Chapter 4: Relations & Digraphs

January 15, 2020

1 Product Sets and Partitions

1. Ordered pair are listing in the form of (a, b)
   (a) It is a **sequence** of **length 2**
   (b) Commonly used to represent coordinates of a point in a plane.
      i. $A = \mathbb{R}$ and $B = \mathbb{R}$ (note: $\mathbb{R}$ represents all real numbers)
      ii. $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$
      iii. $\mathbb{R}^2$ represents set of all points in plane
2. Cartesian product: "$\times$" sign
   (a) Represents set of all ordered pairs (aka tuples) (a, b), where $a \in A$ and $b \in B$.
   (b) $A \times B = \{(a, b) \,|a \in A \cup b \in B\}$
      i. Basically, the cartesian product of A and B, is the set of ordered pairs (a, b), where a is an element of the set A, and b is an element of the set B.
      ii. Table form:
         A.

	B	b_1	b_2
A			
a_1		(a_1, b_1)	(a_1, b_2)
a_2		(a_2, b_1)	(a_2, b_2)

3. **E.g. 1**
   (a) Let $A = \{1, 2, 3\}$ and $B = \{r, s\}$, find $A \times B$ and $B \times A$.
   (b) $A \times B = \{(1, r), (1, s), (2, r), (2, s), (3, r), (3, s)\}$
4. **Theorem**: For any 2 finite, non-empty sets, the distributive law applies on modulo.
$$|A \times B| = |A| \times |B|$$
5. Cartesian products can also span more than 2 sets. In this case,
$$A_1 \times A_2 \times ... \times A_m = \{(a_1, a_2, ..., a_m) \,|a_1 \in A_i, i = 1, 2, .., m\}$$

(a) This reads as the cartesian product of the non-empty sets $A_1, ...A_m$ is the set of all ordered m-tuples $(a_1, a_2, ...a_m)$ where a_i is an element of A_i, and i is a counter going from 1 to m.

6. The total elements/tuples in a cartesian product, is the product of the size of each set as per follows:

   Elements in cartesian product $= eleInSetA * eleInSetB * ... * eleInSetM$

   (a) **Category:** Each element in cartesian product (E.g.: (a_1, b_1, c_1))
   (b) **Classification scheme:** All the categories (its a number, mkay?)

7. **Partition**

   (a) AKA **quotient set**
   (b) Consists of **blocks** or **cells**
      i. Every element in the set being partitioned, must belong to one block/cell
      ii. No block/cell overlap another $A_1 \cap A_2 = \varnothing$
   (c) Each block or cell is a subset derived from another set by an **equivalence relation** (quote: Quotient Set - Art of Problem Solving)
   (d) **Equivalence relation:**
      i. A **relation** between elements of a set, which is reflexive, symmetric, and transitive.
      ii. **Relation (Symbol: $\sim$ or R):** The way in which 2 (or more elements) are connected. For example: $a \sim b$ if $a - b \in \mathbb{Z}$. a and b are related if their condition, $a - b$ is an integer, is satisfied. The condition is the "way".
      iii. **Reflexive:** $a \sim a$. a is related to itself.
      iv. **Symmetric:** if $a \sim b$, then $b \sim a$.
      v. **Transitive:** if $a \sim b$, and $b \sim c$, then $a \sim c$.
   (e) Let S be a set, let R be an equivalence relation.
      i. The quotient of S by R (aka S/R) is the set of **equivalence classes** of S with respect to R.
         A. **Equivalence classes**: A subset S which includes all elements equivalent to each other with respect to a certain relation R. Equivalence Class: Definition - DataScienceCentral
      ii. Partitions can be considered as a $\wp(S)$, or the **power set** of S
         A. **Power set** of S: The set of all subsets of S.

2 Relations and Digraphs

1. If we have a set $A \times B$ (cartesian product of set A and set B)
   (a) A relation of R from A to B is a subset of $A \times B$
2. If $R \subseteq A \times B$ and $(a, b) \in R$, then aRb
   (a) If R is a subset of $A \times B$ and the tuple (a, b) is related. Then a is related to b by R.
   (b) Else, $a \not R b$.
3. Example, let $A = \{1, 2, 3\}$ and $B = \{r, s\}$
   (a) $R = \{(1, r), (2, s)\}$ is a relation from A to B. It does not contain everything, but it satisfies point 2 above.
4. To define relations, usually we write: aRb iff (a condition involving a and b)
   (a) E.g.: aRb iff a divides b. Then,
      i. $(a, b) = (4, 12)$

$$\begin{aligned} a|b &= \frac{b}{a} \\ &= \frac{12}{4} \\ &= 3 \in \mathbb{Z} \end{aligned}$$

         A. $\therefore 4R12$

      ii. $(a, b) = (5, 7)$

$$\begin{aligned} a|b &= \frac{b}{a} \\ &= \frac{7}{5} \notin \mathbb{Z} \end{aligned}$$

         A. $\therefore 5 \not R 7$

5. Finding R in circles.
   (a) Say if you are given $xRy \iff \frac{x^2}{4} + \frac{y^2}{9} = 1$
   (b) To find set R, simply:
      i. Substitute $x = 0$ to find all possible values of y on the x-axis

$$\begin{aligned} \frac{y^2}{9} &= 1 \\ y &= \pm 3 \end{aligned}$$

ii. Substitute $y = 0$ to find all possible values of x on the y-axis

$$\frac{x^2}{4} = 1$$
$$x = \pm 2$$

iii. Therefore, we can say R is the set of tuple (x, y) where:

$$-2 \leq x \leq 2$$
$$-3 \leq y \leq 3$$

iv. If you want to find the exact values of infinite possible combinations, go ahead, but I'm stopping here.

