MATE 5150: Exam 01 Review

Alejandro Ouslan

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Contents

1 Determine if T is a Vector Space

1.1 Conditions

1. $x + y = y + x$
2. $x + (y + z) = (x + y) + z$
3. $\exists 0 \in V \quad \text{s.t.} \quad x + 0 = x$
4. $\exists 0 \in V \quad \text{s.t.} \quad x + y = 0$
5. $1x = x$
6. $a(bx) = (ab)x$

7. $a(x+y) = ax + ay$
8. $(a+b)x = ax + bx$

1.2 Summery of the Conditions

1. $0x = 0$
2. $(-a)x = -ax = a(-x)$
3. $a0 = 0$

1.3 Example

Let $S = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$ For $(a_1, a_2), (b_1, b_2) \in S$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2)$$

Property 1: (fails to be a vector space)

$$\begin{aligned}(a_1, a_2) + (b_1, b_2) &= (b_1, b_2) + (a_1, a_2)\\ (a_1 + b_1, a_2 - b_2) &\neq (b_1 + a_1, b_2 - a_2)\end{aligned}$$

2 Determine if T is in the Span of S

2.1 Definition

Definition 1. *The **Span** of the set S denoted by Span(S), is the smallest subspace of V that contains S. That is,*

- *Span(S) is a subspace of V*
- *For any subspaces $W \subseteq V$ such that $S \subseteq W \implies$ Span(S) $\subset W$*

Definition 2. *Let S be a subset of a vector space V.*

- *If $S = \{v_1, v_2, \ldots, v_n\}$, then Span($S$) is the set of all combinations $r_1, r_2, \ldots, r_n \in \mathbb{R}$*
- *If S is an infinite set then Span(S) is the set of all linear combinations $r_1v_1 + r_2v_2 + \ldots + r_nv_n$ where $v_1, v_2, \ldots, v_n \in S$ and $r_1, r_2, \ldots, r_n \in \mathbb{R}$, $n \geq 1$*

2.2 Example 1

Let $v_1 = (1, 2, 0)$, $v_2 = (3, 1, 1)$, and $w = (4, -7, 3)$. Determine whether w belongs to $\text{Span}(v_1, v_2)$.

$$\begin{aligned} w &= r_1v_1 + r_2v_2\\ (4, -7, 3) &= r_1(1, 2, 0) + r_2(3, 1, 1)\\ (4, -7, 3) &= (r_1 + 3r_2, 2r_1 + r_2, r_2)\end{aligned}$$

$$\begin{cases} 4 & = r_1 + 3r_2\\ -7 & = 2r_1 + r_2\\ 3 & = 0r_1 + r_2\end{cases}$$

Thus $w = -5v_1 + 3v_2 \in \text{Span}(v_1, v_2)$

2.3 Example 2

Let $v_1 = (2,5)$, $v_2 = (1,3)$, show that $\{v_1, v_2\}$ is a Span for $\mathbb{R}^2$.

Take any vector $W = (a,b) \in \mathbb{R}^2$. We have to chack thatg there exist $r_1, r_2 \in \mathbb{R}$ such that

$$w = r_1 v_1 + r_2 v_2 \iff \begin{cases} a = 2r_1 + r_2 \\ b = 5r_1 + 3r_2 \end{cases}$$

Coeficients matrix $= \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, $det = 1 \neq 0$. Since the matrix is invertible, the system hasx a unique solution for any a and b. Thus $\{v_1, v_2\}$ is a Span for $\mathbb{R}^2$.

2.4 Example 2 (Alternative)

Same as before,

First let us show that vectors $e_1 = (1,0)$ and $e_2 = (0,1)$ are a Span(v_1, v_2).

$$e_1 = 2v_1 + v_2 \iff \begin{cases} 1 = 2r_1 + r_2 \\ 0 = 5r_1 + 3r_2 \end{cases} \iff \begin{cases} r_1 = 3 \\ r_2 = -5 \end{cases}$$

$$e_2 = v_1 + 3v_2 \iff \begin{cases} 0 = 2r_1 + 3r_2 \\ 1 = 5r_1 + 9r_2 \end{cases} \iff \begin{cases} r_1 = -1 \\ r_2 = 2 \end{cases}$$

Thus $e_1 = 3v_1 - 5v_2$ and $e_2 = -v_1 + 2v_2$. Then for any $w = (a,b) \in \mathbb{R}^2$ we have

$$w = ae_1 + be_2 = a(3v_1 - 5v_2) + b(-v_1 + 2v_2) = (3a - b)v_1 + (-5a + 2b)v_2$$

3 Proove that the pair of vectors is a basis for vector space V

Definition 3. *Let V be a vector space. A linearly independent set spanning set for V is called a **basis**.*

Definition 4. *A set of vectors $S = \{v_1, v_2, \ldots, v_n\}$ is **linearly independent** if the only solution to the equation*

$$r_1 v_1 + r_2 v_2 + \ldots + r_n v_n = 0$$

is $r_1 = r_2 = \ldots = r_n = 0$.

