Calc I : Chapter 4 - Applications of Diff.

August 29, 2019

1 Example

1. Find the formula for the equation of gradient $\frac{dy}{dx}$

$$y = x^2$$
$$\frac{dy}{dx} = 2x$$

2. Find the gradient

$$\frac{dy}{dx} = 2\,(3)$$
$$= 6$$

2 Example

1. Find the formula for the equation of gradient $\frac{dy}{dx}$

$$\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

2. The point where the curve crosses the x-axis is when $y = 0$

$$0 = 1 - \sqrt{x}$$
$$-1 = -\sqrt{x}$$
$$\left(\sqrt{x}\right)^2 = 1^2$$
$$x = 1$$

3. Find the slope, $\frac{dy}{dx}$ at $x = 1$

$$\frac{dy}{dx} = -\frac{1}{2\sqrt{1}}$$
$$= -\frac{1}{2}$$

3 Example

1. $x^3 + 2y^3 + 3xy = 0$

2.

$$\begin{aligned}
3x^2 + 6y^2\frac{dy}{dx} + 3\left(\frac{d}{dx}[x] \cdot y + x\frac{d}{dx}[y]\right) &= \frac{d}{dx}[0] \\
3x^2 + 6y^2\frac{dy}{dx} + 3\left(1 \cdot y + x\frac{dy}{dx}\right) &= 0 \\
3x^2 + 6y^2\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} &= 0 \\
3x^2 + 3y &= -\left(6y^2\frac{dy}{dx} + 3x\frac{dy}{dx}\right) \\
&= -\frac{dy}{dx}\left(6y^2 + 3x\right) \\
-\left(3x^2 + 3y\right) &= \frac{dy}{dx}\left(6y^2 + 3x\right) \\
\frac{dy}{dx} &= -\frac{\left(3x^2 + 3y\right)}{\left(6y^2 + 3x\right)} \\
&= -\frac{3x^2 + 3y}{3x + 6y^2} \\
&= -\frac{3\left(x^2 + y\right)}{3\left(x + 2y^2\right)} \\
&= -\frac{x^2 + y}{x + 2y^2}
\end{aligned}$$

3. At $(2, -1)$

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{(2)^2 + (-1)}{(2) + 2(-1)^2} \\
&= -\frac{4 - 1}{2 + 2} \\
\boldsymbol{\frac{dy}{dx}} &\boldsymbol{= -\frac{3}{4}}
\end{aligned}$$

4 Example

$$4x^2 - y^2 = 7, \frac{dy}{dx} = \frac{8}{3}$$

1. Find the $\frac{dy}{dx}$ formula

(a) Lets try to solve for $\frac{dy}{dx}$

$$\begin{aligned}\frac{d}{dx}\left[4x^2 - y^2\right] &= \frac{d}{dx}[7] \\ 8x - 2y \cdot \frac{dy}{dx} &= 0 \\ -2y \cdot \frac{dy}{dx} &= -8x \\ \frac{dy}{dx} &= \frac{-8x}{-2y} \\ &= \frac{4x}{y}\end{aligned}$$

2. Lets substitute $\frac{dy}{dx}$ in

$$\begin{aligned}\frac{8}{3} &= \frac{4x}{y} \\ 8y &= 4x\,(3) \\ 8y &= 12x \\ 2y &= 3x \\ y &= \frac{3}{2}x\end{aligned}$$

3. Oh, and we're still stuck with x and y, but hey, now we have 2 equations with x and y, remember simultaneous equations

$$\begin{aligned}y = \frac{3}{2}x &\Longrightarrow \alpha \\ 4x^2 - y^2 = 7 &\Longrightarrow \beta\end{aligned}$$

(a) Substitute α into β

$$\begin{aligned}4x^2 - \left(\frac{3}{2}x\right)^2 &= 7 \\ 4x^2 - \frac{9}{4}x^2 &= 7 \\ 16x^2 - 9x^2 &= 28 \\ 7x^2 &= 28 \\ x^2 &= \frac{28}{7} \\ &= 4 \\ x &= \pm 2\end{aligned}$$

(b) Oh now we found x, lets try finding y by substituting x into α

i. When $x = 2$

$$y = \frac{3}{2}(2) \\ = 3$$

ii. When $x = -2$

$$y = \frac{3}{2}(-2) \\ = -3$$

(c) We found 2 points

$$(2, -3), (-2, -3)$$

5 Example

1. $y = x^3 - 2x^2 + 3$, $x = 2$
2. Find the equation of the tangent
   (a) Find the equation of the $\frac{dy}{dx}$

   $$\frac{dy}{dx} = 3x^2 - 4x$$

   (b) Find $\frac{dy}{dx}$ at $x = 2$

   $$\begin{aligned} \frac{dy}{dx} &= 3(2)^2 - 4(2) \\ &= 12 - 8 \\ \boldsymbol{\frac{dy}{dx}} &\boldsymbol{= 4} \end{aligned}$$

   (c) Find the equation of the tangent line
   i. When $x = 2$, $y = (2)^3 - 2(2)^2 + 3 = 3$
   ii. So the point is $(2, 3)$
   iii. $y - y_1 = m(x - x_1)$

   $$\begin{aligned} y - 3 &= 4(x - 2) \\ y &= 4x - 8 + 3 \\ \boldsymbol{y} &\boldsymbol{= 4x - 5} \end{aligned}$$

3. Find the equation of the normal

(a) Find the gradient of the normal m_n

$$\begin{aligned} m_n m_t &= -1 \\ m_n (4) &= -1 \\ m_n &= -\frac{1}{4} \end{aligned}$$

(b) Find the equation of the normal line

$$\begin{aligned} y - 3 &= -\frac{1}{4}(x-2) \\ y &= -\frac{1}{4}x + \frac{1}{2} + 3 \\ &= -\frac{1}{4}x + \frac{7}{2} \end{aligned}$$

