MATE 5150: Asignacion #2

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Problem 1

Solve the following systems of linear equations by the method introduced introduces in this section.

$$\begin{aligned} x_1 &+ 2x_2 && +2x_3 && &&= 2 \\ x_1 & && +8x_3 && +5x_4 &&= -6 \\ x_1 &+ x_2 && +5x_3 && +5x_4 &&= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & 0 & 8 & 5 \\ 1 & 1 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & 1 & 5 & 5 \\ 1 & 0 & 8 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & -1 & 3 & 5 \\ 0 & -2 & 6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 8 & 20 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -20 \\ 9 \\ 2 \end{bmatrix}$$

$$(x_1, x_2, x_3, x_4) = (-20, 9, 0, 2)$$

Problem 2

In each part, determine whether the given vector is in the span of S.(change is wrohng)

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}, S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

$$S = a\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} a & 0 \\ -a & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix} + \begin{pmatrix} c & c \\ 0 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} a+c & b+c \\ -a & b \end{pmatrix}$$

Problem 2 continued on next page...

Given $a = 3$, $b = 4$, and $c = -2$ we have that the given vector is in the span of S.

Problem 3

Show that the vectors $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$ generate $\mathbb{F}^3$

Proof.

$$\begin{aligned} &\text{Let } v = (x, y, z) \in \mathbb{F}^3 \\ &\text{Then } v = a(1, 1, 0) + b(1, 0, 1) + c(0, 1, 1) \\ &\text{Then } v = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \text{Then determinate } &\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(0 - 1) - 1(1 - 0) = -1 \neq 0 \end{aligned}$$

Thus the vectors $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$ generate $\mathbb{F}^3$. □

Problem 4

Let S_1 and S_2 be subsets of a vector space V. Prove that $span(S_1 \cap S_2) \subseteq span(S_1) \cap span(S_2)$. Give an example in which $span(S_1 \cap S_2)$ and $span(S_1) \cap span(S_2)$ are equal and one in which they are not equal.

Proof.

$$\begin{aligned} &\text{Let } x \in span(S_1 \cap S_2) \\ &\text{Then } x = c_1v_1 + c_2v_2 + \ldots + c_nv_n \text{ where } v_i \in S_1 \cap S_2 \\ &\text{Since } x \in S_1 \cap S_2 \implies v_i \in S_1 \text{ and } v_i \in S_2 \text{ for all } i \\ &\text{Thus } x = c_1v_1 + c_2v_2 + \ldots + c_nv_n \text{ where } v_i \in S_1 \text{ and } v_i \in S_2 \end{aligned}$$

Therefore $x \in span(S_1) \cap span(S_2)$, thus $span(S_1 \cap S_2) \subseteq span(S_1) \cap span(S_2)$. □

Problem 5

Determine whether the following sets are linearly dependent or linearly independent.

$$\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix} \right\}$$

$$v = c_1 \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} + c_3 \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{aligned} c_1 - c_3 + 2c_4 &= 0 \quad (1) \\ -c_2 + 2c_3 + c_4 &= 0 \quad (2) \\ -2c_1 + c_2 + c_3 - 4c_4 &= 0 \quad (3) \\ c_1 + c_2 + 4c_4 &= 0 \quad (4) \end{aligned}$$

Problem 6

In $M_{m\times n}(\mathbb{F})$, let E_{ij} denote the matrix whose only nonzero entry is a 1 in the ith row and jth column. Prove that $\{E_{ij} : 1 \le i \le m, 1 \le j \le n\}$ is linearly independent.

