ESMA 6787: Exam 2

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Problem 1: Acceptance of Syllabus

I have read the syllabus, understand its contents, and have no questions.

Problem 2: Definitions

(a) **Sample Space:** The sample space is the set of all possible outcomes of an experiment. It is to say everything that could happen in the experiment

(b) **Kolmgorov Axioms of Probability:** It is a set of rules that define the probability of an event. Those rules can be summarized by the sum of the probability of all the events is 1, the probability of a given event should be non negative and if the events are disjoint then the probability of the union of the events is the sum of the probability of the events

(c) **Exponential family:** A family of distributions is said to be an exponential family if the pdf can be written in the form

(d) **Convergence in distribution:**

(e) **Convergence in Probability:**

(f) **Almost sure convergence (or convergence with probability 1):**

(g) **Weak law of large numbers:**

(h) **Strong law of large numbers:**

(i) **Characteristics functions:**

Problem 3:

Show that each of the following families of distributions is an exponential family,

(a) The family of Bernoulli distribution with a unknown value of the parameter p.

$$f(x;p) = p^x(1-p)^{1-x}, \quad x \in \{0,1\}$$

$$f(x;p) = \exp\left(x\log(p) + (1-x)\log(1-p)\right)$$

Hence, it is in the form of an exponential family with natural parameter $\theta = \log\left(\frac{p}{1-p}\right)$, sufficient statistic $T(x) = x$, and log-partition function $A(\theta) = \log(1+\exp(\theta))$.

(b) The family of Poisson distributions with an unknown mean λ

$$f(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x \in \{0,1,2,\dots\}$$

$$f(x;\lambda) = \exp\left(x\log(\lambda) - \lambda - \log(x!)\right)$$

Hence, it is in the form of an exponential family with natural parameter $\theta = \log(\lambda)$, sufficient statistic $T(x) = x$, and log-partition function $A(\theta) = e^\theta$.

(c) The family of negative binomial distributions for which the value of r is known and the value of p is unknown.

$$f(x;p) = \binom{x+r-1}{x}p^x(1-p)^r, \quad x = 0,1,2,\dots$$

$$f(x;p) = \exp\left(x\log(p) + (r-1)\log(1-p) + \log\left(\binom{x+r-1}{x}\right)\right)$$

Thus, it is in the form of an exponential family with natural parameter $\theta = \log\left(\frac{p}{1-p}\right)$, sufficient statistic $T(x) = x$, and log-partition function $A(\theta) = -r\log(1-\exp(\theta))$.

(d) The family of normal distributions with an unknown mean and a known variance σ^2.

$$f(x;\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$f(x;\mu) = \exp\left(-\frac{1}{2\sigma^2}\left(x^2 - 2\mu x + \mu^2\right)\right)$$

Hence, it is in the form of an exponential family with natural parameter $\theta = \frac{\mu}{\sigma^2}$, sufficient statistic $T(x) = x$, and log-partition function $A(\theta) = \frac{\mu^2}{2\sigma^2}$.

(e) The family of normal distributions with an unknown variance and a known mean.

$$f(x;\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$f(x;\sigma^2) = \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2 - \frac{1}{2}\log(2\pi\sigma^2)\right)$$

Hence, it is in the form of an exponential family with natural parameter $\theta = -\frac{1}{2\sigma^2}$, sufficient statistic $T(x) = (x-\mu)^2$, and log-partition function $A(\theta) = -\frac{1}{2}\log(-2\pi\theta)$.

(f) The family of gamma distributions for which the value of α is unknown and the value of β is known.

$$f(x;\alpha) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}, \quad x > 0$$

$$f(x;\alpha) = \exp\left((\alpha-1)\log(x) - \frac{x}{\beta} - \log(\Gamma(\alpha)) - \alpha\log(\beta)\right)$$

Hence, it is in the form of an exponential family with natural parameter $\theta = -\frac{1}{\beta}$, sufficient statistic $T(x) = x$, and log-partition function $A(\theta) = -\alpha\log(-\theta\beta)$.

