ESMA 5015: Simulaciones Estocasticas

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Contents

1 notes for 2025-03-06

$$\begin{aligned} &E(w) = \alpha\beta \\ w \sim gamma(\alpha,\beta) \quad &var(w) = \alpha\beta^2 \end{aligned}$$

$$\begin{aligned} F_y(g) &= \frac{1}{2}e^{-y} + \frac{1}{2}ye^{-y} \\ E[y] &= \int_0^\infty y f_y(y) dy \\ &= \frac{1}{2}E[y_1] + \frac{1}{2}E[y_2] \\ y_1 &\sim gamma(1,1) \\ y_2 &\sim gamma(2,1) \\ &= \frac{1}{2}1 + \frac{1}{2}2 \\ &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} var(y) &= E[y^2] - E[y]^2 \\ E[y^2] &= \int_0^\infty y^2 f_y(y) dy \\ &= .5E[y_1^2] + .5E[y_2^2] \\ &= \frac{1}{2}2 + \frac{1}{2}6 \\ &= 4 \end{aligned}$$

$$\begin{aligned} E[y_1^2] &= var(y_1) + E[y_1]^2 \\ &= 1 + 1^2 \\ &= 2 \end{aligned}$$

$$
\begin{aligned}
E[y_2^2] &= var(y_2) + E[y_2]^2 \\
&= 2 + 2^2 \\
&= 6
\end{aligned}
$$

$$
\begin{aligned}
&= 4 - \frac{3}{2}^2 \\
&= 4 - \frac{9}{4} \\
&= \frac{7}{4}
\end{aligned}
$$

2 Consideren un algoritmo Accept-Reject para simular $X \sim N(0,1)$

1. usando $Y \sim cauchy(0,1)$

$$F_Y(y) = \frac{1}{\pi}\frac{1}{1+y^2} \quad y \in \mathbb{R}$$

2. usando $Y \sim double - exponential(0,1)$

$$f_Y(y) = \frac{1}{2}e^{-|y|} \quad y \in \mathbb{R}$$

2.1 theory

1. calculate $M = \sup_x \frac{f_X(x)}{f_Y(y)}$

$$
\begin{aligned}
&= \sup_x \frac{\frac{1}{\sqrt{2\pi}}e^{-x^2/2}}{\frac{1}{\pi}\frac{1}{1+x^2}} \\
&= \sup_x \frac{\pi}{\sqrt{2}}e^{-x^2/2}(1+x^2)
\end{aligned}
$$

sea $h(x) = e^{-x^2/2}(1+x^2)$

$$
\begin{aligned}
\frac{dh(x)}{dx} &= -xe^{-x^2/2}(1+x^2) + e^{-x^2/2}2x \\
&= \vdots \\
&= -xe^{-x^2/2}(x^2-1)
\end{aligned}
$$

$\frac{dh(x)}{dx} = 0 \Rightarrow x = 0, \pm 1$ puede demostra maxiomo occurra en $x = \pm 1 \implies M = \frac{w\sqrt{\pi}}{\sqrt{2}}e^{-1/2} = 1.52$

2.2 algorithm

1. simular $Y \sim cauchy(0,1)$ y $U \sim U(0,1)$ independientes.
2. si

$$
\begin{aligned}
U < \frac{f_X(Y)}{Mf_Y(Y)} \\
&= \frac{1}{\sqrt{2\pi}}e^{-1/2}\frac{\sqrt{\pi}}{\sqrt{\pi}}(1+Y^2) \\
&= \frac{1}{2}(1+Y^2)e^{\frac{-Y^2+1}{2}}
\end{aligned}
$$

aceptar $X = Y$

3. si no, regresar a paso 1.

2.3 steps

1.

$$
\begin{aligned}
M = \sup_x \frac{f_X(x)}{f_Y(y)} &= \sup_x \frac{\int \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}{\frac{1}{2} e^{-|x|}} \\
&= \sup_x \frac{\sqrt{2}}{\sqrt{\pi}} e^{-x^2/2+|x|}
\end{aligned}
$$

$$
h(x) = \ln\left(e^{-x^2/2+|x|}\right)
$$

$$
\frac{dh(x)}{dx} = \frac{d}{dx}\left(-\frac{x^2}{2} + |x|\right) = \begin{cases} -x-1 & x<0 \\ -x+1 & x>0 \end{cases}
$$

$$
\frac{dh(x)}{dx} = 0 \Rightarrow x = \pm 1
$$

debe demostrar que el supremo ocure en $x = \pm 1$

2.4 algorithm

1. $M = \frac{\sqrt{2}}{\sqrt{\pi}} e^{-1/2+1} = 1.31$
2. simular $Y \sim double - exponential(0,1)$ y $U \sim U(0,1)$ independientes.
3. si

$$
\begin{aligned}
U < \frac{f_X(Y)}{M f_Y(Y)} \\
&= \frac{\sqrt{\pi}}{\sqrt{2} e^1} e^{\frac{-y^2}{2}+|y|} \\
&= e^{\frac{-y^2}{2}+|y|-1/2}
\end{aligned}
$$

   entonces aceptar $X = Y$

4. si no, regresar a paso 2.