MATE 5150: Asignacion #7

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Problem 1

Compute the determinants of the following matrices in $M_{2x2}(C)$.

$$\begin{bmatrix} 2i & 3 \\ 4 & 6i \end{bmatrix}$$

$$\begin{aligned} |C| &= 2i \cdot 6i - 3 \cdot 4 \\ |C| &= 12i^2 - 12 \\ |C| &= -12 - 12 \\ |C| &= -24 \end{aligned}$$

Problem 2

The classical adjoint of a $2x2$ matrix $A \in M_{2x2}(F)$ is the matrix

$$C = \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$$

Prove that

- $CA = AC = [det(A)]I$

$$\begin{aligned} CA &= \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{22}A_{11} - A_{12}A_{21} & 0 \\ 0 & -A_{21}A_{12} + A_{11}A_{22} \end{bmatrix} \\ AC &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} = \begin{bmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & -A_{21}A_{12} + A_{11}A_{22} \end{bmatrix} \\ \det(A) &= (A_{22}A_{11} - A_{12}A_{21}) \cdot (-A_{21}A_{12} + A_{11}A_{22}) \\ \det(A)I &= \begin{bmatrix} A_{22}A_{11} - A_{12}A_{21} & 0 \\ 0 & -A_{21}A_{12} + A_{11}A_{22} \end{bmatrix} \end{aligned}$$

- $det(C) = det(A)$

$$\begin{aligned} det(C) &= A_{22}A_{11} - A_{12}A_{21} \\ det(A) &= A_{11}A_{22} - A_{12}A_{21} \end{aligned}$$

- The classical adjoint of A^T is C^T

$$\begin{aligned} A^T &= \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} \\ C^T &= \begin{bmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{bmatrix} \end{aligned}$$

- If A is invertible, then $A^{-1} = [det(A)]^{-1}C$

$$\begin{aligned} A^{-1} &= \frac{1}{det(A)} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} \\ A^{-1} &= \begin{bmatrix} \frac{A_{22}}{det(A)} & \frac{-A_{12}}{det(A)} \\ \frac{-A_{21}}{det(A)} & \frac{A_{11}}{det(A)} \end{bmatrix} \end{aligned}$$

Problem 3

Find the value of k such that satisfies the following equation.

$$det\begin{bmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix} = k\cdot det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$det\begin{bmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix} = 2a_1b_2c_3+2a_2b_3c_1+2a_3b_1c_2-2a_3b_2c_1-2a_2b_1c_3-2a_1b_3c_2$$

$$det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2$$

$$k=2$$

Problem 4

Evaluate the determinant of the given matrix by cofactor expansion along the indicated row.

$$\begin{bmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{bmatrix}$$

along the fourth row.

$$(-1)^{4+1}\cdot -2\cdot\begin{vmatrix} -1 & 2 & -1 \\ 4 & 1 & -1 \\ -5 & -3 & 8 \end{vmatrix} + (-1)^{4+2}\cdot 6\cdot\begin{vmatrix} 1 & 2 & -1 \\ -3 & 1 & -1 \\ 2 & -3 & 8 \end{vmatrix}$$

$$+(-1)^{4+3}\cdot -4\cdot\begin{vmatrix} 1 & -1 & 2 \\ -3 & 4 & 1 \\ 2 & -5 & -3 \end{vmatrix} + (-1)^{4+4}\cdot 1\cdot\begin{vmatrix} 1 & -1 & 2 \\ -3 & 4 & 1 \\ 2 & -5 & -3 \end{vmatrix}$$

$$2\cdot\left((-1)^{1+1}\cdot -1\cdot\begin{vmatrix} 1 & -1 \\ -3 & 4 \end{vmatrix} + (-1)^{1+2}\cdot 2\cdot\begin{vmatrix} 4 & -1 \\ -5 & 8 \end{vmatrix} + (-1)^{1+3}\cdot -1\cdot\begin{vmatrix} 4 & 1 \\ -5 & -3 \end{vmatrix}\right)$$

$$+6\cdot\left((-1)^{2+1}\cdot 1\cdot\begin{vmatrix} -3 & 1 \\ 2 & -3 \end{vmatrix} + (-1)^{2+2}\cdot 2\cdot\begin{vmatrix} 1 & -1 \\ 2 & 8 \end{vmatrix} + (-1)^{2+3}\cdot -1\cdot\begin{vmatrix} 1 & -1 \\ 2 & -5 \end{vmatrix}\right)$$

$$-4\cdot\left((-1)^{3+1}\cdot 1\cdot\begin{vmatrix} -3 & 4 \\ -5 & -3 \end{vmatrix} + (-1)^{3+2}\cdot 2\cdot\begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix} + (-1)^{3+3}\cdot -1\cdot\begin{vmatrix} 1 & -1 \\ 2 & -5 \end{vmatrix}\right)$$