(g) The family of gamma distributions for which the value of α is known and the value of β is unknown.

$$f(x;\beta) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}, \quad x > 0$$

$$f(x;\beta) = \exp\left((\alpha-1)\log(x) - \frac{x}{\beta} - \log(\Gamma(\alpha)) - \alpha\log(\beta)\right)$$

Hence, it is in the form of an exponential family with natural parameter $\theta = -\frac{1}{\beta}$, sufficient statistic $T(x) = x$, and log-partition function $A(\theta) = \alpha\log(-\theta)$.

(h) The family of beta distributions for which the value of α is unknown and the value of β is known.

$$f(x;\alpha) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \quad x \in [0,1]$$

$$f(x;\alpha) = \exp\left((\alpha-1)\log(x) + (\beta-1)\log(1-x) - \log(B(\alpha,\beta))\right)$$

Hence, it is in the form of an exponential family with natural parameter $\theta_1 = \alpha - 1$ and $\theta_2 = \beta - 1$, sufficient statistic $T(x) = (\log(x), \log(1-x))$, and log-partition function $A(\theta_1,\theta_2) = \log(B(\alpha,\beta))$.

(i) The family of beta distributions for which the value of α is known and the value of β is unknown.

$$f(x;\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \quad x \in [0,1]$$

$$f(x;\beta) = \exp\left((\alpha-1)\log(x) + (\beta-1)\log(1-x) - \log(B(\alpha,\beta))\right)$$

Hence, it is in the form of an exponential family with natural parameter $\theta_1 = \alpha - 1$ and $\theta_2 = \beta - 1$, sufficient statistic $T(x) = (\log(x), \log(1-x))$, and log-partition function $A(\theta_1,\theta_2) = \log(B(\alpha,\beta))$.

Problem 4:

Let X be a random variable with a Student's t distribution with p degrees of freedom.

(a) Derive the mean and variance of X.

(b) Show that X^2 has an $F_{1,p}$ distribution.

(c) Let $f(x|p)$ denote the pdf of X. Show that

$$\lim_{p\to\infty} f(x|p) \to \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\}$$

at each value of x, $-\infty < x < \infty$. This correctly suggests that as $p \to \infty$, X converges in distributions to a $N(0,1)$ random variable (Hint: Use Stirirling's Formula).

(d) Use the results of parts (a) and (b) to argue that, as $p \to \infty$, X converges in distribution to a X_1^2 random variable.

(e) What might you conjecture about the distributional limit, as $p \to \infty$, of $F_{p,q}$

Optional Problem 1:

Suppose that X has the log-normal distribution with parameters μ and σ^2. Find the distribution of $\frac{1}{X}$.
Let X follow a lognormal distribution, so:

$$X \sim \text{Lognormal}(\mu, \sigma^2) \quad \text{implies} \quad \log(X) \sim N(\mu, \sigma^2)$$

This means that:

$$X = \exp(Z) \quad \text{where} \quad Z \sim N(\mu, \sigma^2)$$

Now, we are asked to find the distribution of $Y = \frac{1}{X}$.
We can write Y as:

$$Y = \frac{1}{X} = \frac{1}{\exp(Z)} = \exp(-Z)$$

Since $Z \sim N(\mu, \sigma^2)$, the random variable $-Z$ follows a normal distribution with parameters $-\mu$ and σ^2, i.e.,

$$-Z \sim N(-\mu, \sigma^2)$$

Thus, $Y = \exp(-Z)$ follows a lognormal distribution, and we conclude:

$$Y \sim \text{Lognormal}(-\mu, \sigma^2)$$

Therefore, the distribution of $Y = \frac{1}{X}$ is a lognormal distribution with parameters $-\mu$ and σ^2:

$$Y \sim \text{Lognormal}(-\mu, \sigma^2)$$

Optional Problem 2:

Suppose $\bar{X}$ is the mean of 100 observations from a population with mean μ and variance $\sigma^2 = 9$. Find limits between which $\bar{X} - \mu$ will lie with probability at least 90%. Use both Chebyshev's inequality and the Central Limit Theorem, and comment on each.