6. Finding R using set notation
   (a) Draw out the sets in ellipses and connect the nodes if they satisfy the condition. The set R is simply the tuples of connected nodes.
7. Finding R using tables
   (a) List down all values in tables. The set R is simply each (x, y) combination which satisfies the condition.
8. Domain & Range of R
   (a) Say if we have a relation set R, such that R is a subset of the cartesian product between set A and set B, or, $R \subseteq A \times B$.
      i. **Domain** of R, or $Dom\,(R)$ is set of elements in set A inside set R.
      ii. **Range** of R, or $Ran\,(R)$ is set of elements in set B inside set R.
   (b) If a group elements y inside a set B , is related to a group of elements x inside set A, according to the condition we want, R. Then we can say,
      i. $R\,(x) = \{y \in B | xRy\}$, or,
      ii. The R-relative set of x is the set of all y in B such that x is R-related to y.
   (c) Example
      i. Let $A = \{1, 2, 3\}$, $B = \{r, s\}$, and $R = \{(1, r), (2, s), (3, r)\}$ is a relation from A to B. Determine Dom (R) and Ran (R).
         A. Dom$(R) = \{1, 2, 3\}$
         B. Ran$(R) = \{r, s\}$
9. Theorems
   (a) **Theorem 1**:
      i. Let R be relation from A to B

ii. Let A_1 and A_2be subsets of A

iii. $A_1 \subseteq A_2 \implies R(A_1) \subseteq R(A_2)$.

A. If set A_1 is a subset of set A_2, then R-related set of A_1 should also be a subset of R-related set of A_2.

iv. $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$

A. To simplify, $R(A_1 + A_2) = R(A_1) + R(A_2)$

B. This reads as, if the subsets A_1 and A_2were combined, the R-related set should be the same, as if they split off, then R-related, then combined.

v. $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$

A. This reads as, if we first find the intersection between subset A_1 and subset A_2, then find their R-related set, and call that relation set B.

B. Then, we take each subset, find their R-related set, and then find the intersection, then call that relation set C.

C. Relation set C, should be a subset (or an equal set) of relation set B.

D. The reason being when we find an intersection first, then we have less elements on hand, to match with the R-relation. Therefore, it must be contain an equal or less elements in relation set C.

E. **Note**: $R(A_1 \cap A_2) \neq R(A_1) \cap R(A_2)$, it's not always wrong, but generally wrong.

(b) **Theorem 2**:

i. Let R and S be relations from A to B.

ii. If $R(a) = S(a)$ for all a in A, then $R = S$.

10. Matrix of a Relation

(a) Let A be a set containing $\{a_1, a_2, ..., a_m\}$ and B be a set containing $\{b_1, b_2, ..., b_n\}$. Both of them are finite. So set A has m elements, set B has n elements.

(b) Let R be a relation set from A to B.

(c) We can represent R:

i. First by taking the cartesian product of $m \times n$,

ii. Then creating a matrix,$M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if} (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

iii. The matrix M_R is called matrix of R, easy to check properties.

(d) Example:

i. $A = \{1, 2\}$

ii. $B = \{r, s\}$

iii. $R = \{(1, r), (2, s), (2, r)\}$

iv. M_R

B	r	s
A		
1	1	0
2	1	1

11. Digraphs of Relation

(a) **Digraph**: Directed graph. Geometrical representation of R relation set.

(b) Step:

i. Draw a **vertex**: a small circle, each with label according to corresponding element of the set A.

ii. Draw an **edge:** An arrow, from vertex a_i to vertex a_j iff $a_i R a_j$

(c) **Vertices**: same number as elements in R

(d) **Edges**: same number as pairs in R

(e) **In-degree:** Number of edges going "in" a vertex

i. $b \in A$ such that $(b, a) \in R$

(f) **Out-degree:** Number of edges going "out" of a vertex

i. $b \in A$ such that $(a, b) \in R$

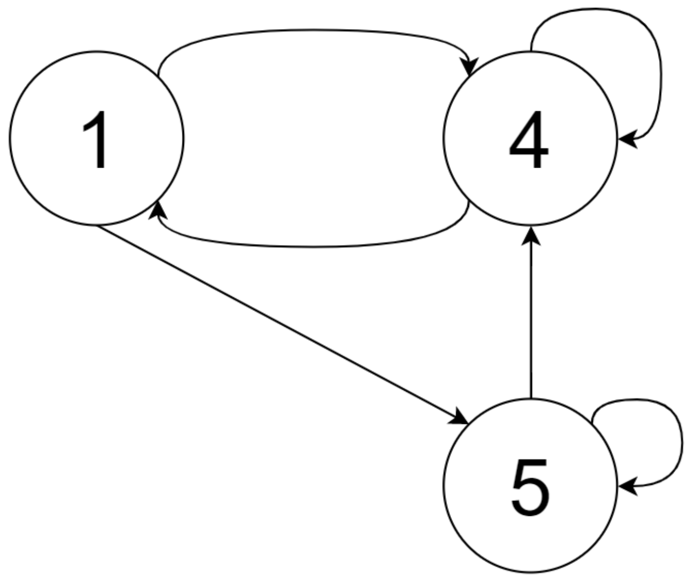
(g) Sum of all in-degrees always same as sum of out-degrees