6 Example

1. $y = x^2 - 6x + 15$, $(4, 7)$

   (a) Gradient at P, and equation at of normal at P

      i. Gradient at P

$$\begin{aligned} \frac{dy}{dx} &= 2x - 6 \\ \frac{dy}{dx}|_{x=4} &= 2(4) - 6 \\ &= 2 \end{aligned}$$

      ii. Equation of normal at P

         A. Gradient of normal at P

$$\begin{aligned} m_n m_t &= -1 \\ m_n &= -\frac{1}{m_t} \\ &= -\frac{1}{2} \end{aligned}$$

         B. Equation of normal at P

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 7 &= -\frac{1}{2}(x-4) \\ y &= -\frac{1}{2}x + 2 + 7 \\ \boldsymbol{y} &\boldsymbol{= -\frac{1}{2}x + 9} \end{aligned}$$

(b) Find the coordinates where the normal cuts the curve again

i. Well since both of them intersect, they have the same coordinates, its simultaneous equations again!

$$y = x^2 - 6x + 15 \implies \alpha$$
$$y = -\frac{1}{2}x + 9 \implies \beta$$

A. Lets substitute β inside α

$$\begin{aligned} -\frac{1}{2}x + 9 &= x^2 - 6x + 15 \\ x^2 - 6x + 15 + \frac{1}{2}x - 9 &= 0 \\ x^2 - \frac{11}{2}x + 6 &= 0 \\ 2x^2 - 11x + 12 &= 0 \\ (2x-3)(x-4) &= 0 \\ x &= \frac{3}{2}, 4 \end{aligned}$$

B. Find all the y's by substituting them into β

C. When $x = \frac{3}{2}$

$$\begin{aligned} y &= -\frac{1}{2}\left(\frac{3}{2}\right) + 9 \\ &= -\frac{3}{4} + 9 \\ &= \frac{33}{4} \end{aligned}$$

D. We will ignore $x = 4$ because the question ask us to find ANOTHER point other than when $x = 4$, $y = 7$

ii. Therefore we conclude that the other point is $\frac{33}{4}$

7 Example

$x^2y + xy^2 = 12, (1,3)$

$$
\begin{aligned}
\frac{d}{dx}\left[x^2y + xy^2\right] &= \frac{d}{dx}[12] \\
\frac{d}{dx}\left[x^2\right]\cdot y + x^2\frac{dy}{dx} + y^2 + x\frac{dy}{dx}\cdot 2y + x\cdot\frac{d}{dx}\left[y^2\right] &= 0 \\
2x\cdot y + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} &= 0 \\
2xy + y^2 &= -2xy\frac{dy}{dx} - x^2\frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{2xy + y^2}{-2xy - x^2} \\
\frac{dy}{dx} &= \frac{2(1)(3) + (3)^2}{-2(1)(3) - (1)^2} \\
\frac{dy}{dx} &= \frac{6+9}{-6-1} \\
&= -\frac{15}{7}
\end{aligned}
$$

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 3 &= -\frac{15}{7}(x-1) \\
y &= -\frac{15}{7}x + \frac{15}{7} + 3 \\
\boldsymbol{y} &\boldsymbol{= -\frac{15}{7}x + \frac{36}{7}} \\
7y &= -15x + 36 \\
7y + 15x &= 36
\end{aligned}
$$

8 (Rate of Change) Example

1. Given:
$$\frac{dr}{dt} = 0.2cm/s, \frac{dA}{dt} =?$$

2. Let:

   (a) A = area

   (b) r = radius

   (c) t = time

3. Find the formula for area:

$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r$$

4. When $r = 5$

$$\begin{aligned}\frac{dA}{dr} &= 2\pi\,(5) \\ &= 10\pi\end{aligned}$$

5. Find $\frac{dA}{dt}$, when $r = 5$

$$\begin{aligned}\frac{dA}{dt} &= \frac{dA}{dr} * \frac{dr}{dt} \\ &= 10\pi * 0.2 \\ &= 2\pi cm^2/s\end{aligned}$$

9 Example

1. Let:
   (a) $V = volume$
   (b) $r = radius$
   (c) $t = time$
2. List down terms
   (a) $\frac{dV}{dt} = 10cm^3s^{-1}$
   (b) $r = 5$
   (c) $\frac{dr}{dt} =?$
3. Find $\frac{dV}{dr}$, when $r = 5$

$$V = \frac{4}{3}\pi r^3$$

$$\begin{aligned}\frac{dV}{dr} &= 4\pi r^2 \\ &= 4\pi\,(5)^2 \\ &= 100\pi\end{aligned}$$

4. Find $\frac{dr}{dt}$

$$\begin{aligned}\frac{dr}{dt} &= \frac{dr}{dV} * \frac{dV}{dt} \\ &= \frac{1}{100\pi} * 10 \\ &= \frac{1}{10\pi} cms^{-1}\end{aligned}$$

10 Example

1. List down the terms

   (a) $x^2 + y^2 = 5^2$

   (b) $\frac{dx}{dt} = 0.2$

   (c) $y = 4$

   (d) $\frac{dy}{dt} =?$

2. Find x

$$\begin{aligned}x^2 + (4)^2 &= 5^2 \\ x^2 &= 25 - 16 \\ &= 9 \\ x &= 3\end{aligned}$$

3. Find $\frac{dy}{dt}$, when

$$\begin{aligned}x^2 + y^2 &= 5 \\ \frac{d}{dt}\left[x^2 + y^2\right] &= 0 \\ 2x\frac{dx}{dt} + 2y \cdot \frac{dy}{dt} &= 0 \\ \frac{dy}{dt} &= \frac{-2x}{2y}\frac{dx}{dt} \\ &= -\frac{x}{y} \cdot \frac{dx}{dt} \\ \frac{dy}{dt}|_{x=3,y=4} &= -\frac{3}{4} \cdot \frac{2}{10} \\ &= -\frac{3}{20} \\ &= -0.15ms^{-1}\end{aligned}$$

