MATE 5150: Asignacion #3

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Problem 1

Use the proof of Theorem 3.2 to obtain the inverse of each of the following matrices:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1} = E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 2

Let A be an $m \times n$ matrix. Prove that if B can be obtained from A by an elementary row [column] operation, then B^T can be obtained from A^T by the corresponding elementary column [row] operation.

Proof.

$$\begin{aligned} \text{Let } E_i \text{ be the elementary row operation that transforms } A \text{ into } B \\ B &= E_i \cdots E_2 E_1 A \\ B^T &= (E_i \cdots E_2 E_1 A)^T \\ B^T &= A E_1^T E_2^T \cdots E_i^T \\ \text{Let } E_j \text{ be the elementary column operation that transforms } A^T \text{ into } B^T \\ B &= A E_1 E_2 \cdots E_j \\ B^T &= (A E_1 E_2 \cdots E_j)^T \\ B^T &= E_j^T \cdots E_2^T E_1^T A^T \end{aligned}$$

Therefore, B^T can be obtained from A^T by the corresponding elementary column operation. □

Problem 3

Prove that any elementary row [column] operation of type 2 can be obtained by dividing some row [column] by a nonzero scalar.

Proof. Let A be an $m \times n$ matrix and B be the matrix obtained by dividing the i-th row of A by a nonzero scalar c which would be equivalent to multiplying the i-th row of A by some scalar k

$$c = \frac{1}{\frac{1}{k}}$$
$$c = k$$

Therefore, the elementary row operation of type 2 can be obtained by dividing some row by a nonzero scalar. □

Problem 4

Find the rank of the following matrix:

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

The rank of the matrix is 1

Problem 5

For each of the following matrices, compute the rank and the inverse if exists:

$$F = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$FE_1 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The rank of the matrix is 2 and the inverse does not exist.

Problem 6

For each of the following linear transformations T, determine whether T is invertible, and compute T^{-1} if it exists:

$$T = P_2(\mathbb{R}) \to P_2(\mathbb{R}) \quad \text{defined by} \quad T(f(x)) = (x+1)f'(x)$$

$$\text{Let } f'(x) = 2ax + b$$

$$\text{Then } T(f(x)) = (x+1)(2ax+b) = 2ax^2 + (2a+b)x + b$$

$$\text{Let } g(x) = 2dx + e$$

$$f(x) = ax^2 + bx + c \text{ and } g(x) = dx^2 + ex + f$$

Since $f(x)$ is constant when $a = 0$ and $b = 0$, then T is not injective. Therefore, T is not invertible.

Problem 7

For each of the following linear transformations T, determine whether T is invertible, and compute T^{-1} if it exists:

$$T = \mathbb{R}^3 \to \mathbb{R}^3 \quad \text{defined by} \quad T(a_1, a_2, a_3) = (a_1 + 2a_2 + 3a_3, -a_1 + a_2 + 2a_3, a_1 + a_3)$$

Problem 7 continued on next page. . .

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$Rank(A) = 3$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{3}{4} & -\frac{1}{2} & -\frac{5}{4} \\ -\frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Problem 8

Let $T, U : V \to W$ be linear transformations.

- Prove that $R(T+U) \subseteq R(T) + R(U)$.

$$\begin{aligned} &\text{Let } v \in V \\ (T+U)(v) &= T(v) + U(v) \\ \text{Since } T(v) \in R(T) &\text{ and } U(v) \in R(U) \\ T(v) + U(v) &\in R(T) + R(U) \\ R(T+U) &\subseteq R(T) + R(U) \end{aligned}$$

- Prove that if W if finite-dimensional, then $rank(T+U) \leq rank(T) + rank(U)$.

$$\begin{aligned} R(T+U) &\subseteq R(T) + R(U) \\ dim(R(T+U)) &\leq dim(R(T) + R(U)) \\ dim(R(T+U)) &\leq dim(R(T)) + dim(R(U)) \\ rank(T+U) &\leq rank(T) + rank(U) \end{aligned}$$

- Deduce from (b) that if $rank(A+B) \leq rank(A) + rank(B)$ for any $m \times n$ matrices A and B.