2 Basic Structures: Sets, Functions, Sequences , Sums and Matrices 【集合、函数、序列、和、矩阵】

Introduction to Set Theory 【集合论】

- **Set【集合】: A *set* is a new type of structure, representing an *unordered* collection (group, plurality) of zero or more *distinct* (different) objects.**
- **Set theory【集合论】: Set theory deals with *operations* between, *relations* among, and *statements* about sets.**
- **Applications【应用】: Computer software systems, *All* of mathematics can be defined in terms of some form of set theory.【表示】**

Basic notations for sets

1. *notation*【表示】: For sets, we'll use variables S, T, U, … .
2. *Roster method notation*【排列方法表示】: We can denote a set S in writing by listing all of its elements in curly braces（）: $\{a, b, c\}$.
3. *Set -builder notation*【集合构建表示】: For any proposition $P(x)$ over any universe of discourse,

$\{x \mid P(x)\}$ is *the set of all x such that* $P(x)$.

Basic properties of sets

1. *unordered* : Sets are inherently *unordered*:
   - No matter what objects a, b, and c denote, {a, b, c} = {a, c, b} = {b, a, c} = {b, c, a} = {c, a, b} = {c, b, a}.
2. *distinct* :All elements are *distinct* (unequal); multiple listings make no difference!
   - If a=b, then {a, b, c} = {a, c} = {b, c} = {a, a, b, a, b, c, c, c, c}.
   - This set contains (at most) 2 elements!

Definition of Set Equality
集合相等

- **Set Equality :** Two sets are declared to be equal *if and only if* they contain exactly the same elements.
- **For example:**
- The set $\{1, 2, 3, 4\}$

  $= \quad \{x \mid x$ is an integer where $x>0$ and $x<5\}$

  $= \quad \{x \mid x$ is a positive integer whose square is >0 and $<25\}$

集合相等

1. **定义：** 设A,B为集合,如果$A\subseteq B$且$B \subseteq A$,则称A与B相等,记作**A=B**.符号化表示为$A=B \Leftrightarrow A\subseteq B \land B \subseteq A$. 如果A和B不相等,则记作**$A\neq B$**.

2. **例题：** 两个集合相等的充分必要条件是他们具有相同的元素.例如 $A=\{x|x$是小于等于3的素数$\}$,$B=\{x|x=2 \lor x=3\}$. 则A=B.

Infinite Sets 【无限集】

1. *definition*【定义】: Conceptually, sets may be *infinite* (*i.e.*, not *finite*, without end, unending).【无穷、不可数、可数的概念不同】
2. symbols: Symbols for some special infinite sets:
   N = {0, 1, 2, …} The **N**atural numbers.
   Z = {…, -2, -1, 0, 1, 2, …} The **Z**ntegers.
   R = The "**R**eal" numbers, such as 374.1828471929498181917281943125…

Basic Set Relations: Member of 【成员】

1. **member** :$x \in S$ ("x is in S") is the proposition that object x is an $\in$*lement* or *member* of set S.
   1. *e.g.* $3 \in \mathbf{N}$, "a" $\in \{x \mid x$ is a letter of the alphabet$\}$
2. $x \notin S$ or $\neg(x \in S)$ "x is not in S"

The Empty Set 【空集】

1. $\varnothing$ ("*null*", "the empty set") is the unique set that contains no elements whatsoever.
2. $\varnothing = \{\} = \{x \mid \mathbf{False}\} = \{x|x \neq x\}$.
3. No matter the domain of discourse, we have the axiom $\neg\exists x(x \in \varnothing)$.

空集

1. **空集是一切集合的子集.**
   - 证明:任何集合A,由子集定义有
   $\varnothing \subseteq A \Leftrightarrow \forall x(x \in \varnothing \to x \in A)$
2. **推论:空集是唯一的.**
   - 假设存在空集$\varnothing_1$和$\varnothing_2$,由定理有$\varnothing_1 \subseteq \varnothing_2$和$\varnothing_2 \subseteq \varnothing_1$,根据集合相等定义可知$\varnothing_1 = \varnothing_2$.

Subset 【子集】

1. $S \subseteq T$ ("S is a subset of T") means that every element of S is also an element of T.
2. $S \subseteq T \Leftrightarrow \forall x\ (x \in S \rightarrow x \in T)$
3. $\emptyset \subseteq S$, $S \subseteq S$.

