Notations

- $\mathcal{L}(A) \subseteq \Sigma^\omega$: the accepted language of an automaton A with the alphabet Σ, namely, the set of all infinite words accepted by A.

Claim Let $B = (X, x_{init}, \Sigma, \delta, \mathcal{F})$ and $\bar{B} = (\bar{X}, \bar{x}_{init}, \bar{\Sigma}, \bar{\delta}, \bar{\mathcal{F}})$ be an arbitrary tLDGBA and its augmentation, respectively. Then, we have $\mathcal{L}(B) = \mathcal{L}(\bar{B})$.

Proof

1. First, we show $\mathcal{L}(B) \subseteq \mathcal{L}(\bar{B})$. Consider any $w = \sigma_0\sigma_1 \ldots \in \mathcal{L}(B)$. Then, there exists a run $r = x_0\sigma_0x_1\sigma_1x_2 \ldots \in X(\Sigma X)^\omega$ of B such that $x_0 = x_{init}$ and $inf(r) \cap F_j \neq \emptyset$ for each $F_j \in \mathcal{F}$. For the run r, we construct a sequence $\bar{r} = \bar{x}_0\bar{\sigma}_0\bar{x}_1\bar{\sigma}_1\bar{x}_2 \ldots \in \bar{X}(\bar{\Sigma}\bar{X})^\omega$ satisfying $\bar{x}_i = (x_i, \bar{v}_i)$ and $\bar{\sigma}_i = \sigma_i$ for any $i \in \mathbb{N}$, where
$$\bar{v}_0 = \mathbf{0} \text{ and } \forall i \in \mathbb{N},\ \bar{v}_{i+1} = reset\Big(Max\big(\bar{v}_i, visitf((x_i, \sigma_i, x_{i+1}))\big)\Big).$$
Clearly from the construction, we have $(\bar{x}_i, \bar{\sigma}_i, \bar{x}_{i+1}) \in \bar{\delta}$ for any $i \in \mathbb{N}$. Thus, $\bar{r}$ is a run of $\bar{B}$ starting from $\bar{x}_0 = (x_{init}, \mathbf{0}) = \bar{x}_{init}$. We now show that $inf(\bar{r}) \cap \bar{F}_j \neq \emptyset$ for any $\bar{F}_j \in \bar{\mathcal{F}}$.

   Suppose that there exists $\bar{F}_{\bar{j}} \in \bar{\mathcal{F}}$ such that $inf(\bar{r}) \cap \bar{F}_{\bar{j}} = \emptyset$. Since $inf(r) \cap F_j \neq \emptyset$ for each $F_j \in \mathcal{F}$, we have
$$\forall F_j \in \mathcal{F},\ inf(\bar{r}) \cap \{\big((x, \bar{v}), \bar{\sigma}, (x', \bar{v}')\big) \in \bar{\delta} : (x, \bar{\sigma}, x') \in F_j\} \neq \emptyset, \tag{1}$$
which implies that
$$\forall k \in \mathbb{N},\ \exists l \geq k,\ \bar{v}_l = \mathbf{0}. \tag{2}$$
On the other hand, $inf(\bar{r}) \cap \bar{F}_{\bar{j}} = \emptyset$ means that
$$\exists k' \in \mathbb{N},\ \forall l' \geq k',\ \neg\Big((x_{l'}, \bar{\sigma}_{l'}, x_{l'+1}) \in F_{\bar{j}} \wedge (\bar{v}_{l'})_{\bar{j}} = 0 \wedge visitf\big((x_{l'}, \bar{\sigma}_{l'}, x_{l'+1})\big)_{\bar{j}} = 1\Big).$$
By Eq. (1), after some time step $\bar{l} \in \mathbb{N}$, $\bar{v}_{\bar{j}}$ keeps the same value 1, which contradicts Eq. (2). Therefore, for the word w, there exists a run $\bar{r} = \bar{x}_0\bar{\sigma}_0\bar{x}_1\bar{\sigma}_1\bar{x}_2 \ldots$ of $\bar{B}$ such that $\bar{x}_0 = \bar{x}_{init}$ and $inf(\bar{r}) \cap \bar{F}_j \neq \emptyset$ for each $\bar{F}_j \in \bar{\mathcal{F}}$. We conclude that $w \in \mathcal{L}(\bar{B})$.

2. Next, we show $\mathcal{L}(B) \supseteq \mathcal{L}(\bar{B})$. Consider any $\bar{w} \in \bar{\sigma}_0\bar{\sigma}_1 \ldots \in \mathcal{L}(\bar{B})$. Then, there exists a run $\bar{r} = \bar{x}_0\bar{\sigma}_0\bar{x}_1\bar{\sigma}_1\bar{x}_2 \ldots \in \bar{X}(\bar{\Sigma}\bar{X})^\omega$ of $\bar{B}$ such that $\bar{x}_0 = \bar{x}_{init}$ and $inf(\bar{r}) \cap \bar{F}_j \neq \emptyset$ for each $\bar{F}_j \in \bar{\mathcal{F}}$, i.e.,
$$\forall k \in \mathbb{N},\ \exists l \geq k,\ ([\![\bar{x}_l]\!]_X, \bar{\sigma}_l, [\![\bar{x}_{l+1}]\!]_X) \in F_j \wedge (\bar{v}_l)_j = 0 \wedge visitf\big((x_l, \bar{\sigma}_l, x_{l+1})\big)_{\bar{j}} = 1, \tag{3}$$
where $[\![(x, v)]\!]_X = x$ for each $(x, v) \in \bar{X}$. For the run $\bar{r}$, we construct a sequence $r = x_0\sigma_0x_1\sigma_1x_2 \ldots \in X(\Sigma X)^\omega$ such that $x_i = [\![\bar{x}_i]\!]_X$ and $\sigma_i = \bar{\sigma}_i$ for any $i \in \mathbb{N}$. It is clear that r is a run of B starting from $x_0 = x_{init}$. Also, it holds by Eq. (3) that $inf(r) \cap F_j \neq \emptyset$ for each $F_j \in \mathcal{F}$, which implies $\bar{w} \in \mathcal{L}(B)$.