Category A: Equivalence Relation

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There are two ways to define the equivalence relation.

- Set Theory: Consider the equivalence relation on X as a **subset** of the Cartesian product $X \times X$;
- Type Theory: Consider the equivalence relation on X as a **term** of $X \to X \to \text{Prop}$.

I decide to adopt the second approach, though I have done both on lean. There are two reasons for preferring the way of Type Theory:

- More like human-style. For comparison,

  Set Theory: $(a, b) \in S$ Type Theory: $S\ a\ b$
- If we define a relation in this way:

  def some_relation $(a : X)(b : X) := \cdots$

  then "some_relation" has the type $X \to X \to \text{Prop}$ automatically.

1 Basic Definitions

Definition 1. is_refl $\{X : \text{Type}\}(S : X \to X \to \text{Prop}) := \forall a : X, S\ a\ a$

Definition 2. is_symm $\{X : \text{Type}\}(S : X \to X \to \text{Prop}) :=$
$\forall a\ b : X, S\ a\ b \to S\ b\ a$

Definition 3. is_trans $\{X : \text{Type}\}(S : X \to X \to \text{Prop}) :=$
$\forall a\ b\ c : X, S\ a\ b \to S\ b\ c \to S\ a\ c$

Definition 4. is_equiv $\{X : \text{Type}\}(S : X \to X \to \text{Prop}) :=$
is_refl $S \land$ is_symm $S \land$ is_trans S

Definition 5. equiv_class $\{X : \text{Type}\}(S : X \to X \to \text{Prop})(a : X) :=$
$\{x : X \mid S\ x\ a\}$

Definition 6. quotient $\{X : \text{Type}\}(S : X \to X \to \text{Prop}) :=$
$\{x : \text{set } X \mid \exists a : X, x = \text{equiv\_class } S\ a\}$

2 Properties

Here I write lemmas in the mathematical way to make them better understood. See the lean file for the codes.

In the following lemmas, $\sim$ is an equivalence relation on X.

Lemma 7. $a \in [a]$

Lemma 8. $[a] \subseteq X$

Lemma 9. $a \sim b \leftrightarrow [a] = [b]$

Lemma 10. $a \sim b \leftrightarrow [a] \cap [b] \neq \emptyset$

Lemma 11. $[a] = [b] \leftrightarrow [a] \cap [b] \neq \emptyset$

Lemma 12. $[a] \neq [b] \leftrightarrow [a] \cap [b] = \emptyset$

Lemma 13. $\bigcup_{a \in X} [a] = X$

3 Canonical Map and Section

Lemma 14. $[a] \in X/\sim$

Thus, we can define the canonical map $X \to X/\sim$ sending a to $[a]$.

Definition 15. can: $X \to X/\sim, a \mapsto [a]$

Lemma 16. $\forall E \in X/\sim, E \neq \emptyset$

Proof. Since $E \in X/\sim$, $\exists a \in X$ such that $E = [a]$. By Lemma 7, $a \in [a] = E$. Thus $E \neq \emptyset$. □

Therefore, for any $E \in X/\sim$, there exists an element $a \in E \subseteq X$ by the fact that E is nonempty. This gives us a particular section.

Definition 17. particular_sec $:= X/\sim \to X, E \mapsto$ (an element of E)

(We construct such element by set.nonempty.some on LEAN.)

Definition 18. can_sec $sec :=$ can $\circ sec$

Lemma 19. *If sec and sec' are two sections (i.e.* $\mathrm{can} \circ sec = \mathrm{can} \circ sec' = \mathrm{id}$*), then for any $E \in X/\sim$, $sec(E) \sim sec'(E)$.*

Lemma 20. particular_sec$(E) \in E$

Lemma 21. can $\circ$ particular_sec $=$ id

Lemma 22. particular_sec(can(a)) $\sim a$

Lemma 23. $a \sim b \leftrightarrow \mathrm{can}(a) = \mathrm{can}(b)$

4 Operations

If we have an operation $op : X \times X \to X$ and a section $sec : X/\sim \to X$, then an operation $i : X/\sim \times X/\sim \to X/\sim$ is induced by the following definition.

Definition 24. induced_op_by_sec($sec : X/\sim \to X$)($op : X \times X \to X$) $:= i : X/\sim \times X/\sim \to X/\sim, (e_1, e_2) \mapsto \mathrm{can} \circ op(sec(e_1), sec(e_2)))$

induced_op is the operation induced by a particular section.

Definition 25. induced_op(op) := induced_op(particular_sec)(op)

Definition 26. is_well_defined(op) $:= \forall a, b, c, d : X$,
$a \sim c \ \wedge \ b \sim d \to op(a, b) = op(c, d)$

Lemma 27. *If* is_well_defined(op), *then for any* $a, b : X$,
we have induced_op(op)($\mathrm{can}(a), \mathrm{can}(b)) = op(a, b)$.

5 Progress

All definitions and lemmas have been implemented on Lean.
Thanks for Thomas's remarkable advice on converting the type.