(h) **Loop**: Edge from vertex a to a. $(a, a) \in R$

(i) If B is a subset of A ($B \subseteq A$), and R is a relation on set A. R to B is restricted to $R \cap B \times B$.

i. To put it simply, if you have an R relation, from A to B. Then R must be restricted to only the cross product of B (AKA all the possible tuples in B).

(j) Examples of a digraph:

i. 

3 Paths in Relations and Digraphs

1. **Path** (of length n) is a finite sequence of elements, starting from a certain element a, ending at another b element. When laid out, it looks like:

$$aRx_1, x_1Rx_2, ...x_{n-1}Rb$$

   (a) **Length** is the number of edges in path

      i. Length n involves $n+1$ elements (might not be distinct)

2. **Cycle:** path that ends at the starting vertex

3. A relation, R^n on set A

   (a) xR^ny means that there is a path of length n from x to y in R.

   (b) **Connectivity relation** for R:

      i. $xR^\infty y$ means there is **some** path from x to y.

4. Cool example:

   (a) If aR^nb, where a is John , b is Jane, and R is the relation of mutual acquaintance.

      i. aR^2b means that John and Jane have 1 acquaintance in common.

      ii. $aR^\infty b$ means that they are a chain of acquaintances, where 1 acquaintance in chain knows John and 1 knows Jane (they can be the same).

5. Computing path using matrices
    (a) When $|R|$ is large, we use M_R to compute R^2 and R^∞
6. Theorem Time!
    (a) **Theorem 1**:
        i. Let R be a relation on finite set $A = \{a_1, a_2, ..., a_n\}$.
        ii. Let M_R be the $n \times n$ matrix, representing R. (Basically, the cartesian product of the elements related by R)
        iii. To put it simply, to get the "R-related by 2 vertices away" elements (and above), we will do a regular matrix multiplication. Because we are only checking "present" (1) or "absent" (0) like boolean, and not "how many" (0,1,2,...), we must use a special symbol to denote this operation. In this case, we use the o-dot, $\bigodot$ sign. This is EXTREMELY misleading because the same sign is also used for Hadamard product, but we are not performing a Hadamard product, we just "borrow" the sign. Sorry, Hadamard fellow...
        iv. $M_{R^2} = M_R \bigodot M_R$ or $(M_R)^2_{\bigodot}$.
        v. Similarly, $M_{R^3} = M_R \bigodot (M_R \bigodot M_R) = (M_R)^3_{\bigodot}$.
    (b) **Theorem 2: (CONTINUE HERE, pg 60)**
        i. For $n \geq 2$ and R is a relation on a finaite set A,
            A. $M_{R^n} = M_R \bigodot M_R \bigodot ... \bigodot M_R$ (n factors)
            B. $M_{R^\infty} = M_R \vee (M_R)^2_{\bigodot} \vee (M_R)^3_{\bigodot} \vee ...$
7. R^*, **reachability relation** of R.
    (a) xR^*y means that $x = y$ or $xR^\infty y$
        i. Either x is the same as y, OR,
        ii. There is some path connecting x and y
    (b) $M_{R^*} = M_{R^\infty} \vee I_n$,where I_n is the $n \times n$ identity matrix. Thus,
        i. $M_{R^*} = I_n \vee M_R \vee (M_R)^2_{\bigodot} \vee (M_R)^3_{\bigodot} \vee ...$
8. **Composition of paths**
    (a) If we have path $A_1 : a, ..., b$, and another path, $A_2 : b, ..., c$.
    (b) If both of them are from the same relation, R.
    (c) The composition of them should be $A_2 \circ A_1$. Yes, I know, it is backwards. It reads as "a path to A_2 from A_1"
    (d) A composition can intersect each other, but the element in the TAIL of one must form the HEAD of another.

4 Properties of Relations

4.1 Reflexivity

1. Note: The set arriving from applying the equality relation, Δ on A, is called the **equality set.** In here, we use Δ to represent the equality set.

Reflexive	Differenc
$(a, a) \in R$ for all $a \in R$	Symbolic Me
the tuple element, (a, a), exists in the relation set for every element a inside the relation set.	Literal mea
every element is related to itself	TL;DR
All 1's on main diagonal	Matrix
Have length-1 loops at every vertex.	Digraph
$Dom\,(R) = Ran\,(R) = A$	Domain & R
Reflexive $\iff$ $\Delta \subseteq R$	Equality se

1. **Not reflexive** if does not satisfy **reflexivity**
2. **Not irreflexive** if does not satisfy **irreflexivity**
3. Not reflexive $\neq$ irreflexive

4.2 Symmetricity (yes, that's a real word)

Symmetric	Differences	
For all (a, b) in R, $aRb \iff bRa$	Symbolic Meaning	
If one element is related to another, then the vice versa must be true	TL;DR	**If o**
If $m_{ij} = 1$, then $m_{ji} = 1$, same for 0	Matrix	
Edges must be bidirectional, "two-way-street"	Digraph	

1. **Asymmetric**: A graph that only have one way streets
2. Note, in some questions regarding symmetricity, you might see the following:
   (a) $a \mid b$ but sometimes $b \nmid a$
   (b) The "(vert)ical line", or "pipe" or just "(mid)line", can mean:
      i. "such that"
      ii. "divides"
      iii. or "pipe"
   (c) In this context, its "divides".

