Multivariable Calculus: A Lot of Integrals

mostly from Vector Calculus, 6th Edition - Jerrold Marsden & Class Notes, Fall 2019

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Contents

1 Review

Recall the change of variables stuff:

- The Jacobian matrix measures how much a transformation is changing by. Partials with respect to du in first column, dv in second column, etc.
- Change of variables formula: [do]

- Polar coordinates change of variables formula:

$$\int\int_D f(x,y)dxdy = \int\int_{D*} f(rcos\theta, rsin\theta)rdrd\theta$$

- Cylindrical coordinates change of variables:

$$\int\int\int_W f(x,y,z)dxdydz = \int\int\int_{W*} f(rcos\theta, rsin\theta, z)rdrd\theta dz$$

- Spherical coordinates change of variables:

$$\int\int\int_W f(x,y,z)dxdydz = \int\int\int_{W^*} f(psin\phi cos\theta, psin\phi sin\theta, pcos\phi)p^2 sin\phi dpd\theta d\phi dx$$

2 Early Stuff: Limits, Continuity, Gradient, etc.

A **vector valued** function is one that returns a vector. Otherwise, scalar valued. A **graph** is points of R^{n+1} with all points $(x_1, ...x_n, f(x_1, ...x_n))$

An **open set / ball** $U \subset R^n$ is an open set if for every point $x_0 \in U$ if $\forall x_0 \exists r > 0$ such that $D_r(x_0) \in U$
Theorem: For each $x_0 \in R^n$ and $r > 0$, $D_r(x_0)$ is an open set.
A **neighborhood** of $x \in R^n$ is an open set ...something....

Let U be an open set in R^n and $f : U \subset \mathbb{R}^n \to R^m$. f is differentiable at $x_0 \in U$ if partials of f exist at x_0 and if

$$\lim_{x \to x_0} \frac{||f(x) - f(x_0) - T(x - x_0)||}{||x - x_0||} = 0$$

where $T = Df(x_0)$ is partials matrix and $T(x - x_0)$ is $T \cdot (x - x_0)$
where partials are $\frac{df}{dx}$,

$$\frac{df}{dx} = \lim \frac{f(a+h) - f(a,b)}{h}$$

, $h = dx$ if you imagine x y plane where a is on x axis, b is on y axis, (a, b) is a point on a concave up graph. Rise is df, run is df.
IMPORTANT WHEN TALKING ABOUT DERICTIONAL DERIVITIVES: derivative is [[something????]] and slope is rise over run, so you have to normalize directional derivative is you want the slope.

c is a path $(x, y, ...)$

3 Path & Line Integrals

[add problems]
PATH INTEGRALS

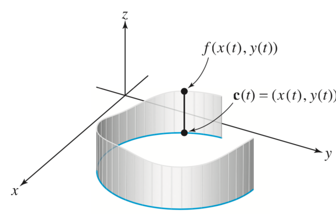


Figure 1: Path integral example; fence idea.

If you want to integrate f over some path c so that you get an area of a "fence," you'd take the sum of all (infinitely many) lengths of subdivisions of the path c multiplied by the height, f, which makes a rectangle of area.
Each subdivision of c is the arc length, aka $\sqrt{[x']^2+[y']^2+[z']^2} = \sqrt{([c']^2)} = ||c'(t)||$
Then, multiply this by the height to get: $\int_c f ds = \int_a^b f(x(t), y(t), z(t))||c'(t)||dt$
$= \int_a^b f(c(t))||c'(t)||dt$
You can divide by $\int ||c'(t)||dt$ to get the average distance between $f(c(t))$ & $c(t)$

LINE INTEGRALS

Whenever people say "work," it is traditionally, in simple physics, work times distance, or work over some distance. So work here, similarly, is essentially finding the force over some path (the distance).

