

ESMA 6787: Exam 1

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Israel Almodovar

Alejandro Ouslan

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Problem 1: Definitions

Define the followings:

- (a) p-value: It is the probability that we would see that our results given that we assume that the null hypothesis is true
- (b) Projection Matrix: It is the map of our estimated parameter and data onto the response variable. In other words is the estimation of our fitted line
- (c) Complete randomized Design: It is an experiment design where we randomly assign treatments to a sample.
- (d) Complete Randomized Block Design: Similar to the CRD but with the distinction of partitioning the sample in shared characteristic

Problem 2: Dough Experiment

In the experiment to study the effects of the amount of baking powder in a biscuit dough upon the rise heights of the biscuits, four levels of baking powder were tested and four replicate biscuits were made with each level in a random order.

Table 1: Training Methods Data			
0.25 tsp	0.5 tsp	0.75 tsp	1 tsp
11.4	27.8	47.6	61.6
11.0	29.2	47.0	62.4
11.3	26.8	47.3	63.0
9.5	26.0	45.5	63.9

- (a) What is the experimental unit?
Answer: The biscuits dough
- (b) Under this model show why each of the following is estimable or non-estimable:

$$\tau_1, \tau_2, \tau_3, \tau_4, \quad \tau_1 + \tau_2 - (\tau_3 + \tau_4), \mu + \tau_1 + \tau_2$$

The overall model is given by the following:

$$Y_{ij} = \mu + \tau_i + \epsilon_{ij}$$

Thus:

- (a) $\tau_1 + \tau_2 - (\tau_3 + \tau_4)$ Is also estimable given it is also a linear combination
- (b) $\mu + \tau_1 + \tau_2$ is also estimable given it is also a linear combination
- (c) Perform the analysis of variance to test the hypothesis of no treatment effect.

Table 2: ANOVA for the Effect of Baking Powder on Biscuit Rise Height					
Source	df	Sum of Squares	Mean Square	F-statistic	p-value
Baking Powder	3	6145.73	2048.58	1822.65	3.23×10^{-16}
Residual	12	13.49	1.12	-	-

- (d) Formulate a contrast to test the hypothesis that increase in rise height is a linear function of the increase in baking powder in the dough, and test this hypothesis.
Choosing $L = -3\tau_1 - 1\tau_2 + 1\tau_3 + 3\tau_4$ with the hypothesis:

- (a) H_0 : no linear
 (b) H_1 : linear

Table 3: ANOVA for the Effect of Baking Powder on Biscuit Rise Height and Linear Trend

Source	df	Sum of Squares	Mean Square	F-statistic	p-value
Treatment (BakingPowder)	3	6145.732	2048.577	1822.645	3.23×10^{-16}
Linear Trend	1	6137.256	6137.256	3912.062	1.54×10^{-18}
Residual	12	13.488	1.124	-	-

With the p-value we reject the null hypothesis and say that there exists a linear relationship between Rise and amount of baking powder

- (e) If the dough were made in batches and the four replicate biscuit rise heights in each column (Table 1) were all from the same batch, would your answer to (a) be different? How could the data be analyzed if this were the case?

Answered: The answer would be the same if we now have a batch effect. We could use latin squares for this experiment if it is structured in batches

Problem 3: Randomize group

Consider a completely randomized design with four treatment groups, in which a total of $N = 100$ units are to be used. Although it won't be explicitly used in the analysis model, treatments 1 through 5 actually represent increasing concentrations of one component in an otherwise standard chemical compound, and the primary purpose of the experiment is to understand whether certain measurable properties of the compound change with this concentration. The investigator decides to address these questions by estimating these quantities

$$\tau_2 - \tau_1, \tau_2 - \tau_3, \tau_2 - \tau_4, \tau_3 - \tau_4, \tau_3 - \tau_1$$

where each τ_i is a parameter in the standard effects model. Find the optimal allocation for the 100 available units (i.e., values for n_1, \dots, n_5) that minimizes the average variance of estimates of the five contrasts of interest. Do this as a constrained, continuous optimization problem, then round the solution to integer values that are consistent with the required constraints.

