ESMA 6787: Exam 2

Due on December 14, 2025

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Problem 1: Definitions

(a) **Interaction plot:** An interaction plot as a visual tool to determine if there interaction between 2 factors with each other. You determine there is interaction when you see the line intersect with each other.

(b) **Random effect:** When we work with block designs a random effect is the unobserved effects within the blocks, that is to say each group contains its own random effect $N(0, \sigma)$

(c) **Factorial design:** Factorial design is a method to combare multiple factors with each other and determine its effects on the response variable. Example we al looking at the shrinkage of type of cloth and water temperature, in this experiments the type of cloth and water temperature are the factors.

(d) **Intraclass correlation:** This is the correlation that groups can have with each other.

(e) **Split-plot design:** The split-plot are design method where we mesure the effects of factors. This is done by first dividing the entire group in section where each will revive a specific treatment the each group is divided in sub groups that recave the other factor. This results in a multiple big groups that contain a factor and inside those groups there is a sub group that are randomly assigned the factors.

Problem 2: Marketing Problem

A marketing research consultant evaluated the effects of fee schedule, scope of work, and type of supervisory control on the quality of work performed under contract by independent marketing research agencies. The quality of work performed was measured by an index taking into account several characteristics of quality. Four agencies were chosen for each factor level combination and the quality of their work evaluated. See **marketing.txt** files for this dataset.

(a) Create interaction plots of the factors for the fee schedule and scope of the work of the estimated treatment means $\bar{Y}_{ij}$. Does it appear that any interactions are present? Any main effects?
Answer: There appears to no interaction between schedule. The main effect appears to be that as the fee of the fee increases the quality of work decrease.

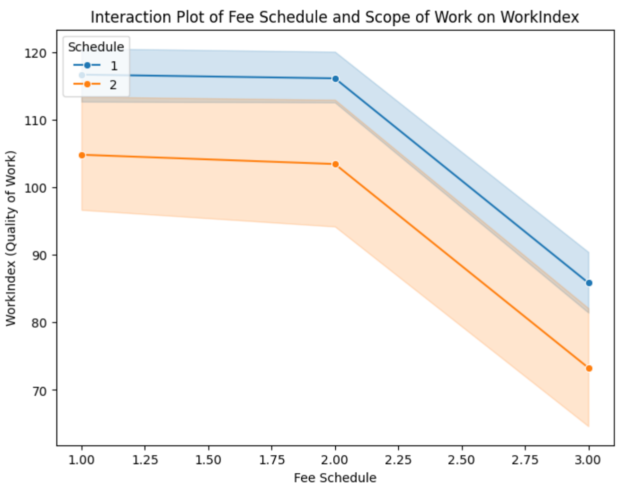

Figure 1: Interaction Graph

Code for Interaction Graph

```
import pandas as pd
import seaborn as sns
import matplotlib.pyplot as plt


def main() -> None:
    df = pd.read_csv("marketing.txt", sep=r"\s+")
    plt.figure(figsize=(8, 6))

    sns.lineplot(data=df, x="Fee", y="WorkIndex", hue="Schedule", palette="tab10")
    plt.title("Interaction Plot of Fee Schedule and Scope of Work on WorkIndex")
    plt.xlabel("Fee Schedule")
    plt.ylabel("WorkIndex (Quality of Work)")
    plt.legend(title="Schedule", loc="upper left")
    plt.show()


if __name__ == "__main__":
    main()
```

(b) Create interaction plots of the factors fee schedule and type of supervisory control of the estimated treatment means $\bar{Y}_{ijk}$. Do your plots convey the same information as those in part (a)?
Answer: There appears to no interaction between supervisory. The main effect appears to be that as the fee of the fee increases the quality of work decrease. This seems to be the same effect of part a

