Lemma 1 *For any policy π and any recurrent class $R^{\otimes i}_{\pi}$ in the Markov chain $MC^{\otimes}_{\pi}$, $MC^{\otimes}_{\pi}$ satisfies one of the following conditions.*

1. $\delta^{\otimes}_{\pi,i} \cap \bar{F}^{\otimes}_{j} \neq \emptyset$, $\forall j \in \{1, \dots, n\}$,
2. $\delta^{\otimes}_{\pi,i} \cap \bar{F}^{\otimes}_{j} = \emptyset$, $\forall j \in \{1, \dots, n\}$.

Let $\mathcal{SV}^*$ be the set of optimal supervisors. Let $D^{\otimes}_{SV^*}$ denote the product DES $D^{\otimes}$ controlled by the optimal supervisor SV^*.

For a Markov chain $MC^{\otimes}_{SV}$ induced by a product MDP $D^{\otimes}$ with a supervisor SV, let $S^{\otimes}_{SV} = T^{\otimes}_{SV} \sqcup R^{\otimes 1}_{SV} \sqcup \ldots \sqcup R^{\otimes h}_{SV}$ be the set of states in $MC^{\otimes}_{SV}$, where $T^{\otimes}_{SV}$ is the set of transient states and $R^{\otimes i}_{SV}$ is the recurrent class for each $i \in \{1, \dots, h\}$, and let $R(MC^{\otimes}_{SV})$ be the union of all recurrent classes in $MC^{\otimes}_{SV}$. Let $\delta^{\otimes}_{SV,i}$ be the set of transtions in a recurrent class $R^{\otimes i}_{SV}$, namely $\delta^{\otimes}_{SV,i} = \{(s^{\otimes}, e, s^{\otimes\prime}) \in \delta^{\otimes} ; s^{\otimes} \in R^{\otimes i}_{SV},\ P^{\otimes}_{T}(s^{\otimes\prime}|s^{\otimes}, e) > 0, P^{\otimes}_{E}(e|s^{\otimes}, SV(s^{\otimes})) > 0\}$, and let $P^{\otimes}_{SV} : S^{\otimes}_{SV} \times S^{\otimes}_{SV} \rightarrow [0,1]$ such that $P^{\otimes}_{SV} = \sum_{e \in SV(s^{\otimes})} P^{\otimes}_{T}(s^{\otimes\prime}|s^{\otimes}, e) P^{\otimes}_{E}(e|s^{\otimes}, SV(s^{\otimes}))$ be the transition probability under SV.

Theorem 1 *Let $D^{\otimes}$ be the product DES corresponding to a DES D and an LTL formuula φ. If there exists a supervisor SV satisfying φ, then there exist a discount factor γ^* and a positive reward r_p such that any algorithm that maximizes the expected discounted reward with $\gamma > \gamma^*$ and $r_p > ||\mathcal{R}_1||_{\infty}$ will find a supervisor satisfying φ.*

Proof 1 *Suppose that SV^* be an optimal supervisor but does not satisfy the LTL formula φ. Then, for any recurrent class $R^{\otimes i}_{SV^*}$ in the Markov chain $MC^{\otimes}_{SV^*}$ and any accepting set $\bar{F}^{\otimes}_{j}$ of the product DES $D^{\otimes}$, $\delta^{\otimes}_{SV^*,i} \cap \bar{F}^{\otimes}_{j} = \emptyset$ holds by Lemma 1. Thus, the agent under the policy π^* can obtain rewards only in the set of transient states. We consider the best scenario in the assumption. Let $p^k(s, s')$ be the probability of going to a state s' in k time steps after leaving the state s, and let $Post(T^{\otimes}_{\pi^*})$ be the set of states in recurrent classes that can be transitioned from states in $T^{\otimes}_{\pi^*}$ by one event occurrence. For the initial state $s^{\otimes}_{init}$ in the set of transient states, it holds that*

$$
\begin{aligned}
V^{SV^*}(s^{\otimes}_{init}) = &\sum_{k=0}^{\infty} \sum_{s^{\otimes} \in T^{\otimes}_{\pi^*}} \gamma^k p^k(s^{\otimes}_{init}, s^{\otimes}) \\
&\sum_{s^{\otimes\prime} \in T^{\otimes}_{\pi^*} \cup Post(T^{\otimes}_{\pi^*})} \sum_{e \in SV(s^{\otimes})} P^{\otimes}_{T}(s^{\otimes\prime}|s^{\otimes}, e) P^{\otimes}_{E}(e|s^{\otimes}, SV(s^{\otimes})) \mathcal{R}(s^{\otimes}, SV(s^{\otimes}), e, s^{\otimes\prime}) \\
\leq &\ r_p \sum_{k=0}^{\infty} \sum_{s^{\otimes} \in T^{\otimes}_{\pi^*}} \gamma^k p^k(s^{\otimes}_{init}, s^{\otimes}).
\end{aligned}
$$

By the property of the transient states, for any state $s^{\otimes}$ in $T^{\otimes}_{\pi^*}$, there exists a bounded positive value m such that $\sum_{k=0}^{\infty} \gamma^k p^k(s^{\otimes}_{init}, s^{\otimes}) \leq \sum_{k=0}^{\infty} p^k(s^{\otimes}_{init}, s^{\otimes}) <$

m [1]. Therefore, there exists a bounded positive value $\bar{m}$ such that $V^{\pi^*}(s^{\otimes}_{init}) < \bar{m}$. Let $\bar{SV}$ be a supervisor satisfying φ. We consider the following two cases.