4.3 Transitivity in Relations

1. A subset from applying relation R on set A is transitive if whenever aRb, and bRc, then aRc.

2. **Matrix** for transitivity: If $m_{ij} = 1$ and $m_{jk} = 1$, then $m_{ik} = 1$

   (a) Steps:

      i. Find $(M_R)^2_{\bigodot}$ (the cartesian product. $M_R \times M_R$).

      ii. If m_{ik} $(M_R)^2_{\bigodot}$ is 1, then same thing should happen inM_R

   (b) To understand why (source: Matrix transitivity - Stackexchange),

      i. First note that M_R tells you how many 'one-step paths' in $\{1, 2, 3\}$ that are in the relation set R.

      ii. So for example, we have

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

         A. M_R shows that we have 3 'one-step' paths, which are $\{(1, 2), (2, 2), (2, 3)\}$

      iii. So by multiplying M_R by M_R, we are finding how many '2-step' paths in M_R, so we have:

$$\left(M_R^2\right)_{\odot} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 \end{bmatrix}$$

$$\left(M_R^2\right)_{\odot} = \begin{bmatrix} 0 & \mathbf{1} & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

         A. If you didn't notice, lets look at the first row and column in M_R, in the 2nd cell inside $\left(M_R^2\right)_{\odot}$we multiplied the matrix coordinates as per follows:

$$m_{ij} * m_{jk}$$

         B. Essentially, we did

$$\begin{aligned} m_{00} * m_{10} + m_{10} * m_{11} + m_{20} * m_{12} &= (0, 0) * (1, 0) + (1, 0) * (1, 1) + (2, 0) * (1, 2) \\ &= (0 * 1) + (1 * 1) + (0 * 1) \\ &= 1 \end{aligned}$$

         C. So, if we have a 1 in a column inside M_R^2, then that means there are "at least" a 2-link that involves $m_{ij} * m_{jk}$. Thus, that column, m_{ik} in M_R must also have "at least" a 1-link, to satisfy $m_{ik} = m_{ij} * m_{jk}$ and be **transitive**.

         D. In many cases, we only write 1 even if there are more links in same cell. This is because additional links does not concern us as we only want to know if it satisfies $m_{ik} = m_{ij} * m_{jk}$.

3. **Digraph** for transitivity

(a) If there's an edge directed from a to b, and one from b to c, then there MUST be an edge a to c.

(b) **Theorem 1:** If there is a path length > 1 from a to b, then there must be a path of length 1 from a to b.

  i. $R^n \subseteq R$ for all $n \geq 1$.

(c) **Theorem 2**: Let R be a relation on set A. Then,

  i. **Reflexivity** of r: $a \in R(a)$ for all a in A

  ii. **Symmetry** of R: $a \in R(b)$ only if $b \in R(a)$

  iii. **Transitivity** of R: if $b \in R(a)$, and $c \in R(b)$, then $c \in R(a)$.

5 Equivalence Relations

1. **Equivalence relation:** relation that is reflexive, symmetric, and transitive.

   (a) Why are they called as so? Check out the reference [4]

     i. Clearly some object has to have the same value as itself; thus any equivalence relation fulfils aRa for all a (reflexivity).

     ii. Just as clearly, if a has the same value as b, then b has the same value as a. That is $aRb \implies bRa$ (symmetry).

     iii. And of course, if a has the same value as b, and b has the same value as c, then a has the same value as c, that is $a \sim b \wedge b \sim c \implies a \sim c$ (transitivity).

6 Computer Representation

6.1 Matrix representation

1. Represent relations in matrices inside computers

2. Represented as 2-dimensional array, such as: $MAT[0,0] = 0, MAT[0,1] = 1, \ldots$

6.2 Linked Lists

1. Each edge represented by two arrays:

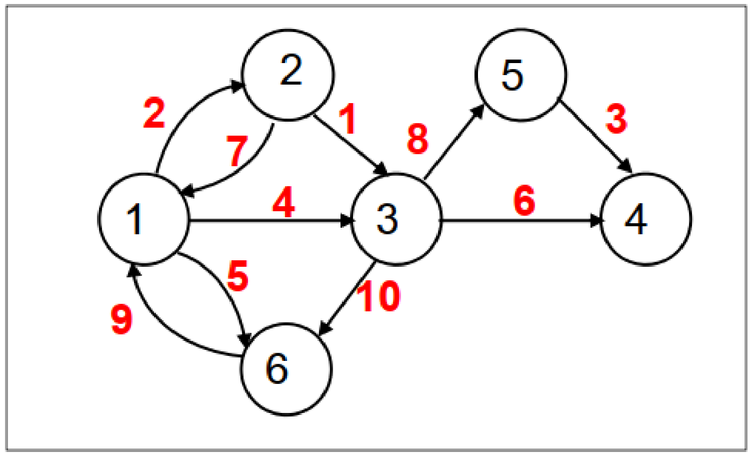
   (a) **TAIL**, give beginning (or tail) of the arrow

