}{4} \\
&= 0.175
\end{aligned}
$$

1.2 For a continuous uniform distribution, $X \sim U(a, b)$

$\Rightarrow$ **Mean**, $\mu = \frac{a+b}{2}$. **Variance,** $\sigma^2 = \frac{(b-a)^2}{12}$

1.2.1 Proof

First part of the proof, going from the original mean formula (similar to $\frac{\sum fx}{\sum f}$, where $\sum f = 1$) to the continuous uniform distribution Note: The first line below reads as, "the mean is the sum of the value of the random variable X"

$$
\begin{aligned}
\mu &= E(X) \\
&= \int_{-\infty}^{+\infty} x f(x)\, dx \\
&= \int_{a}^{b} x \left(\frac{1}{b-a}\right) dx \\
&= \frac{1}{b-a} \int_{a}^{b} x\, dx \\
&= \frac{1}{b-a} \left[\frac{x^2}{2}\right]_{a}^{b} \\
&= \frac{1}{b-a} \left(\frac{b^2}{2} - \frac{a^2}{2}\right) \\
&= \frac{1}{b-a} \left(\frac{b^2 - a^2}{2}\right) \\
&= \frac{b^2 - a^2}{2(b-a)} \\
&= \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}} \\
\boldsymbol{\mu} &= \boldsymbol{\frac{(b+a)}{2}}
\end{aligned}
$$

Second part, going from the original variance formula $E(X^2) - [E(X)]^2$ **where** $E(X)$ **is actually** μ**, the mean.** Note: the first line below reads as, "the variance is the sum of the values squared minus the mean squared"

!!! TODO !!!

2 Exponential Distribution

The continuous random variable X has an exponential distribution, with parameter µ , if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{\mu} e^{-\frac{x}{\mu}} & x > 0 \\ 0 & otherwise \end{cases}$$

Where $\mu > 0$. We write $X \sim Exp(\mu)$.

$\Rightarrow$ **Mean,** $E(X) = \mu$**. Variance,** $Var(X) = \mu^2$

2.1 Example 2

The lifetime of a certain brand of light bulbs is a random variable X, distributed exponentially with mean time to failure µ = 200 hours. What is the probability that such light bulb will last

(i) at most 100 hours,

(ii) between 190 and 240 hours,

(iii) longer than 150 hours?

Answer:

1. Let X be he lifetime of a certain brand of light bulbs with $X \sim Exp(200)$.

$$\begin{aligned} P(X \leq 100) &= \int_0^{100} \frac{1}{200} e^{-\frac{x}{200}} dx \\ &= \left[-e^{-\frac{x}{200}}\right]_0^{100} dx \text{ Note that: } \int e^{kx} dx = \left(\frac{1}{k}\right) e^{kx} + C \\ &= -e^{-\frac{1}{2}} - (-1) \\ &= 1 - e^{-\frac{1}{2}} \\ &= 0.3935 \end{aligned}$$

2. !!!TODO!!!

3. Longer than 150 hours?

$$\begin{aligned} P(X > 150) &= \int_{150}^{200} \frac{1}{200} e^{-\frac{x}{200}} dx \\ &= \left[-e^{-\frac{x}{200}}\right]_{150}^{200} \\ &= -e^{-1} - \left(-e^{-\frac{3}{4}}\right) \\ &= e^{-\frac{3}{4}} - e^{-1} \\ &= 0.1045 \end{aligned}$$

3 Normal Distribution

The probability distribution of a normal random variable is called a normal distribution.

3.1 Properties of a Normal Distribution

A normally distributed random variable X with mean µ and variance σ^2 $\left[X \sim N\left(\mu, \sigma^2\right)\right]$ has the following properties:

1. Probability density function

$$f\left(x\right)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}for\ -\infty<x<+\infty$$

   (a) Where $\pi = 3.14159...$ and $e = 2.71828...$

2. The graph of the probability density function of a normal distribution is a continuous bell-shaped curve with the mean µ at the center of the normal curve and the curve is symmetrical about µ

3. The probability distribution for Z which has **mean** µ = 0and **variance** $\sigma^2 = 1$ is called the standard normal distribution and the random variable Z is called the **standard normal random variable**

!!! TODO !!!

3.2 Example 3

Find the following probabilities for the standard Normal curve. (Use the table for "tail of the normal distribution" to help)

(a) P (Z > 1.5)

$$P(Z > 1.5) = 0.0668$$

(b) P (Z > − 2)

$$\begin{aligned} P\left(Z>-2\right) &= 1-P\left(Z>2\right)\\ &= 1-0.2275 \end{aligned}$$

$$\boldsymbol{P\left(Z>-2\right)=0.7725}$$

(c) $P(Z < 0.8)$

$$!!!TODO!!!$$

(d) P (Z < − 1.5)

$$!!!TODO!!!$$

(e) P(− 1 < Z < − 0.5)

$$!!!TODO!!!$$

(f) P (− 2 < Z < 1)

$$!!!TODO!!!$$

(g) P (1.19 < Z < 2.12)

$$
\begin{aligned}
P(1.19 < Z < 2.12) &= 1 - P\left(Z > 1.19\right) - P\left(Z > 2.12\right) \\
&= 1 - 0.1170 - 0.01700 \\
\boldsymbol{P(1.19 < Z < 2.12)} &\boldsymbol{= 0.866}
\end{aligned}
$$

3.3 Example 4

If Z ~ N (0,1), find the value of a if

(a) P(Z > a) = 0.3783
(b) P(Z > a) = 0.7823
(c) P(Z < a) = 0.0793
(d) P(Z < a) = 0.9693
(e) P(| Z | < a) = 0.9

$$
\begin{aligned}
P\left(-a < Z < a\right) &= 0.9 \\
P\left(Z > a\right) &= 0.05 \\
\boldsymbol{a} &\boldsymbol{= 1.6449}
\end{aligned}
$$

