0-ary functions leaves us with an expression that involves only the operators $\oplus$, $\wedge$, and a sequence of 2^n constants. Furthermore, those constants can usually be simplified away, because we have

$$x\wedge 0 = 0 \qquad \text{and} \qquad x\wedge 1 = x\oplus 0 = x. \tag{18}$$

linear
multilinear representation
Boole
development
minterms
truth table
pi, as "random" example

After applying the associative and distributive laws, we end up needing the constant 0 only if $f(x_1,...,x_k)$ is identically zero, and the constant 1 only if $f(0,...,0)=1$.

We might have, for instance,

$$\begin{aligned} f(x,y,z) &= ((1\oplus 0\wedge x)\oplus(0\oplus 1\wedge x)\wedge y)\oplus((0\oplus 1\wedge x)\oplus(1\oplus 1\wedge x)\wedge y)\wedge z \\ &= (1\oplus x\wedge y)\oplus(x\oplus y\oplus x\wedge y)\wedge z \\ &= 1\oplus x\wedge y\oplus x\wedge z\oplus y\wedge z\oplus x\wedge y\wedge z. \end{aligned}$$

And by rule (5), we see that we're simply left with the polynomial

$$f(x,y,z) = (1{+}xy{+}xz{+}yz{+}xyz) \bmod 2, \tag{19}$$

because $x\wedge y = xy$. Notice that this polynomial is linear(of degree ≤ 1) in each of its variables. In general, a similar calculation will show that *any* Boolean function $f(x_1,...,x_n)$ jas a unique representation such as this, called its *multilinear representation*, which is a sum (modulo 2) of zero or more of the 2^n possible terms 1, x_1, x_2, x_1x_2, x_3, x_1x_3, x_2x_3, $x_1x_2x_3$,..., $x_1x_2...x_n$.

George Boole decomposed Boolean functions in a different way, which is often simpler for the kinds of functions that arise in practice. Instead of (16), he essentially wrote

$$f(x_1,...,x_n) = (g(x_1,...,x_{n-1})\wedge\bar{x})\vee(h(x_1,...,x_{n-1})\wedge x_n) \tag{20}$$

and called it the "law of development," where we now have simply

$$\begin{aligned} g(x_1,...,x_{n-1}) &= f(x_1,...,x_{n-1},0), \\ h(x_1,...,x_{n-1}) &= f(x_1,...,x_{n-1},1), \end{aligned} \tag{21}$$

instead of (17). Repeatedly iterating Boole's procedure, using the distributive law (1), and eliminating constants, leaves us with a formula that is a disjunction of zero or more *minterms*, where each minterm is a conjunction such as $x_1\wedge\bar{x}_2\wedge\bar{x}_3\wedge x_4\wedge\bar{x}_5$ in which every variable or its complement is present. Notice that a minterm is a Boolean function that is true at exactly one point.

For example, let's consider the more-or-less random function $f(w,x,y,z)$ whose truth table is

$$1100\ 1001\ 0000\ 1111. \tag{22}$$

When this function is expanded by repeatedly applying Boole's law (20), we get a disjunction of eight minterms, one for each of the 1s in the truth table:

$$\begin{aligned} f(w,x,y,z) = {} & (\bar{w}\wedge\bar{x}\wedge\bar{y}\wedge\bar{z})\vee(\bar{w}\wedge\bar{x}\wedge\bar{y}\wedge z)\vee(\bar{w}\wedge x\wedge\bar{y}\wedge\bar{z})\vee(\bar{w}\wedge x\wedge y\wedge z) \\ & \vee(w\wedge x\wedge\bar{y}\wedge\bar{z})\vee(w\wedge x\wedge\bar{y}\wedge z)\vee(w\wedge x\wedge y\wedge\bar{z})\vee(w\wedge x\wedge y\wedge z). \end{aligned} \tag{23}$$