ECON 219: Problem Set #5

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Dr. Jasmine N Fuller

Alejandro Ouslan

Problem 1

Suppose that a firm's fixed production function is given by

$$q = \min(5k, 10l)$$

1. Calculate the firm's long-run total, average, and marginal cost function.

$$q = \min(5k, 10l) \Rightarrow 5k = 10l \Rightarrow k = 2l$$

$$q = 10l \Rightarrow l = \frac{q}{10}, \quad k = \frac{q}{5}$$

$$C(q) = vk + wl = v \cdot \frac{q}{5} + w \cdot \frac{q}{10} = \left(\frac{v}{5} + \frac{w}{10}\right) q$$

Therefore:

$$\text{Total Cost (TC)} : C(q) = \left(\frac{v}{5} + \frac{w}{10}\right) q$$
$$\text{Average Cost (AC)} : AC(q) = \frac{C(q)}{q} = \frac{v}{5} + \frac{w}{10}$$
$$\text{Marginal Cost (MC)} : MC(q) = \frac{dC}{dq} = \frac{v}{5} + \frac{w}{10}$$

2. Suppose that k is fixed at 10 in the short run. Calculate the firm's short run total, average, and marginal cost function.

$$q = \min(5 \cdot 10, 10l) = \min(50, 10l) \Rightarrow q \leq 50, \quad l \geq \frac{q}{10}$$

$$C_{\text{SR}}(q) = 10v + \frac{w}{10}q$$

Hence:

$$\text{Short-Run Total Cost (TC)} : C_{\text{SR}}(q) = 10v + \frac{w}{10}q$$
$$\text{Short-Run Average Cost (AC)} : AC_{\text{SR}}(q) = \frac{10v}{q} + \frac{w}{10}$$
$$\text{Short-Run Marginal Cost (MC)} : MC_{\text{SR}}(q) = \frac{dC_{\text{SR}}}{dq} = \frac{w}{10}$$

3. Suppose $v = 1$ and $w = 3$. Calculate this firm's long-run and short-run average and marginal cost curves. *Long Run:*

$$C(q) = \left(\frac{1}{5} + \frac{3}{10}\right) q = 0.5q$$
$$\text{LR TC}(q) = 0.5q$$
$$\text{LR AC}(q) = 0.5$$
$$\text{LR MC}(q) = 0.5$$

Short Run:

$$C_{\text{SR}}(q) = 10 \cdot 1 + \frac{3}{10}q = 10 + 0.3q, \quad \text{for } q \leq 50$$
$$\text{SR TC}(q) = 10 + 0.3q$$
$$\text{SR AC}(q) = \frac{10}{q} + 0.3$$
$$\text{SR MC}(q) = 0.3$$

Problem 2

An enterprising entrepreneur purchases two factories to produce widgets. Each factory produces identical products, and each has a production function given by

$$q = \sqrt{k_i l_i}, i = 1, 2$$

The factories differ, however, in the amount of capital equipment each has. In particular, factory 1 has $k_1 = 25$, whereas factory 2 has $k_2 = 100$. Rental rates for k and l are given $w = v = \$1$.

1. If the entrepreneur wishes to minimize short-run total costs of widget production, how should output be allocated between the two factories?

$$\min_{l_1, l_2} C = l_1 + l_2 \quad \text{s.t.} \quad q = 5\sqrt{l_1} + 10\sqrt{l_2}$$

$$\mathcal{L} = l_1 + l_2 + \lambda(q - 5\sqrt{l_1} - 10\sqrt{l_2})$$

$$\frac{\partial \mathcal{L}}{\partial l_1} = 1 - \lambda \cdot \frac{5}{2\sqrt{l_1}} = 0 \Rightarrow \lambda = \frac{2\sqrt{l_1}}{5}$$

$$\frac{\partial \mathcal{L}}{\partial l_2} = 1 - \lambda \cdot \frac{10}{2\sqrt{l_2}} = 0 \Rightarrow \lambda = \frac{\sqrt{l_2}}{5}$$

$$\frac{2\sqrt{l_1}}{5} = \frac{\sqrt{l_2}}{5} \Rightarrow l_2 = 4l_1$$

$$q = 5\sqrt{l_1} + 10\sqrt{l_2} = 5\sqrt{l_1} + 10\sqrt{4l_1} = 25\sqrt{l_1} \Rightarrow \sqrt{l_1} = \frac{q}{25} \Rightarrow l_1 = \left(\frac{q}{25}\right)^2$$

$$l_2 = 4l_1 = 4\left(\frac{q}{25}\right)^2$$

2. Given that output is optimally allocated between the two factories, calculate the short-run total, average, and marginal cost curves. What is the marginal cost of the 100th widget? The 125th widget? The 200th widget?

