Chapter 1: Random Variables and Probability Distributions

Random Variable

- A variable whose value is determined by the outcome of a random experiment.
- Can be discrete or continuous.

Discrete Random Variable

- A random variable that assumes countable values.
- Examples: 1. Number of cars sold
  2. Number of complaints received
  3. Number of shoes pairs a person own

Continuous Random Variable

- A random variable that can assume any value contained in one or more intervals.
- Examples: 1. Height of a person
  2. Time taken to complete an examination
  3. Weight of a baby

Example 1

Three cards are selected without replacement from a deck of 52 cards.
A random variable may be defined as

$$X = \text{number of aces obtained}$$

Then X can assume the values 0, 1, 2 or 3.
Since X can assume only 4 values, it is a discrete random variable.

Example 2

Two dice are tossed. Let the random variable, X be the sum of the spots appearing on the two dice. Then X is a discrete random variable taking values 2, 3, 4, … , 12.

Example 3

Get on a scale and weigh yourself. Let the random variable, X be defined as your weigh in kilogram. Then X could be any of the infinitely many values between 40kg and 90kg, i.e. $40 < X < 90$.
$\Rightarrow$ X is continuous random variable.

Discrete Probability Distribution

Definition: A probability distribution function for a discrete random variable is a correspondence which assigns probabilities to the values of the random variable. The probabilities of the values of the random variable is denoted by

$$f(x) = P[X = x]$$

Properties: 1. $0 \le P[X = x] \le 1$ 2. $\sum f(x) = \sum P[X = x] = 1$

Example 4

Each of the following tables lists certain values of *X* and their probabilities. Determine whether or not each table represents a valid probability distribution.

(a)

x	*f(x)*
0	0.08
1	0.11
2	0.39
3	0.27

(b)

x	*f(x)*
2	0.25
3	0.34
4	0.28
5	0.13

(c)

x	*f(x)*
7	0.7
8	0.5
9	–0.2

Example 5

Toss 2 fair dice. Let *X* denotes the sum of the spots on the 2 dice. Find the probability distribution of the r. v. *X*.

		Second Die					
		1	2	3	4	5	6
	1						
	2						
First	3						
Die	4						
	5						
	6						

Probability Distribution

x	2	3	4	5	6	7	8				
$P[X = x]$	1/36										

Example 6

A coin is tossed 3 times. Let *X* be the number of heads that come up in all 3 tosses of the coin. Find the probability distribution of *X*.

Example 7

The following table lists the probability distribution of the number of breakdowns per week, X for a machine based on past data.

Breakdowns per week, x	0	1	2	3
Probability, $P(X=x)$	0.15	0.20	0.35	0.30

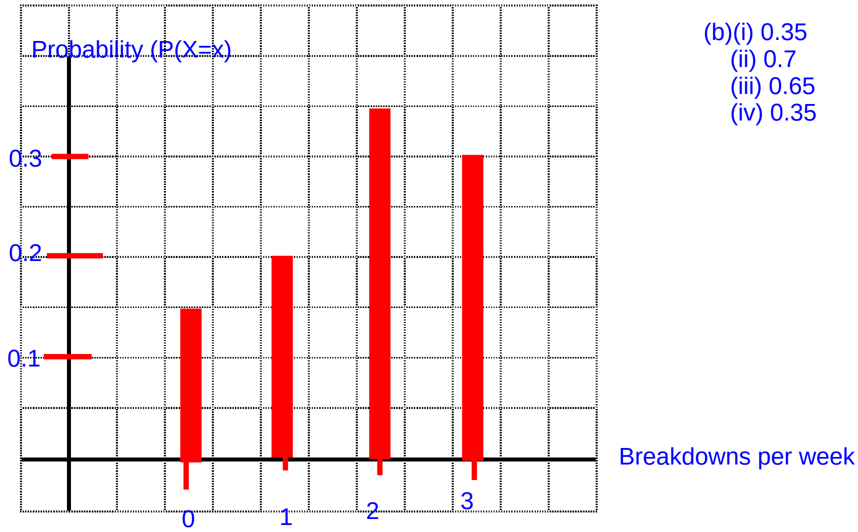
(a) Present this probability distribution graphically.