1. Proove that the set S is linearly independent.
2. Proove that the set S spans V.

3.1 Example

Determine which of the following sets of vectors is a basis for $\mathbb{R}^3$.

1. $S_1 = \{(2,-4,1), (0,3,-1), (6,0,-1)\}$

$$r_1(2,-4,1) + r_2(0,3,-1) + r_3(6,0,-1) = (0,0,0)$$

$$\begin{cases} 2r_1 + 0r_2 + 6r_3 & = 0 \\ -4r_1 + 3r_2 + 0r_3 & = 0 \\ r_1 - r_2 - r_3 & = 0 \end{cases} \implies \begin{cases} r_1 = -3r_3 \\ r_2 = -4r_3 \\ 0 = -3r_3 - 4r_3 - r_3 \end{cases}$$

Thus S_1 is not linearly independent and therefore not a basis for $\mathbb{R}^3$, given that $0 = -3r_1 - 4r_2 + r_3$

2. $S_2 = \{(1,-3,-2), (-3,1,3), (-2,-10. -2)\}$

4 Generate a polynomial of Lagrange Interpolation of degree 3

$$P(x) = \sum_{i=0}^{n} y_i \mathscr{L}_i(x)$$

$$\mathscr{L}_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j} = \frac{(x - x_0) \cdots (x - x_i) \cdots (x - x_n)}{(x_i - x_0) \cdots (x_i - x_{i-1}) \cdots (x_i - x_n)}$$

4.1 Example for $n = 3$

$$\{(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3))\}$$

$$\begin{aligned}
P(x) = & \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0) + \\
& + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1) + \\
& + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2) +
\end{aligned}$$

5 Determine if T is bijective

1. Proove that T is linear.
2. Find the kernel of T.
3. Find the Rank of T.
4. Determine if it is 1-1.

6 Change of Basis

The strategy is to find the change of basis matrix P such that $[v]_\beta = P[v]_\alpha$.

6.1 Example

Let $\beta = \{1 + x - x^2, x + x^2, -x + 3x^2\}$ and $\alpha = \{1 + x + x^2, 1 - 2x^2, 4x\}$. and let $p(x) \in P_2$ be such that $[p(x)]_\alpha = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Find $[p(x)]_\beta$.

$$\begin{aligned}
[p(x)]_\alpha &= 3(1 + x + x^2) + 2(1 - 2x^2) + 1(4x) \\
&= 3 + 3x + 3x^2 + 2 - 4x^2 + 4x + 4x \\
&= 5 + 5x - x^2 \\
[p(x)]_\beta &= 5 + 5x - x^2 = c_1(1 + x - x^2) + c_2(x + x^2) + c_3(-x + 3x^2) \\
[p(x)]_\beta &= \begin{cases} 5 = c_1 + 0 + 0 \\ 5 = c_1 + c_2 - c_3 \\ -1 = -c_1 + c_2 + 3c_3 \end{cases} \implies \begin{cases} c_1 = 5 \\ c_2 = 1 \\ c_3 = 1 \end{cases} \\
[p(x)]_\beta &= \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}$$

7 Answer of Given Questions

In R^2, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $T(x, y)$, where

1. T is the reflection of R^2 about L.

$$M_x = \frac{x_1 + x_2}{2} \quad \text{and} \quad M_y = \frac{y_1 + y_2}{2}$$
$$AB = \frac{y_2 - y_1}{x_2 - x_1} = -1/m$$

2. T is the projection on L along the line perpe qqndicular to L. (See the definition of projection in the exercises of Section 2.1.)

8 Answer of Given Questions 2

Let V be a vector space and S a subset of V with the property that whenever $v_1, v_2, \cdots, v_n \in S$ and $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$, then $a_1 = a_2 = \cdots = a_n = 0$. Prove that every vector in the span of S can be uniquely writen as a linear combination of vectors of S.

9 Answer of Given Questions 3

The combinattion $W = \{f(x) \in P(F) : f(x) = 0$ or $f(x)$ is a polynomial of degree n $\}$ is a subspace of $P(F)$ if $n \geq 1$. Justify your answer.

10 Answer of Given Questions 4

Let $T \in L(V, W)$. Since each subspace has a complement, we can write $V = N(T) \oplus N(T)^C$, where $N(T)^C$ is the complement of $N(T)$ in V. Show that any complement of $N(T)$ is a isomorphic to $R(T)$.