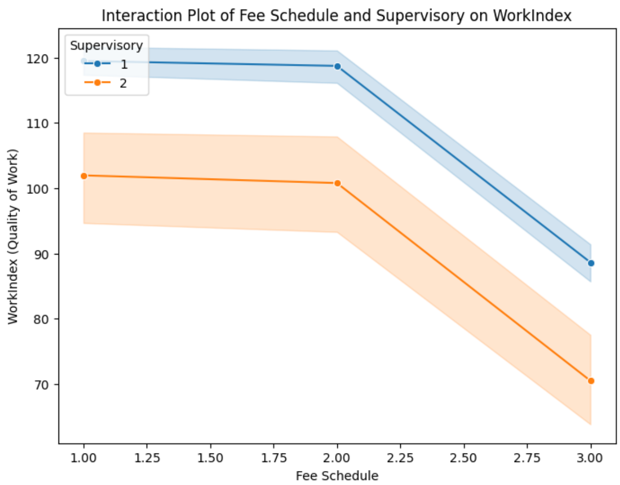

Figure 2: Interaction Graph

Code for Interaction Graph

```python
import matplotlib.pyplot as plt
import pandas as pd
import seaborn as sns


def main() -> None:
    df = pd.read_csv("marketing.txt", sep=r"\s+")
    plt.figure(figsize=(8, 6))

    sns.lineplot(data=df, x="Fee", y="WorkIndex", hue="Supervisory")
    plt.title("Interaction Plot of Fee Schedule and Supervisory on WorkIndex")
    plt.xlabel("Fee Schedule")
    plt.ylabel("WorkIndex (Quality of Work)")
    plt.legend(title="Supervisory", loc="upper left")
    plt.show()


if __name__ == "__main__":
    main()
```

(c) Obtain the analysis of variance table.

Table 1: ANOVA for the Effect of Fee and Schedule on Work Index

Source	**df**	**Sum of Squares**	**Mean Square**	**F-statistic**	**p-value**
C(Fee)	2.0	10044.27	5022.14	45.09	3.50×10^{-11}
C(Schedule)	1.0	1833.98	1833.98	16.47	2.11×10^{-4}
C(Fee):C(Schedule)	2.0	1.60	0.80	0.01	9.93×10^{-1}
Residual	42.0	4678.04	111.38	-	-

Code for Anova Table

```python
import pandas as pd
import statsmodels.formula.api as smf
from statsmodels.stats.anova import anova_lm


def main() -> None:
    df = pd.read_csv("marketing.txt", sep=r"\s+")
    model = smf.ols("WorkIndex ~ C(Fee) * C(Schedule)", data=df).fit()
    anova_results = anova_lm(model)
    print(anova_results)


if __name__ == "__main__":
    main()
```

(d) Test for three-factor interactions: using $\alpha = 0.01$. State the alternatives, division rule, and conclusion. What is the p-value of the test?

(a) H_0: There is no three way interaction

(b) H_a: There is three way interaction

Table 2: ANOVA for the Effect of Fee, Schedule, and Supervisory on Work Index (Part 2)

Source	**F-statistic**	**p-value**
C(Fee)	679.34	2.59×10^{-29}
C(Schedule)	248.08	1.00×10^{-17}
C(Supervisory)	518.40	5.74×10^{-23}
C(Fee):C(Schedule)	0.11	8.98×10^{-1}
C(Fee):C(Supervisory)	0.05	9.48×10^{-1}
C(Schedule):C(Supervisory)	77.75	1.60×10^{-10}
C(Fee):C(Schedule):C(Supervisory)	0.27	7.67×10^{-1}
Residual	-	-

Answer: Given that C(Fee):C(Schedule):C(Supervisory) as a p-value of 0.767 we don't have enough evidence to reject the null hypothesis thous we conclude there is not sufficient evidence to conclude that there is a three way interaction

Code for Anova Table

```python
from statsmodels.stats.anova import anova_lm
from statsmodels.formula.api import ols
import pandas as pd


def main() -> None:
    df = pd.read_csv("marketing.txt", sep=r"\s+")
    model = ols("WorkIndex ~ C(Fee) * C(Schedule) * C(Supervisory)", data=df).fit()
    anova_results = anova_lm(model)
    print(anova_results)


if __name__ == "__main__":
    main()
```

(e) Test for factor fee schedule main effects; use $\alpha = 0.05$. State the alternatives, decision rule, and conclusion. Hint: You can use the confidence interval.

