ECON 219: Problem Set #5

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Problem 1

Asume the following smooth production function:

$$Q = Q(K, L)$$

with positive marginal productivities. Let w and r the prices of labor and capital, respectively.

1. Formulate the problem of minimizing costs subject to the technology.

$$\begin{aligned} \min_{K,L} \quad & C = wL + rK \\ \text{s.t.} \quad & Q(K, L) = \bar{Q} \end{aligned}$$

2. Explain under what conditions you might have to consider the case of corner solution (optimal labor or capital equal to zero). Provide an example.
   A corner solution arises when either $K = 0$ or $L = 0$. This occurs when:
   - The marginal productivity of one input is too low relative to its price.
   - The isoquants are linear (perfect substitutes) or L-shaped (perfect complements).

   Example: For a linear production function:

$$Q = aL + bK$$

   If $\frac{w}{r} > \frac{a}{b}$, then the firm chooses $L = 0$ (uses only capital).
3. Assuming interior solution present the first order conditions. Provide an economic interpretation to the optimality condition. In you answer, refer to the Lagrange multiplier.

$$\mathcal{L} = wL + rK + \lambda(\bar{Q} - Q(K, L))$$

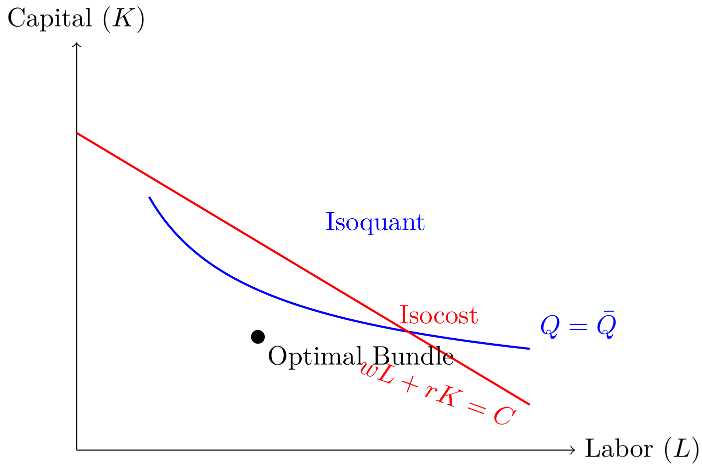
$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial L} &= w - \lambda \frac{\partial Q}{\partial L} = 0 \\ \frac{\partial \mathcal{L}}{\partial K} &= r - \lambda \frac{\partial Q}{\partial K} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= \bar{Q} - Q(K, L) = 0 \end{aligned}$$

$$\frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = \frac{w}{r} \quad \Rightarrow \quad MRTS_{L,K} = \frac{w}{r}$$

   Interpretation:
   - λ is the marginal cost of producing one more unit of output.
   - At the optimum, the marginal rate of technical substitution equals the input price ratio.
   - The firm equalizes the marginal product per dollar across inputs.
4. Provide a graphical representation of the resulting optimal input combination.
   The optimal bundle lies at the tangency point between the isoquant ($Q = \bar{Q}$) and the isocost line ($wL + rK = C$), where:

$$MRTS = \frac{w}{r}$$

Problem 1 continued on next page...



At this tangency point, the marginal rate of technical substitution equals the input price ratio:

$$\frac{MP_L}{MP_K} = \frac{w}{r}$$

5. Present the second order condition. Use the bordered Hessian:

$$H = \begin{bmatrix} 0 & Q_L & Q_K \\ Q_L & Q_{LL} & Q_{LK} \\ Q_K & Q_{KL} & Q_{KK} \end{bmatrix}$$

   The second-order condition requires that H is positive semi-definite at the optimum to ensure a minimum.

6. Explain how the strict convexity of the isoquants would ensure a minimum cost. Strict convexity of isoquants implies that the cost minimization problem has:
   - A unique solution.
   - No corner solutions.
   - A globally cost-minimizing input bundle.
7. Explain how quasi-concave production function can generate everywhere strictly convex, downward-sloping isoquants. A strictly quasi-concave production function has convex upper contour sets. This ensures:
   - Isoquants are strictly convex.
   - Isoquants are downward-sloping.
8. Now, assume $Q = AL^\alpha K^\beta$. Show that the expansion path (optimal combinations of capital and labor for different isocosts) is characterized by a linear combination.

$$\min_{K,L} \quad wL + rK \quad \text{s.t.} \quad Q = AL^\alpha K^\beta = \bar{Q}$$

$$\frac{MP_L}{MP_K} = \frac{w}{r} \Rightarrow \frac{\alpha}{\beta} \cdot \frac{K}{L} = \frac{w}{r} \quad \Rightarrow \quad \frac{K}{L} = \frac{\beta}{\alpha} \cdot \frac{w}{r}$$

$$K = \left(\frac{\beta}{\alpha} \cdot \frac{w}{r} \right) L$$

9. Show the previous result holds for all homogeneous production functions. Let $Q(K, L)$ be homogeneous of degree $d > 0$.

Because MRTS depends only on K/L, solving:

$$MRTS = \frac{w}{r}$$

yields a constant ratio:

$$\frac{K}{L} = \phi(w, r) \quad \Rightarrow \quad K = \phi(w, r) \cdot L$$

Therefore, the expansion path is again a **straight line** through the origin.

Problem 2

Consider the following model:

$$Y = X\beta + \epsilon$$

where the standard assumption securing OLS delivers BLUE estimators hold. Assume the error terms is normally distributed with mean 0 and variance σ^2.

1. Present the likelihood function and optimization problem
2. Present the first and second order conditions.
3. Generate a sample of 1000 observations under the following parameterization:

   $$Y = X\beta + \epsilon = \beta_0 + \beta_1 X + \epsilon$$

   where $\beta_0 = 0.5, \quad \beta_1 = -0.75, \quad X \sim (0.5, 2)$, and $\epsilon \sim N(0, 1)$.Present summary statistics
4. Implement the Newton-Raphson algorithm for the MLE problem.
   (a) Report the estimated values for the three parameters.
   (b) Compute the Hessian at the estimated values. How is this connected to the estimators' variance covariance MLE matrix?