11 Example

1. Define terms

   (a) Revenue $= R(x)$

   (b) $Sales = x = 1000$

   (c) Formula: $R(x) = 200x - 3x^{\frac{2}{3}}$

   (d) $\frac{dx}{dt} = 10$

   (e) $\frac{dR}{dt}|_{x=1000} = ?$

2. Find $\frac{dR}{dx}$, when $x = 1000$

$$\begin{aligned}
\frac{d}{dt}\left[R(x)\right] &= \frac{d}{dt}\left[200x - 3x^{\frac{2}{3}}\right] \\
\frac{dR}{dt} &= 200\frac{dx}{dt} - 2x^{-\frac{1}{3}} \cdot \frac{dx}{dt} \\
&= \frac{dx}{dt}\left[200 - 2x^{-\frac{1}{3}}\right] \\
\frac{dR}{dt}|_{x=1000,\frac{dx}{dt}=10} &= 10\left[200 - 2(1000)^{-\frac{1}{3}}\right] \\
&= 10\left[200 - \frac{2}{10}\right] \\
&= 10\left[199.8\right] \\
&= 1998 dollars/month
\end{aligned}$$

12 Example (Linear Approximation & Differentials)

1. List down the terms

   (a) $r = 5$

   (b) $\delta r = 0.01$

   (c) $\delta A = ?$

2. Find $\frac{dA}{dr}$

$$\begin{aligned}
A &= \pi r^2 \\
\frac{dA}{dr} &= 2\pi r \\
\frac{dA}{dr}|_{r=5} &= 10\pi
\end{aligned}$$

3. Since δr is small,

$$\frac{dA}{dr} \approx \frac{\delta A}{\delta r}$$
$$\delta A \approx \frac{dA}{dr} * \delta r$$

4. Find δA

$$\delta A|_{r=5,\delta r=0.01} \approx 10\pi * 0.01$$
$$\delta A = 0.1\pi cm^2$$

5. Conclusion

   (a) The approximate increase in the area is 0.1 πcm^2.

13 Example

1. List down terms

   (a) Let $x = sides = 10cm$

   (b) Let $V = volume$

   (c) Let $\delta x = change\, in\, sides = -0.1$

   (d) Find δV, change in volume

2. Find $\frac{dV}{dx}$, when $x = 10$

$$\begin{aligned} V &= x^3 \\ \frac{dV}{dx} &= 3x^2 \\ \frac{dV}{dx}|_{x=10} &= 3\,(10)^2 \\ &= 300 \end{aligned}$$

3. Find δV

   (a) Since δx is small,

$$\begin{aligned} \delta V &\approx \frac{dV}{dx} \cdot \delta x \\ \delta V|_{\frac{dV}{dx}=300,\delta x=-0.1} &= 300 \cdot (-0.1) \\ &= -30cm^3 \end{aligned}$$

4. Conclusion

   (a) The approximate decreas in the volume is 30 cm^3.

14 Example

1. Find linearization

$$\begin{aligned} f(x) &= x^{\frac{1}{2}} \\ f'(x) &= \frac{1}{2\sqrt{x}} \\ f(x+\delta x) &= \sqrt{x+\delta x} \end{aligned}$$

2. $\sqrt{3.98}$

   (a) Let $x = 4, \delta x = -0.02$

$$\begin{aligned} f(x+\delta x) &\approx f(x) + f'(x)\,\delta x \\ &= x^{\frac{1}{2}} + \frac{1}{2\sqrt{x}} \cdot \delta x \\ f(4+(-0.02)) &= 4^{\frac{1}{2}} + \frac{1}{2\sqrt{4}} \cdot (-0.02) \\ &= 2 + \frac{1}{4} \cdot (-0.02) \\ \sqrt{3.98} &= 1.995 \end{aligned}$$

3. $\sqrt{4.05}$

   (a) Let $x = 4, \delta x = 0.05$

$$\begin{aligned} f(x+\delta x) &\approx f(x) + f'(x)\,\delta x \\ &= x^{\frac{1}{2}} + \frac{1}{2\sqrt{x}} \cdot \delta x \\ f(4+0.05) &= 2 + \frac{1}{2(2)} \cdot 0.05 \\ \sqrt{4.05} &= 2.0125 \end{aligned}$$

15 Example

1. Find the definitions

$$\begin{aligned} f(x) &= \sqrt[3]{x} \\ f(x+\delta x) &= \sqrt[3]{x+\delta x} \\ f'(x) &= \frac{1}{3x^{\frac{2}{3}}} \end{aligned}$$

2. $\sqrt[3]{8030}$

(a) Let $x = 8000, \delta x = 8030 - 8000 = 30$

$$f(x + \delta x) \approx f(x) + f'(x)\,\delta x$$
$$f(x + \delta x)\,|_{x=8000,\delta x=30} = \sqrt[3]{8000} + \frac{1}{3\,(8000)^{\frac{2}{3}}} \cdot 30$$
$$\sqrt[3]{8030} = 20.025$$

16 Example (How about negative part)

1. List down the terms

   (a) $r = radius = 21$

   (b) $\delta r = maximum\,error = 0.05$

   (c) $\delta V =$ maximum error for volumes $=?$

2. Find the $\frac{dV}{dr}$

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dr} = 4\pi r^2$$
$$dV = 4\pi r^2 dr$$
$$\delta V \approx 4\pi r^2 \delta r$$

3. Find δV

$$\delta V \approx 4\pi\,(21)^2\,(0.05)$$
$$= 277$$

4. Conclusion

   (a) The maximum error in calculated volume is 277 cm^3.