Total Cost:

$$C(q) = l_1 + l_2 = \left(\frac{q}{25}\right)^2 + 4\left(\frac{q}{25}\right)^2 = \frac{q^2}{125}$$

Average Cost:

$$AC(q) = \frac{C(q)}{q} = \frac{q}{125}$$

Marginal Cost:

$$MC(q) = \frac{dC}{dq} = \frac{2q}{125}$$

Values:

$$MC(100) = \frac{200}{125} = 1.6 \quad MC(125) = \frac{250}{125} = 2 \quad MC(200) = \frac{400}{125} = 3.2$$

3. How should the entrepreneur allocate widget production between the two factories in the long run? Calculate the long-run total, average, and marginal cost curves for widget production In the long run, both capital and labor are variable. Given:

$$q_i = \sqrt{k_i l_i} \Rightarrow k_i l_i = q_i^2 \Rightarrow l_i = \frac{q_i^2}{k_i} \Rightarrow C_i = k_i + \frac{q_i^2}{k_i}$$

$$\frac{dC_i}{dk_i} = 1 - \frac{q_i^2}{k_i^2} = 0 \Rightarrow k_i = q_i \Rightarrow l_i = q_i \Rightarrow C_i = 2q_i$$

$$C(q) = 2q \quad \Rightarrow \quad AC(q) = 2, \quad MC(q) = 2$$

Problem 2 continued on next page...

4. How would your answer to part (c) change if both factories exhibited diminishing returns to scale?

$$q_i = (k_i l_i)^{\alpha}, \quad \text{with } 0 < \alpha < \frac{1}{2}$$

Then:

- Marginal product of inputs declines more rapidly
- Long-run cost curves are increasing and convex

Problem 3

Consider the production function $q = f(k, l) = l^{0.8}k^{0.2}$. The cost of labor and capital are w and v respectively. Aside from the cost of labor and capital, there are no costs. Initially assume k is fixed at k_1.

1. Find the short run cost function $C_{SR}(q)$.

$$q = l^{0.8}k_1^{0.2} \Rightarrow l = \left(\frac{q}{k_1^{0.2}}\right)^{\frac{1}{0.8}} = q^{1.25}k_1^{-0.25}$$

$$C_{SR}(q) = w \cdot l + v \cdot k_1 = w \cdot q^{1.25}k_1^{-0.25} + v \cdot k_1$$

2. Find short run marginal cost.

$$MC_{SR}(q) = \frac{dC_{SR}}{dq} = w \cdot 1.25 \cdot q^{0.25}k_1^{-0.25}$$

3. Find the short run average cost.

$$AC_{SR}(q) = \frac{C_{SR}(q)}{q} = w \cdot q^{0.25}k_1^{-0.25} + \frac{v \cdot k_1}{q}$$

4. Demostrare that cost is minimized when $AC = MC$.

$$AC_{SR}(q) = w \cdot q^{0.25}k_1^{-0.25} + \frac{v \cdot k_1}{q}$$

$$MC_{SR}(q) = 1.25 \cdot w \cdot q^{0.25}k_1^{-0.25}$$

$$w \cdot q^{0.25}k_1^{-0.25} + \frac{v \cdot k_1}{q} = 1.25 \cdot w \cdot q^{0.25}k_1^{-0.25}$$

$$\Rightarrow \frac{v \cdot k_1}{q} = 0.25 \cdot w \cdot q^{0.25}k_1^{-0.25}$$

now asume all inputs are variable.

5. Find the long run cost function $C_{LR}(q)$. Minimize cost:

$$C = w \cdot l + v \cdot k \quad \text{subject to} \quad q = l^{0.8}k^{0.2}$$

$$\mathcal{L} = w \cdot l + v \cdot k + \lambda(q - l^{0.8}k^{0.2})$$

$$\frac{\partial \mathcal{L}}{\partial l} = w - \lambda \cdot 0.8 \cdot l^{-0.2}k^{0.2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial k} = v - \lambda \cdot 0.2 \cdot l^{0.8}k^{-0.8} = 0$$

Problem 3 continued on next page...

$$\frac{w}{v} = \frac{0.8 \cdot l^{-0.2}k^{0.2}}{0.2 \cdot l^{0.8}k^{-0.8}} = 4 \cdot \frac{k}{l} \Rightarrow k = \frac{w}{4v} \cdot l$$

$$q = l^{0.8}\left(\frac{w}{4v}l\right)^{0.2} = \left(\frac{w}{4v}\right)^{0.2} \cdot l \Rightarrow l = \left(\frac{4v}{w}\right)^{0.2} \cdot q$$

$$k = \frac{w}{4v} \cdot l = \left(\frac{w}{4v}\right)^{0.8} \cdot q$$

$$C_{LR}(q) = w \cdot l + v \cdot k = w\left(\frac{4v}{w}\right)^{0.2} q + v\left(\frac{w}{4v}\right)^{0.8} q = A \cdot q$$

6. Find the long run marginal Cost.

$$MC_{LR}(q) = \frac{dC_{LR}}{dq} = A$$

7. Find the long run average cost.

$$AC_{LR}(q) = \frac{C_{LR}(q)}{q} = A$$