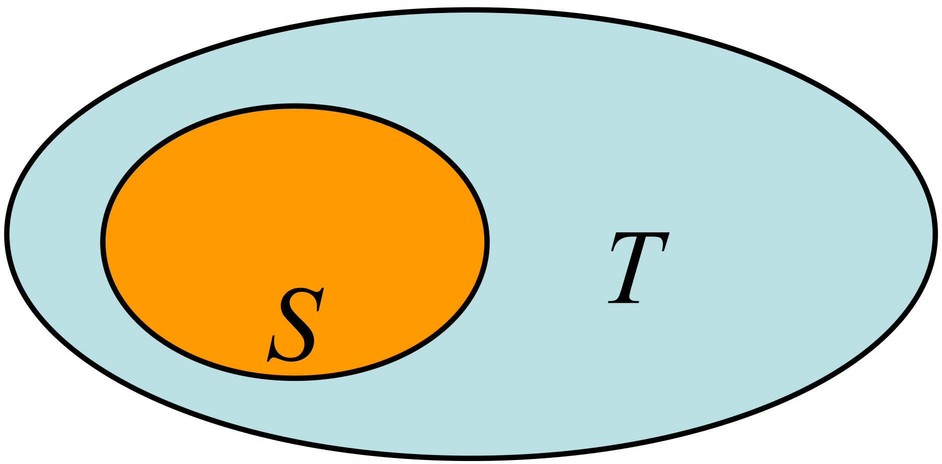
子集个数

- A={a,b,c},求A的全部子集.
  - 0元子集,即空集,只有1个∅.
  - 1元子集,即单元集, C_3^1个 {a},{b},{c}
  - 2 元子集 C_3^2个 {a,b},{a,c}{b,c}
  - 3元子集1个 {a,b,c}
- 子集个数为:

$$C_n^0 + C_n^1 + ... + C_n^n = 2^n$$

Proper (Strict) Subsets 【真子集】

- $S \subset T$ ("S is a proper subset of T") means that $S \subseteq T$ but $S \neq T$



Example:
$\{1,2\} \subset \{1,2,3\}$

Venn Diagram equivalent of $S \subset T$

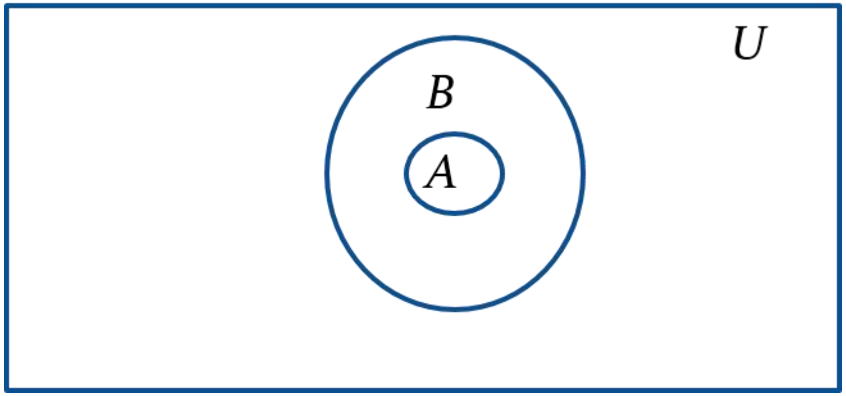
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x \wedge (x \in A \rightarrow x \in B) \wedge \exists x (x \in B \wedge x \notin A)$$

is true.

Venn Diagram



Sets Can be Objects, Too! 【集合也可以是集合元素】

- The objects that are elements of a set may *themselves* be sets.
- *E.g.* let $S=\{x \mid x \subseteq \{1,2,3\}\}$
  then $S=\{\emptyset,$
  $\{1\}, \{2\}, \{3\},$
  $\{1,2\}, \{1,3\}, \{2,3\},$
  $\{1,2,3\}\}$
- Note that $1 \neq \{1\} \neq \{\{1\}\}$!!!!

Cardinality and Finiteness 【基和有限性】

- $|S|$ (read "the *cardinality* of S") is a measure of how many different elements S has.
- *E.g.*, $|\varnothing|=0$, $|\{1,2,3\}| = 3$, $|\{a,b\}| = 2$,
  $|\{\{1,2,3\},\{4,5\}\}| =$ ____
- If $|S|\in\mathbf{N}$, then we say S is *finite*.
  Otherwise, we say S is *infinite*.

The *Power Set* 【幂集】 Operation

- The *power set* P(S) of a set S is the set of all subsets of S. $\mathrm{P}(S) = \{x \mid x \subseteq S\}$.
- *E.g.* P({a,b}) = {∅, {a}, {b}, {a,b}}.
- Sometimes P(S) is written $\mathbf{2}^S$.
  Note that for finite S, $|\mathrm{P}(S)| = 2^{|S|}$.
- It turns out $\forall S(|\mathrm{P}(S)|>|S|)$, *e.g.* $|\mathrm{P}(\mathbf{N})| > |\mathbf{N}|$.
  There are different sizes of infinite sets!

