ECON 124: Problem Set #1

Due on Jun 5, 2025

Dr. Deniz Baglan

Alejandro Ouslan

Problem 1

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -3 & -2 & -1 \\ 1 & 2 & 3 \end{bmatrix}$. Find the following:

1. $A - C'$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}'$$
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 4 & 5 \end{bmatrix}$$

2. $C' + 3D$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + 3 \begin{bmatrix} -3 & -2 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} -9 & -6 & -3 \\ 3 & 6 & 9 \end{bmatrix}$$
$$\begin{bmatrix} -10 & -5 & -3 \\ 3 & 7 & 10 \end{bmatrix}$$

3. BA

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$
$$\begin{bmatrix} -5 & -3 & -3 \\ 4 & 5 & 6 \end{bmatrix}$$

4. CB

$$\begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & -2 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

5. $D'D$

$$\begin{bmatrix} -3 & -2 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ -2 & 2 \\ -1 & 3 \end{bmatrix}$$

Problem 2

Given the square matrices:

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 3 \\ 3 & -2 & -5 \end{bmatrix}, B = \begin{bmatrix} 3 & -6 & -3 \\ 7 & -14 & -7 \\ -1 & 2 & 1 \end{bmatrix}$$

Problem 2 continued on next page. . .

Verify that $AB = 0$

$$
\begin{aligned}
(AB)_{11} &= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = 3 \cdot 3 + (-1) \cdot 7 + 2 \cdot (-1) = 0 \\
(AB)_{12} &= a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = 3 \cdot (-6) + (-1) \cdot (-14) + 2 \cdot 2 = 0 \\
(AB)_{13} &= a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} = 3 \cdot (-3) + (-1) \cdot (-7) + 2 \cdot 1 = 0 \\
(AB)_{21} &= a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} = 1 \cdot 3 + 0 \cdot 7 + 3 \cdot (-1) = 0 \\
(AB)_{22} &= a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} = 1 \cdot (-6) + 0 \cdot (-14) + 3 \cdot 2 = 0 \\
(AB)_{23} &= a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = 1 \cdot (-3) + 0 \cdot (-7) + 3 \cdot 1 = 0 \\
(AB)_{31} &= a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} = 3 \cdot 3 + (-2) \cdot 7 + (-5) \cdot (-1) = 0 \\
(AB)_{32} &= a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} = 3 \cdot (-6) + (-2) \cdot (-14) + (-5) \cdot 2 = 0 \\
(AB)_{33} &= a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} = 3 \cdot (-3) + (-2) \cdot (-7) + (-5) \cdot 1 = 0
\end{aligned}
$$

Problem 3

The following table shows the number of personnel, in thousands, in three branches of the U.S. Army in 2001, and the changes in 2002 and 2003.

	Active Duty	Reserve	National Guard
2001	75	35	60
Change in 2002	5	-15	2
2003	-12	5	-17

Use matrix algebra to find the number of personnel in each branch

1. personnel 2001

$$
\begin{bmatrix} 75 \\ 35 \\ 60 \end{bmatrix} + \begin{bmatrix} 5 \\ -15 \\ 2 \end{bmatrix} = \begin{bmatrix} 80 \\ 20 \\ 62 \end{bmatrix}
$$

2. Personnel 2002

$$
\begin{bmatrix} 75 \\ 35 \\ 60 \end{bmatrix} + \begin{bmatrix} 5 \\ -15 \\ 2 \end{bmatrix} + \begin{bmatrix} -12 \\ 5 \\ -17 \end{bmatrix} = \begin{bmatrix} 68 \\ 25 \\ 45 \end{bmatrix}
$$

Problem 4

1. Suppose b_1 is the least squares estimator of the slope coefficient in a regression of Y on X and b_2 is the slope coefficient estimator in a "reverse" regression of X on Y. Show that $R^2 = b_1 b_2$ where R is

Problem 4 continued on next page...

the correlation between Y and X.

$$
\begin{aligned}
b_1 &= \frac{Cor(X,Y)}{Var(X)} \\
b_2 &= \frac{Cor(X,Y)}{Var(Y)} \\
b_1 b_2 &= \frac{Cor(X,Y)^2}{Var(X)Var(Y)} \\
&= \left(\frac{Cor(X,Y)}{\sqrt{Var(X)Var(Y)}}\right)^2 \\
&= R^2
\end{aligned}
$$

2. From a sample of 200 observation the following quantities were calculated:

$$\sum X = 11, \quad \sum Y = 20, \quad \sum X^2 = 12, \quad \sum XY = 22, \quad \sum Y^2 = 84$$