$$+1\cdot\left((-1)^{4+1}\cdot 1\cdot\begin{vmatrix} -3 & 4 \\ -5 & -3 \end{vmatrix} + (-1)^{4+2}\cdot 2\cdot\begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix} + (-1)^{4+3}\cdot -1\cdot\begin{vmatrix} 1 & -1 \\ 2 & -5 \end{vmatrix}\right)$$

$$2((-1)^{1+1}\cdot -1\cdot(4-3)+(-1)^{1+2}\cdot 2\cdot(4+15)+(-1)^{1+3}\cdot -1\cdot(-12-5))$$

$$+6((-1)^{2+1}\cdot 1\cdot(-9+2)+(-1)^{2+2}\cdot 2\cdot(-3-16)+(-1)^{2+3}\cdot -1\cdot(-8-2))$$

$$-4((-1)^{3+1}\cdot 1\cdot(-9+20)+(-1)^{3+2}\cdot 2\cdot(-3-8)+(-1)^{3+3}\cdot -1\cdot(-4-10))$$

$$+1((-1)^{4+1}\cdot 1\cdot(-27+20)+(-1)^{4+2}\cdot 2\cdot(-3-8)+(-1)^{4+3}\cdot -1\cdot(-4+10))$$

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Problem 5

Compute $det(E_i)$ if E_i is an elementary matrix of type i.

- E_1 is obtained by interchanging two rows of I_n.
$$det(E_1) = -1$$
- E_2 is obtained by multiplying a row of I_n by a nonzero scalar.
$$det(E_2) = k$$
- E_3 is obtained by adding a multiple of one row of I_n to another row.
$$det(E_3) = 1$$

Problem 6

Use Cramer's rule to solve the given system of linear equations.

$$\begin{aligned} x_1 - x_2 + 4x_3 &= -4 \\ -8x_1 + 3x_2 + x_3 &= 8 \\ 2x_1 + x_2 + x_3 &= 0 \end{aligned}$$

$$det(A) = \begin{bmatrix} 1 & -1 & 4 \\ -8 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix} = 32$$

$$det(A_1) = \begin{bmatrix} -4 & -1 & 4 \\ 8 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 32$$

$$det(A_2) = \begin{bmatrix} 1 & -4 & 4 \\ -8 & 8 & 1 \\ 2 & 0 & 1 \end{bmatrix} = -96$$

$$det(A_3) = \begin{bmatrix} 1 & -1 & -4 \\ -8 & 3 & 8 \\ 2 & 1 & 0 \end{bmatrix} = 32$$

$$x_1 = \frac{det(A_1)}{det(A)} = \frac{32}{-64} = -\frac{1}{2}$$

$$x_2 = \frac{det(A_2)}{det(A)} = \frac{-96}{-64} = \frac{3}{2}$$

$$x_3 = \frac{det(A_3)}{det(A)} = \frac{32}{-64} = -\frac{1}{2}$$

Problem 7

A matrix $M \in M_{nxn}(F)$ is called nipotent if, some positive interrger k, $M^k = 0$, where 0 is the nxn zero matrix. Prove that if M is nilpotent, then $det(M) = 0$.

Problem 7 continued on next page. . .

Proof. Let $M \in M_{nxn}(F)$ be a nilpotent matrix. Then there exists a positive integer k such that $M^k = 0$. We will prove that $det(M) = 0$ using properties of determinants Given that $M^k = 0$, then $det(M^k) = det(0) = 0$. Using the property of determinants that $det(M^k) = det(M)^k$, then $det(M)^k = 0$. Since the only solution that satisfies is when $det(M) = 0$ Therefore, if M is nilpotent, then $det(M) = 0$ $\square$

Problem 8

Use determinants to prove that if $A, B \in M_{nxn}(F)$ are sucht that $AB = I$, then A is invertible (and hence $B = A^{-1}$).

Proof.

If A is invertible, then

$$AB = I$$

Given the only solution to the equation is when $B = A^{-1}$

$$\det(A) \cdot \det(A^{-1}) = \det(AA^{-1}) = \det(I_n) = 1$$

Therefore, $\det(A^{-1}) = \frac{1}{\det(A)}$ if A is invertible and $\det(A) \neq 0$ and $B = A^{-1}$. $\square$

Problem 9

Let $A \in M_{nxn}(F)$ have the form

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{bmatrix}$$

Compute $det(A + tI)$ where I is the nxn identity matrix.

$$A + tI = \begin{vmatrix} t & 0 & 0 & \cdots & 0 & a_0 \\ -1 & t & 0 & \cdots & 0 & a_1 \\ 0 & -1 & t & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & ta_{n-1} \end{vmatrix}$$

$$det(A_{2x2}) = \begin{vmatrix} t & 0 \\ -1 & t \end{vmatrix} = t^2$$

$$det(A_{3x3}) = \begin{vmatrix} t & 0 & 0 \\ -1 & t & 0 \\ 0 & -1 & t \end{vmatrix} = t^3 + t + 1$$

$$det(A_{nxn}) = t^n + t^{n-2} + t^{n-3} + \ldots + t + 1$$

Problem 10