幂集

- 设A为集合,把A的全体子集构成的集合叫做A的**幂集**,记作P(A),符号化成 $P(A)=\{x|x\subseteq A\}$
- 设 $A=\{a,b,c\}$,则 $P(A)=\{\varnothing,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\{a,b,c\}\}$
- 计算以下幂集
  - $P(\varnothing) =\{\varnothing\}$
  - $P(\{\varnothing\})=\{\varnothing,\{\varnothing\}\}$
  - $P(\{\varnothing,\{\varnothing\}\})=\{\varnothing,\{\varnothing\},\{\{\varnothing\}\},\{\varnothing,\{\varnothing\}\}\}$

Review: Set Notations So Far

- Variable objects x, y, z; sets S, T, U.
- Literal set {a, b, c} and set-builder $\{x \mid P(x)\}$.
- $\in$ relational operator, and the empty set $\emptyset$.
- Set relations $=$, $\subseteq$, $\supseteq$, $\subset$, $\supset$, $\not\subset$, etc.
- Venn diagrams.
- Cardinality $|S|$ and infinite sets **N**, **Z**, **R**.
- Power sets P(S).

Ordered *n*-tuples 【有序*n*元组】

- These are like sets, except that duplicates matter, and the order makes a difference. Ordered pairs 序偶
- For $n \in \mathbf{N}$, an *ordered n-tuple* or a *sequence* or *list of length n* is written $(a_1, a_2, \ldots, a_n)$. Its *first* element is a_1, *etc.*
- Note that $(1, 2) \neq (2, 1) \neq (2, 1, 1)$.
- Empty sequence, singlets, pairs, triples, quadruples, quin<u>tuples</u>, …, *n*-tuples.

Cartesian Products of Sets 【迪卡尔集】

- For sets A, B, their *Cartesian product* $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$.
- *E.g.* $\{a,b\} \times \{1,2\} = \{(a,1),(a,2),(b,1),(b,2)\}$
- Note that for finite A, B, $|A \times B| = |A||B|$.
- $A \times B \neq B \times A$.
- Extends to $A_1 \times A_2 \times \ldots \times A_n$...

Section Summary

Set Operations

- Union
- Intersection
- Complementation
- Difference

More on Set Cardinality

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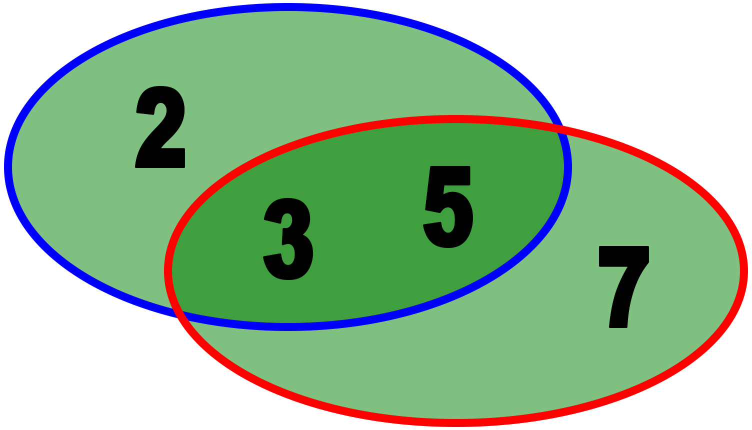
集合运算The Union Operator并

- For sets A, B, their $\cup$*nion* $A \cup B$ is the set containing all elements that are either in A, **or** ("$\vee$") in B (or, of course, in both).
- Formally, $\forall A,B$: $A \cup B = \{x \mid x \in A \vee x \in B\}$.
- Note that $A \cup B$ is a **superset** of both A and B (in fact, it is the smallest such superset):

$$\forall A, B: (A \cup B \supseteq A) \wedge (A \cup B \supseteq B)$$

Union Examples

- $\{a,b,c\} \cup \{2,3\} = \{a,b,c,2,3\}$
- $\{2,3,5\} \cup \{3,5,7\} = \{2,3,5,3,5,7\} = \{2,3,5,7\}$



