Ginwidth=

Tutorial 7: Calc 1

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1.

(a) $y = \sqrt{x}e^{x^2}\left(x^2+1\right)^{10}$

$$\begin{aligned}
y &= \sqrt{x}e^{x^2}\left(x^2+1\right)^{10} \\
\ln(y) &= \ln\left(\sqrt{x}e^{x^2}\left(x^2+1\right)^{10}\right) \\
&= \ln\sqrt{x}e^{x^2} + \ln\left(x^2+1\right)^{10} \\
&= \ln\sqrt{x} + \ln e^{x^2} + \ln\left(x^2+1\right)^{10} \\
&= \frac{1}{2}\ln x + x^2\ln e + 10\ln\left(x^2+1\right) \\
&= \frac{1}{2}\ln x + x^2 + 10\ln\left(x^2+1\right) \\
\frac{1}{y}\left(\frac{dy}{dx}\right) &= \frac{1}{2}\cdot\frac{1}{x} + 2x + 10\cdot\frac{2x}{x^2+1} \\
\frac{dy}{dx} &= \left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right)\cdot y \\
&= \left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right)\cdot\sqrt{x}e^{x^2}\left(x^2+1\right)^{10} \\
&= \sqrt{x}e^{x^2}\left(x^2+1\right)^{10}\left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right)
\end{aligned}$$

(b) $y = (\sin x)^x$

$$\begin{aligned}
\ln y &= x\ln(\sin x) \\
\frac{1}{y}\left(\frac{dy}{dx}\right) &= x\cot x + \ln(\sin x) \\
\frac{dy}{dx} &= [x\cot + \ln(\sin x)]\left[(\sin x)^x\right]
\end{aligned}$$

$$
\begin{aligned}
y &= \ln(\sin x) \\
y' &= \frac{1}{\sin x} \cdot \cos x \\
&= \frac{\cos x}{\sin x} \\
&= \cot x
\end{aligned}
$$

(c) $y = x^{\ln x}$

$$
\begin{aligned}
y &= x^{\ln x} \\
\ln y &= \ln x \cdot \ln x \\
\ln y &= (\ln x)^2 \\
\frac{d}{dx}[\ln y] &= 2\ln x \cdot \frac{d}{dx}[\ln x] \\
\frac{1}{y} \cdot \frac{dy}{dx} &= 2\ln x \cdot \frac{1}{x} \\
\frac{dy}{dx} &= \frac{2\ln x}{x} \cdot y \\
&= \frac{2\ln x}{x} \cdot x^{\ln x}
\end{aligned}
$$

2. Find y' if $x^y = y^x$

$$
\begin{aligned}
\ln x^y &= \ln y^x \\
y \ln x &= x \ln y \\
\frac{d}{dx}[y \ln x] &= \frac{d}{dx}[x \ln y] \\
\frac{dy}{dx} * \ln x + \frac{y}{x} &= \ln y + \frac{x}{y} \cdot \frac{dy}{dx} \\
\frac{dy}{dx} * \ln x - \frac{x}{y} \cdot \frac{dy}{dx} &= \ln y - \frac{y}{x} \\
\frac{dy}{dx}\left[\ln x - \frac{x}{y}\right] &= \ln y - \frac{y}{x} \\
y' &= \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}} \\
y' &= \frac{y(x \ln y - y)}{x(y \ln x - x)}
\end{aligned}
$$

3. Linearlization, find straight line equation

$$
L(x) = g(a) + g'(a)(x - a), a = 0
$$

(a) Find $g'(x)$

$$\begin{aligned}
g(x) &= \sqrt[3]{1+x} \\
&= (1+x)^{\frac{1}{3}} \\
g'(x) &= \frac{1}{3}(1+x)^{-\frac{2}{3}} \\
&= \frac{1}{3\sqrt[3]{(1+x)^2}}
\end{aligned}$$

(b) Find $L(x)$

$$\begin{aligned}
L(x) &= g(0) + g'(0)(x-0) \\
&= \sqrt[3]{1+0} + \frac{1}{3\sqrt[3]{(1+0)^2}}(x) \\
&= 1 + \frac{1}{3}x
\end{aligned}$$

(c) Approximate $\sqrt[3]{0.95} = \sqrt[3]{1-0.05}$, $x = -0.05$

$$\begin{aligned}
L(-0.05) &= 1 - \frac{1}{3}(0.05) \\
&= 0.983
\end{aligned}$$

(d) Approximate $\sqrt[3]{1.1} = \sqrt[3]{1+0.1}, x = 0.1$

$$\begin{aligned}
L(0.1) &= 1 + \frac{1}{3}(0.1) \\
&= 1.03
\end{aligned}$$

4. $y = \sqrt{1+x^3}$

$$\begin{aligned}
y &= \left(1+x^3\right)^{\frac{1}{3}} \\
\frac{dy}{dx} &= \frac{1}{2}\left(1+x^3\right)^{-\frac{1}{2}} 3x^2 \\
&= \frac{3x^2}{2}\left(1+x^3\right)^{-2} \\
&= \frac{3x^2}{2\sqrt{1+x^3}} \\
\frac{dx}{dt} &= \frac{dx}{dy} \cdot \frac{dy}{dt} \\
&= \frac{2\sqrt{1+x^3}}{3x^2} \cdot 4 \\
When\, x = 3 &= \frac{2\sqrt{1+(2)^3}}{3(2)^2} \\
&= 2
\end{aligned}$$

5.

(a) List down terms

i. $\frac{dV}{dt} = 50$

ii. $\frac{dr}{dt} = ?$

iii. $r = radius = 10$

(b) Find formula to get $\frac{dr}{dV}$ or something similar

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

i. When $r = 10$

$$\begin{aligned} \frac{dV}{dr} &= 4\pi\left(10\right)^2 \\ &= 400\pi \end{aligned}$$

(c) Use chain rule to eliminate dV

$$\begin{aligned} \frac{dr}{dt} &= \frac{dr}{dV} * \frac{dV}{dt} \\ &= \frac{1}{\frac{dV}{dt}} * 50 \\ &= \frac{1}{400\pi} * 50 \\ &= \frac{1}{8\pi} cms^{-1} \end{aligned}$$

6.

$$\begin{aligned} \frac{dx}{dt} &= 1.5, \frac{dy}{dt} = ? \\ \frac{5}{y} &= \frac{2}{y-x} \\ 5y - 5x &= 2y \\ y &= \frac{5}{3}x \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt} &= \frac{dy}{dx} * \frac{dx}{dt} \\ &= \frac{5}{3} * 1.5 \\ &= 2.5m/s \end{aligned}$$

7.

$$\begin{aligned}\frac{dx}{dt} &= 1.6 \\ \frac{dy}{dt} &= ?\end{aligned}$$

$$\begin{aligned}\frac{y}{12} &= \frac{2}{x} \\ y &= \frac{24}{x} \\ &= 24x^{-1}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx} * \frac{dx}{dt} \\ &= -\frac{24}{x^2} * 1.6 \\ &= -\frac{25}{8^2} * 1.6 \\ &= -0.6 m/s\end{aligned}$$