ESMA 6787: Examen 3

Due on Mayo 12, 2025

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1 problem 1

Define using your own words:

1. Estimate $c'\beta$
2. Power
3. Null hypothesis
4. Alternative hypothesis
5. Test Statistics
6. Non-centrality parameter
7. Details about the non-centrality parameter

2 Problem 2

Consider a completely randomized design with four treatment groups, with $n_i > 0$ units assigned to treatment $i = 1, 2, 3, 4$.

1. One way to model data from such an experiment is with the effect model:

   $$y_{ij} = \alpha + \tau_i + \epsilon ij, \quad i = 1, 2, 3, 4; \quad j = 1 \ldots, n_i$$

   Under this model show why each of the following is estimable or nonestimable:

   $$\tau_3, \tau_3 - \tau_2, \tau_3 + \tau_2$$

2. Now define a different model for the same experiment, as:

   $$y_{1j} = \mu_1 + \epsilon_{1j}, \quad j = 1, \ldots, n_i$$

   $$y_{ij} = \mu_1 + \theta_i + \epsilon_{ij} \quad i = 2, 3, 4; \quad j = 1, \ldots, n_i$$

   Under this model, show why each of the followings is estimable or nonestimable:

   $$\theta_3, \theta_3 - \theta_2, \theta_3 + \theta_2$$

3 problem 3

A chemical engineer is interested in comparing three different versions of a reaction process, labeled A, B, and C, with respect to "percent conversion of feedstock." In a preliminary experiment, she applied each process to four batches of raw material, using appropriate randomization of the 12 available batches to the three treatments, and collected the percent conversion values presented in the following table.

A	B	C
27.3	41.9	36.8
31.6	36.8	39.2
34.6	38.9	36.1
29.4	37.5	38.0

Assuming the data are independent and can be reasonably modeled as:

$$y_{ij} = \mu_i + \epsilon ij, \quad E(e_{ij}) = 0, \quad Var(e_{ij}) = \sigma^2$$

1. Estimate σ^2 and test the hypothesis: $\mu_i = \mu_2 = \mu_3$ against H_1: at least one of μ_i is different.
2. Using your estimate of σ as if it were the true parameter value, how large would a follow-up experiment (with equal sample sizes) have to be so that the 0.05-level confidence interval for $\mu_1 - \mu_3$ would have expected width (5%).

4 problem 4

Consider a completely randomized design with five treatment groups, in which a total of $N = 50$ units are to be used. Although it won't be explicitly used in the analysis model, treatments 1 through 5 actually represent increasing concentrations of one component in an otherwise standard chemical compound, and the primary purpose of the experiment is to understand whether certain measurable properties of the compound change with this concentration. The investigator decides to address these questions by estimating four quantities:

$$\tau_2 - \tau_1, \tau_3 - \tau_2, \tau_4 - \tau_3, \tau_5 - \tau_4$$

where each τ_i is a parameter in the standard effects model. Find the optimal allocation for the 50 available units (i.e., values for $n_1, \ldots, n_5$) that minimizes the average variance of estimates of the four contrasts of interest. Do this as a constrained, continuous optimization problem, then round the solution to integer values that are consistent with the required constraint.

5 problem 5

Continue working with the experimental design described in problem 2. Suppose the experiment- specific treatment means in this problem, as would be expressed in the cell means model, are actually:

μ_1	μ_2	μ_3	μ_4	μ_5
10	11	12	12	12

and $\sigma = 2$. What is the power of the standard F-test for the hypothesis

$$\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau_5$$

at $\alpha = 0.05$:

1. if all $n_i = 10$?
2. under the optimal sample sample allocation you found in problem 2?
3. Derive an optimal allocation for the F-test of equal treatment effects, i.e, the sample size (totaling 50) that would result in the greatest power, if in reality the experiment-specific means are $\mu_1 = 10$ and $\tau_2 = \tau_3 = \tau_4 = \tau_5 = 8$

6 problem 6

The entire class of ESMA 6616 wanted to study how the power of the F changes with the non- centrality parameter λ^2. To achieve this we will do as follow. Consider the model

$$y_{ij} = \mu_i + \epsilon_{ij}$$

Where $\epsilon_{ij} \sim N(0, \sigma^2)$, with $i = 1, \ldots, k$ and $j = 1, ..., n_i$.
The null and alternative hypotheses for the ANOVA F-test are

$$H_0 : \mu_1 = \ldots = \mu_k$$
$$H_1 : \text{at least one is different}$$

The test rejects H_0 if the $F^\star$-statistic, defined as $F^\star = \frac{MS_{model}}{MS_{error}}$, exceeds a critical value.
The power of the ANOVA F-test, which measures the probability of rejecting H_0 when H_1 is true, is given by $P(F^\star > \alpha|H_1)$. To calculate the power, we must know the distribution of $F\star$. Under H_0, $F^\star \sim$

$F_{k-1,\sum_{k}^{i=1} n_i - k}$ degrees of freedom. The distribution under H_1 depends on the true differences among the group means.
Consider the following example $k = 5$ treatment groups with group sizes $n_1 = n_2 = n_3 = 10, n_4 = 8$, and $n_5 = 6$. Further, assume the standard deviation $\sigma = 1.5$. The task is to find the power of the test when H_1 is true, given the group means $\mu_1 = 2, \mu_2 = 3, \mu_3 = 2.5, \mu_4 = 0$, and $\mu_5 = 1$. Assume we are using a significance level of 0.05.
The overall mean is defined $\bar{\mu} = \sum_5^{i=1} n_i * \frac{\mu_i}{\sum_5^{i=1} * n_i}$
The non-centrality parameter is $\frac{\sum_5^{i=1} n_i(\mu_i - \bar{\mu})^2}{\sigma^2}$
In R, the argument ncp stands for the non-centrality parameter in the desity functions. In this example, the test statisic have the following distribution $F^\star \sim F_{5-1,40-5}(\lambda^2 = 0)$ under H_0 and $F^\star \sim F_{5-1,40-5}(\lambda^2 = 15.32222)$ under H_1.
Figure 1 showed the difference between the F distribution under the null hypothesis(solid red) with 4 and 35 with non-centrality parameter of 0 and the F under the alternative (dashed blue) with the same degrees of freedom but non-centrality parameter $\lambda^2 = 15.3222$. We can observed as λ^2 increases the density under the alternative get farther from the null. The critical value to reject H_0 keeping the significance level 2.6415. If H_1 is true and $F^\star \sim F_{5-1,40-5}(\lambda^2 = 15.32222)$, the power to reject H_0 is the