   (b) **HEAD**, give end (or head) of the arrow

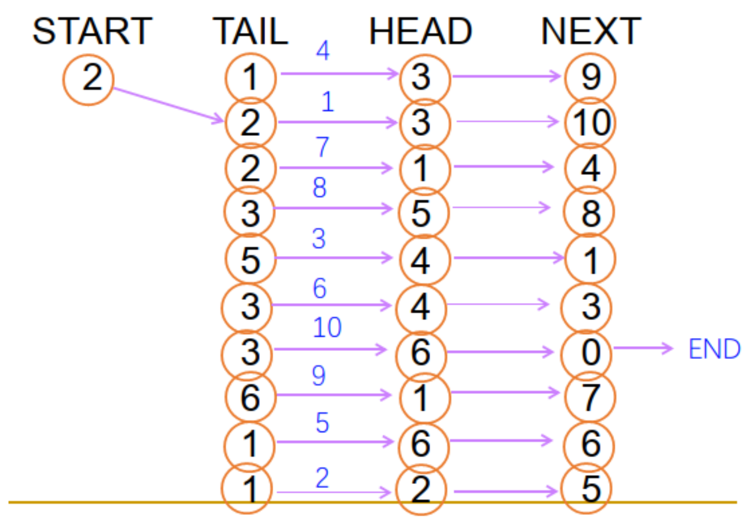
2. To combine all edge data into linked list, we use a **NEXT** array to indicate the next edge in path.

3. Very pretty example

(a) Image



(b) Linked List



i.

ii. Steps:

A. Mark each edges on the digraph. Order doesn't matter as long as each is unique.

B. List down all the edges in ordered pairs in the $TAIL$ and $HEAD$ array. Order doesn't matter

C. Pick a starting $TAIL$ node, and join it with the same index to the $START$ node.

D. Write down all the "next edge" **indexes** inside the $NEXT$ array. The total count should be the same as the number of elements inside the $TAIL$ and $HEAD$ array. Ideally you want to finish one node before moving to another, but order doesn't matter (you can jump from one edge to another completely unrelated edge). For the last one, use '0' to mark END.

(c) As you can see, there are more than 1 ways to start and end the linked list. However, their "in-between" contents don't change a lot.

6.3 "Better" Linked-List

1. Improve efficiency, particularly when we want to find the edges that begin/end with a certain vertex. (Example, I challenge you to find edges beginning with vertex 6 in 3 seconds)

2. We add a $VERT$ linear array

   (a) For each "starting" $TAIL$ vertex, I, their index is $VERT[I]$, where I is the offset.

   (b) $VERT$ contains pointers like $NEXT$.

   (c) For each vertex I, we using $VERT$ to "start", and use "NEXT" to "link together" edges leaving I.

   (d) The last vertex of a chain leads to 0 in $NEXT$.

   (e) In a sense, this linked list contains multiple links compared to the single link in the previous one.

   (f) There are many possible solutions, therefore, as long as you can join all, even if more/less links, it doesn't matter.

   (g) Steps

      i. Mark each edges on the digraph. Order doesn't matter as long as each is unique.

      ii. List down all the edges in ordered pairs in the $TAIL$ and $HEAD$ array. Order doesn't matter. Join them together horizontally.

      iii. Identify the "chains" in the diagram

      iv. For each chain:

         A. Create an entry in $VERT$ to point to the correct **index** where the vertex resides in $TAIL$

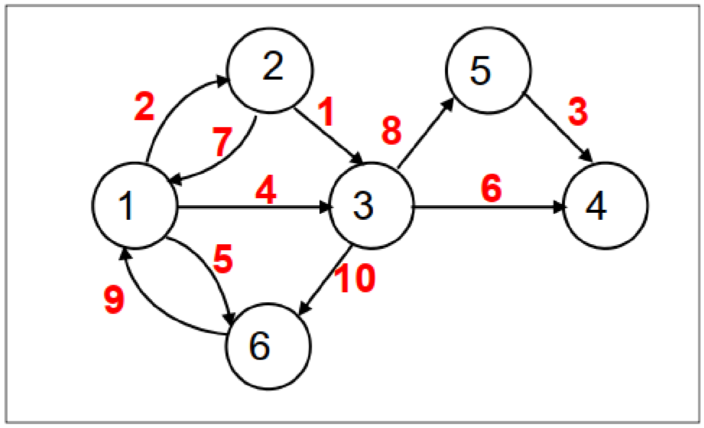
         B. Create an entry in $NEXT$ **directly beside** the entry, directing towards the next **index** inside the array where the next chain resides.

