ESMA 6787: Exam 2

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Problem 1: Acceptance of Syllabus

I have read the syllabus, understand its contents, and have no questions.

Problem 2: Definitions

(a) **Sample Space:** The sample space is the set of all possible outcomes of an experiment. It is to say everything that could happen in the experiment

(b) **Kolmgorov Axioms of Probability:** It is a set of rules that define the probability of an event. Those rules can be summarized by the sum of the probability of all the events is 1, the probability of a given event should be non negative and if the events are disjoint then the probability of the union of the events is the sum of the probability of the events

(c) **Exponential family:** A family of distributions is said to be an exponential family if the pdf can be written in the form

(d) **Convergence in distribution:** Convergence of a distributions shows that as the size of given set of random outcome it the shape or distribution of that set will converge to a given distribution

(e) **Convergence in Probability:** Convergence in Probability tells that as your sample gets larger the probability of a a outcome will get close or more accurate compared to the original probability distribution

(f) **Almost sure convergence (or convergence with probability 1):** Almost sure convergence tells if you were to follow an outcome repeated times you will almost always see it converge to something.

(g) **Weak law of large numbers:** It tells that as you sample size increases your estimator will be closer to true estimator

(h) **Strong law of large numbers:** If had infinite samples that your estimator will converge to the true estimator.

(i) **Characteristics functions:** They are representations of the distributions.

Problem 3:

Show that each of the following families of distributions is an exponential family,

(a) The family of Bernoulli distribution with a unknown value of the parameter p.

$$f(x;p) = p^x(1-p)^{1-x}, \quad x \in \{0,1\}$$

$$f(x;p) = \exp\left(x\log(p) + (1-x)\log(1-p)\right)$$

Hence, it is in the form of an exponential family with natural parameter $\theta = \log\left(\frac{p}{1-p}\right)$, sufficient statistic $T(x) = x$, and log-partition function $A(\theta) = \log(1+\exp(\theta))$.

(b) The family of Poisson distributions with an unknown mean λ

$$f(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x \in \{0,1,2,\dots\}$$

$$f(x;\lambda) = \exp\left(x\log(\lambda) - \lambda - \log(x!)\right)$$

Hence, it is in the form of an exponential family with natural parameter $\theta = \log(\lambda)$, sufficient statistic $T(x) = x$, and log-partition function $A(\theta) = e^\theta$.

(c) The family of negative binomial distributions for which the value of r is known and the value of p is unknown.

$$f(x;p) = \binom{x+r-1}{x} p^x(1-p)^r, \quad x = 0,1,2,\dots$$

$$f(x;p) = \exp\left(x\log(p) + (r-1)\log(1-p) + \log\left(\binom{x+r-1}{x}\right)\right)$$

Thus, it is in the form of an exponential family with natural parameter $\theta = \log\left(\frac{p}{1-p}\right)$, sufficient statistic $T(x) = x$, and log-partition function $A(\theta) = -r\log(1-\exp(\theta))$.

(d) The family of normal distributions with an unknown mean and a known variance σ^2.

$$f(x;\mu) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$f(x;\mu) = \exp\left(-\frac{1}{2\sigma^2}\left(x^2 - 2\mu x + \mu^2\right)\right)$$

Hence, it is in the form of an exponential family with natural parameter $\theta = \frac{\mu}{\sigma^2}$, sufficient statistic $T(x) = x$, and log-partition function $A(\theta) = \frac{\mu^2}{2\sigma^2}$.

(e) The family of normal distributions with an unknown variance and a known mean.

$$f(x;\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$f(x;\sigma^2) = \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2 - \frac{1}{2}\log(2\pi\sigma^2)\right)$$

Hence, it is in the form of an exponential family with natural parameter $\theta = -\frac{1}{2\sigma^2}$, sufficient statistic $T(x) = (x-\mu)^2$, and log-partition function $A(\theta) = -\frac{1}{2}\log(-2\pi\theta)$.