(a) H_0: The fee has no impact on work productivity.

(b) H_a: The fee has an impact on work productivity.

(c) H_0: The schedule has no impact on work productivity.

(d) H_a: The schedule has an impact on work productivity.

Table 3: ANOVA for the Main Effects of Fee and Schedule on Work Index

Source	**df**	**Sum of Squares**	**Mean Square**	**F-statistic**	**p-value**
C(Fee)	2.0	10044.27	5022.14	47.22	1.12×10^{-11}
C(Schedule)	1.0	1833.98	1833.98	17.24	1.49×10^{-4}
Residual	44.0	4679.65	106.36	-	-

Answer: Looking at the p-values both the fee and the schedule and the fee have a negative impact of the

Table 4: Confidence Intervals for the Factors in the ANOVA Model

Factor	Lower Bound	Upper Bound
Intercept	110.91	122.91
C(Fee)[T.2]	-8.31	6.39
C(Fee)[T.3]	-38.50	-23.81
C(Schedule)[T.2]	-18.36	-6.36

work productivity. Looking at the confidence intervals the fee 3 seems to have the negative impact while level 2 does not seem to have an impact given that it does not include 0.

Code for Anova Table

```
from statsmodels.stats.anova import anova_lm
from statsmodels.formula.api import ols
import pandas as pd


def main() -> None:
    df = pd.read_csv("marketing.txt", sep=r"\s+")
    model = ols("WorkIndex ~ C(Fee) + C(Schedule)", data=df).fit()

    anova_results = anova_lm(model)
    print(anova_results)

    conf_int = model.conf_int(alpha=0.05)
    print("Confidence Intervals for the factors:")
    print(conf_int)


if __name__ == "__main__":
    main()
```

Problem 3: Calculator

To test the efficiency of its new programmable calculator, a computer company selected at random six engineers who were proficient in the use of both this calculator and an earlier model and asked them to work out two problems on both calculators. One of the problems was statistical in nature, the other was an engineering problem. The ercer of the four calculations was randomized independently for each engineer. The length of time (in minutes) required to solve each problem was observed. See Table 1

Table 5: The length of time (in minutes) required to solve each problem

Programmer	Statistical Problem		Engineering Problem	
	New Model	Earlier Model	New Model	Earlier Model
1	3.1	7.5	2.5	5.1
2	3.8	8.1	2.8	5.3
3	3.0	7.6	2.0	4.9
4	3.4	7.8	2.7	5.5
5	3.3	6.9	2.5	5.4
6	3.6	7.8	2.4	4.8

(a) Create interaction plots of the estimated treatment means. Does it appear that treatment interaction effects are present?

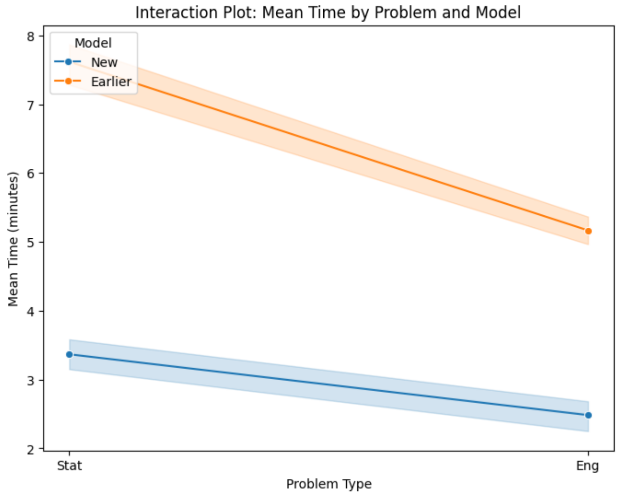

Figure 3: Interaction Graph

Answer: There apear to be no interaction between model type and problem. There apear to be a reduction in the completion time from a stats problem to a engineering problem. The overall effect is that the new model reduced the completion time.