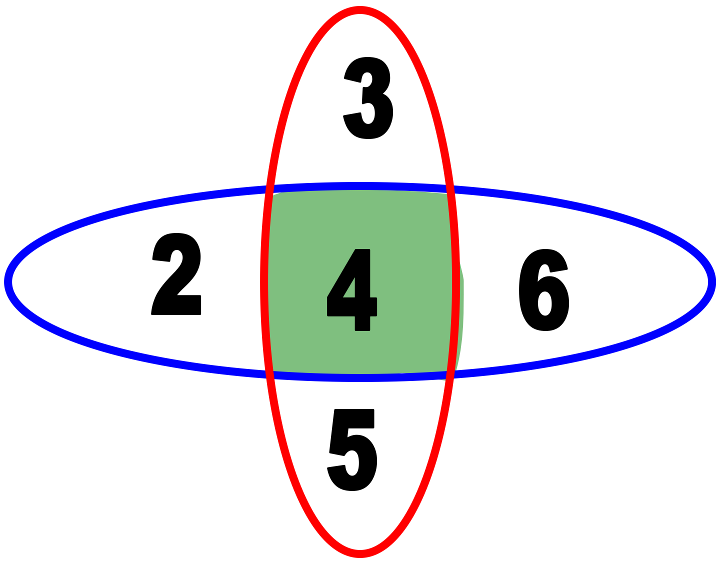
The Intersection Operator交

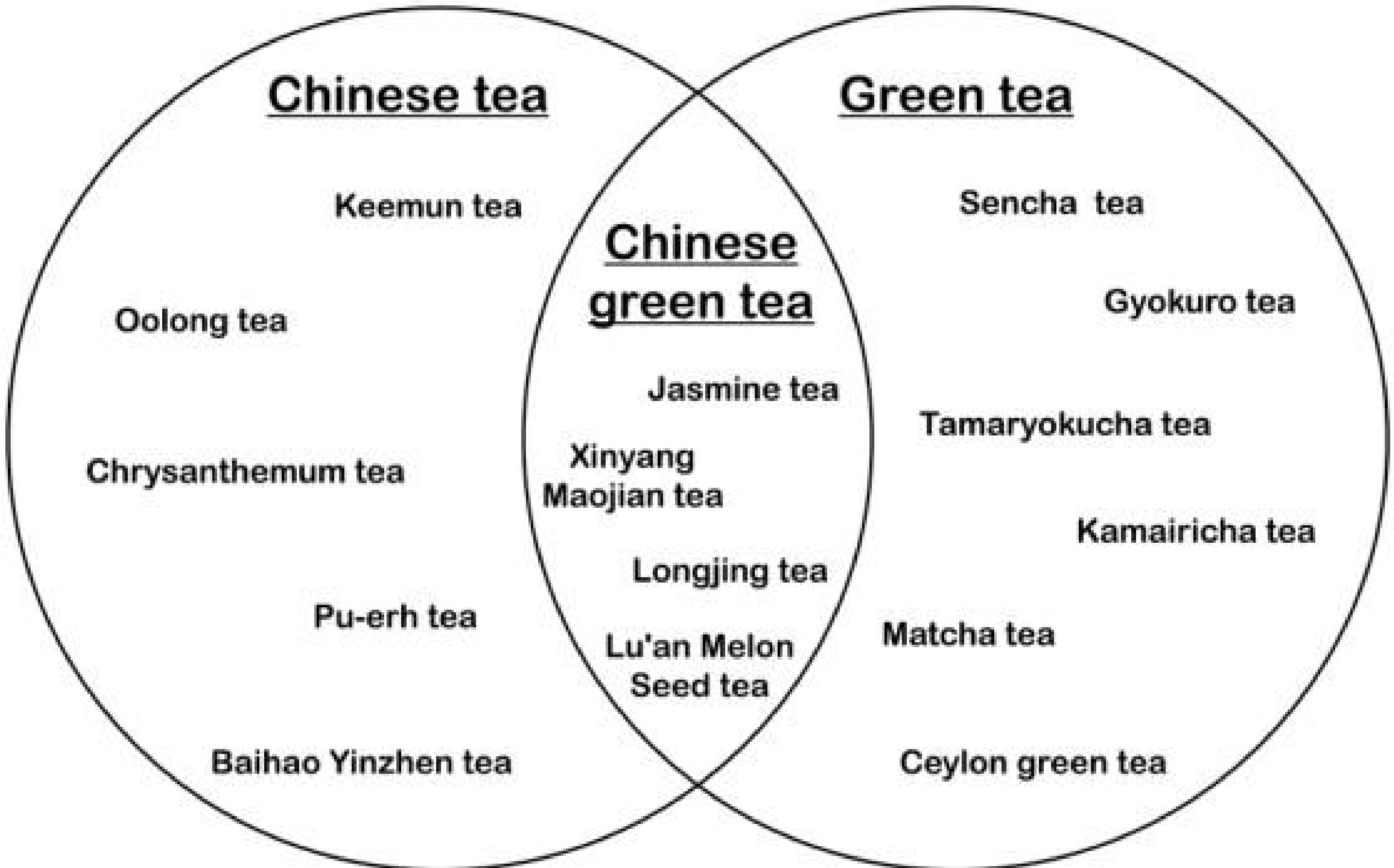
- For sets A, B, their *intersection* $A \cap B$ is the set containing all elements that are simultaneously in A **and** ("$\wedge$") in B.
- Formally, $\forall A,B: A \cap B = \{x \mid x \in A \wedge x \in B\}$.
- Note that $A \cap B$ is a **subset** of both A and B (in fact it is the largest such subset):