Estimate both regression equations an calculate R^2. Calculate the standard error of b_1.

$$
\begin{aligned}
\bar{X} = \frac{11}{200} = 0.055, \quad \bar{Y} &= \frac{20}{200} = 0.1 \\
S_{XX} = \sum X^2 - n\bar{X}^2 &= 12 - 200 \cdot (0.055)^2 = 11.395 \\
S_{YY} = \sum Y^2 - n\bar{Y}^2 &= 84 - 200 \cdot (0.1)^2 = 82 \\
S_{XY} = \sum XY - n\bar{X}\bar{Y} &= 22 - 200 \cdot 0.055 \cdot 0.1 = 20.9 \\
b_1 = \frac{S_{XY}}{S_{XX}} &= \frac{20.9}{11.395} \approx 1.834 \\
a_1 = \bar{Y} - b_1\bar{X} &= 0.1 - 1.834 \cdot 0.055 \approx -0.00087 \\
\hat{Y} &= -0.00087 + 1.834X \\
b_2 = \frac{S_{XY}}{S_{YY}} &= \frac{20.9}{82} \approx 0.255 \\
a_2 = \bar{X} - b_2\bar{Y} &= 0.055 - 0.255 \cdot 0.1 = 0.0295 \\
\hat{X} &= 0.0295 + 0.255Y \\
R^2 = b_1 b_2 &= 1.834 \cdot 0.255 \approx 0.467 \\
\text{RSS} = S_{YY} - b_1 S_{XY} &= 82 - 1.834 \cdot 20.9 \approx 43.6594 \\
\text{SE}_{b_1} &= \sqrt{\frac{1}{n-2} \cdot \frac{\text{RSS}}{S_{XX}}} \\
&= \sqrt{\frac{1}{198} \cdot \frac{43.6594}{11.395}} \\
&\approx \sqrt{0.01936} \approx 0.139
\end{aligned}
$$

Problem 5

Consider the equation $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2$, where $X_1 \sim (\mu_{X_1}, \sigma^2_{X_1})$ is independent of $X_2 \sim (\mu_{X_2}, \sigma^2_{X_2})$ with this information answer the following questions:

1. What is the expected value of X_1? What is the expected value of X_2?
2. What is the variance of X_1? what is the variance of X_2.
3. What is the expected value of Y?

4. What is the variance of Y?
5. What is the marginal effect of X_1 on Y? What is the marginal effect of X_2 on Y?

Problem 6

Consider an independent random sample of data of size n drawn from the continuous distribution of $X \sim N(\mu, \sigma^2)$. Suppose for whatever reason, we do not like the fist observtion and we propose the following estimator of μ:

$$\bar{X} = \frac{x_2 + x_3 + \ldots + x_n}{n}$$

1. what is expected value of $\bar{x}$

$$\mathbb{E}[\bar{X}] = \mathbb{E}\left[\frac{x_2 + x_3 + \cdots + x_n}{n}\right]$$
$$\mathbb{E}[\bar{X}] = \frac{1}{n}\sum_{i=2}^{n}\mathbb{E}[X_i]$$
$$\mathbb{E}[\bar{X}] = \frac{1}{n} \cdot (n-1) \cdot \mu = \frac{n-1}{n}\mu$$
$$\mathbb{E}[\bar{X}] = \frac{n-1}{n}\mu$$

2. What is the bias of $\bar{x}$?

$$\text{Bias}(\bar{X}) = \mathbb{E}[\bar{X}] - \mu$$
$$\text{Bias}(\bar{X}) = \frac{n-1}{n}\mu - \mu = \mu\left(\frac{n-1}{n} - 1\right) = \mu \cdot \left(\frac{-1}{n}\right)$$
$$\text{Bias}(\bar{X}) = -\frac{\mu}{n}$$

3. What is the variance of $\bar{x}$?

$$\text{Var}(\bar{X}) = \frac{1}{n^2} \cdot \sum_{i=2}^{n}\text{Var}(X_i)$$
$$\text{Var}(\bar{X}) = \frac{1}{n^2} \cdot (n-1) \cdot \sigma^2 = \frac{n-1}{n^2}\sigma^2$$
$$\text{Var}(\bar{X}) = \frac{n-1}{n^2}\sigma^2$$