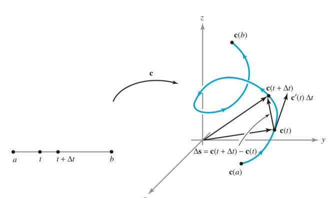


Figure 2: Line Integral.

To get the vector field and direction over this path, you need $F(c(t))$. But to this is just at some point: to get a length (which will get infinitly small), you want the force from $c(t)$ to $c(t + \Delta t)$. The difference in the input is Δt.
Recalling the definition of the derivative,
$f'(t) = \lim_{\Delta t \to \infty} \frac{f(t+\Delta t)-f(t)}{\Delta t}$ and for small distances, $f'(t) \approx \frac{f(t+\Delta t)-f(t)}{\Delta t}$. So we know $f'(t)\Delta t \approx f(t+\Delta t) - f(t)$ and applying this,
$c'(t)\Delta t \approx c(t+\Delta t) - c(t)$
So calculating the force over the path becomes $F(c(t)) \cdot [c(t+\Delta t) - c(t)]$ (why

dotting?) or $F(c(t)) \cdot c'(t)\Delta t$. Then you take the integral to get the sum over infinitely small $\Delta t's$.

$$\int_c F \cdot ds = \int_a^b F(c(t)) \cdot c'(t)dt$$

Logically, if the path moves along the opposite direction, the work will be different. So
$\int_p F \cdot ds = \int_c F \cdot ds$ when orientation preserving and $\int_p F \cdot ds = -\int_c F \cdot ds$ when orientation reversing (because opposite path is moving in the opposite direction of the vector field
Also, with math, the answer is changing because you're not dotting $c(t)$ but $c'(t)\delta t$, which is a vector going in the direction your path is going. Naturally, if the path is going in the opposite direction, $F(c) \cdot c'$ is in the opposite direction, so negative
The integral is the total force of the vector field on a path and dividing by arc length, $\int ||c'||$ will give you the average work of F on c. Also,

$$\int_c \nabla f \cdot ds = f(c(b)) - f(c(a))$$

by FTC if $f : \mathbb{R}^3 \to \mathbb{R}$ (helps when F is the gradient of a function because you can just do this instead fo calculating more stuff). Recall:

3.1 Fundamental Theorem of Calculus

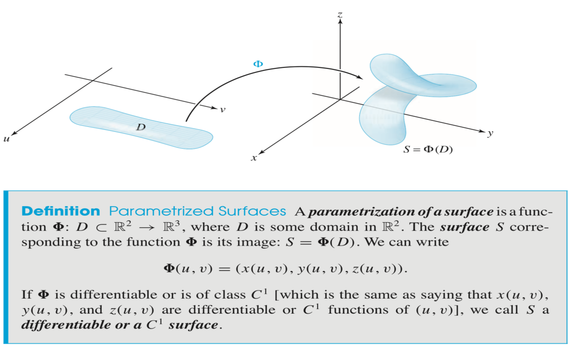
yeet

3.2 Exercises

1. Evaluate the following path integrals $\int_c f(x, y, z)ds$, where
   (a) $f(x, y, z) = exp\sqrt{z}$, and $c : \mapsto (1, 2, t^2), t \in [0, 1]$
   (b) $f(x, y, z) = yz$, and $c : \mapsto (t, 3t, 3t), t \in [1, 3]$
2. Evaluate each of the following line integrals:
   (a) $\int_c xdy - ydx$, $c(t) = (cos(t), sin(t))$, $0 \leq t \leq 2\pi$
   (b) $\int_c xdx - ydy$, $c(t) = (cos(\pi t), sin(\pi t))$, $0 \leq t \leq 2$
   (c) $\int_c yzdx + xzdy + xydz$, where c consists of straight line segments joining $(1, 0, 0)$ to $(0, 1, 0)$ to $(0, 0, 1)$
   (d) $\int_c x^2dx - xydy + dz$, where c is the parabola $z = x^2$, $y = 0$ from $(-1, 0, 1)$ to $(1, 0, 1)$

4 Parametrized Surfaces & Calculating Surface Areas

Parameterizing surfaces changes a region D to a surface S and allows you to integrate surfaces that aren't graphs (surfaces that folds over itself).