Consider a completely randomized design with five treatments and a total of $N = 100$ experimental units. We are interested in the following five contrasts:

$$\tau_2 - \tau_1, \quad \tau_2 - \tau_3, \quad \tau_2 - \tau_4, \quad \tau_3 - \tau_4, \quad \tau_3 - \tau_1$$

Where τ_i are the treatment effects in the standard effects model.

$$\begin{aligned} \text{Var}(\tau_2 - \tau_1) &= \sigma^2 \left(\frac{1}{n_1} + \frac{1}{n_2} \right) \\ \text{Var}(\tau_2 - \tau_3) &= \sigma^2 \left(\frac{1}{n_2} + \frac{1}{n_3} \right) \\ \text{Var}(\tau_2 - \tau_4) &= \sigma^2 \left(\frac{1}{n_2} + \frac{1}{n_4} \right) \\ \text{Var}(\tau_3 - \tau_4) &= \sigma^2 \left(\frac{1}{n_3} + \frac{1}{n_4} \right) \\ \text{Var}(\tau_3 - \tau_1) &= \sigma^2 \left(\frac{1}{n_3} + \frac{1}{n_1} \right) \end{aligned}$$

$$V_{\text{avg}} \propto \frac{2}{n_1} + \frac{3}{n_2} + \frac{3}{n_3} + \frac{2}{n_4}.$$

$$f(n_1, n_2, n_3, n_4) = \frac{2}{n_1} + \frac{3}{n_2} + \frac{3}{n_3} + \frac{2}{n_4}$$

subject to

$$n_1 + n_2 + n_3 + n_4 + n_5 = 100, \quad n_i > 0.$$

$$n_1 = n_4 = \sqrt{\frac{2}{\lambda}}, \quad n_2 = n_3 = \sqrt{\frac{3}{\lambda}},$$

$$2\sqrt{\frac{2}{\lambda}} + 2\sqrt{\frac{3}{\lambda}} = 100 \implies \sqrt{\lambda} = \frac{\sqrt{2} + \sqrt{3}}{50}.$$

$$n_1 = n_4 \approx 22.5, \quad n_2 = n_3 \approx 27.5, \quad n_5 \approx 0.$$

Treatment	1	2	3	4	5
Units n_i	22	28	28	22	0

1 Problem 4

The effect of plant growth regulators and spear bud scales on spear elongation in asparagus was investigated by Yang-Gyu and Woolley (2006). Elongation rate of spears is an important factor determining final yield of asparagus in many temperate climatic conditions. Spears were harvested from 6-year-old Jersey Giant asparagus plants grown in a commercial planting at Bulls (latitude 40.2S, longitude 175.4E), New Zealand. Spears were harvested randomly and transported from field to lab for investigation. After trimming to 80mm length, spears were immersed completely for 1 h in aqueous solutions of 10 mg l⁻¹ concentration of indole-3-acetic acid (IAA), abscisic acid (ABA), GA3, or CPPU (Sitofex EC 2.0%; SKW, Trostberg, Germany) in test tubes. Control spears were submerged in distilled water for 1 h. The experiment was a completely randomized design with five replications (spears) per treatment

Table 4: Spear length in mm

Control	IAA	ABA	GA3	CPPU
94.7	89.9	96.8	99.1	104.4
96.1	94.0	87.8	95.3	98.9
86.5	99.1	89.1	94.6	98.9
98.5	92.8	91.1	93.1	106.5
94.9	99.4	89.4	95.7	104.8

- (a) What are the experimental units? **Answer:** spear bud scales on spear elongation in asparagus are the experimental unit
- (b) Proposed a linear model in this case. Explain each of the variable carefully. Explain each of the terms carefully in context of the problem

$$Y_{ij} = \mu + \tau_i + \epsilon_{ij}$$

- (a) Y_{ij} : The spear length
- (b) μ : the overall mean of the spear length
- (c) τ_i : effect of the treatments

- (d) ϵ : is the error term o unobserved effects
- (c) Test the hypothesis of no treatment effect. State the alternative, decision rule, and conclusion.
- (a) H_0 : All treatments have the same effects of the spear length
- (b) H_1 : At least one treatment has an effect

Table 5: ANOVA for the effect of treatments on asparagus spear length

Source	Sum of Squares	df	F	p-value
C(Treatment)	377.4936	4	6.9837	0.001092
Residual	270.2680	20	-	-

Given that the p-value is 0.001 we reject the alternative hypothesis. Thus there sufficient evidence that there is an effect from the treatments to the length of the spear

- (d) Test all pairwise comparisons of treatment means. You can use either Bonferroni, Turkeyk, Scheffe or Dunnett, assume that $\alpha = 0.01$.