4. What is the mean square error of $\bar{x}$?

$$\text{MSE}(\bar{X}) = \mathbb{E}[\bar{X}^2] - (\mathbb{E}[\bar{X}])^2$$
$$\mathbb{E}[\bar{X}^2] = \text{Var}(\bar{X}) + (\mathbb{E}[\bar{X}])^2$$
$$\mathbb{E}[\bar{X}] = \frac{n-1}{n}\mu, \quad \text{Var}(\bar{X}) = \frac{n-1}{n^2}\sigma^2$$
$$\mathbb{E}[\bar{X}^2] = \frac{n-1}{n^2}\sigma^2 + \left(\frac{n-1}{n}\mu\right)^2$$
$$= \frac{n-1}{n^2}\sigma^2 + \frac{(n-1)^2}{n^2}\mu^2$$
$$\text{MSE}(\bar{X}) = \frac{n-1}{n^2}\sigma^2 + \frac{(n-1)^2}{n^2}\mu^2 - \left(\frac{n-1}{n}\mu\right)^2$$
$$\text{MSE}(\bar{X}) = \frac{n-1}{n^2}\sigma^2$$

Problem 6 continued on next page...

5. What happens to the results in parts (a-d) when the sample size n tends to infinity? what can be said about this estimator in this scenario?
   Answer: Given the law of large number even with the bias as the number of observations increases the parameter will converge to the true parameter

Problem 7

Consider the following multiple linear regression model

$$y = X\beta + u$$

where y is $n \times 1$, X is $n \times k$ and u is $n \times 1$ such that $u|x \sim N(0, \sigma^2 I_n)$. Write $Y = \hat{Y} + \hat{u}$,where $\hat{y} = X\hat{\beta}$ is the least squares predicted values.

1. Show that $(\hat{\beta} - \beta) = Au$ and $\hat{u} = Mu$, what are your A and M?

$$\begin{aligned}
\hat{\beta} &= (X'X)^{-1}X'y \\
\hat{\beta} &= (X'X)^{-1}X'(X\beta + u) = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'u \\
\hat{\beta} &= \beta + (X'X)^{-1}X'u \\
\hat{\beta} - \beta &= (X'X)^{-1}X'u \\
A &= (X'X)^{-1}X' \\
\hat{\beta} - \beta &= Au \\
\hat{y} = X\hat{\beta} &= X\left(\beta + (X'X)^{-1}X'u\right) \\
\hat{y} &= X\beta + X(X'X)^{-1}X'u \\
\hat{u} = y - \hat{y} &= X\beta + u - \left(X\beta + X(X'X)^{-1}X'u\right) \\
\hat{u} &= u - X(X'X)^{-1}X'u \\
M &= I_n - X(X'X)^{-1}X' \\
\hat{u} &= Mu
\end{aligned}$$

2. Show that $\bar{y} =$ the mean of the predicted values $\hat{y}$ The mean of the observed values y is:

$$\begin{aligned}
\bar{y} &= \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}\mathbf{1}'y \\
\hat{y} &= X\hat{\beta} \\
\bar{\hat{y}} &= \frac{1}{n}\mathbf{1}'\hat{y} = \frac{1}{n}\mathbf{1}'X\hat{\beta} \\
\bar{\hat{y}} &= \frac{1}{n}\mathbf{1}'X(X'X)^{-1}X'y \\
\bar{\hat{y}} &= \bar{y}
\end{aligned}$$

3. Show that $X'\hat{u} = 0, \hat{y}\hat{u} = 0$

$$\begin{aligned}
\hat{u} &= y - \hat{y} = y - X\hat{\beta} \\
X'\hat{u} &= X'(y - X\hat{\beta}) = X'y - X'X\hat{\beta} \\
X'\hat{u} &= X'y - X'y = 0 \\
\hat{y}'\hat{u} &= (X\hat{\beta})'(y - X\hat{\beta}) \\
\hat{y}'\hat{u} &= \hat{\beta}'X'(y - X\hat{\beta}) = \hat{\beta}'X'y - \hat{\beta}'X'X\hat{\beta} \\
\hat{y}'\hat{u} &= \hat{\beta}'X'y - \hat{\beta}'X'X(X'X)^{-1}X'y \\
&= \hat{\beta}'X'y - \hat{\beta}'X'y = 0
\end{aligned}$$

Problem 7 continued on next page...

4. Derive R^2 for the model where the first column of X has a constant.

$$\begin{aligned} R^2 &= 1 - \frac{\text{SSR}}{\text{SST}} \\ \text{SST} &= (y - \bar{y})'(y - \bar{y}) = y'y - n\bar{y}^2 \\ \text{SSR} &= \hat{u}'\hat{u} = u'Mu \\ \text{SSE} &= \hat{y}'\hat{y} - n\bar{y}^2 \\ R^2 &= 1 - \frac{\hat{u}'\hat{u}}{y'y - n\bar{y}^2} \end{aligned}$$