Definition Parametrized Surfaces A ***parametrization of a surface*** is a function $\mathbf{\Phi}: D \subset \mathbb{R}^2 \to \mathbb{R}^3$, where D is some domain in $\mathbb{R}^2$. The ***surface*** S corresponding to the function $\mathbf{\Phi}$ is its image: $S = \mathbf{\Phi}(D)$. We can write

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

If $\mathbf{\Phi}$ is differentiable or is of class C^1 [which is the same as saying that $x(u, v)$, $y(u, v)$, and $z(u, v)$ are differentiable or C^1 functions of (u, v)], we call S a ***differentiable or a C^1 surface***.

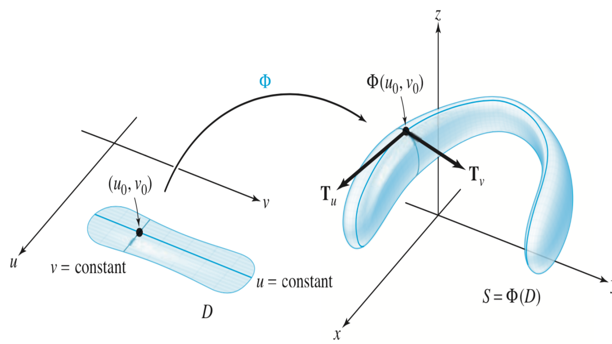


Figure 3: Parameterizing surfaces & tangent vectors

A surface is smooth or regular at $\Phi(u, v)$ if $T_u \times T_v \neq 0$ where
$T_u = \frac{\partial \Phi}{\partial u} = \frac{\partial x}{\partial u}(u_0, v_0)i + \frac{\partial y}{\partial u}(u_0, v_0)j + \frac{\partial z}{\partial u}(u_0, v_0)k$
$T_v = \frac{\partial \Phi}{\partial v} = \frac{\partial x}{\partial v}(u_0, v_0)i + \frac{\partial y}{\partial v}(u_0, v_0)j + \frac{\partial z}{\partial v}(u_0, v_0)k$

T_u, T_v are tangent vectors at the parameterized point. if you can't find the tangent there, like at $\Phi(u, v, |u|)$, then [??] it equals 0 and $T_u \times T_v = 0$.
To imagine the tangents T_u, T_v at some point, imagine if the point was the origin and make the x and y axis there; then the tangent vectors are the lines tangent to those new axes.
Because $T_u \times T_v$ tells you whether or not the surface is regular, you can compute $(x - x_0, y - y_0, z - z_0) \cdot n = 0$ where $n = T_u \times T_v$
to find the plane tangent at the point (x_0, y_0, z_0)

Same idea as the process for finding the arc length. You get one tiny area of the surface that you can compute and integrate that on some interval. Because $||T_u \times T_v||$ is a plane, multiplying by du and dv (very small numbers), gets the area of a small portion of that plane, $||T_u \times T_v||dudv$. Integrating between bounds (over the region D from the bounds of u and v) gives the area of the surface between those bounds.
Also, recall from single variable that the area for a surface revolution about the x axis is $A = 2\pi \int_a^b |f(x)|\sqrt{1 + [f'(x)']^2}dx$ and about the y axis is $A = 2\pi \int_a^b |x|\sqrt{1 + [f'(x)']^2}dx$. Derive these formulas from our new method (no solutions).