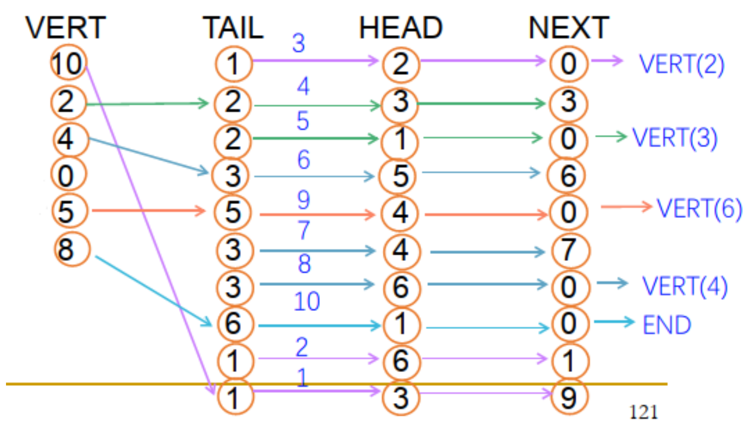
         C. Repeat point A-B, until you reach the end of the chain.

         D. Once so, repeat B, but put 0 as $NEXT$ index

      v. Retrace it to check

   (h) One solution

i. 

ii. 

7 Operations on Relations

7.1 Complementary Relation, $\bar{R}$

1. Complement of R, or relation.
2. Expression: $a\bar{R}b$ iff $a \not{R} b$
3. **Matrix**, $M_{\bar{R}}$: Obtained from M_R, interchange every 1 and 0 in M_R.

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$M_{\bar{R}} = \overline{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}$$

$$M_{\bar{R}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

7.2 Intersection Relation, $R \cap S$

1. R and S are both relations. aRb AND aSb
2. **Expression**: $R \cap S = aRb \wedge aSb$
3. **Matrix:** $M_{R \cap S} = M_R \wedge M_S$

$$\begin{aligned}
M_R &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\
M_S &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
M_{R \cap S} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \wedge 0 & 0 \wedge 1 \\ 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} \\
M_{R \cap S} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\end{aligned}$$

7.3 Union Relation, $R \cup S$

1. aRb OR aSb
2. **Expression:** $R \cup S = aRb \vee aSb$
3. **Matrix:** $M_{R \cup S} = M_R \vee M_S$

7.4 Inverse Relation, R^{-1}

1. Relation from set B to set A (reverse order of R)
2. **Expression:** $bR^{-1}a$ iff aRb
   (a) $\left(R^{-1}\right)^{-1} = R$
   (b) $Dom\left(R^{-1}\right) = Ran\left(R\right)$ and vice versa.
3. **Matrix:** $M_{R^{-1}} = \left(M_R\right)^T$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

7.5 Equality & Universal Relation

1. Let $A = R$. Let R be relation "$\leq$" on A and S be relation $\geq A$.
2. Complement of R: $>$
3. Complement of S: $<$

4. $R^{-1} = S, S^{-1} = R$

5. **Equality Relation:** $R \cap S$

6. **Universal Relation:** Any a related to any b

7.6 Theorem 1: Relationship between relations

1. R and S are relations from A to B.

2. If $R \subseteq S$, then $R^{-1} \subseteq S^{-1}$: If relation R is a subset of relation S, then their inverses must also be the same. (WHAT"S THIS!)

3. If $R \subseteq S$, then $\bar{S} \subseteq \bar{R}$: If relation R is a subset of relation S, then their complement must also be the same. (WHAT"S THIS!)

4. $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$, $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$: The inverse ("route-backs"), of the intersection between the set arising from the relation R and relation S must be the same as the intersection of the set arising from inverses ("route-backs") of R and S . The same applies for unions.

5. $\overline{R \cap S} = \bar{R} \cup \bar{S}$, $\overline{R \cup S} = \bar{R} \cap \bar{S}$

   (a) If you don't have R AND S, means you don't have R, or you don't have S. The opposite must be true too. (Note: DeMorgan's Law apply here)

7.7 Theorem 2: Reflexivity of relations

1. If R is reflexive, so is R^{-1}.

2. If R and S are reflexive, so are $R \cap S$ and $R \cup S$.

3. R is reflexive iff $\bar{R}$ is irreflexive. R is reflexive if and only if its complement is irreflexive.

7.8 Theorem 3: Symmetry, Antisymmetry, and Asymmetry of R

1. R is symmetric iff $R = R^{-1}$. R is symmetric if and only if it is the same as its inverse.

2. R is antisymmetric iff $R \cap R^{-1} \subseteq \Delta$. R is antisymmetric if and only if its intersection with its inverse is a subset of the equality set, where $\Delta = \{(a, a), (b, b), (c, c), ...\}$

   (a) AKA if its the intersection between edges that exists and edges that don't exist forms a symmetry

3. R is asymmetric iff $R \cap R^{-1} = \varnothing$. R is asymmetric if there only exists "one-way" streets.

7.9 Theorem 4: Symmetry of R and S

1. If R is symmetric, then $R^{-1}$and $\bar{R}$ are also symmetric. To understand why, check [2][4]
2. If a relation set is symmetric, then $R \cap S$ and $R \cup S$ are also symmetric.