(b) Find the probability that the number of breakdowns for this machine during a given week is

(i) exactly 2 (ii) 0 to 2, both inclusive

(iii) more than 1 (iv) at most 1

(a)



(b)(i) 0.35
(ii) 0.7
(iii) 0.65
(iv) 0.35

Definition: The cumulative distribution $F(x)$ of a discrete random variable X with probability distribution $f(x)$ is given by

$$F(x) = P[X \le x] = \sum_{t \le x} f(t)$$

The r. v. X in Example 6 has a distribution function given by

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ 1/8 & , \quad 0 \le x < 1 \\ 4/8 & , \quad 1 \le x < 2 \\ 7/8 & , \quad 2 \le x < 3 \\ 1 & , \quad x \ge 3 \end{cases}$$

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Continuous Probability Distribution

- The probability distribution curve of a continuous random variable is also called as *probability density function* (p.d.f).
- The probability density function of a continuous r. v. *X* is such that the total area under the probability curve $f(x)$ is equal to 1. Thus the probability that a continuous r. v. falls in an interval, say between *a* and *b*, is equal to the area under the curve over the interval *a* to *b*.
- This probability is

$$P[a < X < b] = \int_a^b f(x)\,dx$$

Graph of the density function of a continuous r. v. X:

Properties

1. $f(x) \geq 0$ for all values of *X*
2. Total area under $f(x)$ is 1, i.e. $\int_{-\infty}^{\infty} f(x)\,dx = 1$
3. $P[a < X < b] = \int_a^b f(x)\,dx$

Note: $P[a < X < b] = P[a < X \leq b] = P[a \leq X < b] = P[a \leq X \leq b]$

<u>Example 8</u>

A continuous random variable X that can assume values between $x = 2$ and $x = 4$ has a density function given by

$$f(x) = \frac{x+1}{8}, \quad 2 < x < 4$$

(a) Show that $P(2 < X < 4) = 1$ (b) Find $P(X < 3.5)$

(c) Find $P[2.4 < X < 3.5]$

(a)

Definition: The cumulative distribution $F(x)$ of a continuous r. v. *X* with density function $f(x)$ is given by

$$F(x) = P[X \le x] = \int_{-\infty}^{x} f(t)\ dt$$

Remark: From the above definition:

$$P[a < X < b] = \int_{a}^{b} f(x)\,dx = F(b) - F(a)$$

and

$$f(x) = \frac{d}{dx}F(x) \text{ if the derivative exists}$$

<u>Example 9</u>

$$f(x) = \begin{cases} x^2/3 & , \quad -1 < x < 2 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Verify that $f(x)$ is a p.d.f. and find $F(x)$ then evaluate $P[0 < X \le 1]$.

<u>Example 10</u>

X is a continuous r. v. with the probability density function $f(x) = 3x^2$, $0 < x < 1$, zero elsewhere. Find

(i) $P[0 < X < 1/2]$

(ii) $P[-2 < X < 1/2]$

(iii) $P[1/4 < X < 1/2]$

(iv) $P[1/2 < X < 1.6]$

The Mean and Variance of a Probability Distribution

Definition: The mean or mathematical expectation or expected value of a random variable *X* is defined by μ .

★ If *X* is discrete, then the mean of *X*, μ is defined as

$$\mu = E[X] = \sum x\, f(x) \quad \text{for all values of } X$$

★ If *X* is continuous, then the mean of *X*, μ is defined as

$$\mu = E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

Example 11

Find the expected value of *X*, where *X* represents the outcome when a die is tossed.

Example 12

The following table gives the probabilities that customers will give tips of varying amounts of money to a waiter.

Amount of Money (cents), x	30	35	40	45	50	55	60
Probability, $f(x) = P[X = x]$	0.45	0.25	0.12	0.08	0.05	0.03	0.02

Find the expected tips of the waiter.

Example 13

In a gambling game a man is paid RM5 if he gets all heads or all tails when 3 coins are tossed, otherwise he will pay out RM3. What is his expected gain?