Table 6: Tukey HSD pairwise comparisons of treatment means ($\alpha = 0.01$) for asparagus spear length

Group 1	Group 2	Mean Diff	p-adj	Lower	Upper	Significant
ABA	CPPU	11.86	0.0005	3.158	20.562	Yes
ABA	Control	3.30	0.623	-5.402	12.002	No
ABA	GA3	4.72	0.2881	-3.982	13.422	No
ABA	IAA	4.20	0.3973	-4.502	12.902	No
CPPU	Control	-8.56	0.0114	-17.262	0.142	No
CPPU	GA3	-7.14	0.0425	-15.842	1.562	No
CPPU	IAA	-7.66	0.0266	-16.362	1.042	No
Control	GA3	1.42	0.9717	-7.282	10.122	No
Control	IAA	0.90	0.9948	-7.802	9.602	No
GA3	IAA	-0.52	0.9994	-9.222	8.182	No

- (e) Can we test if there's difference between the control group with any of the treatment group? Discuss your results bases on part (c) and (d). **Answer:** There does not seem to be sufficient evidence to say that there is difference between the control group and the treatment group. Although there seem to be a difference between ABA and CPPU.
- (f) Perform a non-parametric approach to test the hypothesis of no treatment effect. State the alternative, decision rule, conclusion and asume that $\alpha 0.01$.

Table 7: Kruskal-Wallis test for treatment effect on asparagus spear length ($\alpha = 0.01$)

Test	H-statistic	p-value	Decision at $\alpha = 0.01$
Kruskal-Wallis H	11.734	0.0194	Fail to reject H_0

- (a) H_0 : no treatment effect
- (b) H_1 : there is a treatment effect
- (g) Preform non-parametric pairwise comparisons of treatment effect. Discuss your results comparing the control group whit the treatment groups. Further, asume that $\alpha = 0.01$.
- Answer:** There is not sufficient evidence to reject the null hypothesis. This the same for the Turkey test. We might not be able to detect the changes because of the error or the effects between the treatment is very minimal.

2 Problem 5

An accounting firm prior to introducing in the firm widespread training in statistical sampling for auditing tested three training methods: (1) study at home with programmed training materials, (2) training sessions at local offices conducted by local staff, and (3) training sessions in Chicago conducted by national staff. Thirty auditors were grouped into 10 blocks of three according to time elapsed since college graduation and the auditors in each block were randomly assigned to the three training methods. At the end of the training each auditor was asked to analyze a complex case involving statistical applications proficiency measure based on this analysis was obtained for each auditor.

Table 8: Training Methods Data

Block i	Training Method (1)	Training Method (2)	Training Method (3)
1	73	81	92
2	76	78	89
3	72	80	87
4	74	79	90
5	76	71	88
6	75	75	86
7	68	72	88
8	72	84	87
9	65	73	81
10	62	69	78

- (a) Why do you think the blocking variable "time elapsed since college graduation" was employed?

Answer: This is to control for experience as people who have spend more time since graduation are more likely to have more experience and hence more likely to preform better in the tests

- (b) Obtain the residuals for the randomized block model and plot them against the fitted values. Also, prepare a normal probability plot of the residuals. What are your findings?