17 Example

1. Find coordinates of turning points, which mean $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 6x^2 + 6x - 12$$
$$6x^2 + 6x - 12 = 0$$
$$x^2 + x - 2 = 0$$
$$(x + 2)\,(x - 1) = 0$$

(a) $x = -2, x = 1$

(b) When $x = -2$

$$y = 2\left(-2\right)^3 + 3\left(-2\right)^2 - 12\left(-2\right) + 7$$
$$y = 27$$

i. $(-2, 27)$

(c) When $x = 1$

$$\begin{aligned} y &= 2\left(1\right)^3 + 3\left(1\right)^2 - 12\left(1\right) + 7 \\ &= 2 + 3 - 12 + 7 \\ &= 0 \end{aligned}$$

i. $(1, 0)$

2. Finding the nature, method 1

(a)

x	-3	-2	0	1	2
$f'(x)$	+ve	0	-ve	0	+ve

i. $(-2, 27)$is a local maximum point

ii. $(1, 0)$ is a local minimum point.

3. Finding the nature, method 2

(a) Find $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[6x^2 + 6x - 12\right]$$
$$\frac{d^2y}{dx^2} = 12x + 6$$

(b) When $x = -2$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 12x + 6 \\ &= 12\left(-2\right) + 6 \\ &= -18 < 0, \therefore \textit{local max} \end{aligned}$$

(c) When $x = 1$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 12x + 6 \\ &= 12\left(1\right) + 6 \\ &= 18 > 0 \therefore \textit{local min} \end{aligned}$$

18 Example

$$y = 27x + \frac{4}{x^2}$$

1. Find stationary points

$$\frac{dy}{dx} = \frac{d}{dx}\left[27x + \frac{4}{x^2}\right]$$
$$= 27 - \frac{8}{x^3}$$

   (a) When $\frac{dy}{dx} = 0$

$$0 = 27 - \frac{8}{x^3}$$
$$27x^3 = 8$$
$$x^3 = \frac{8}{27}$$
$$x = \frac{2}{3}$$

   (b) Find y when $x = \frac{2}{3}$

$$y = 27\left(\frac{2}{3}\right) + \frac{4}{\left(\frac{2}{3}\right)^2}$$
$$= 27$$

   (c) Find $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[27 - \frac{8}{x^3}\right]$$
$$\frac{d^2y}{dx^2} = \frac{24}{x^4}$$

      i. When $x = \frac{2}{3}$

$$\frac{d^2y}{dx^2} = \frac{24}{\left(\frac{2}{3}\right)^4}$$
$$= \frac{24}{\left(\frac{2}{3}\right)^4}$$
$$= +ve; \therefore min\, point$$

   (d) Conclusion

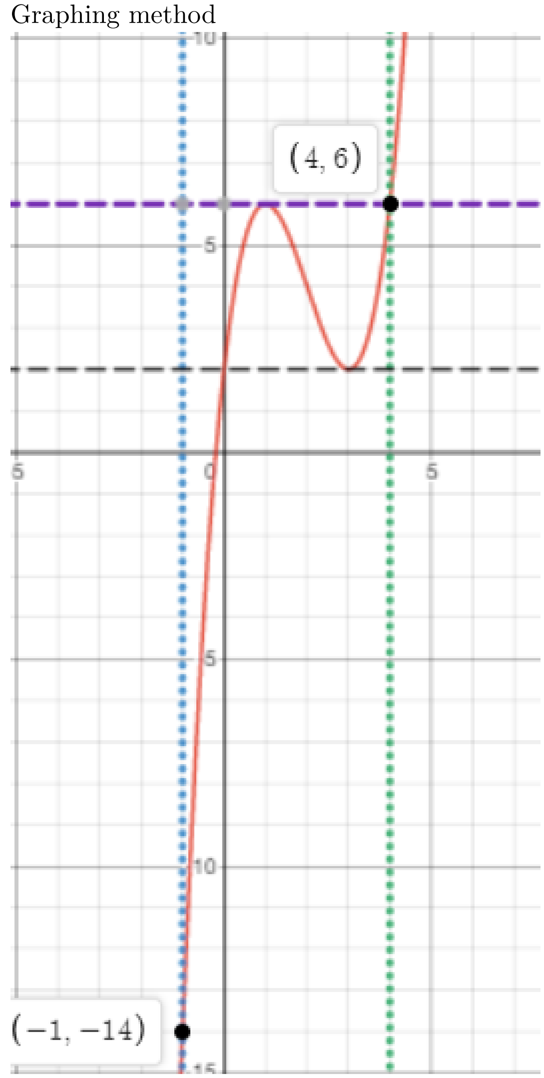
$$\left(\frac{2}{3}, 27\right)$$

   is a minimum point.

19 Example (Extreme Values)

1. $y = x^3 - 6x^2 + 9x + 2, -1 \leq x \leq 4$

   (a) Graphing method

   (b) Closed-interval method

      i. According the extreme value theorem, since f is continuous on the closed interval $[-1, 4]$, it attains abs. max and abs. min at some numbers.

      ii. Find the values of f at the critical numbers $\left(\frac{dy}{dx} = 0\right)$ of f in

$[-1, 4]$

$$
\begin{aligned}
y &= x^3 - 6x^2 + 9x + 2 \\
\frac{dy}{dx} &= 3x^2 - 12x + 9
\end{aligned}
$$

$$
\begin{aligned}
3x^2 - 12x + 9 &= 0 \\
x^2 - 4x + 3 &= 0 \\
(x-3)(x-1) &= 0 \\
x = 3, x &= 1
\end{aligned}
$$