7.10 Theorem 5: Transitivity, Equivalence between Relations

1. Let R and S be relations on A
2. $(R \cap S)^2 \subseteq R^2 \cap S^2$.
   (a) $S^2 = S \times S = \{(x_1, x_2) : x_1, x_2 \in S\}$
   (b) If R and S are transitive, so is $R \cap S$
      i. If they are both transitive, then a subset of them must also be transitive
   (c) If R and S are equivalence relations, then so is $R \cap S$.
      i. If they are both equivalent, their subset must also be equivalent

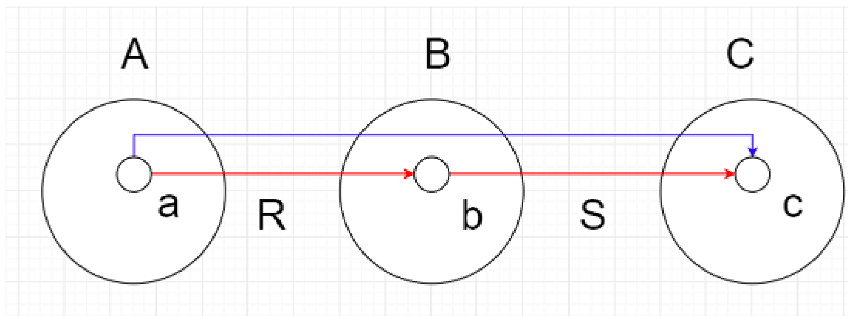
8 Reflexive Closure, Symmetric Closure and Transitive Closure

8.1 Closures

1. Some relation R on A lacks important relational properties (RefSym-Trans).
   (a) We add pairs to R until we get the property.
2. **Closure of R:** Smallest relation R_1on A, which is a superset of R, possessing the property we desire.
   (a) For the below, assume that R is neither reflexive nor symmetric.
   (b) **Reflexive closure:** $R \cup \Delta$.
      i. R union with the equality set $\{(a, a), (b, b), ...\}$
      ii. **Digraph:** Add missing loops to each vertex
   (c) **Symmetric closure:** $R \cup R^{-1}$
      i. **Digraph:** All edges made bidirectional

8.2 Compositions

1. Let's say we have 3 sets, set A, set B, and set C
2. Then we have a relation R from set A to set B, and relation S from set B to set C.
3. **Composition** of $R \wedge S = S \circ R$. Composition to S from R.
4. If $a \in A$ and $c \in C$, then $a\,(S \circ R)\,c$ iff for some $b \in B$, we have aRb and bSc.
   (a) Take it simply, if we have a composition of relations between three sets, means we have relations among 3 elements in 3 different sets.



5. **Digraph:**
   (a) The red line are the individual relations, the blue line is the $a\,(S \circ R)\,c$
6. Matrices:
   (a) M_R has size of $n * p$ and M_S has size $p * m$
   (b) $M_R \odot M_S = M_{S \circ R}$

8.3 Theorem 6: Composition of Subsets

Let R be a relation from A to B and let S be a relation from B to C.

1. If $A_1$is a subset of A , then we have

$$(S \circ R)\,(A_1) = S\,(R\,(A_1))$$

8.4 Theorem 7: Associative law on compositions

$$T \circ (S \circ R) = (T \circ S) \circ R$$

8.5 Theorem 8: Inverses on compositions

$$(S \circ R)^{-1} = R^{-1} \circ S^{-1}$$

1. In general, $S \circ R \neq R \circ S$, to understand why: [4]

8.6 Finding Compositions: Sets

1. **Question**: Let $A = \{1, 2, 3, 4\}$, $R = \{(1, 2), (1, 1), (1, 3), (2, 4), (3, 2)\}$, and $S = \{(1, 4), (1, 3), (2, 3), (3, 1), (4, 1)\}$.
   (a) Find $S \circ R$.
2. **Answer**:
   (a) You want to start from R, lets go with the first element in R, labelled r_0,
   $$r_0 = (1, 2)$$
   (b) So, we know that this leads to the 2. So, we find the (x, y) pair in S with $x = 2$. So, we found
   $$s_2 = (2, 3)$$
   (c) Since this entry's $y = 3$, we thus conclude that
   $$(S \circ R)_0 = (1, 3)$$
   (d) We can then continue and make the following tuple list
   $$S \circ R = \{(1, 3), (1, 4), (1, 1), (2, 1), (3, 3)\}$$

8.7 Finding compositions: Matrices

Let $A = \{a, b, c\}$ and let R and S be relations on A whose matrices are

$$M_R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, M_S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

1. $M_{S \circ R}$

$$\begin{aligned} M_{S\circ R} &= M_R \odot M_S \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0*1+1*1 & 1*1+1*0 \\ 1*1+0*1 & 1*1+0*0 \end{bmatrix} \\ M_{S\circ R} &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

8.8 Transitive Closure

1. R^∞ is transitive closure of R.
2. **Transitive closure**: R^∞, all vertices are connected (by paths) to other vertices.
3. **(Geometric point of view) Connectivity relation:** R^∞, specify which vertices are connected to other vertices.
4. R^* reachability relation $= R^\infty \cup \Delta$

8.8.1 Theorem 2

1. Let A be a set with $|A| = n$, and let R be a relation on A. Then, $R^\infty = R \cup R^2 \cup ... \cup R^n$.

9 Transitive Closure using Warshall's Algorithm

1. Use to compute transitive closure efficiently.
2. Define $W_0 = M_R$, then we have sequence $W_0, W_1, ..., W_n$ whose first term is M_R and last term is M_{R^∞}.