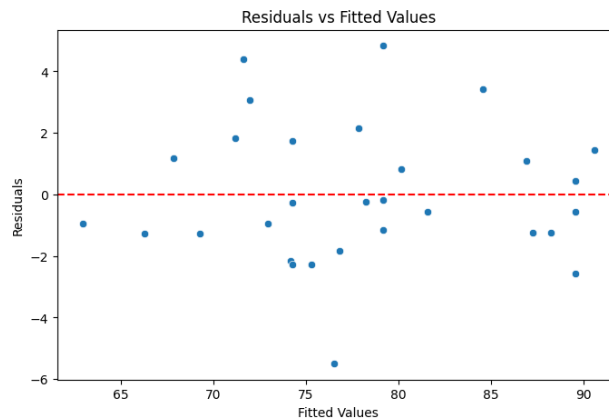


Figure 1: Interaction Graph

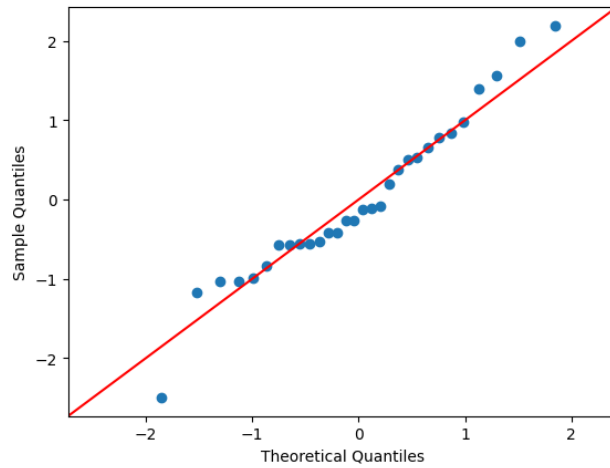


Figure 2: Interaction Graph

Those show that the errors comes from a normal distribution with mean 0 and contrant variance. The qqp shows that the residuals are normally distributed

- (c) Plot the responses Y_{ij} by blocks. What does this plot suggest about the appropriateness of the no-interaction assumption here?

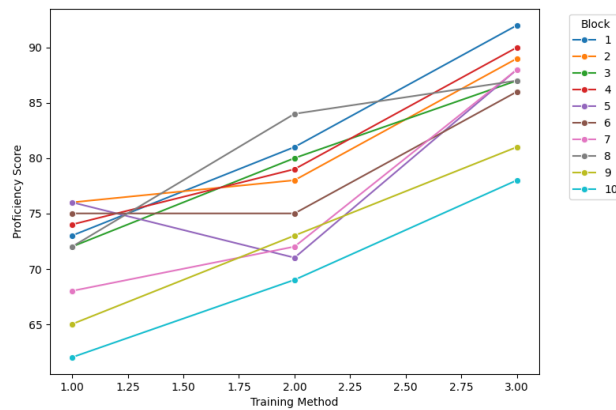


Figure 3: Interaction Graph

The graph shows that there is interaction between the training methods

Assume that the randomized block model is appropriate.

- (a) Test whether or not the mean proficiency is the same for the three training methods. Use a level of significance $\alpha = 0.05$. State the alternatives, decision rule, and conclusion. What is the p-value of the test?

Table 9: ANOVA Table for Randomized Block Design

	sum_sq	df	mean_sq	F	PR(>F)
C(Method)	794.9333	2.0	397.4667	64.4565	0.0000
C(Block)	579.9000	9.0	64.4333	10.4541	0.0000
Residual	111.0000	18.0	6.1667		

(a) H_0 : All treatments have the same effects

(b) H_1 : At least one treatment has an effect

Given that the p-value is near 0 we reject the null hypothesis and conclude that there is significant differences between training methods

- (b) Make all pairwise comparisons between the training method means; use the Tukey procedure with a 90% confidence level coefficient. State your findings.

Table 10: Tukey HSD Pairwise Comparisons Among Training Methods (90% Confidence Level)

Group 1	Group 2	Mean Diff	p-adj	Lower	Upper	Reject
1	2	4.9	0.0639	0.458	9.342	Yes
1	3	15.3	0.0000	10.858	19.742	Yes
2	3	10.4	0.0001	5.958	14.842	Yes

(a) H_0 : There are no differences

(b) H_1 : The pair is different

This table show that there are significant evidence to conclude that there is an effect on training method use and the score results. The order of the effect is studding in Chicago then studding with the local office then studying at home. Overall this results make sense, this might be due to the level effort or investment the employee feels from the company.

- (c) Test whether or not blocking effects are present; use $\alpha = 0.05$. State the alternatives, decision rule, and conclusion. What is the p-value of the test?
- H_0 There is no blocking effects
 - H_1 There is blocking effects

Since the p-value is less than 0.05, we reject the null hypothesis and conclude that there is a blocking effect