$$P(F^\star > F_{k-1,\sum_{i=1}^{k} n_i - k, \alpha} | H_1) = 0.8489175$$

To interpret this value, the probability of rejecting the null hypothesis (non difference across the means) when in fact there's difference across the means is 0.8489. To compute the critical value and the power in R you can do the following,

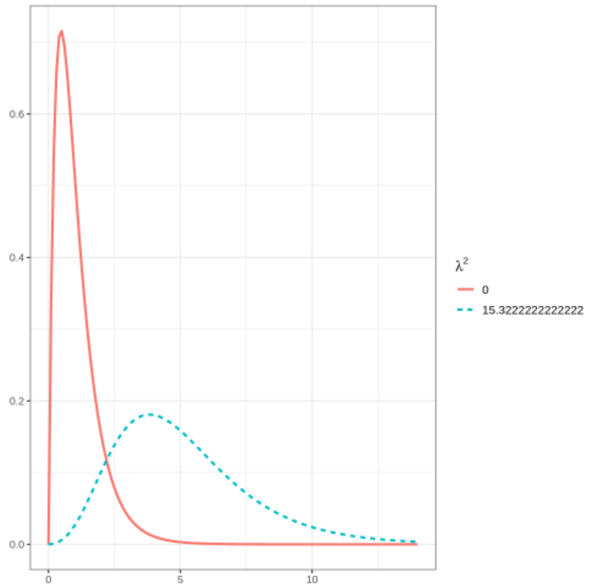

Figure 1: Comparison of the central (solid red) and non-central (dashed blue) F-distribution.

1. Calculate the overall mean.
2. Calculate the non-centrality parameter. Hint: The non-centrality parameter should be a function of n.
3. Find the critical value at the 5% level. Note that the degrees of freedom should be a function of the sample size n as well.

4. Compute the power for each sample size n, i.e., $n = 2, 3, 4, \ldots, n_{min}$ where $nmin$ is the minimum sample size such that power ≥ 0.95.
5. Use the following code to show your figure. This code will create the smae figure in this document, but you must modify it to include your information.

Example Code

```
library("ggplot2")
# Define parameters
k <- length(nk)
df1 <- k - 1 # Numerator degrees of freedom
df2 <- sum(nk) - k # Denominator degrees of freedom
lambda2_values <- c(0, lambda2) # Non-centrality parameters
x <- seq(0, 14, by = 0.1)
fvalues <- data.frame(
  x = rep(x, times = length(lambda2_values)),
  density = unlist(lapply(lambda2_values, function(lambda2) {
    df(x, df1, df2, ncp = lambda2)
  })), lambda2 = factor(rep(lambda2_values, each = length(x)))
)
ggplot(fvalues, aes(x = x, y = density, color = lambda2, linetype = lambda2)) +
  geom_line(linewidth = 1) +
  labs(
    x = NULL,
    y = NULL,
    color = expression(lambda^2),
    linetype = expression(lambda^2)
  ) +
  theme_bw()
```

7 problem 7

The entire class of ESMA 6616 wanted to study how the power of the F -test changes with the non-centrality parameter λ^2. To achieve this, we will proceed as follows. Consider the model

$$y_{ij} = \mu_i + \epsilon_{ij}$$

where $eij \sim N(0, \sigma^2)$ with $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$. The null and alternative hypotheses for the ANOVA F-test are

$$H_0 : \mu_i = \cdots = \mu_k \quad H_1 : \text{at least one } \mu_i \text{ is different}$$

The test rejects H_0 if the statistic

$$F^\star = \frac{MS_{model}}{MS_{error}}$$

exceeds a critical value. The power of the ANOVA F -test, which measures the probability of rejecting H0 when H1 is true, is given by

$$P(F^\star > \text{crtitical value}|H_1)$$

To calculate the power, we must know the distribution of $F^\star$. Under H_0, we have

$$F^\star \sim F_{k-1, \sum_{i=1}^{k} n_i - k}$$

The distribution under H_1 depends on the true differences among the group means. Example: Consider $k = 5$ treatment groups with sample sizes $n_1 = n_2 = n_3 = 10, n_4 = 8$, and $n_5 = 6$. Assume the standard deviation $\sigma = 1.5$. The task is to find the power of the test when H_1 is true, given the group means

$$\mu_1 = 2, \mu_2 = 3, \mu_3 = 2.5, \mu_4 = 0, \mu_5 = 1$$

with a significance level $\alpha = 0.05$

1. Calculate the overall mean.
2. Calculate the non-centrality parameter. (Note: it should be a function of n).
3. Find the critical value at the 5% level. Note that the degrees of freedom should be a function of the sample size n as well.
4. Compute the power for each sample size n, i.e., $n = 2, 3, 4, \ldots, n_{min}$ where $nmin$ is the minimum sample size such that power ≥ 0.95.
5. se the provided R code to produce your figure. Modify the code to include your information