9.1 Steps

1. Transfer to W_k all 1's in W_{k-1}. (For W_1 you are simply moving W_0 inside)
2. List down the rows & colmns where the entry is 1 in two lists.
3. Find all possible combinations of row & columns from the list your obtained in step 2
4. Make it 1 for each tuple (x, y) that exists both inside step 3, and W_k (if not already 1)
5. Repeat until the last row & column inside the matrix.

10 Theorem 3: Equivalence relations between relations

1. The smallest equivalence relation containing both R and S is $(R \cup S)^\infty$

$$M_{R \cup S} = M_{(R \cup S)^\infty}$$

11 Example on Warshall + Theorem 3

11.1 Question

1. Let:
   (a) $A = \{1, 2, 3, 4, 5\}$
   (b) $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\}$ and
   (c) $S = \{(1, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$
2. Find the smallest equivalence relations containing R and S, and compute the partition of A it produces.

11.2 Answer

1. Find the two matrices

   (a) $M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

   (b) $M_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

2. Note: Some of the W_n are the same as the previous ones, therefore, there is no need to keep redrawing them.

   (a) $M_{R \cup S} = M_R \vee M_S$

      i. $W_0 = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & 1 & 0 & 0 & 0 \\ \mathbf{0} & 0 & 1 & 1 & 0 \\ \mathbf{0} & 0 & 1 & 1 & 1 \\ \mathbf{0} & 0 & 0 & 1 & 1 \end{bmatrix}$

         A. We start by finding the row & column list for the first row and column (bolded).

         B. $C_1 : 1, 2$

         C. $R_1 : 1, 2$

         D. Then, we find all possible combinations (that do not already exist inside W_0)

         E. Add: $\{(1,1), (1,2), (2,1), (2,2)\}$

         F. $W_1 = W_0$

      ii. $W_1 = \begin{bmatrix} 1 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{0} & 1 & 1 & 0 \\ 0 & \mathbf{0} & 1 & 1 & 1 \\ 0 & \mathbf{0} & 0 & 1 & 1 \end{bmatrix}$

         A. Note: no need to redraw, purpose is to highlight rows & columns selected. In exam, you can just write $W_1 = W_0$ as above.

         B. $C_1 : 1, 2$

         C. $R_1 : 1, 2$

         D. Add $\{(1,1), (1,2), (2,1), (2,2)\}$

         E. $W_2 = W_1$

iii. $W_2 = \begin{bmatrix} 1 & 1 & \mathbf{0} & 0 & 0 \\ 1 & 1 & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & \mathbf{1} & 1 & 1 \\ 0 & 0 & \mathbf{0} & 1 & 1 \end{bmatrix}$

A. $C_3 = 3, 4$

B. $R_3 = 3, 4$

C. Add: $\{(3,3), (3,4), (4,3), (4,4)\}$

D. $W_3 = W_2$

iv. $W_3 = \begin{bmatrix} 1 & 1 & 0 & \mathbf{0} & 0 \\ 1 & 1 & 0 & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{1} & 1 \\ 0 & 0 & 0 & \mathbf{1} & 1 \end{bmatrix}$

A. $C_4 = 3, 4, 5$

B. $R_4 = 3, 4, 5$

C. Add: $\{(3,3), (3,4), \mathbf{(3,5)}, (4,3), (4,4), (4,5), \mathbf{(5,3)}, (5,4), (5,5)\}$

D. Tuples added are highlighted: $W_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \mathbf{1} \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & \mathbf{1} & 1 & 1 \end{bmatrix}$

v. $W_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & \mathbf{0} \\ 1 & 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & 1 & \mathbf{1} \\ 0 & 0 & 1 & 1 & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}$

A. $C_5 = 3, 4, 5$

B. $R_5 = 3, 4, 5$

C. Add $(3,3)$ ...

D. $W_5 = W_4 = M_{(R \cup S)^\infty}$, The smallest equivalence relation containing R and S

References

[1] Mathematics Stack Exchange. 2020. terminology - why are equivalence relations called so? - Mathematics Stack Exchange. [ONLINE] Available at: https://math.stackexchange.com/questions/1262216/why-are-equivalence-relations-called-so. [Accessed 14 January 2020].

[2] Mathematics Stack Exchange. 2020. logic - Proof: if R is symmetric then so is R^{-1} - Mathematics Stack Exchange. [ONLINE] Available at: https://math.stackexchange.com/questions/1170879/proof-if-r-is-symmetric-then-so-is-r-1. [Accessed 15 January 2020].

[3] Mathematics Stack Exchange. 2020. discrete mathematics - Is the complement of a symmetric relation symmetric? - Mathematics Stack Exchange. [ONLINE] Available at: https://math.stackexchange.com/questions/1810687/is-the-complement-of-a-symmetric-relation-symmetric. [Accessed 15 January 2020].

[4] Mathematics Stack Exchange. 2020. elementary set theory - Inverse of composition of relation - Mathematics Stack Exchange. [ONLINE] Available at: https://math.stackexchange.com/questions/234891/inverse-of-composition-of-relation. [Accessed 15 January 2020].