4.1 Exercises

1. Match the following parameterizations to the surfaces shown in the figures.
(a) $\Phi(u,v) = ((2\sqrt{1+u^2}cosv, (2\sqrt{1+u^2})sinv, u)$
(b) $\Phi(u,v) = (3cosusinv, 2sinusinv, cosv)$
(c) $\Phi(u,v) = (u, v, u^2)$
(d) $\Phi(u,v) = (ucosv, usinv, u)$

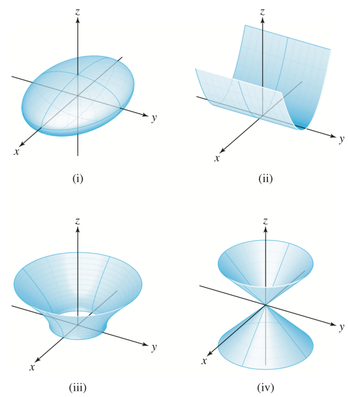


Figure 4: 7.3 Question 7

2. Let $\Phi(u,v) = (e^u cosv, e^u sinv, v)$ be a mapping from $D = [0,1] \times [0,\pi]$ in the uv plane onto a surface S in the xyz space.
(a) Find $T_u \times T_v$.
(b) Find the equation for the tangent plane to S when $(u,v) = (0, \frac{\pi}{2})$
(c) Find the area of $\Phi(D)$

5 Integrating Scalar Functions over Surfaces & Surface Integrals of Vector Fields

$\int\int_S f(x,y,z)dS = \int\int_D f(\Phi(u,v))||T_u \times T_v||dudv$
This formula comes from $||T_u \times T_v||dudv$ being some small square ($T_u \times T_v$ is a vector, $||T_u \times T_v||$ is a plane, $||T_u \times T_v||dudv$ is the plane scaled down to be really small. f would be some other surface in $\mathbb{R}^3$ and $f(\Phi)$ is the part of f that is directly above / below the surface being parameterized. When you multiply these two, you get the area between them. So any one $f(\Phi(u,v))||T_u \times T_v||dudv$ is a small box going from the small square $||T_u \times T_v||dudv$ to the height of $f(\Phi)$.

Apparently, "the physical interpretation is not as important/intuitive as the realistic applications." For example, if f is the temperature and $f(\Phi)$ is the temperature over the surface of Earth, $\int\int f(\Phi)||T_u \times T_v||dudv$ divided by

$\int\int ||T_u \times T_v|| dudv$ will give the average temperature over the surface. Another example: if you have a piece of metal made of different elements so that it's copper in some places, aluminum or gold in others, but it has different proportions of each element in different places, you can let $f(\Phi)$ be the mass density of the surface and dividing $\int\int f(\Phi)||T_u \times T_v||dudv$ by $\int\int ||T_u \times T_v||dudv$ will give you the average density, where $\int\int f(\Phi)||T_u \times T_v||dudv$ itself is the total mass density of your peice of metal.

Vector Fields

Recall that T_u and T_v are tangent vectors and that $T_u \times T_v$ is a vector pointing [[somewhere]]. Similar to a line integral, dotting these vectors by a third vector, the vector field F, at some point

gives the projection of F onto $T_u \times T_v$. Physically, it's like the work being done by the vector field to the surface. So maybe if there was particle moving on the surface, it would be the work done by the vector field to that particle...

also gives you the volume of a parallelepiped???

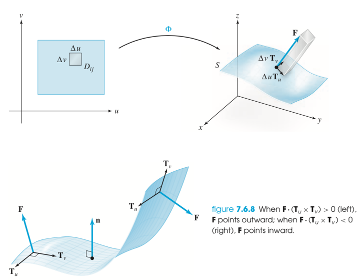


Figure 5: Parallelepiped formed by $F \cdot (T_u \times T_v) du\ dv$ and effect of difference faces; On the other side of the surface, you have some negative vector, making the volume of the parallelepiped negative. .