(f) The family of gamma distributions for which the value of α is unknown and the value of β is known.

$$f(x;\alpha) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}, \quad x > 0$$

$$f(x;\alpha) = \exp\left((\alpha-1)\log(x) - \frac{x}{\beta} - \log(\Gamma(\alpha)) - \alpha\log(\beta)\right)$$

Hence, it is in the form of an exponential family with natural parameter $\theta = -\frac{1}{\beta}$, sufficient statistic $T(x) = x$, and log-partition function $A(\theta) = -\alpha\log(-\theta\beta)$.

(g) The family of gamma distributions for which the value of α is known and the value of β is unknown.

$$f(x;\beta) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}, \quad x > 0$$

$$f(x;\beta) = \exp\left((\alpha-1)\log(x) - \frac{x}{\beta} - \log(\Gamma(\alpha)) - \alpha\log(\beta)\right)$$

Hence, it is in the form of an exponential family with natural parameter $\theta = -\frac{1}{\beta}$, sufficient statistic $T(x) = x$, and log-partition function $A(\theta) = \alpha\log(-\theta)$.

(h) The family of beta distributions for which the value of α is unknown and the value of β is known.

$$f(x;\alpha) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \quad x \in [0,1]$$

$$f(x;\alpha) = \exp\left((\alpha-1)\log(x) + (\beta-1)\log(1-x) - \log(B(\alpha,\beta))\right)$$

Hence, it is in the form of an exponential family with natural parameter $\theta_1 = \alpha - 1$ and $\theta_2 = \beta - 1$, sufficient statistic $T(x) = (\log(x), \log(1-x))$, and log-partition function $A(\theta_1,\theta_2) = \log(B(\alpha,\beta))$.

(i) The family of beta distributions for which the value of α is known and the value of β is unknown.

$$f(x;\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \quad x \in [0,1]$$

$$f(x;\beta) = \exp\left((\alpha-1)\log(x) + (\beta-1)\log(1-x) - \log(B(\alpha,\beta))\right)$$

Hence, it is in the form of an exponential family with natural parameter $\theta_1 = \alpha - 1$ and $\theta_2 = \beta - 1$, sufficient statistic $T(x) = (\log(x), \log(1-x))$, and log-partition function $A(\theta_1, \theta_2) = \log(B(\alpha,\beta))$.

Problem 4:

Let X be a random variable with a Student's t distribution with p degrees of freedom.

(a) Derive the mean and variance of X.

$$f(x) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{p\pi}\,\Gamma\left(\frac{p}{2}\right)}\left(1+\frac{x^2}{p}\right)^{-\frac{p+1}{2}}$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$g(-x) = (-x)\cdot f(-x) = -x \cdot f(x) = -g(x)$$

$$E[X] = 0 \quad \text{for } p > 1$$

(b) Show that X^2 has an $F_{1,p}$ distribution.

$$X = \frac{Z}{\sqrt{V/p}}$$

where $Z \sim N(0,1)$ and $V \sim \chi^2(p)$ are independent.

$$X^2 = \frac{Z^2}{V/p}$$

Since the square of a standard normal variable is a Chi-squared variable with 1 degree of freedom, $Z^2 \sim \chi^2(1)$.

$$X^2 = \frac{Z^2/1}{V/p}$$

The definition of the F-distribution with d_1 and d_2 degrees of freedom is:

$$F_{d_1,d_2} = \frac{U_1/d_1}{U_2/d_2}$$

where $U_1 \sim \chi^2(d_1)$ and $U_2 \sim \chi^2(d_2)$ are independent.
Identifying $U_1 = Z^2$, $d_1 = 1$, $U_2 = V$, and $d_2 = p$, we get:

$$X^2 \sim F_{1,p}$$

(c) Let $f(x|p)$ denote the pdf of X. Show that

$$\lim_{p\to\infty} f(x|p) \to \frac{1}{\sqrt{2\pi}}\exp\{-\frac{x^2}{2}\}$$

at each value of x, $-\infty < x < \infty$. This correctly suggests that as $p \to \infty$, X converges in distributions to a $N(0,1)$ random variable (Hint: Use Stirirling's Formula).

