ECON 219: Problem Set #4

Due on May 29, 2025

Dr. Jasmine N Fuller

Alejandro Ouslan

Problem 1

As we have seen in many places, the general Cobb-Douglas production function for two inputs is given by

$$q = f(k, l) = AK^{\alpha} l^{\beta}$$

where $0 < \alpha < 1$ and $0 < \beta < 1$. For this production function:

1. Show that $f_k > 0$, $f_1 > 0$, $f_{kk} < 0$, $f_{ll} < 0$, and $f_{kl} > 0$.

$$\begin{aligned} f_k &= \frac{\partial f}{\partial k} = A\alpha k^{\alpha-1} l^{\beta} > 0 \\ f_l &= \frac{\partial f}{\partial l} = A\beta k^{\alpha} l^{\beta-1} > 0 \\ f_{kk} &= \frac{\partial^2 f}{\partial k^2} = A\alpha(\alpha-1) k^{\alpha-2} l^{\beta} < 0 \\ f_{ll} &= \frac{\partial^2 f}{\partial l^2} = A\beta(\beta-1) k^{\alpha} l^{\beta-2} < 0 \\ f_{kl} &= \frac{\partial^2 f}{\partial k \partial l} = A\alpha\beta k^{\alpha-1} l^{\beta-1} > 0 \end{aligned}$$

2. Show that $e_{q,k} = \alpha$ and $e_{q,l} = \beta$.

$$\begin{aligned} e_{q,k} &= \frac{\partial f}{\partial k} \cdot \frac{k}{f(k,l)} = \frac{A\alpha k^{\alpha-1} l^{\beta} \cdot k}{Ak^{\alpha} l^{\beta}} = \alpha \\ e_{q,l} &= \frac{\partial f}{\partial l} \cdot \frac{l}{f(k,l)} = \frac{A\beta k^{\alpha} l^{\beta-1} \cdot l}{Ak^{\alpha} l^{\beta}} = \beta \end{aligned}$$

3. In footnote 5, we defined the scale elasticity as

$$e_{q,t} = \frac{\partial f(tk, tl)}{\partial t} \cdot \frac{t}{f(tk, tl)}$$

where the expression is to be evaluated at $t = 1$. Show that, for this Cobb-Douglas function, $e_{q,t} = \alpha + \beta/$ Hence in this case the elasticity and the returns to scale of the production function agree (for more on this concept see Problem 9.9).

$$\begin{aligned} \frac{dF}{dt} &= A(\alpha+\beta) t^{\alpha+\beta-1} k^{\alpha} l^{\beta} \\ f(tk, tl) &= At^{\alpha+\beta} k^{\alpha} l^{\beta} \end{aligned}$$

$$e_{q,t} = \frac{\partial f(tk, tl)}{\partial t} \cdot \frac{t}{f(tk, tl)}\bigg|_{t=1} = \frac{A(\alpha+\beta) t^{\alpha+\beta-1} k^{\alpha} l^{\beta} \cdot t}{At^{\alpha+\beta} k^{\alpha} l^{\beta}}\bigg|_{t=1} = \alpha + \beta$$

4. Show that this function is quasi-concave.

$$\log f(k, l) = \log A + \alpha \log k + \beta \log l$$

Answer: This function is concave in $\log k$ and $\log l$, which implies f is log-concave, and hence quasi-concave.

5. Show that the function is concave for $\alpha + \beta \geq 1$ but not concave for $alpha + \beta > 1$.

$$H = \begin{bmatrix} f_{kk} & f_{kl} \\ f_{lk} & f_{ll} \end{bmatrix} = A \begin{bmatrix} \alpha(\alpha-1) k^{\alpha-2} l^{\beta} & \alpha\beta k^{\alpha-1} l^{\beta-1} \\ \alpha\beta k^{\alpha-1} l^{\beta-1} & \beta(\beta-1) k^{\alpha} l^{\beta-2} \end{bmatrix}$$

Problem 1 continued on next page...

$$\det(H) = A^2 k^{2\alpha-2} l^{2\beta-2} \left[(\alpha-1)(\beta-1) - \alpha\beta\right]$$

$$(\alpha-1)(\beta-1) \geq \alpha\beta \quad \Rightarrow \quad \alpha+\beta \leq 1$$

Answer: Thus, the function is concave when $\alpha+\beta \leq 1$, and not concave when $\alpha+\beta > 1$.

Problem 2

Consider a generalization of the production function in Example 9.3:

$$q = \beta_0 + \beta_1\sqrt{kl} + \beta_2 k + \beta_3 l$$

where

$$0 \geq \beta_i \quad i = 0, \ldots, 3$$

1. if this function is to exhibit constant returns to scale, what restrictions should be placed on the parameters $\beta_0, \cdots, \beta_3$?

$$q(tk,tl) = \beta_0 + \beta_1\sqrt{tk \cdot tl} + \beta_2(tk) + \beta_3(tl) = \beta_0 + \beta_1 t\sqrt{kl} + \beta_2 tk + \beta_3 tl$$

$$tq(k,l) = t(\beta_0 + \beta_1\sqrt{kl} + \beta_2 k + \beta_3 l) = t\beta_0 + t\beta_1\sqrt{kl} + t\beta_2 k + t\beta_3 l$$

Equality holds if and only if $\beta_0 = 0$. Hence, the condition for CRS is:

$$\boxed{\beta_0 = 0}$$

2. Show that, in the constant returns-to-scale case, this function exhibits diminishing marginal productivities and that marginal productivity function ar homogeneous of degree 0.