$$[From class. December 4th]$$

A surface is orientable if it has 2 sides. More precisely, if one can draw a nonzero normal vector continuously for each point on the surface S. (He also said the second more precise part is somewhat useless because you can't prove a surface is orientable with this, i think, only that it's not – not 100 percent sure he said that, but something about the second part being somewhat useless). For example, a sphere is orientable because you can have an inside and outside; you can draw normal vectors at some point on one side and you won't be able to get to the other side without reducing the vector to 0 and passing through. For a sphere, if the parameterization give normal vectors that point outside on the surface, the parameterization has positive orientation; if the tangent vectors point inwards, the parameterization has negative orientation. This is because

of the whole thing about it being convention that the the "parameterization is positively orientated when $T_u \times T_v$ (or also the normal vecotrs?] point outside of the closed surface."
Some facts stated in class:

- Graphs are always orientable.
- Closed surfaces in $\mathbb{R}^3$ are orientable.
- If you can parameterize S, then the orientations "one." [dont know what this means and if he actually said this]

Class example: $\Phi(\theta, \phi) = (cos\theta sin\phi, sin\theta sin\phi, cos\phi)$, $F = (x, y, z)$
sphere, so has determined orientation (an orientation exists because fact 3), as opposed to a surface like a piece of paper, where you can choose the orientation (bc there is no "outwards." You can choose which side is outwards because it's not closed)
Find orientation by taking $T_\theta \times T_\phi$ at some point
$T_\theta = (-sin\theta sin\phi, cos\theta sin\phi, 0)$
$T_\phi = (cos\theta cos\phi, cos\phi sin\theta, -sin\theta)$
$T_\theta(0, \frac{\pi}{2}) \times T_\phi(0, \frac{\pi}{2}) = (0, 1, 0) \times (0, 0, -1) = (-1, 0, 0)$ so negative orientation

5.1 Exercises

1. Evaluate the surface integral $\int\int_S F \cdot dS$ where $F(x, y, z) = xi + yj + z^2k$ and S is the surface parameterized by $\Phi(u, v) = (2sinu, 3cosu, v)$, with $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$.

6 Exercise Solutions

6.1 Path Integrals

1. $\int_c fds = \int_a^b f(x(t), y(t), z(t))||c'(t)||dt = \int_0^1 e^t||(0, 0, 2t)||dt = \int_0^1 e^t\sqrt{4t^2}dt$
   $= \int_0^1 e^t 2tdt$
   By integration by parts, $\int udv = uv - \int uvdu$ and LIATE (Log Inverse trig Algebraic Trig Exponential)
   $\int_0^1 e^t 2tdt$, $u = 2t$, $dv = e^t$ and $du = 2$, $v = e^t$
   $\int_0^1 e^t 2tdt = 2te^t - \int_0^1 2e^t dt = 2e - [2e - 2] = 2$
   (b) $52\sqrt{14}$

6.2 Line Integrals

1. $\int_c xdy - ydx$ and $\int_a^b F(c(t))c'(t)dt$
   $\int_0^{2\pi} = (cost \cdot costdt) - (sint \cdot -sintdt) = \int_0^{2\pi}(cos^2t + sin^2t)dt = \int_0^{2\pi} 1dt$
   $= 2\pi$

(b) 0
(c) 0
(d) $\frac{2}{3}$

6.3 Surface Areas

1. This question is easy so no solution, only answers:
(a) $(e^u sinv, e^u cosv, 1)$
(b)$x + y = \frac{\pi}{2}$
(c) Not a hundred percent sure, but I think it's $\frac{\pi}{3}(1 + e^2)^{3/2} - \frac{\pi}{3}(2)^{3/2}$

6.4 Integrating Scalar Functions over Surfaces & Surface Integrals of Vector Fields

1. $T_u = (2cosu, -3sinv, 0)$, $T_v = (0, 0, 1)$
$T_u \times T_v = (-3sinu, -2cosu, 0)$, $(x, y, z^2) = (2inu, 3cosu, v^2)$
$(2inu, 3cosu, v^2) \cdot (-3sinu, -2cosu, 0) = (-6sin^2u + -6cos^2u + 0) = -6$
$\int_0^1 \int_0^{2\pi} -6dudv = \int_0^1 -12\pi dv = -12\pi$