ESMA 5015: Accept Reject

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Contents

1 notes for 2025-03-06

$$
\begin{aligned}
& E(w) = \alpha\beta \\
w \sim gamma(\alpha, \beta) \quad & var(w) = \alpha\beta^2
\end{aligned}
$$

$$
\begin{aligned}
F_y(g) &= \frac{1}{2}e^{-y} + \frac{1}{2}ye^{-y} \\
E[y] &= \int_0^\infty y f_y(y) dy \\
&= \frac{1}{2}E[y_1] + \frac{1}{2}E[y_2] \\
y_1 &\sim gamma(1, 1) \\
y_2 &\sim gamma(2, 1) \\
&= \frac{1}{2}1 + \frac{1}{2}2 \\
&= \frac{3}{2}
\end{aligned}
$$

$$
\begin{aligned}
var(y) &= E[y^2] - E[y]^2 \\
E[y^2] &= \int_0^\infty y^2 f_y(y) dy \\
&= .5E[y_1^2] + .5E[y_2^2] \\
&= \frac{1}{2}2 + \frac{1}{2}6 \\
&= 4
\end{aligned}
$$

$$
\begin{aligned}
E[y_1^2] &= var(y_1) + E[y_1]^2 \\
&= 1 + 1^2 \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
E[y_2^2] &= var(y_2) + E[y_2]^2 \\
&= 2 + 2^2 \\
&= 6
\end{aligned}
$$

$$
\begin{aligned}
&= 4 - \frac{3}{2}^2 \\
&= 4 - \frac{9}{4} \\
&= \frac{7}{4}
\end{aligned}
$$

2 Consideren un algoritmo Accept-Reject para simular $X \sim N(0,1)$

1. usando $Y \sim cauchy(0,1)$

$$F_Y(y) = \frac{1}{\pi}\frac{1}{1+y^2} \quad y \in \mathbb{R}$$

2. usando $Y \sim double-exponential(0,1)$

$$f_Y(y) = \frac{1}{2}e^{-|y|} \quad y \in \mathbb{R}$$

2.1 theory

1. calculate $M = \sup_x \frac{f_X(x)}{f_Y(y)}$

$$
\begin{aligned}
&= \sup_x \frac{\frac{1}{\sqrt{2\pi}}e^{-x^2/2}}{\frac{1}{\pi}\frac{1}{1+x^2}} \\
&= \sup_x \frac{\pi}{\sqrt{2}}e^{-x^2/2}(1+x^2)
\end{aligned}
$$

sea $h(x) = e^{-x^2/2}(1+x^2)$

$$
\begin{aligned}
\frac{dh(x)}{dx} &= -xe^{-x^2/2}(1+x^2) + e^{-x^2/2}2x \\
&= \vdots \\
&= -xe^{-x^2/2}(x^2-1)
\end{aligned}
$$

$\frac{dh(x)}{dx} = 0 \Rightarrow x = 0, \pm 1$ puede demostra maxiomo occurra en $x = \pm 1 \implies M = \frac{w\sqrt{\pi}}{\sqrt{2}}e^{-1/2} = 1.52$

2.2 algorithm

1. simular $Y \sim cauchy(0,1)$ y $U \sim U(0,1)$ independientes.
2. si

$$\begin{aligned} U < \frac{f_X(Y)}{Mf_Y(Y)} \\ &= \frac{1}{\sqrt{2\pi}}e^{-1/2}\frac{\sqrt{\pi}}{\sqrt{\pi}}(1+Y^2) \\ &= \frac{1}{2}(1+Y^2)e^{\frac{-Y^2+1}{2}} \end{aligned}$$

   aceptar $X = Y$

3. si no, regresar a paso 1.