Example 14

Let X be a continuous r. v. having probability density function, $f(x) = 3x^2$, $0 < x < 1$. Find the mean of this distribution.

Definition:

The variance of a probability distribution, denoted by σ^2 is defined to be

★ $\sigma^2 = E[(X-\mu)^2] = \sum (x-\mu)^2 f(x)$ if X is discrete

where Σ sum over all values of X

★ $\sigma^2 = E[(X-\mu)^2] = \int (x-\mu)^2 f(x)\,dx$ if X is continuous

The standard deviation of a probability distribution is $\sigma = \sqrt{\text{variance}}$.

Computing formula of σ^2:

$\sigma^2 = E(X^2) - \mu^2 = \sum x^2 f(x) - \mu^2$ if X is discrete,

$\sigma^2 = E(X^2) - \mu^2 = \int x^2 f(x)\,dx - \mu^2$ if X is continuous.

Example 15

The random variable X, representing the number of defective missiles when 3 missiles are fired, has the following probability distribution:

x	0	1	2	3
$P[X=x]$	0.51	0.38	0.10	0.01

What is the variance of this distribution?

Example 16

Refer to Example 12, find variance and standard deviation of *X*.

Example 17

X is a continuous r. v. with p.d.f. $f(x) = 3x^2$, $0 < x < 1$. Find the variance of X.

Summary

Description	Discrete	Continuous
Random Variable	Assume countable values	Assume any value contained in one or more intervals
Probability distribution	$f(x) = P[X = x]$	$P[a < X < b] = \int_a^b f(x)\,dx$
Cumulative distribution	$F(x) = P[X \le x] = \sum_{t \le x} f(t)$	$F(x) = P[X \le x] = \int_{-\infty}^{x} f(t)\,dt$
Mean	$\mu = E[X] = \sum x\,f(x)$	$\mu = E[X] = \int_{-\infty}^{\infty} x\,f(x)\,dx$
Variance	$\sigma^2 = E(X^2) - \mu^2$ where $E(X^2) = \sum x^2 f(x)$	$\sigma^2 = E(X^2) - \mu^2$ where $E(X^2) = \int x^2 f(x)\,dx$

Example 18

Two fair coins are tossed and X = number of heads observed. Find

(i) the probability distribution of X.

(ii) the mean and standard deviation of X.

(iii) the probability of obtaining at least one head.

Example 19

The number of automobiles rented per hour, *X*, at the Rent-A-Car counter at KLIA has the probability distribution given in the following table.

Automobiles Rented, *x*	0	1	2	3	4
Probability, *f(x)*	0.10	0.25	0.40	0.20	0.05

(i) Why is this a probability distribution?
(ii) Find the probability that there will be at most 2 cars rented in 1 hour.
(iii) Find the probability that there will be at least 3 cars rented in an hour?

Example 20

John sells Proton new cars. He usually sells the largest number of cars on Saturday. He has established the following probability distribution for the number of cars he expects to sell on a particular Saturday.

Number of cars sold, *x*	0	1	2	3	4
Probability, $P[X = x]$	0.1	0.2	0.3	0.3	0.1

(i) What type of distribution is this?
(ii) On a particular Saturday, how many cars would John expect to sell? Interpret the result.
(iii) What is the variance of this distribution?

Example 21

In a lottery conducted to benefit the local fire company, 8000 tickets are to be sold at RM 15 each. The prize is a RM 36,000 automobile. If you purchase two tickets, what is your expected gain?

Example 22

Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} 20000/x^3 & , \quad x > 100 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Find the expected life of this type of electronic device.

Example 23

A hospital receives a pharmaceutical delivery each morning at a time that varies between 7:00 and 8:00 AM with a probability density function described by $f(x) = 1$, 7.00 < x < 8.00.

(i) Find the probability that the delivery on a given morning will occur between 7:15 and 7:30 AM.

(ii) What is the expected time of delivery?

(iii) Find the probability that the time of delivery will be within 1 standard deviation of the expected time. That is, within the interval $\mu - \sigma \le X \le \mu + \sigma$.