1. *Assume that the initial state $s^{\otimes}_{init}$ is in a recurrent class $R^{\otimes i}_{\bar{\pi}}$ for some $i \in \{1, \dots, h\}$. For any accepting set $\bar{F}^{\otimes}_j$, $\delta^{\otimes}_{\bar{\pi},i} \cap \bar{F}^{\otimes}_j \neq \emptyset$ holds by the definition of $\bar{\pi}$. The expected discounted reward for $s^{\otimes}_{init}$ is given by*

$$V^{\bar{SV}}(s^{\otimes}_{init}) = \mathbb{E}^{SV}[\sum_{k=0}^{\infty} \gamma^k \mathcal{R}(s_k, \pi_k, e_k, s_{k+1}) | s_0 = s^{\otimes}_{init}] \tag{1}$$

Since $s^{\otimes}_{init}$ is in $R^{\otimes i}_{\bar{\pi}}$, there exists a set of positive numbers $K = \{k \ ; \ k \geq n, p^k(s^{\otimes}_{init}, s^{\otimes}_{init}) > 0\}$ [1]. We consider the worst scenario of returning the initial state in this case. For the stopping time k of first returning to the initial state, it holds that

$$\begin{aligned} V^{\bar{\pi}}(s^{\otimes}_{init}) >& \mathbb{E}^{\bar{SV}}[\gamma^k r_p - (1 + \dots + \gamma^k) ||\mathcal{R}_1||_{\infty} + \gamma^k V^{\bar{\pi}}(s^{\otimes}_{init}) | s_0 = s^{\otimes}_{init}] \\ \geq& \gamma^{\mathbb{E}^{\bar{SV}}[k|s_0 = s^{\otimes}_{init}]} r_p - (1 + \dots + \gamma^{\mathbb{E}^{\bar{SV}}[k|s_0 = s^{\otimes}_{init}]}) ||\mathcal{R}_1||_{\infty} + \gamma^{\mathbb{E}^{\bar{SV}}[k|s_0 = s^{\otimes}_{init}]} V^{\bar{\pi}}(s^{\otimes}_{init}) \\ =& \frac{\gamma^{\mathbb{E}^{\bar{SV}}[k|s_0 = s^{\otimes}_{init}]} r_p - (1 + \dots + \gamma^{\mathbb{E}^{\bar{SV}}[k|s_0 = s^{\otimes}_{init}]}) ||\mathcal{R}_1||_{\infty}}{1 - \gamma^{\mathbb{E}^{\bar{SV}}[k|s_0 = s^{\otimes}_{init}]}}, \end{aligned}$$

where second inequality holds since it holds that $\mathbb{E}^{\bar{SV}}[\gamma^k | s_0 = s^{\otimes}_{init}] \geq \gamma^{\mathbb{E}^{\bar{SV}}[k|s_0 = s^{\otimes}_{init}]}$ and $\frac{1 - \gamma^{\mathbb{E}^{\bar{SV}}[k+1|s_0 = s^{\otimes}_{init}]}}{1 - \gamma} \leq \mathbb{E}^{\bar{SV}}[\frac{1 - \gamma^{k+1}}{1 - \gamma} | s_0 = s^{\otimes}_{init}]$ by Jensen's inequality. Therefore, for any $\bar{m} \in (V^{SV^*}(s^{\otimes}_{init}), \infty)$ and any reward function $\mathcal{R}_1$, there exist $\gamma^* < 1$ and a positive reward r_p such that $\gamma > \gamma^*$ and $\gamma^{\mathbb{E}^{\bar{SV}}[k|s_0 = s^{\otimes}_{init}]} r_p - (1 + \dots + \gamma^{\mathbb{E}^{\bar{SV}}[k|s_0 = s^{\otimes}_{init}]}) ||\mathcal{R}_1||_{\infty}$ imply $V^{\bar{SV}}(s^{\otimes}_{init}) > \bar{m} > V^{SV^*}(s^{\otimes}_{init})$.

2. *Assume that the initial state $s^{\otimes}_{init}$ is in the set of transient states $T^{\otimes}_{\bar{SV}}$. $P^{M^{\otimes}}_{\bar{SV}}(s^{\otimes}_{init} \models \varphi) > 0$ holds by the definition of $\bar{SV}$. For a recurrent class $R^{\otimes i}_{\bar{SV}}$ such that $\delta^{\otimes}_{\bar{SV},i} \cap \bar{F}^{\otimes}_j \neq \emptyset$ for each accepting set $\bar{F}^{\otimes}_j$, there exist a number $\bar{l} > 0$, a state $\hat{s}^{\otimes}$ in $Post(T^{\otimes}_{\bar{SV}}) \cap R^{\otimes i}_{\bar{SV}}$, and a subset of transient states $\{s^{\otimes}_1, \dots, s^{\otimes}_{\bar{l}-1}\} \subset T^{\otimes}_{\bar{SV}}$ such that $p(s^{\otimes}_{init}, s^{\otimes}_1) > 0$, $p(s^{\otimes}_i, s^{\otimes}_{i+1}) > 0$ for $i \in \{1, ..., \bar{l} - 2\}$, and $p(s^{\otimes}_{\bar{l}-1}, \hat{s}^{\otimes}) > 0$ by the property of transient states. Hence, it holds that $p^{\bar{l}}(s^{\otimes}_{init}, \hat{s}^{\otimes}) > 0$ for the state $\hat{s}^{\otimes}$. Thus, for the stopping time k of first returning to the state $\hat{s}^{\otimes}$, by ignoring positive rewards in $T^{\otimes}_{\bar{\pi}}$ and assuming the system incurs the full costs regarding events prohibition, we have*

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