ECON 219: Problem Set #4

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Problem 1

As we have seen in many places, the general Cobb-Douglas production function for two inputs is given by

$$q = f(k, l) = AK^{\alpha}l^{\beta}$$

where $0 < \alpha < 1$ and $0 < \beta < 1$. For this production function:

1. Show that $f_k > 0$, $f_1 > 0$, $f_{kk} < 0$, $f_{ll} < 0$, and $f_{kl} > 0$.

$$\begin{aligned} f_k &= \frac{\partial f}{\partial k} = A\alpha k^{\alpha-1}l^{\beta} > 0 \\ f_l &= \frac{\partial f}{\partial l} = A\beta k^{\alpha}l^{\beta-1} > 0 \\ f_{kk} &= \frac{\partial^2 f}{\partial k^2} = A\alpha(\alpha-1)k^{\alpha-2}l^{\beta} < 0 \\ f_{ll} &= \frac{\partial^2 f}{\partial l^2} = A\beta(\beta-1)k^{\alpha}l^{\beta-2} < 0 \\ f_{kl} &= \frac{\partial^2 f}{\partial k \partial l} = A\alpha\beta k^{\alpha-1}l^{\beta-1} > 0 \end{aligned}$$

2. Show that $e_{q,k} = \alpha$ and $e_{q,l} = \beta$.

$$\begin{aligned} e_{q,k} &= \frac{\partial f}{\partial k} \cdot \frac{k}{f(k,l)} = \frac{A\alpha k^{\alpha-1}l^{\beta} \cdot k}{Ak^{\alpha}l^{\beta}} = \alpha \\ e_{q,l} &= \frac{\partial f}{\partial l} \cdot \frac{l}{f(k,l)} = \frac{A\beta k^{\alpha}l^{\beta-1} \cdot l}{Ak^{\alpha}l^{\beta}} = \beta \end{aligned}$$

3. In footnote 5, we defined the scale elasticity as

$$e_{q,t} = \frac{\partial f(tk, tl)}{\partial t} \cdot \frac{t}{f(tk, tl)}$$

where the expression is to be evaluated at $t = 1$. Show that, for this Cobb-Douglas function, $e_{q,t} = \alpha + \beta/$ Hence in this case the elasticity and the returns to scale of the production function agree (for more on this concept see Problem 9.9).

$$\begin{aligned} \frac{dF}{dt} &= A(\alpha+\beta)t^{\alpha+\beta-1}k^{\alpha}l^{\beta} \\ f(tk, tl) &= At^{\alpha+\beta}k^{\alpha}l^{\beta} \end{aligned}$$

$$e_{q,t} = \frac{\partial f(tk, tl)}{\partial t} \cdot \frac{t}{f(tk, tl)}\bigg|_{t=1} = \frac{A(\alpha+\beta)t^{\alpha+\beta-1}k^{\alpha}l^{\beta} \cdot t}{At^{\alpha+\beta}k^{\alpha}l^{\beta}}\bigg|_{t=1} = \alpha + \beta$$

4. Show that this function is quasi-concave.

$$\log f(k, l) = \log A + \alpha \log k + \beta \log l$$

Answer: This function is concave in $\log k$ and $\log l$, which implies f is log-concave, and hence quasi-concave.

5. Show that the function is concave for $\alpha + \beta \geq 1$ but not concave for $alpha + \beta > 1$.

$$H = \begin{bmatrix} f_{kk} & f_{kl} \\ f_{lk} & f_{ll} \end{bmatrix} = A \begin{bmatrix} \alpha(\alpha-1)k^{\alpha-2}l^{\beta} & \alpha\beta k^{\alpha-1}l^{\beta-1} \\ \alpha\beta k^{\alpha-1}l^{\beta-1} & \beta(\beta-1)k^{\alpha}l^{\beta-2} \end{bmatrix}$$

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$$\det(H) = A^2 k^{2\alpha-2} l^{2\beta-2} \left[(\alpha-1)(\beta-1) - \alpha\beta\right]$$

$$(\alpha-1)(\beta-1) \geq \alpha\beta \quad \Rightarrow \quad \alpha+\beta \leq 1$$

Answer: Thus, the function is concave when $\alpha+\beta \leq 1$, and not concave when $\alpha+\beta > 1$.