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}E_{ij} = 0 \quad \text{for scalars } c_{ij} \in \mathbb{F}$$

$$\Rightarrow \quad \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}E_{ij} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\Rightarrow \quad c_{ij} = 0 \quad \text{for all } i, j$$

Problem 7

Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the functions defined by $f(x) = e^{rx}$ and $g(x) = e^{sx}$, where $r \neq s$. Prove that f and g are linearly independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

$$c_1 f(x) + c_2 g(x) = 0 \quad \text{for all } x \in \mathbb{R}$$

$$\Rightarrow \quad c_1 e^{rx} + c_2 e^{sx} = 0$$

$$\Rightarrow \quad e^{rx}(c_1 + c_2 e^{(s-r)x}) = 0$$

$$\Rightarrow \quad c_1 + c_2 e^{(s-r)x} = 0 \quad \text{for all } x$$

$$\Rightarrow \quad c_1 = 0 \quad \text{and} \quad c_2 = 0$$

Problem 8

Determine which of the following sets are bases for $\mathbb{R}^3$.

$$\{(1, -3, 1), (-3, 1, 3), (-2, -10, 2)\}$$

Construct the matrix:

$$A = \begin{pmatrix} 1 & -3 & 1 \\ -3 & 1 & 3 \\ -2 & -10 & 2 \end{pmatrix}$$

Row reduce:

$$\begin{pmatrix} 1 & -3 & 1 \\ 0 & -8 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

Rank = 2.
The set $\{(1, -3, 1), (-3, 1, 3), (-2, -10, 2)\}$ is **not a basis** for $\mathbb{R}^3$.

Problem 9

To find a basis for $\mathbb{R}^3$ from the set $\{u_1, u_2, u_3, u_4, u_5\}$, we can examine the linear independence of the vectors. We will form a matrix A with the vectors as rows and perform row reduction:

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 4 & -2 \\ -8 & 12 & -4 \\ 1 & 37 & -17 \\ -3 & -5 & 8 \end{bmatrix}$$

Performing row reduction, we find:

$$\begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From this row echelon form, we see that u_1 and u_2 are linearly independent and span $\mathbb{R}^3$ with an additional vector.
We can choose:

$$\{u_1, u_2, u_5\}$$

Thus, a basis for $\mathbb{R}^3$ is given by:

$$\{(2, -3, 1), (1, 4, -2), (-3, -5, 8)\}$$

Problem 10

To express the vector $v = (a_1, a_2, a_3, a_4)$ as a linear combination of the basis vectors u_1, u_2, u_3, u_4:

$$v = c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4$$

From the equations:
1. $c_1 = a_1$ 2. $c_2 = a_2 - a_1$ 3. $c_3 = a_3 - a_2$ 4. $c_4 = a_4 - a_3$
The unique representation is:

$$v = a_1u_1 + (a_2 - a_1)u_2 + (a_3 - a_2)u_3 + (a_4 - a_3)u_4$$

Problem 11

For the points $(-2, 3), (-1, -6), (1, 0), (3, -2)$:

$$P(x) = 3\left(-\frac{(x+1)(x-1)(x-3)}{15}\right) - 6\left(\frac{(x+2)(x-1)(x-3)}{12}\right) - 2\left(\frac{(x+2)(x+1)(x-1)}{40}\right)$$

For the points $(-2, -6), (-1, 6), (1, 0), (3, -2)$:

Problem 11 continued on next page. . .

$$P(x) = -6\left(-\frac{(x+1)(x-1)(x-3)}{15}\right) + 6\left(\frac{(x+2)(x-1)(x-3)}{12}\right) - 2\left(\frac{(x+2)(x+1)(x-1)}{40}\right)$$

The specific polynomials can be computed and simplified from these expressions.

Problem 12

The set of all upper triangular $n \times n$ matrices can be expressed as:

$$W = \left\{ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} : a_{ij} \in \mathbb{F},\ i \leq j \right\}$$

A basis for W consists of matrices that have a single entry of 1 in the upper triangular position and 0 elsewhere. Specifically, these matrices can be represented as follows:

1. $E_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

2. $E_{12} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$

3. $E_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 & \end{pmatrix}$

4. $\vdots$

5. $E_{nn} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$

The total number of these basis matrices corresponds to the number of entries in the upper triangular part of the matrix, which is:

$$\text{Dimension of } W = \frac{n(n+1)}{2}$$

Thus, the dimension of the subspace W of upper triangular $n \times n$ matrices is:

$$\boxed{\frac{n(n+1)}{2}}$$

A basis for W consists of the matrices E_{ij} where $1 \leq i \leq j \leq n$.