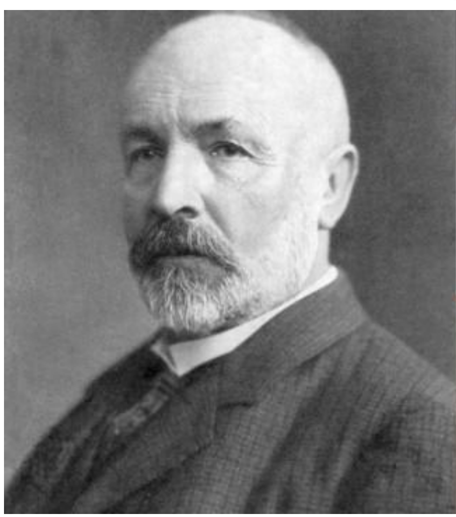
$$\forall A, B: (A \cap B \subseteq A) \wedge (A \cap B \subseteq B)$$

Intersection Examples

- $\{a,b,c\} \cap \{2,3\} = \underline{\ \emptyset\ }$
- $\{2,4,6\} \cap \{3,4,5\} = \underline{\ \{4\}\ }$







A Venn diagram showing a union of the Chinese tea set and the Green tea set. At the intersection of these sets are different types of Chinese green tea.

Disjointedness 【互斥】

- Two sets A, B are called *disjoint* (*i.e.*, unjoined) iff their intersection is empty. ($A \cap B = \emptyset$)
- Example: the set of even integers is disjoint with the set of odd integers.

Set Difference【差】

- For sets A, B, the *difference of A and B*, written $A{-}B$, is the set of all elements that are in A but not B. Formally:

$$A - B :\equiv \{x \mid x{\in}A \wedge \mathrm{x}{\notin}B\}$$
$$= \{x \mid \neg(x{\in}A \rightarrow x{\in}B)\ \}$$

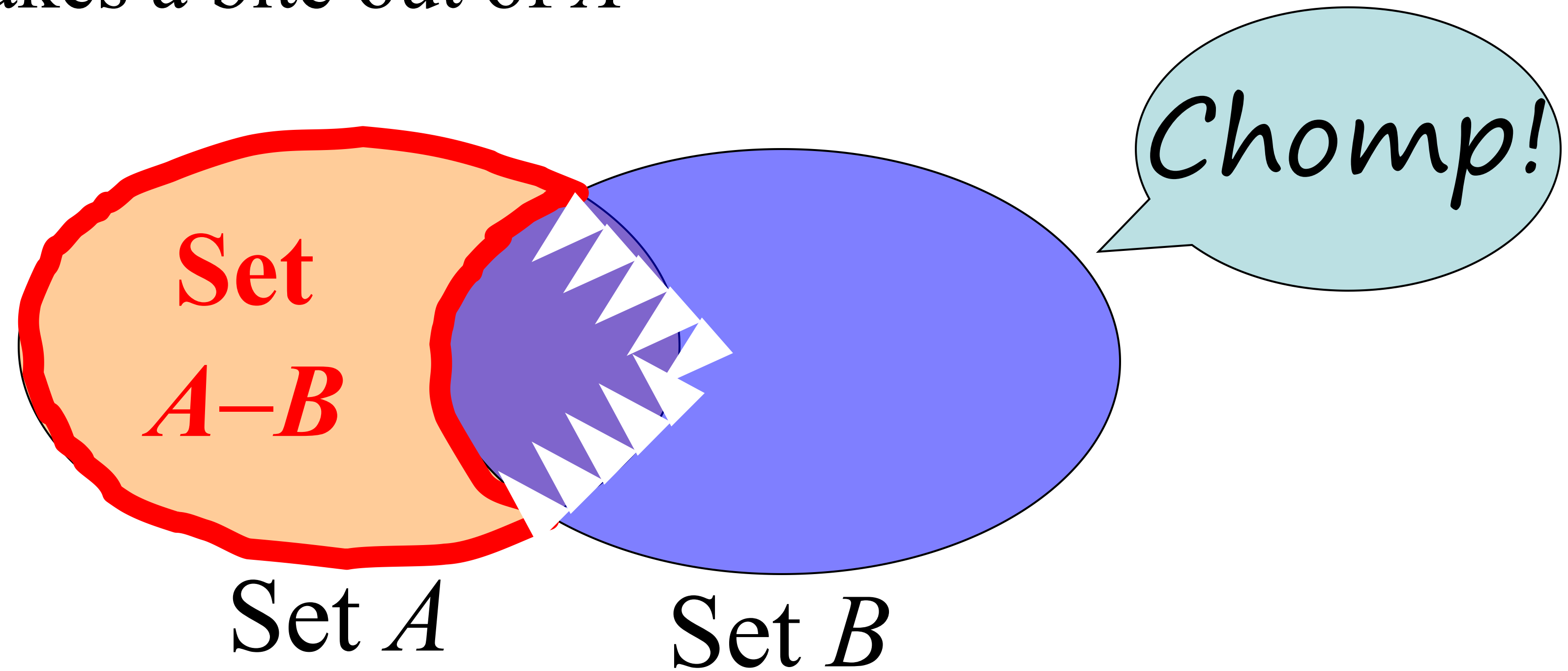
- Also called:
The *complement of B with respect to A*.