Code for Anova Table

```python
import matplotlib.pyplot as plt
import pandas as pd
import seaborn as sns


def main():
    data = {
        "Programmer": [1, 2, 3, 4, 5, 6],
        "Stat_New": [3.1, 3.8, 3.0, 3.4, 3.3, 3.6],
        "Stat_Earlier": [7.5, 8.1, 7.6, 7.8, 6.9, 7.8],
        "Eng_New": [2.5, 2.8, 2.0, 2.7, 2.5, 2.4],
        "Eng_Earlier": [5.1, 5.3, 4.9, 5.5, 5.4, 4.8],
    }

    df = pd.DataFrame(data)
    long_df = pd.melt(
        df,
        id_vars="Programmer",
        value_vars=["Stat_New", "Stat_Earlier", "Eng_New", "Eng_Earlier"],
        var_name="Condition",
        value_name="Time",
    )
    long_df["Problem"] = long_df["Condition"].apply(
        lambda x: "Stat" if "Stat" in x else "Eng"
    )
    long_df["Model"] = long_df["Condition"].apply(
        lambda x: "New" if "New" in x.split("_")[1] else "Earlier"
    )
    plt.figure(figsize=(8, 6))
    sns.lineplot(
        data=long_df, x="Model", y="Time", hue="Problem", marker="o", palette="tab10"
    )
    plt.title("Interaction Plot: Mean Time by Problem and Model")
    plt.xlabel("Problem Type")
    plt.ylabel("Mean Time (minutes)")
    plt.legend(title="Model", loc="upper left")
    plt.show()


if __name__ == "__main__":
    main()
```

(b) Create the skeleton ANOVA. If there are any random effects mark them.

Table 6: Skeleton ANOVA Table for Mixed Effects Model

Source of Variation	df	SS	MS	F
Fixed Effects				
Model (C(Model))	1	SS_M	MS_M	F_M
Problem (C(Problem))	1	$SS_{Problem}$	MS_P	F_P
Model × Problem (C(Model):C(Problem))	1	$SS_{M\times P}$	$MS_{M\times P}$	$F_{M\times P}$
Random Effects (R)				
Programmer (random)	$b-1=5$	SS_{Prog}	MS_{Prog}	—
Residual/Error (within Programmer)	$ab-a-b+1=18$	SS_R	MS_R	—
Total	$N-1=23$	SS_{Total}	—	—

(c) Obtain the analysis of variance table.

Table 7: Mixed Linear Model Fixed Effects Results

Effect	Coef.	Std. Err.	z-value	p-value	95% CI
Intercept	5.167	0.131	39.402	0.000	[4.910, 5.424]
C(Model)[T.New]	-2.683	0.150	-17.911	0.000	[-2.977, -2.390]
C(Problem)[T.Stat]	2.450	0.150	16.354	0.000	[2.156, 2.744]
C(Model)[T.New]:C(Problem)[T.Stat]	-1.567	0.212	-7.394	0.000	[-1.982, -1.151]

Code for Anova Table

```
import pandas as pd
import statsmodels.formula.api as smf


def main():
    data = {
        "Programmer": [1, 2, 3, 4, 5, 6],
        "Stat_New": [3.1, 3.8, 3.0, 3.4, 3.3, 3.6],
        "Stat_Earlier": [7.5, 8.1, 7.6, 7.8, 6.9, 7.8],
        "Eng_New": [2.5, 2.8, 2.0, 2.7, 2.5, 2.4],
        "Eng_Earlier": [5.1, 5.3, 4.9, 5.5, 5.4, 4.8],
    }

    df = pd.DataFrame(data)
    long_df = pd.melt(
        df,
        id_vars="Programmer",
        value_vars=["Stat_New", "Stat_Earlier", "Eng_New", "Eng_Earlier"],
        var_name="Condition",
        value_name="Time",
    )
    long_df["Problem"] = long_df["Condition"].apply(
        lambda x: "Stat" if "Stat" in x else "Eng"
    )
    long_df["Model"] = long_df["Condition"].apply(
```

```
        lambda x: "New" if "New" in x.split("_")[1] else "Earlier"
    )
    model = smf.mixedlm(
        "Time ~ C(Model) * C(Problem)", long_df, groups=long_df["Programmer"]
    ).fit()

    print(model.summary())


if __name__ == "__main__":
    main()
```