(f) P(| Z | > a) = 0.0602

3.4 Example 5

The mean weight of 200 people is 67 kg and the standard deviation is 7 kg. Assuming that the weights are **Normally distributed**, determine how many people have a weight

Let X be the weight of the people with $X \sim N\left(67, 7^2\right)$

(a) between 60 and 74 kg
(b) more than 81kg
(c) between 53 and 88 kg

$$
\begin{aligned}
P\left(53 \leq X \leq 88\right) &= P\left(\frac{53-67}{7} \leq \frac{X-\mu}{\sigma} \leq \frac{88-67}{7}\right) \\
&= P\left(-2 \leq Z \leq 3\right) \\
&= 1 - P\left(Z > 2\right) - P\left(Z > 3\right) \\
&= 1 - 0.02275 - 0.00135 \\
\boldsymbol{P\left(53 \leq X \leq 88\right)} &\boldsymbol{= 0.9759}
\end{aligned}
$$

3.5 Example 6

The score on a final examination was **normally distributed** with mean 72 and the standard deviation 9. The top 10% of the students are receive ‘A’s. What is the minimum score that a student must get in order to receive an ‘A’?

Let X be the student's score on a final examination with $X \sim N\left(72, 9^2\right)$

$$
\begin{aligned}
P\left(X > a\right) &= 0.1 \\
P\left(Z > \frac{a-72}{\sqrt{9^2}}\right) &= 0.1 \\
\frac{a-72}{\sqrt{9^2}} &= 1.2816 \\
\boldsymbol{a} &= \boldsymbol{83.5344 marks}
\end{aligned}
$$

3.6 Example 7

$$
\begin{aligned}
P\left(Z < \frac{106-100}{\sigma}\right) &= 0.8849 \\
P\left(Z > \frac{106-100}{\sigma}\right) &= 0.1151 \\
\frac{6}{a} &= 1.20 \\
a &= \frac{6}{1.2} \\
\boldsymbol{a} &= \boldsymbol{5}
\end{aligned}
$$

3.7 Example 8

4 The Normal Approximation to the Binomial Distribution

4.1 Criteria

1. n is large (generally > 30)
2. p is not too small or large (closer to 0.5 the better)
3. $np \geq 5, nq \geq 5$

If all the criteria are met, then: The Normal distribution with mean µ = np and variance = npq 2 σ can be used to approximate the Binomial distribution X ˜ B (n , p) ≈ X ˜ N (= np , = npq) 2 µ σ when np ≥ 5 and nq ≥ 5

4.2 Continuity correction factor

⇒ The addition and/or subtraction of 0.5 from the value(s) of x when the Normal distribution is used as an approximation to the Binomial distribution, where x is the number of successes in n trials.

Basically, make the size of the measurement larger

4.3 Example 9

4.4 Example 10

4.5 Example 11

4.6 Example 12

The number of calls received by an office switchboard per hour follows a Poisson distribution with parameter 30. Using the normal approximation to the Poisson distribution, find the probability that in one hour, there are

1.

   (a) $X=$ no. of calls in one hour $\sim$Possion with $\lambda = 30$, big $\sim$Approximately normal with $\mu = \lambda = 30$, $\sigma^2 = \lambda = 30$

      i.

$$\begin{aligned} P(X > 23) = P(X > 23.5) &= P\left(\frac{x-\mu}{\sigma} > \frac{23.5-30}{\sqrt{30}}\right) \\ &= P(Z > -1.19) \\ &= 0.8830 \end{aligned}$$

   (b) $P(25 \leq X \leq 28) = P(25 \leq X \leq 28)$

$$\begin{aligned} P\left(\frac{24.5-30}{\sqrt{30}} < \frac{x-\mu}{\sigma} < \frac{28.5-30}{\sqrt{30}}\right) &= P(0.29 < Z < 1) \\ &= 0.2349 \end{aligned}$$

   (c) $P(X = 34) = P(33.5 \leq X \leq 34.5)$

$$\begin{aligned} P\left(\frac{33.5-30}{\sqrt{30}} \leq Z \leq \frac{34.5-30}{\sqrt{30}}\right) &= P(0.64 \leq Z \leq 0.82) \\ &= 0.0550 \end{aligned}$$

4.7 Example 13

The reaction time to a certain psychological experiment is normally distributed with a mean of 20 seconds and standard deviation of 4 seconds. What is the reaction time for which only 1% of all subjects is faster?

$$\mu = 20, \sigma = 4$$

$$\begin{aligned} P(X \leq a) &= 0.01 \\ P\left(Z \leq \frac{a-20}{4}\right) &= 0.01 \\ \frac{a-20}{4} &= -2.3263 \\ a &= 10.6948 seconds \end{aligned}$$

4.8 Example 14

$$p = 0.36$$

$$n = 400$$

X $\sim B\left(n=400, p=0.36\right)$
Since n is very big (n $>$ 50)
np $=$ 144, nq $=$ 256 (both $\geq$ 5)
We can use a normal approximation, with $X \sim N\left(144\left(np\right), 92.16\left(npq\right)\right)$

$$\begin{aligned}
P\left(X125\right) &\approx P\left(X > 125.5\right) \\
&= P\left(Z > \frac{125.5 - 144}{\sqrt{92.16}}\right) \\
&= P\left(Z > \frac{125.5 - 144}{\sqrt{92.16}}\right) \\
&= P\left(Z > -1.93\right) \\
&= 0.9372
\end{aligned}$$