2.3 steps

1.

$$\begin{aligned} M = \sup_x \frac{f_X(x)}{f_Y(y)} &= \sup_x \frac{\int \frac{1}{\sqrt{2\pi}}e^{-x^2/2}}{\frac{1}{2}e^{-|x|}} \\ &= \sup_x \frac{\sqrt{2}}{\sqrt{\pi}}e^{-x^2/2+|x|} \\ h(x) = \ln\left(e^{-x^2/2+|x|}\right) \\ \frac{dh(x)}{dx} = \frac{d}{dx}\left(-\frac{x^2}{2}+|x|\right) = \begin{cases} -x-1 & x<0 \\ -x+1 & x>0 \end{cases} \\ \frac{dh(x)}{dx} = 0 \Rightarrow x = \pm 1 \end{aligned}$$

   debe demostrar que el supremo ocure en $x = \pm 1$

2.4 algorithm

1. $M = \frac{\sqrt{2}}{\sqrt{\pi}}e^{-1/2+1} = 1.31$
2. simular $Y \sim double - exponential(0,1)$ y $U \sim U(0,1)$ independientes.
3. si

$$\begin{aligned} U < \frac{f_X(Y)}{Mf_Y(Y)} \\ &= e^{\frac{-y^2}{2}+|y|-1/2} \end{aligned}$$

   entonces aceptar $X = Y$

4. si no, regresar a paso 2.

3 Generar una $X \sim Beta(\alpha, \beta)$

Donte $0 < x < 1$ y $\alpha, \beta > 0$

Utilizar $y \sim U(0,1)$.

$$M = \sup_x \frac{f_X(x)}{f_Y(y)} = \frac{\frac{\gamma(\alpha+\beta)}{\gamma(\alpha)\gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}}{1}$$
$$h(x) = \ln\left(X^{\alpha-1}(1-x)^{\beta-1}\right)$$
$$\frac{dh(x)}{dx} = \frac{\alpha-1}{x} - \frac{\beta-1}{1-x}$$
$$\frac{dh(x)}{dx} = 0 \Rightarrow x = \frac{\alpha-1}{\alpha+\beta-2} \text{ y } \beta! = 1, \alpha! = 1$$

puede demostrar que en esta X OCURRE UN maxiomo

3.1 algorithm

1. Generar $y \sim U(0,1)$ y $U \sim U(0,1)$ independientes.
2. Si
$$u < \frac{f_X(y)}{Mf_Y(y)} = \frac{\frac{\gamma(\alpha+\beta)}{\gamma(\alpha)\gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}}{\frac{\gamma(\alpha+\beta)}{\gamma(\alpha)\gamma(\beta)} \frac{\alpha-1}{\alpha+\beta-2}^{\alpha-1}(1-\frac{\alpha-1}{\alpha+\beta-2})^{\beta-1}}$$
define $X = y$
3. si no, regresar a paso 1.

4 Ejemplo generar $X \sim Beta(\alpha = 2.7, \beta = 6.3)$

$M = 2.6667444$ que ocurre en $x = \frac{2.7-1}{2.7+6.3-2} = 0.2428$

4.1 algorithm

1. Generar $y \sim U(0,1)$ y $U \sim U(0,1)$ independientes.
2. Si
$$u < \frac{f_X(y)}{Mf_Y(y)} = \frac{\frac{\gamma(\alpha+\beta)}{\gamma(\alpha)\gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}}{2.666744}$$
Define $X = y$
3. si no, regresar a paso 1.

5 Bayesian Inference

Accept y Metropolis Hasting (topico de la proxima clase) surgen naturlament en estadisticas Bayesiana. En el Analisis Bayesiano ademas de espexifixar el modelo de los datos observadpos $X = x_1, x_2, \ldots, x_n$ dado un vector de parametros desxonoxidos θ definido por $f(x|\theta)$, se define θ como una variable aleatoria que tiene una distribucion priori $\pi(\theta)$. El Conocimiento de θ se axtualiza con cm el conocimiento que se obtiene de los datos basadosk la inferencia conxerniente a θ. en la distribucion posterior definida por

$$\pi(\theta|x) = \frac{f(x|\theta)\pi(\theta)}{\int_\Theta f(x|\theta)\pi(\theta)d\theta} \text{ Teorema de Bayes}$$

$$\pi(\theta|x)\alpha f(x|\theta)\pi(\theta)$$

5.1 Ejemplo

$x_1, x_2, \ldots, x_n$ iid $Bernulli(\theta)$

5.1.1 Frecuentista

IC de $(1-\alpha)100\%$ para θ

$$\hat{\theta} \pm z_{\alpha/2}\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}$$

5.1.2 Bayesiana

$$f(x_1, x_2, \ldots, x_n|\theta) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$$

$$\frac{\pi(\theta|x_1, x_2, \ldots, x_n) = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}}{\int_0 \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta}$$