A. When $x = 3$

$$
\begin{aligned}
y &= 3^3 - 6(3)^2 + 9(3) + 2 \\
&= 2
\end{aligned}
$$

B. $\therefore \mathbf{(3, 2)}$

C. When $x = 1$

$$
\begin{aligned}
y &= 1^3 - 6(1)^2 + 9(1) + 2 \\
&= 6
\end{aligned}
$$

D. $\therefore \mathbf{(1, 6)}$

iii. Find the values of f at the endpoints of the interval

$$
\begin{aligned}
y|_{x=-1} &= (-1)^3 - 6(-1)^2 + 9(-1) + 2 \\
&= -14
\end{aligned}
$$

$$
\begin{aligned}
y|_{x=4} &= (4)^3 - 6(4)^2 + 9(4) + 2 \\
&= 6
\end{aligned}
$$

A. $\mathbf{(-1, -14), (4, 6)}$

iv. Conclusion

A. **Absolute minimum: -14**

B. **Absolute maximum: 6**

2. $y = x^3(x-2), -1 \leq x \leq 3$

(a) Graphing method

(b) Closed-interval method

i. According the extreme value theorem, since f is continuous on the closed interval $-1 \leq x \leq 3$, it attains abs. max and abs. min at some numbers.

ii. Find the values of f at the critical numbers $\left(\frac{dy}{dx} = 0\right)$ of f in $[-1 \leq x \leq 3]$

$$
\begin{aligned}
y &= x^3 (x-2) \\
&= x^4 - 2x^3 \\
\frac{dy}{dx} &= 4x^3 - 6x^2 \\
4x^3 - 6x^2 &= 0 \\
2x^3 - 3x^2 &= 0 \\
x^2 (2x-3) &= 0 \\
\boldsymbol{x} &= \mathbf{0}, \mathbf{\frac{3}{2}}
\end{aligned}
$$

A. When $x = 0$

$$y = 0^3 (0 - 2)$$
$$= 0$$

B. $\therefore \mathbf{(0, 0)}$

C. When $x = \frac{3}{2}$

$$y = \left(\frac{3}{2}\right)^3 \left(\frac{3}{2} - 2\right)$$
$$= -1.6875$$

D. $\therefore \left(\mathbf{\frac{3}{2}}, \mathbf{-1.6875}\right)$

iii. Find the values of f at the endpoints of the interval

$$y|_{x=-1} = (-1)^3 (-1 - 2)$$
$$= 3$$

$$y|_{x=3} = (3)^3 (3 - 2)$$
$$= 27$$

A. $\mathbf{(-1, 3), (4, 27)}$

iv. Conclusion

A. **Absolute minimum: -1.6875 or** $-\frac{27}{16}$

B. **Absolute maximum: 27**

20 Example (Mean Value Theorem)

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

1. Find $f'(x)$, then find the range of $f'(x)$ to find the region where it is increasing. The converse of the ranges would be the regions where it is decreasing

$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$Let\, f'(x) > 0$$
$$12x^3 - 12x^2 - 24x > 0$$
$$12x\left(x^2 - x - 2\right) > 0$$
$$12x(x - 2)(x + 1) > 0$$

$x > 0, x > 2, x > -1$are the regions where $f'(x)$is positive

(a) Find the regions where it is increasing, by using the method below. Multiply all the regions together, and check the signs. Note: $-x \cdot -x = +x$.

i. The reason is because if you:

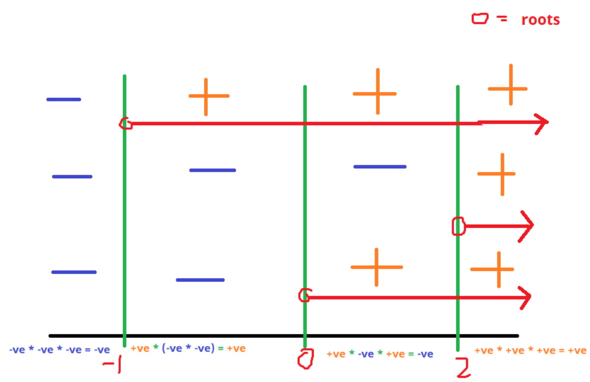
A. split/divide an equation into multiple parts

B. calculate for each part

C. then combine/multiply them together,

D. they should give the same result as calculating directly without splitting them.

ii. Use the tabling method

A. 

iii. Therefore:

A. f is increasing on $(-1, 0)$ and $(2, \infty)$

B. f is decreasing on $(-\infty, -1)$ and $(0, 2)$.

21 Example

$$y = x^4 - 4x^3$$

1. Find y' and y''

$$\begin{aligned} y' &= 4x^3 - 12x^2 \\ &= 4x^2 (x - 3) \\ 4x^2 (x - 3) &= 0 \end{aligned}$$

(a) Find Stationary points

$$x = 0, y = 0$$

$$x = 3, y = -27$$

i. Stationary points: $(0,0)$,$(3,-27)$

(b) Find inflection points

$$\begin{aligned} y'' &= 12x^3 - 24x \\ &= 12x(x-2) \\ &= 0 \end{aligned}$$

i. Inflection points: $(0,2)\,,(0,-16)$

2. Determine nature of stationary points

(a) Substitute $x = 0$ and $x = 3$ into y''