Set Difference Examples

- $\{1,\cancel{2},\cancel{3},4,\cancel{5},6\} - \{2,3,5,7,9,11\} =$
  $\underline{\quad\{1,4,6\}\quad}$
- $\mathbf{Z} - \mathbf{N} = \{\ldots, -1, 0, 1, 2, \ldots\} - \{0, 1, \ldots\}$
  $= \{x \mid x \text{ is an integer but not a nat. \#}\}$
  $= \{x \mid x \text{ is a negative integer}\}$
  $= \{\ldots, -3, -2, -1\}$

Set Difference - Venn Diagram

- $A-B$ is what's left after B "takes a bite out of A"



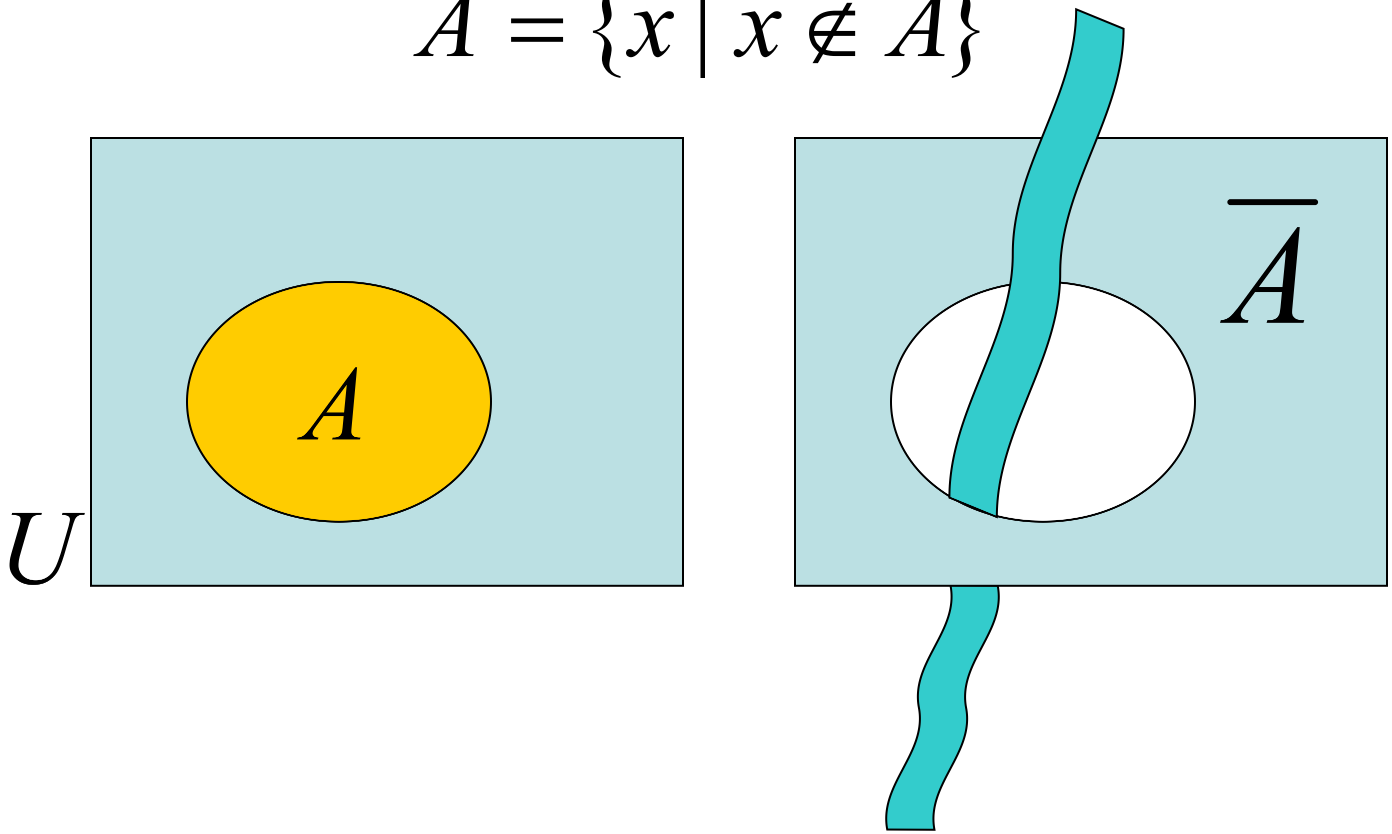
Set Complements 【补】

- The *universe of discourse* can itself be considered a set, call it U.
- When the context clearly defines U, we say that for any set $A \subseteq U$, the *complement* of A, written $\overline{A}$, is the complement of A w.r.t. U, *i.e.*, it is $U-A$.
