ECON 219: Problem Set #5

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Problem 1

Asume the following smooth production function:

$$Q = Q(K, L)$$

with positive marginal productivities. Let w and r the prices of labor and capital, respectively.

1. Formulate the problem of minimizing costs subject to the technology.

$$\begin{aligned} \min_{K,L} \quad & C = wL + rK \\ \text{s.t.} \quad & Q(K, L) = \bar{Q} \end{aligned}$$

2. Explain under what conditions you might have to consider the case of corner solution (optimal labor or capital equal to zero). Provide an example.
   A corner solution arises when either $K = 0$ or $L = 0$. This occurs when:
   - The marginal productivity of one input is too low relative to its price.
   - The isoquants are linear (perfect substitutes) or L-shaped (perfect complements).

   Example: For a linear production function:

   $$Q = aL + bK$$

   If $\frac{w}{r} > \frac{a}{b}$, then the firm chooses $L = 0$ (uses only capital).
3. Assuming interior solution present the first order conditions. Provide an economic interpretation to the optimality condition. In you answer, refer to the Lagrange multiplier.

$$\mathcal{L} = wL + rK + \lambda(\bar{Q} - Q(K, L))$$

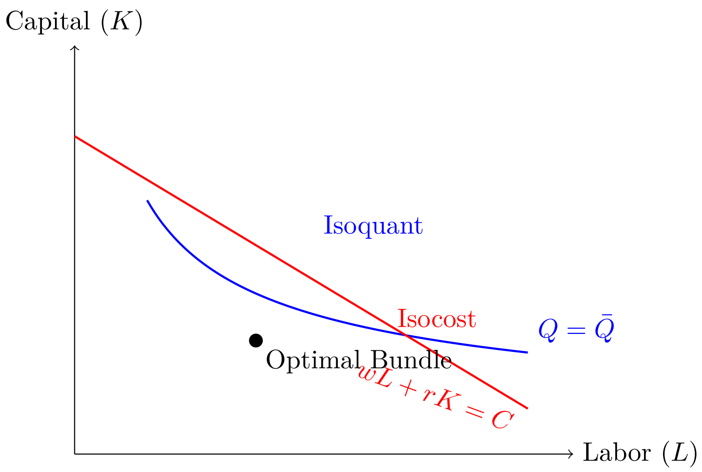
$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial L} &= w - \lambda \frac{\partial Q}{\partial L} = 0 \\ \frac{\partial \mathcal{L}}{\partial K} &= r - \lambda \frac{\partial Q}{\partial K} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= \bar{Q} - Q(K, L) = 0 \end{aligned}$$

$$\frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = \frac{w}{r} \quad \Rightarrow \quad MRTS_{L,K} = \frac{w}{r}$$

   Interpretation:
   - λ is the marginal cost of producing one more unit of output.
   - At the optimum, the marginal rate of technical substitution equals the input price ratio.
   - The firm equalizes the marginal product per dollar across inputs.
4. Provide a graphical representation of the resulting optimal input combination.
   The optimal bundle lies at the tangency point between the isoquant ($Q = \bar{Q}$) and the isocost line ($wL + rK = C$), where:

$$MRTS = \frac{w}{r}$$

Problem 1 continued on next page...



At this tangency point, the marginal rate of technical substitution equals the input price ratio:

$$\frac{MP_L}{MP_K} = \frac{w}{r}$$

5. Present the second order condition. Use the bordered Hessian:

$$H = \begin{bmatrix} 0 & Q_L & Q_K \\ Q_L & Q_{LL} & Q_{LK} \\ Q_K & Q_{KL} & Q_{KK} \end{bmatrix}$$

The second-order condition requires that H is positive semi-definite at the optimum to ensure a minimum.

6. Explain how the strict convexity of the isoquants would ensure a minimum cost. Strict convexity of isoquants implies that the cost minimization problem has:
   - A unique solution.
   - No corner solutions.
   - A globally cost-minimizing input bundle.
7. Explain how quasi-concave production function can generate everywhere strictly convex, downward-sloping isoquants. A strictly quasi-concave production function has convex upper contour sets. This ensures:
   - Isoquants are strictly convex.
   - Isoquants are downward-sloping.
8. Now, assume $Q = AL^{\alpha}K^{\beta}$. Show that the expansion path (optimal combinations of capital and labor for different isocosts) is characterized by a linear combination.

$$\min_{K,L} \quad wL + rK \quad \text{s.t.} \quad Q = AL^{\alpha}K^{\beta} = \bar{Q}$$

$$\frac{MP_L}{MP_K} = \frac{w}{r} \Rightarrow \frac{\alpha}{\beta} \cdot \frac{K}{L} = \frac{w}{r} \quad \Rightarrow \quad \frac{K}{L} = \frac{\beta}{\alpha} \cdot \frac{w}{r}$$

$$K = \left(\frac{\beta}{\alpha} \cdot \frac{w}{r}\right) L$$

9. Show the previous result holds for all homogeneous production functions. Let $Q(K, L)$ be homogeneous of degree $d > 0$.