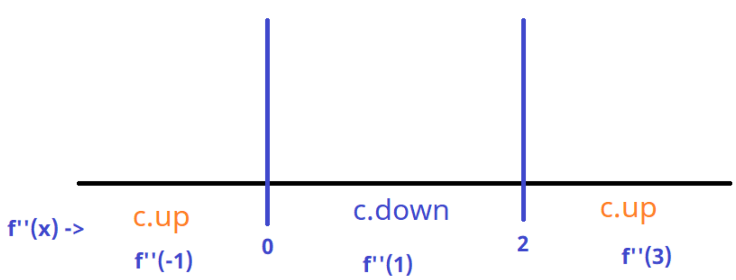
$$y'' = 12(0)^2 - 24(0) = 0$$

i. $\therefore (0,0)$ is an infleciton point

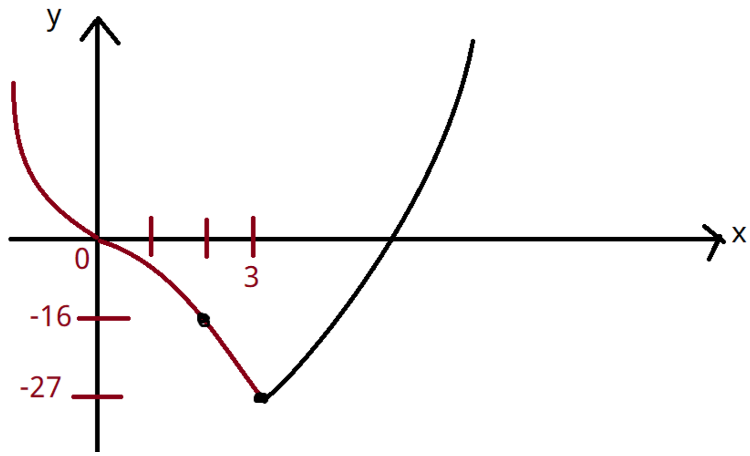
(b) Subsitute $x = 3$ into y''

i. $y''|_{x=3} = 36$, therefore $(3,-27)$ is a local minimum point

3. Determine concavity

(a) 

4. Graph the graph

(a) 

22 Example

1. Understand the problem.

   (a) Unknowns: Dimensions (width, height) of the field

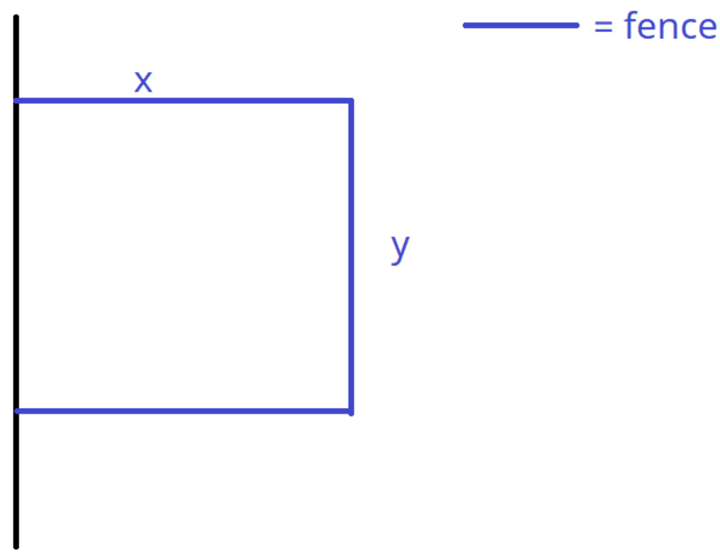
   (b) Given quantities: Total length $= 2400m$, side along the river do not need to be fenced

   (c) Given conditions:

      i. Dimensions of field with largest area,

      ii. $2x + y = 2400$

2. Draw a diagram

(a) 

3. Introduce notation

   (a) Assign symbol to quantity to be maximized/minimized.

      i. Let A be the area. (Above) x be length and y be width.

4. Identify unknown, write equation

$$A = xy$$

5. Find the relationship among the variables (if expressed as more than 1 variable)

$$\begin{aligned} 2x + y &= 2400 \\ y &= 2400 - 2x \end{aligned}$$

6. Find the absolute maximum of f

$$\begin{aligned} A &= x\left(2400 - 2x\right) \\ &= 2400x - 2x^2 \end{aligned}$$

$$\frac{dA}{dx} = -4x + 2400$$

   (a) Absolute maximum is when $\frac{dA}{dx} = 0$

$$\begin{aligned} -4x + 2400 &= 0 \\ 4x &= 2400 \\ x &= 600 \end{aligned}$$

(b) Find y

$$\begin{aligned} y|_{x=600} &= 2400 - 2\,(600) \\ &= 1200 \end{aligned}$$

(c) Conclusions

i. Dimensions of field with largest area = $600m * 1200m$

23 Example

1. Understand the problem.

   (a) Unknowns: Radius, height

   (b) Given quantities: Total volume = $\pi r^2 h = 1\ell/1000cm^3$

   (c) Given conditions: Minimize dimensions (cost)

2. Draw a diagram

3. Introduce notation

   (a) Assign symbol to quantity to be maximized/minimized.

      i. Let A be the area. r be radius and h be height.

4. Identify unknown, write equation

$$A = 2\pi r^2 + 2\pi rh$$

5. Find the relationship among the variables (if expressed as more than 1 variable)

$$\begin{aligned} \pi r^2 h &= 1000 \\ h &= \frac{1000}{\pi r^2} \end{aligned}$$

6. Find the absolute maximum of f

$$\begin{aligned} A &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r^2 + 2\pi r \left(\frac{1000}{\pi r^2}\right) \\ &= 2\pi r^2 + \frac{2000}{r} \\ \frac{dA}{dr} &= 4\pi r - 2000r^{-2} \end{aligned}$$

(a) Absolute minimum for r is when $\frac{dA}{dx} = 0$

$$\begin{aligned}
4\pi r - 2000r^{-2} &= 0 \\
4\pi r - \frac{2000}{r^2} &= 0 \\
4\pi r &= \frac{2000}{r^2} \\
4\pi r^3 &= 2000 \\
r &= \sqrt[3]{\frac{2000}{4\pi}} \\
&= \sqrt[3]{\frac{500}{\pi}}
\end{aligned}$$