(d) Test Whether or not the two treatment factors interact. State the alternatives, decision rule, and conclusion. What is the p-value of the test?

(a) H_0: There is no interaction

(b) H_a: There is interaction

Answer: Given the previous table we see that the p value of 0.00 and it is less than the critical value 0.05 we reject the null hypothesis and conclude that the time to solve the problem depends on the type of problem

Problem 4: Automobile Manufacture

An automobile manufacturer wished to study the effects of difference between drivers and differences between cars on gasoline consumption. Four drivers were selected at random; also live cars of the same model with manual transmission were randomly selected from the assembly line. Each driver drove each car twice over a 40-mile test course and the miles per gallon were recorded. See Table 2

Table 8: Miles per gallon for each driver in each car

Driver	**Car 1**	**Car 2**	**Car 3**	**Car 4**	**Car 5**
1	25.3	28.9	24.8	28.4	27.1
1	25.2	30.0	25.1	27.9	26.6
2	33.6	36.7	31.7	35.6	33.7
2	32.9	36.5	31.9	35.0	33.9
3	27.7	30.7	26.9	29.7	29.2
3	28.5	30.4	26.3	30.2	28.9
4	29.2	32.4	27.7	31.8	30.3
4	29.3	32.4	28.9	30.7	29.9

(a) Test whether or not the two factors interact. State the alternative, decision rule, and conclusion. What is the p-value of the test?

(a) H_0: There is no intereaciton between driver and car

(b) H_a: There is interaciton between driver and car

Answer: Given that the p-value is 0.37 and it is grater than the critical value we don't have sufficient evidence to reject the null hypothesis and say there is interaction present.

Table 9: ANOVA Results for Two-Way Interaction (Driver and Car)

Source	df	Sum of Squares	Mean Squares	F	PR(>F)
C(Driver)	3.0	280.28475	93.428250	531.597440	3.125304e-19
C(Car)	4.0	94.71350	23.678375	134.727596	3.663600e-14
C(Driver):C(Car)	12.0	2.44650	0.203875	1.160028	3.714839e-01
Residual	20.0	3.51500	0.175750	NaN	NaN

(b) Test separately whether or not factor A and factor B main effects are present. For each test, and state the alternatives, decision rule, and conclusion. What is the p-value for each test?

Driver effect

(a) H_0: The miles per gallon for each driver is the same

(b) H_a: At least one is different

Car effect

(a) H_0: The miles per gallon for each car is the same

(b) H_a: At least one is different

Table 10: ANOVA Results for Gasoline Consumption (Driver and Car)

Source	df	Sum of Squares	Mean Squares	F	PR(>F)
C(Driver)	3.0	280.28475	93.42825	501.502	5.728485e-27
C(Car)	4.0	94.71350	23.67838	127.100	3.668485e-19
Residual	32.0	5.96150	0.18630	NaN	NaN

Answer: Looking at the table both Driver and Car have p-values below the critical value of 0.05 thus we reject the null hypothesis and conclude there is both a driver effect and a car effect.

(c) Obtain point estimates of and which factor appears to have the greater effect on gasoline consumption?