Because MRTS depends only on K/L, solving:

$$MRTS = \frac{w}{r}$$

yields a constant ratio:

$$\frac{K}{L} = \phi(w, r) \quad \Rightarrow \quad K = \phi(w, r) \cdot L$$

Therefore, the expansion path is again a **straight line** through the origin.

Problem 2

Consider the following model:

$$Y = X\beta + \epsilon$$

where the standard assumption securing OLS delivers BLUE estimators hold. Assume the error terms is normally distributed with mean 0 and variance σ^2.

1. Present the likelihood function and optimization problem

$$Y = X\beta + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \sigma^2 I)$$

$$L(\beta, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_i - x_i^\top \beta)^2}{2\sigma^2}\right)$$

$$\ell(\beta, \sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i^\top \beta)^2$$

$$\max_{\beta,\sigma^2} \ell(\beta, \sigma^2) \quad \Leftrightarrow \quad \min_{\beta,\sigma^2} \frac{1}{2\sigma^2}(Y - X\beta)^\top (Y - X\beta) + \frac{n}{2}\log\sigma^2$$

2. Present the first and second order conditions.
   Let $Q(\beta) = (Y - X\beta)^\top (Y - X\beta)$.

First-order Conditions

Gradient w.r.t. β:

$$\frac{\partial \ell}{\partial \beta} = \frac{1}{\sigma^2}X^\top (Y - X\beta) \Rightarrow X^\top Y = X^\top X\hat{\beta} \Rightarrow \hat{\beta} = (X^\top X)^{-1}X^\top Y$$

Gradient w.r.t. σ^2:

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}Q(\beta) \Rightarrow \hat{\sigma}^2 = \frac{1}{n}(Y - X\hat{\beta})^\top (Y - X\hat{\beta})$$

Second-order Conditions

$$\frac{\partial^2 \ell}{\partial \beta \partial \beta^\top} = -\frac{1}{\sigma^2}X^\top X \quad \text{(Negative definite)}$$

$$\frac{\partial^2 \ell}{\partial (\sigma^2)^2} = \frac{n}{2\sigma^4} - \frac{2Q(\beta)}{\sigma^6}$$

3. Generate a sample of 1000 observations under the following parameterization:

$$Y = X\beta + \epsilon = \beta_0 + \beta_1 X + \epsilon$$

   where $\beta_0 = 0.5, \quad \beta_1 = -0.75, \quad X \sim (0.5, 2)$, and $\epsilon \sim N(0, 1)$.Present summary statistics
4. Implement the Newton-Raphson algorithm for the MLE problem.
   (a) Report the estimated values for the three parameters.

Problem 2 continued on next page. . .

Statistic	Y	X
count	1000.000000	1000.000000
mean	-0.480080	1.259171
std	1.018840	0.440379
min	-4.134258	0.500444
25%	-1.163643	0.873378
50%	-0.479364	1.277332
75%	0.173818	1.634770
max	3.884550	1.999606

Table 1: Descriptive statistics for variables Y and X

Implementacion de distribución Exponencial

```
import numpy as np
import pandas as pd


rng = np.random.default_rng(787)

n = 1000
X = rng.uniform(0.5, 2, n)
beta_0 = 0.5
beta_1 = -0.75
epsilon = rng.normal(0, 1, n)
Y = beta_0 + beta_1 * X + epsilon
data = pd.DataFrame({"X": X, "Y": Y})
data.describe()
```

(b) Compute the Hessian at the estimated values. How is this connected to the estimators' variance covariance MLE matrix?

Implementacion de distribución Exponencial

```
import numpy as np
import pandas as pd
from statsmodels.tools.tools import add_constant


rng = np.random.default_rng(787)

n = 1000
X = rng.uniform(0.5, 2, n)
beta_0 = 0.5
beta_1 = -0.75
epsilon = rng.normal(0, 1, n)
Y = beta_0 + beta_1 * X + epsilon
data = pd.DataFrame({"X": X, "Y": Y})


X = add_constant(data["X"].values)
```

Problem 2 continued on next page. . .

```python
theta = np.array([0.0, 0.0, 1.0])
max_iter = 100
tol = 1e-6

for i in range(max_iter):
    beta = theta[:2]
    sigma2 = theta[2]
    residuals = Y - X @ beta

    grad_beta = (X.T @ residuals) / sigma2
    grad_sigma2 = -n / (2 * sigma2) + 0.5 * np.sum(residuals**2) / sigma2**2
    grad = np.concatenate((grad_beta, [grad_sigma2]))

    H_beta = -(X.T @ X) / sigma2
    H_cross = -(X.T @ residuals) / sigma2**2
    H_sigma2 = n / (2 * sigma2**2) - np.sum(residuals**2) / sigma2**3

    H = np.zeros((3, 3))
    H[:2, :2] = H_beta
    H[:2, 2] = H_cross
    H[2, :2] = H_cross
    H[2, 2] = H_sigma2

    delta = np.linalg.solve(H, grad)
    theta_new = theta - delta

    if np.linalg.norm(delta) < tol:
        break
    theta = theta_new

print("Estimated parameters:", theta)
```

Estimated parameters: [0.50858469 -0.78517093 0.91755727]