(b) Find h when $r = \sqrt[3]{\frac{500}{\pi}}$

$$\begin{aligned}
h &= \frac{1000}{\pi r^2} \\
&= \frac{1000}{\pi \left(\frac{500}{\pi}\right)^{\frac{2}{3}}} \\
&= \left(\frac{4000}{\pi}\right)^{\frac{1}{3}} \\
&= 2r
\end{aligned}$$

(c) Conclusions

i. $r = \sqrt[3]{\frac{500}{\pi}}$, $h = 2\sqrt[3]{\frac{500}{\pi}}$

24 Example

1. Area of rectangle

$$24xy$$

2. Area of triangle

$$12x * 5x = 60x^2$$

3. Total area

$$A = 24xy + 60x^2$$

4. Total wire length = 240

$$\begin{aligned}
13x * 2 + y * 2 + 24x &= 240 \\
50x + 2y &= 240
\end{aligned}$$

$$
\begin{aligned}
2y &= 240 - 50x \\
y &= \frac{240 - 50x}{2} \\
&= 120 - 25x
\end{aligned}
$$

5. Substitute into area formula

$$
\begin{aligned}
A &= 24x\left(120 - 25x\right) + 60x^2 \\
&= 2880x - 600x^2 + 60x^2 \\
\boldsymbol{A} &\boldsymbol{= 2880x - 540x^2}
\end{aligned}
$$

6. x and y are at their maximum when $\frac{dA}{dx} = 0$

$$
\begin{aligned}
\frac{dA}{dx} &= 2880 - 1080x \\
2880 - 1080x &= 0 \\
2880 &= 1080x \\
x &= \frac{2880}{1080} \\
&= \frac{8}{3}
\end{aligned}
$$

7. Find y when $x = \frac{8}{3}$

$$
\begin{aligned}
y &= 120 - 25x \\
&= 120 - 25\left(\frac{8}{3}\right) \\
&= \frac{160}{3}
\end{aligned}
$$

   (a) The A is at their maximum when $x = \frac{8}{3}$ and $y = \frac{160}{3}$.

8. Find the maximum area, A

$$
\begin{aligned}
A|_{x=\frac{8}{3}} &= 2880\left(\frac{8}{3}\right) - 540\left(\frac{8}{3}\right)^2 \\
&= 3840cm^2
\end{aligned}
$$

25

1. Find the total revenue $R\left(x\right)$

$$
\begin{aligned}
R\left(x\right) &= p * x \\
&= \left(1000 - x\right)x \\
&= 1000x - x^2
\end{aligned}
$$

2. Find the total profit $P(x)$

$$\begin{aligned} P(x) &= R(x) - c(x) \\ &= 1000x - x^2 - (3000 + 20x) \\ &= 980x - x^2 - 3000 \end{aligned}$$

3. How many units to sell to maximize profit, what's the maximum profit.

$$\begin{aligned} \frac{dP}{dx} &= 0 \\ \frac{dP}{dx} &= 980 - 2x \\ 980 - 2x &= 0 \\ 2x &= 980 \\ x &= 490 units \end{aligned}$$

   (a) Maximum profit

$$\begin{aligned} P(490) &= 980(490) - (490)^2 - 3000 \\ &= \boldsymbol{RM237100} \end{aligned}$$

26

1. Notations
   (a) $Volume = V = 320$
   (b) $A = area$
   (c) $x = sides$
   (d) $y = height$
2. Minimize the cost
3. Find the volume

$$\begin{aligned} V &= x^2 y \\ x^2 y &= 320 \end{aligned}$$

4. Find the cost formula

$$\begin{aligned} C &= 15x^2 + 4 * 2.5xy + 10x^2 \\ &= 25x^2 + 10xy \end{aligned}$$

5. Find x in terms of y

$$\begin{aligned} x^2y &= 320 \\ y &= \frac{320}{x^2} \end{aligned}$$

6. Substitute into cost formula

$$\begin{aligned} C &= 25x^2 + 10x\left(\frac{320}{x^2}\right) \\ &= 25x^2 + \frac{3200}{x} \end{aligned}$$

7. Find $\frac{dC}{dx}$and the minimum cost when $\frac{dC}{dx} = 0$

$$\frac{dC}{dx} = 50x - \frac{3200}{x^2}$$

$$\begin{aligned} \frac{dC}{dx} &= 0 \\ 50x - \frac{3200}{x^2} &= 0 \\ 50x^3 - 3200 &= 0 \\ 50x^3 &= 3200 \\ x^3 &= \frac{3200}{50} \\ x &= \sqrt[3]{64} \\ x &= 4 \end{aligned}$$

8. Find y

$$\begin{aligned} y &= \frac{320}{x^2} \\ &= \frac{320}{4^2} \\ &= 20 \end{aligned}$$

9. Dimensions

$$4cm * 4cm * 20cm$$

27 L'Hospital' Rule

1. $\lim_{x\to 0} \frac{x}{\sin x}$

$$\begin{aligned}\lim_{x\to 0} \frac{x}{\sin x} &= \lim_{x\to 0} \frac{1}{\cos x} \\ &= \frac{1}{\cos 0} \\ &= 1\end{aligned}$$

2. $\lim_{x\to 1} \frac{x^3+x-2}{x^2-1}$

$$\begin{aligned}\lim_{x\to 1} \frac{x^3+x-2}{x^2-1} &= \lim_{x\to 1} \frac{\frac{d}{dx}\left[x^3+x-2\right]}{\frac{d}{dx}\left[x^2-1\right]} \\ &= \lim_{x\to 1} \frac{3x^2+1}{2x} \\ &= \frac{3\left(1\right)^2+1}{2\left(1\right)} \\ &= \frac{4}{2} \\ &= 2\end{aligned}$$