Table 11: Driver Means

Driver	Mean MPG
1	26.93
2	34.15
3	28.85
4	30.26

Table 12: Car Means

Car	Mean MPG
Car_1	28.96
Car_2	32.25
Car_3	27.91
Car_4	31.16
Car_5	29.95

Answer: Looking at the means driver 2 and car 2 produce the most MPG consumption. Another way you could determine this is using the OLS and looking at the coaliciones it generates and seeing which has the biggest coef

(d) Use the Satterhwaite procedure to obtain an approximate 95 percent confidence interval. Is your interval estimate reasonably precise? Comment

Parameter	Lower Bound	Upper Bound
Intercept	25.452	26.238
C(Driver)[T.2]	6.827	7.613
C(Driver)[T.3]	1.527	2.313
C(Driver)[T.4]	2.937	3.723
C(Car)[T.Car2]	2.848	3.727
C(Car)[T.Car3]	-1.490	-0.610
C(Car)[T.Car4]	1.760	2.640
C(Car)[T.Car5]	0.548	1.427

Table 13: 95% Confidence Intervals for Model Parameters

Parameter	Lower Bound	Upper Bound
Intercept	25.30	26.39
C(Driver)[T.1]	6.80	7.64
C(Driver)[T.2]	1.44	2.40
C(Driver)[T.3]	2.86	3.80
C(Car)[T.Car1]	2.73	3.85
C(Car)[T.Car2]	-1.58	-0.52
C(Car)[T.Car3]	1.68	2.72
C(Car)[T.Car4]	0.52	1.46

Table 14: 95% Confidence Intervals with Robust Standard Errors

Answer: they are some what more precise but the not as much as i would expect or reasonably argue.

Problem 5: Derive estimators

Some students (Junior, Ariana, Yeily, etc.) mentioned they wanted to derived some of the equations. Consider the following model $Y_{jik} = \mu + \alpha_i + \beta_j + \mu_{ij} + \epsilon_{ijk}; k = 1, \dots, K; i = 1, 2$. Further, assume that $\mu_{ij} \sim N(0, \sigma_u^2)$ and $\epsilon_{ijk} \sim N(0, \sigma_e^2)$

1. Show that,

$$Var(Y_{ijk}) = \sigma_u^2 + \sigma_e^2$$

Answer:

$$\begin{aligned} Var(Y_{ijk}) &= Var(\mu + \alpha_i + \beta_j + \mu_{ij} + \epsilon_{ijk}) \\ &= Var(\mu) + Var(\alpha_i) + var(\beta_j) + var(\mu_{ij}) + var(\epsilon_{ijk}) \\ &= 0 + 0 + 0 + \sigma_u^2 + \sigma_e^2 \\ &= \boxed{\sigma_u^2 + \sigma_e^2} \end{aligned}$$

2. Show that, $k \neq k'$

$$Cov(Y_{ijk}, Y_{ijk'}) = \sigma_u^2$$

Answer:

$$\begin{aligned}
Cov(Y_{ijk}, Y_{ijk'}) &= E[(Y_{ijk} - E[Y_{ijk}])(Y_{ijk'} - E[Y_{ijk'}])] \\
&= E[(\mu_{ij} + \epsilon_{ijk})(\mu_{ij} + \epsilon_{ijk})] \\
&= E[(\mu_{ij}\mu_{ij})] + E[(\mu_{ij}\sigma_{ijk})] + E[(\mu_{ij}\sigma_{ijk'})] + E[(\sigma_{ijk}\sigma_{ijk'})] \\
&= cov(\mu_{ij}) + 0 + 0 + 0 \\
&= \boxed{\sigma_i^2}
\end{aligned}$$

3. Show that, $k \neq k'$

$$Corr(Y_{ijk}, Y_{ijk'}) = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_e^2}$$

Answer:

$$\begin{aligned}
Corr(Y_{ijk}, Y_{ijk'}) &= \frac{cov(Y_{ijk})}{\sqrt{var(Y_{ijk})var(Y_{ijk'})}} \\
&= \frac{\sigma_u^2}{\sqrt{(\sigma_u^2 + \sigma_e^2)(\sigma_u^2 + \sigma_e^2)}} \\
&= \boxed{\frac{\sigma_u^2}{\sigma_u^2 + \sigma_e^2}}
\end{aligned}$$