$$q = \beta_1\sqrt{kl} + \beta_2 k + \beta_3 l$$
$$\frac{\partial q}{\partial k} = \frac{\beta_1}{2}\sqrt{\frac{l}{k}} + \beta_2, \qquad \frac{\partial q}{\partial l} = \frac{\beta_1}{2}\sqrt{\frac{k}{l}} + \beta_3$$
$$\frac{\partial^2 q}{\partial k^2} = -\frac{\beta_1}{4}\sqrt{\frac{l}{k^3}} < 0, \qquad \frac{\partial^2 q}{\partial l^2} = -\frac{\beta_1}{4}\sqrt{\frac{k}{l^3}} < 0$$
$$\frac{\partial q(tk,tl)}{\partial (tk)} = \frac{\beta_1}{2}\sqrt{\frac{tl}{tk}} + \beta_2 = \frac{\beta_1}{2}\sqrt{\frac{l}{k}} + \beta_2 = \frac{\partial q}{\partial k}$$

Answer: Marginal product functions are homogeneous of degree 0

3. Calculate σ in this case. Although σ is not in general constant, for what values of the β's does $\sigma = 0, 1, \infty$

$$\text{MRTS} = \frac{\partial q/\partial k}{\partial q/\partial l} = \frac{\frac{\beta_1}{2}\sqrt{l/k} + \beta_2}{\frac{\beta_1}{2}\sqrt{k/l} + \beta_3}$$

cases:

- If $\beta_1 = 0$, then $q = \beta_2 k + \beta_3 l$ (perfect substitutes): $\sigma = \infty$
- If $\beta_2 = \beta_3 = 0$, then $q = \beta_1\sqrt{kl}$ (Cobb-Douglas): $\sigma = 1$

- If $\beta_1 = \beta_2 = 0$ or $\beta_1 = \beta_3 = 0$: one input is essential, no substitutability: $\sigma = 0$

$$\begin{aligned} \sigma = \infty \quad & \text{if } \beta_1 = 0 \\ \sigma = 1 \quad & \text{if } \beta_2 = \beta_3 = 0 \\ \sigma = 0 \quad & \text{if } \beta_1 = 0 \text{ and only one input matters} \end{aligned}$$

Problem 3

Consider the production function $q = f(k,l) = (\sqrt{k} + \sqrt{l})^2$. The cost of labor and capital are w and v respectively. Aside from the cost of labor and capital, there are no cost.

1. What type of production function is this? Hint: Not Cobb-Douglas

   special case of Constant Elasticity of Substitution production function.

2. Use your answer from part (a) to find the elasticity of substitution. Hint: There is a simple formula for

$$\sigma = \frac{1}{1-r}$$

   $r = \frac{1}{2} \Rightarrow \sigma = \frac{1}{1-\frac{1}{2}} = 2.$

$$\boxed{\sigma = 2}$$

3. Find RTS (l for k).

$$MP_k = \frac{\partial q}{\partial k} = \frac{\sqrt{k}+\sqrt{l}}{\sqrt{k}}, \quad MP_l = \frac{\partial q}{\partial l} = \frac{\sqrt{k}+\sqrt{l}}{\sqrt{l}}$$

$$RTS_{l,k} = \frac{MP_l}{MP_k} = \frac{\sqrt{k}}{\sqrt{l}} = \boxed{\sqrt{\frac{k}{l}}}$$

4. Find the total cost function.

$$C = vx^2 + w(\sqrt{q} - x)^2 = (v+w)x^2 - 2wx\sqrt{q} + wq$$

$$\frac{dC}{dx} = 2(v+w)x - 2w\sqrt{q} = 0 \Rightarrow x = \frac{w}{v+w}\sqrt{q}$$

$$k = \left(\frac{w}{v+w}\right)^2 q, \quad l = \left(\frac{v}{v+w}\right)^2 q$$

$$C(q) = vk + wl = \frac{vw}{v+w}q$$

$$C(q) = \frac{vw}{v+w}q$$

5. Use your answer from part (d) to find the average cost.

$$AC = \frac{C(q)}{q} = \boxed{\frac{vw}{v+w}}$$

6. Use your answer from part (d) to find marginal cost.

$$MC = \frac{dC}{dq} = \boxed{\frac{vw}{v+w}}$$

Problem 3 continued on next page. . .

7. Is the production function homothetic?

$$f(k,l) = (\sqrt{k} + \sqrt{l})^2 \quad \text{is a monotonic transformation of} \quad \sqrt{k} + \sqrt{l}$$

$\boxed{\text{Yes, it is homothetic.}}$

8. Does the production function exhibit increasing, constant, or decreasing return to scale?

$$f(\lambda k, \lambda l) = (\sqrt{\lambda k} + \sqrt{\lambda l})^2 = \lambda(\sqrt{k} + \sqrt{l})^2 = \lambda f(k,l)$$

$\boxed{\text{Constant Returns to Scale}}$

9. Is the expansion path linear?

$$\frac{k}{l} = \left(\frac{w}{v}\right)^2 \Rightarrow \boxed{\text{Yes,linear.}}$$

10. In general, what can you say about AC and MC for homothetic production functions with no fixed cost? For homothetic production functions with no fixed cost:

- If the production function has constant returns to scale, then both AC and MC are constant and equal.
- This is because total cost is linear in output.

$\boxed{\text{AC and MC are constant and equal for CRS, homothetic functions with no fixed cost.}}$