   (a) Or, the non-hospitalized way (its a pun)

$$\begin{aligned}\lim_{x\to 1} \frac{x^3+x-2}{x^2-1} &= \lim_{x\to 1} \frac{\left(x-1\right)\left(x^2+x+2\right)}{\left(x-1\right)\left(x+1\right)} \\ &= \lim_{x\to 1} \frac{\left(x^2+x+2\right)}{\left(x+1\right)} \\ &= \frac{1+1+2}{1+1} \\ &= \frac{4}{2} \\ &= 2\end{aligned}$$

28 Indeterminate form, going to hospital $\frac{\infty}{\infty}$

1. $\lim_{x\to \infty} \frac{x}{1-2x}$

$$\begin{aligned}\lim_{x\to \infty} \frac{x}{1-2x} &= \lim_{x\to \infty} \frac{\frac{d}{dx}\left[x\right]}{\frac{d}{dx}\left[1-2x\right]} \\ &= \lim_{x\to \infty} \frac{1}{-2} \\ &= -\frac{1}{2}\end{aligned}$$

2. $\lim_{x\to\infty} \frac{e^x}{x^n}$ where $n \epsilon Z^+$

$$\begin{aligned}\lim_{x\to\infty} \frac{e^x}{nx^{n-1}} &= \lim_{x\to\infty} \frac{e^x}{n!}\\ &= \infty\end{aligned}$$

3. $\lim_{x\to\infty} \frac{\sinh x}{e^x}$

$$\begin{aligned}\lim_{x\to\infty} \frac{\sinh x}{e^x} &= \lim_{x\to\infty} \frac{\cosh x}{e^x}\\ &= \lim_{x\to\infty} \frac{\sinh x}{e^x}\\ &= (Hospitalization\, failed)\end{aligned}$$

29 The $0 * \infty$ and $\infty - \infty$ Forms

1. $\lim_{x\to 0^+} x \ln x$ $(0 \cdot \infty\, \text{form})$

$$\begin{aligned}\lim_{x\to 0^+} x \ln x &= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \left(\frac{\infty}{\infty} form\right)\\ &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}\\ &= -\lim_{x\to 0^x} x^2\\ &= 0\end{aligned}$$

2. $\lim_{x\to -1^+} \left(\frac{2}{1-x^2} - \frac{1}{1+x}\right)$ $(\infty - \infty)$form

$$\begin{aligned}\lim_{x\to -1^+} \left(\frac{2}{1-x^2} - \frac{1}{1+x}\right) &= \lim_{x\to -1^+} \left(\frac{2-(1-x)}{1-x^2}\right)\\ &= \lim_{x\to -1^+} \frac{1+x}{1-x^2}\\ &= \lim_{x\to -1^+} \frac{1}{-2x}\\ &= \frac{1}{-2\,(-1)}\\ &= \frac{1}{2}\end{aligned}$$

30 Indeterminate Forms 0^0, 1^∞, ∞^0,0^∞

1. $\lim_{x\to 0^+} x^x$

(a) Let $y = x^x$

$$\begin{aligned}
\ln y &= x \ln x \\
\lim_{x\to 0^+} \ln y &= \lim_{x\to 0^+} x \ln x \\
&= 0 \\
\lim_{x\to 0^+} y &= e^0 \\
y &= 1 \\
x^x &= 1
\end{aligned}$$

2. $\lim_{x\to 0^+} x^{\frac{1}{x}}$

$$Let\, y = x^{\frac{1}{x}}$$

$$\begin{aligned}
\lim_{x\to 0} y &= \lim_{x\to 0} x^{\frac{1}{x}} \\
\lim_{x\to 0} \ln y &= \lim_{x\to 0} \frac{1}{x} \ln x \\
&= \lim_{x\to 0} \frac{\ln x}{x} \\
&= \frac{\infty}{0^+} \\
\ln y &= \frac{\infty}{0^+} \\
y &= \ln \infty \\
e^x &= \infty
\end{aligned}$$

31 Newton-Raphson Method

Notes: If question didn't ask until which degree, then do until satisfied

1. Find third aproximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$
2. Find second approximation

$$\begin{aligned}
f(x) &= x^3 - 2x - 5 \\
f'(x) &= 3x^2 - 2
\end{aligned}$$

$$\begin{aligned}
x^2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\
&= 2 - \frac{f(2)}{f'(2)} \\
&= 2.1
\end{aligned}$$

3. Find third approximation

$$\begin{aligned} x^3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ &= 2.1 - \frac{f(2.1)}{f'(2.1)} \\ \boldsymbol{x^3} &\boldsymbol{\approx 2.0946} \end{aligned}$$

32 Example

Find $\sqrt[6]{2}$ correct to 6 d.p.

1. Let $x = \sqrt[6]{2}$

$$\begin{aligned} x^6 &= 2 \\ x^6 - 2 &= 0 \end{aligned}$$

2. Let $f(x) = x^6 - 2$

$$f'(x) = 6x^5$$

3. Since we know the actual value of $\sqrt[6]{2}$ is somewhere close to 1, we choose $x_1 = 1$

4. Find correct to 6 d.p.

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x)} \\ &= 1 - \frac{f(1)}{f'(1)} \\ &\approx 1.16666667 \end{aligned}$$

$$\begin{aligned} x_3 &= 1.16666667 - \frac{f(1.16666667)}{f'(1.16666667)} \\ &\approx 1.1264437 \end{aligned}$$

$$\begin{aligned} x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} \\ &\approx 1.1224971 \end{aligned}$$

$$\begin{aligned} x_5 &= x_4 - \frac{f(x_4)}{f'(x_4)} \\ &\approx 1.1224621 \end{aligned}$$

$$\begin{aligned} x_6 &= x_5 - \frac{f(x_5)}{f'(x_5)} \\ &\approx 1.1224621 \end{aligned}$$

(a) Since x_5 and x_6 agree to six d.p., $\sqrt[6]{2} \approx 1.122462\,(6\,d.p.)$