Problem 2

Consider a generalization of the production function in Example 9.3:

$$q = \beta_0 + \beta_1\sqrt{kl} + \beta_2 k + \beta_3 l$$

where

$$0 \geq \beta_i \quad i = 0, \dots, 3$$

1. if this function is to exhibit constant returns to scale, what restrictions should be placed on the parameters $\beta_0, \cdots, \beta_3$?

$$q(tk, tl) = \beta_0 + \beta_1\sqrt{tk \cdot tl} + \beta_2(tk) + \beta_3(tl) = \beta_0 + \beta_1 t\sqrt{kl} + \beta_2 tk + \beta_3 tl$$

$$tq(k,l) = t(\beta_0 + \beta_1\sqrt{kl} + \beta_2 k + \beta_3 l) = t\beta_0 + t\beta_1\sqrt{kl} + t\beta_2 k + t\beta_3 l$$

Equality holds if and only if $\beta_0 = 0$. Hence, the condition for CRS is:

$$\boxed{\beta_0 = 0}$$

2. Show that, in the constant returns-to-scale case, this function exhibits diminishing marginal productivities and that marginal productivity function ar homogeneous of degree 0.

$$q = \beta_1\sqrt{kl} + \beta_2 k + \beta_3 l$$

$$\frac{\partial q}{\partial k} = \frac{\beta_1}{2}\sqrt{\frac{l}{k}} + \beta_2, \qquad \frac{\partial q}{\partial l} = \frac{\beta_1}{2}\sqrt{\frac{k}{l}} + \beta_3$$

$$\frac{\partial^2 q}{\partial k^2} = -\frac{\beta_1}{4}\sqrt{\frac{l}{k^3}} < 0, \qquad \frac{\partial^2 q}{\partial l^2} = -\frac{\beta_1}{4}\sqrt{\frac{k}{l^3}} < 0$$

$$\frac{\partial q(tk,tl)}{\partial (tk)} = \frac{\beta_1}{2}\sqrt{\frac{tl}{tk}} + \beta_2 = \frac{\beta_1}{2}\sqrt{\frac{l}{k}} + \beta_2 = \frac{\partial q}{\partial k}$$

Answer: Marginal product functions are homogeneous of degree 0

3. Calculate σ in this case. Although σ is not in general constant, for what values of the β's does $\sigma = 0, 1, \infty$

$$\text{MRTS} = \frac{\partial q/\partial k}{\partial q/\partial l} = \frac{\frac{\beta_1}{2}\sqrt{l/k} + \beta_2}{\frac{\beta_1}{2}\sqrt{k/l} + \beta_3}$$

cases:

- If $\beta_1 = 0$, then $q = \beta_2 k + \beta_3 l$ (perfect substitutes): $\sigma = \infty$
- If $\beta_2 = \beta_3 = 0$, then $q = \beta_1\sqrt{kl}$ (Cobb-Douglas): $\sigma = 1$

- If $\beta_1 = \beta_2 = 0$ or $\beta_1 = \beta_3 = 0$: one input is essential, no substitutability: $\sigma = 0$

$$\begin{aligned} \sigma &= \infty \quad \text{if } \beta_1 = 0 \\ \sigma &= 1 \quad \text{if } \beta_2 = \beta_3 = 0 \\ \sigma &= 0 \quad \text{if } \beta_1 = 0 \text{ and only one input matters} \end{aligned}$$

Problem 3

Consider the production function $q = f(k, l) = (\sqrt{k} + \sqrt{l})^2$. The cost of labor and capital are w and v respectively. Aside from the cost of labor and capital, there are no cost.

1. What type of production function is this? Hint: Not Cobb-Douglas
2. Use your answer from part (a) to find the elasticity of substitution. Hint: There is a simple formula for elasticity of substitution related to the functional form.
3. Find RTS (l for k).
4. Find the total cost function.
5. Use your answer from part (d) to find the average cost.
6. Use your answer from part (d) to find marginal cost.
7. Is the production function homothetic?
8. Does the production function exhibit increasing, constant, or decreasing return to scale?
9. Is the expansion path linear?
10. In general, what can you say about AC and MC for homothetic production functions with no fixed cost?

1. **Quasi-Concavity:**