Answer:

$$f(x|p) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{p\pi}\Gamma\left(\frac{p}{2}\right)}\left(1+\frac{x^2}{p}\right)^{-\frac{p+1}{2}}$$

Since $\lim_{n\to\infty}(1+\frac{a}{n})^n = e^a$:

$$\lim_{p\to\infty}\left(1+\frac{x^2}{p}\right)^{-\frac{p+1}{2}} = \lim_{p\to\infty}\left[\left(1+\frac{x^2}{p}\right)^p\right]^{-\frac{p+1}{2p}} = (e^{x^2})^{-1/2} = e^{-x^2/2}$$

Using Stirling's approximation $\frac{\Gamma(n+k)}{\Gamma(n)} \sim n^k$ as $n \to \infty$: Let $n = p/2$ and $k = 1/2$:

$$\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \approx \left(\frac{p}{2}\right)^{1/2}$$

$$\lim_{p\to\infty}\frac{1}{\sqrt{p\pi}}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} = \frac{1}{\sqrt{p\pi}}\sqrt{\frac{p}{2}} = \frac{1}{\sqrt{2\pi}}$$

$$\lim_{p\to\infty} f(x|p) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

(d) Use the results of parts (a) and (b) to argue that, as $p \to \infty$, X converges in distribution to a X_1^2 random variable.
As $p \to \infty$, X converge to a normal distribution standard and normal random variable $Z \sim N(0,1)$. Then, if $X \xrightarrow{d} Z$, then $g(X) \xrightarrow{d} g(Z)$ for any continuous function g. Let $g(x) = x^2$:

$$X^2 \xrightarrow{d} Z^2$$

Since $Z \sim N(0,1)$, its square is distributed as:

$$X^2 \xrightarrow{d} \chi_1^2$$

(e) What might you conjecture about the distributional limit, as $p \to \infty$, of $F_{q,p}$
As the denominator degrees of freedom $p \to \infty$, the distribution of $q \cdot F_{q,p}$ converges in distribution to a Chi-squared random variable with q degrees of freedom:

$$q \cdot F_{q,p} \xrightarrow{d} \chi_q^2$$

$$F_{q,p} \xrightarrow{d} \frac{\chi_q^2}{q}$$

$$F_{q,p} = \frac{U/q}{V/p}$$

$$\frac{V}{p} \xrightarrow{P} E\left[\frac{V}{p}\right] = \frac{p}{p} = 1$$

$$F_{q,p} = \frac{U/q}{V/p} \xrightarrow{d} \frac{U/q}{1} = \frac{U}{q}$$

$$q \cdot F_{q,p} \xrightarrow{d} U \sim \chi_q^2$$

Optional Problem 1:

Suppose that X has the log-normal distribution with parameters μ and σ^2. Find the distribution of $\frac{1}{X}$.
Let X follow a lognormal distribution, so:

$$X \sim \text{Lognormal}(\mu, \sigma^2) \quad \text{implies} \quad \log(X) \sim N(\mu, \sigma^2)$$

This means that:

$$X = \exp(Z) \quad \text{where} \quad Z \sim N(\mu, \sigma^2)$$

Now, we are asked to find the distribution of $Y = \frac{1}{X}$.
We can write Y as:

$$Y = \frac{1}{X} = \frac{1}{\exp(Z)} = \exp(-Z)$$

Since $Z \sim N(\mu, \sigma^2)$, the random variable $-Z$ follows a normal distribution with parameters $-\mu$ and σ^2, i.e.,

$$-Z \sim N(-\mu, \sigma^2)$$

Thus, $Y = \exp(-Z)$ follows a lognormal distribution, and we conclude:

$$Y \sim \text{Lognormal}(-\mu, \sigma^2)$$

Therefore, the distribution of $Y = \frac{1}{X}$ is a lognormal distribution with parameters $-\mu$ and σ^2:

$$Y \sim \text{Lognormal}(-\mu, \sigma^2)$$

Optional Problem 2:

Suppose $\bar{X}$ is the mean of 100 observations from a population with mean μ and variance $\sigma^2 = 9$. Find limits between which $\bar{X} - \mu$ will lie with probability at least 90%. Use both Chebyshev's inequality and the Central Limit Theorem, and comment on each.