- *E.g.*, If $U=\mathbf{N}$, $$\overline{\{3,5\}} = \{0,1,2,4,6,7,\ldots\}$$

More on Set Complements

- An equivalent definition, when *U* is clear:

$$\overline{A} = \{x \mid x \notin A\}$$



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Inclusion-Exclusion Principle
容斥原理

- How many elements are in $A \cup B$?

  $|A \cup B| = |A| + |B| - |A \cap B|$
- Example: How many students are on our class email list? Consider set $E = I \cup M$,

  $I = \{s \mid s$ turned in an information sheet$\}$

  $M = \{s \mid s$ sent the TAs their email address$\}$
- Some students did both!

  $|E| = |I \cup M| = |I| + |M| - |I \cap M|$

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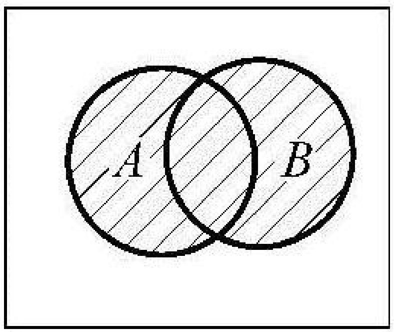
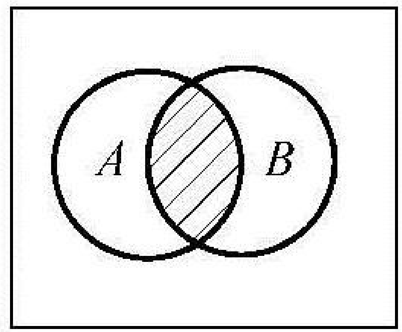
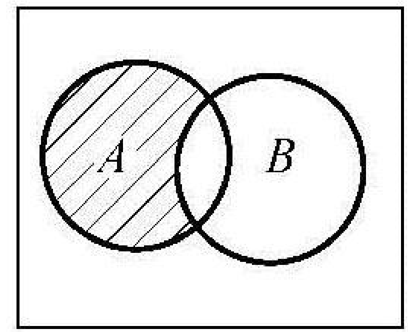
Venn Diagram



