Conjugation pair：選擇一個prior $p(\theta)$ 時，與likelihood $p(D|\theta)$ 計算後所得之posterior $p(\theta|D)$ 與prior 有一樣的 distribution

Proof Beta-Binomial Conjugation

$p \sim Beta(\alpha,\beta)$

$$f(p|\alpha,\beta) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)} \qquad B(\alpha,\beta) = \int_0^1 p^{\alpha-1}(1-p)^{\beta-1}\,dp$$

x 次成功，$n-x$ 次失敗 $p|x \sim Beta(\alpha+x, \beta+n-x)$

$X|p \sim Binomial(n,p)$

$$p(X=k|p) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$p(\theta|D) = \frac{p(D|\theta)\,p(\theta)}{p(D)}$$

∵ $p(D)$ 與 θ 無關可視為常數

∴ 只需 claim $p(\theta|D) \propto \underbrace{p(D|\theta)}_{\text{Binomial}}\overset{\text{Beta}}{p(\theta)}$

$$\underbrace{\binom{n}{k}}_{\text{常數}} p^k(1-p)^{n-k}\, p^{\alpha-1}(1-p)^{\beta-1} \cdot \underbrace{\frac{1}{B(\alpha,\beta)}}_{\text{常數}}$$

bayes' theorem calulate posterior

$$p(p|x=k) \propto p^k(1-p)^{n-k} p^{\alpha-1} (1-p)^{\beta-1}$$

$$p(p|x=k) \propto p^{(\alpha+k)-1} (1-p)^{(\beta+n-k)-1}$$

$$p|x=k \sim Beta(\underset{\text{prior}}{\alpha}+k, \underset{\text{prior}}{\beta}+n-k)$$