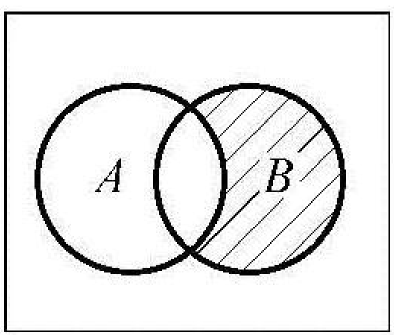
$A \cup B$



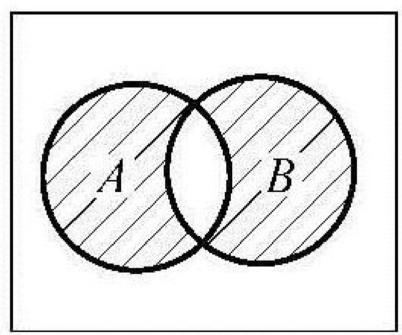
$A \cap B$



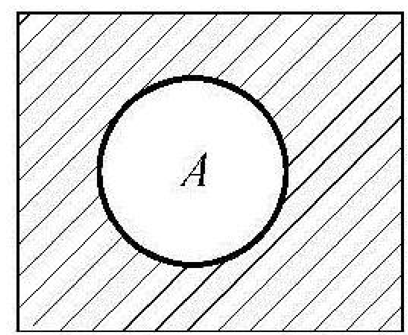
$A - B$



$B - A$



$A \oplus B$



$\overline{A}$

Set Identities

Identity laws

$$A \cup \varnothing = A \qquad A \cap U = A$$

Domination laws

$$A \cup U = U \qquad A \cap \varnothing = \varnothing$$

Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

Complementation law

$$\left(\overline{\overline{A}}\right) = A$$

Set Identities $_2$

Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Set Identities 3

De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \varnothing$$

DeMorgan's Law for Sets

- Exactly analogous to (and provable from) DeMorgan's Law for propositions.

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

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Proving Set Identities

Different ways to prove set identities:

1. Prove that each set (side of the identity) is a subset of the other.
2. Use set builder notation and propositional logic.
3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not

Proof of Second De Morgan Law [1]

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

$$1)\ \overline{A \cap B} \subseteq \overline{A} \cup \overline{B} \qquad \text{and}$$

$$2)\ \overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$$

Proof of Second De Morgan Law 1

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

$1)\ \overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

$2)\ \overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Proof of Second De Morgan Law [2]

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$\equiv x \notin A \cap B$	defn. of complement
$\equiv \neg\big((x \in A) \wedge (x \in B)\big)$	by defn. of intersection
$\equiv \neg(x \in A) \vee \neg(x \in B)$	1st De Morgan law for Prop Logic
$\equiv x \notin A \vee x \notin B$	defn. of negation
$\equiv x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$\equiv x \in \overline{A} \cup \overline{B}$	by defn. of union

Proof of Second De Morgan Law [3]

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$\equiv \left(x \in \overline{A}\right) \vee \left(x \in \overline{B}\right)$	by defn. of union
$\equiv \left(x \notin A\right) \vee \left(x \in \overline{B}\right)$	defn. of complement
$\equiv \neg\left(x \in A\right) \vee \neg\left(x \in B\right)$	defn. of negation
$\equiv \neg\left(\left(x \in A\right) \wedge \neg\left(x \in B\right)\right)$	1st De Morgan law for Prop Logic
$\equiv \neg\left(x \in A \cap B\right)$	defn. of intersection
$\equiv x \in \overline{A \cap B}$	defn. of complement

Set-Builder Notation: Second De Morgan Law

$$
\begin{aligned}
\overline{A \cap B} &= && \text{by defn. of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by defn. of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B\} && \text{by defn. of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by 1st De Morgan law for Prop Logic} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by defn. of not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by defn. of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by defn. of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of notation}
\end{aligned}
$$

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Membership Tables

- Just like truth tables for propositional logic.
- Columns for different set expressions.
- Rows for all combinations of memberships in constituent sets.
- Use “1” to indicate membership in the derived set, “0” for non-membership.
- Prove equivalence with identical columns.

Membership Table Example

Prove $(A \cup B) - B = A - B$.

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
0	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

Membership Table Exercise

Prove $(A \cup B) - C = (A - C) \cup (B - C)$

A B C	$A \cup B$	$(A \cup B) - C$	$A - C$	$B - C$	$(A-C) \cup (B-C)$
0 0 0					
0 0 1					
0 1 0					
0 1 1					
1 0 0					
1 0 1					
1 1 0					
1 1 1					

Generalized Union

- Binary union operator: $A \cup B$
- n-ary union:
  $A \cup A_2 \cup \ldots \cup A_n :\equiv ((\ldots((A_1 \cup A_2) \cup \ldots) \cup A_n)$
  (grouping & order is irrelevant)
- "Big U" notation: $$\bigcup_{i=1}^{n} A_i$$
- Or for infinite sets of sets: $$\bigcup_{A \in X} A$$

Generalized Intersection

- Binary intersection operator: $A \cap B$
- n-ary intersection:
  $A_1 \cap A_2 \cap \ldots \cap A_n \equiv ((\ldots((A_1 \cap A_2) \cap \ldots) \cap A_n)$
  (grouping & order is irrelevant)
- "Big Arch" notation: $\bigcap_{i=1}^{n} A_i$
- Or for infinite sets of sets: $\bigcap_{A \in X} A$

作业

- **2.1　8,12,18,24,34,48**
- **2.2　4,21,32,54,58,62**