MATE 5150: Asignacion #2

Due on Septiembre 18, 2024

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Problem 1

Solve the following systems of linear equations by the method introduced introduces in this section.

$$\begin{aligned} x_1 &\quad + 2x_2 &\quad +2x_3 & & &= 2 \\ x_1 & & +8x_3 &\quad + 5x_4 &\quad &= -6 \\ x_1 &\quad + x_2 &\quad +5x_3 &\quad + 5x_4 & &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & 0 & 8 & 5 \\ 1 & 1 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & 1 & 5 & 5 \\ 1 & 0 & 8 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & -1 & 3 & 5 \\ 0 & -2 & 6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 8 & 20 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -20 \\ 9 \\ 2 \end{bmatrix}$$

$$(x_1, x_2, x_3, x_4) = (-20, 9, 0, 2)$$

Problem 2

In each part, determine whether the given vector is in the span of S.(change is wrohng)

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}, \; S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

$$S = a \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} a & 0 \\ -a & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix} + \begin{pmatrix} c & c \\ 0 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} a + c & b + c \\ -a & b \end{pmatrix}$$

Problem 2 continued on next page...

Given $a = 3$, $b = 4$, and $c = -2$ we have that the given vector is in the span of S.

Problem 3

Show that the vectors $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$ generate $\mathbb{F}^3$

Proof.

$$\begin{aligned}
\text{Let } v &= (x, y, z) \in \mathbb{F}^3 \\
\text{Then } v &= a(1, 1, 0) + b(1, 0, 1) + c(0, 1, 1) \\
\text{Then } v &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
\text{Then determinate } &\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(0 - 1) - 1(1 - 0) = -1 \neq 0
\end{aligned}$$

Thus the vectors $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$ generate $\mathbb{F}^3$. □

Problem 4

Let S_1 and S_2 be subsets of a vector space V. Prove that $span(S_1 \cap S_2) \subseteq span(S_1) \cap span(S_2)$. Give an example in which $span(S_1 \cap S_2)$ and $span(S_1) \cap span(S_2)$ are equal and one in which they are not equal.

Proof.

$$\begin{aligned}
&\text{Let } x \in span(S_1 \cap S_2) \\
&\text{Then } x = c_1v_1 + c_2v_2 + \ldots + c_nv_n \text{ where } v_i \in S_1 \cap S_2 \\
&\text{Since } x \in S_1 \cap S_2 \implies v_i \in S_1 \text{ and } v_i \in S_2 \text{ for all } i \\
&\text{Thus } x = c_1v_1 + c_2v_2 + \ldots + c_nv_n \text{ where } v_i \in S_1 \text{ and } v_i \in S_2
\end{aligned}$$

Therefore $x \in span(S_1) \cap span(S_2)$, thus $span(S_1 \cap S_2) \subseteq span(S_1) \cap span(S_2)$. □

Problem 5

Determine whether the following sets are linearly dependent or linearly independent.

$$\{x^3 - x, 2x^2 + 4, -2x^3 + 3x^2 + 2x + 6\}$$

Problem 6

In $M_{m\times n}(\mathbb{F})$, let E_{ij} denote the matrix whose only nonzero entry is a 1 in the ith row and jth column. Prove that $\{E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is linearly independent.

Problem 7

Let $f, g, \in \mathcal{F}(R, R)$ be the function defined by $f(x) = e^{rt}$ and $g(x) = e^{st}$, where $r \neq s$. Prove that f and g are linearly independent in $\mathcal{F}(R, R)$.

Problem 8

Determine which of the following sets are bases for $\mathbb{R}^3$.

$$\{(1, -3, 1), (-3, 1, 3), (-2, -10, 2)\}$$

Problem 9

The vectors $u_1 = (2, -3, 1)$, $u_2 = (1, 4, -2)$, $u_3 = (-8, 12, -4)$, $u_4 = (1, 37, -17)$ and $u_5 = (-3, -5, 8)$ generate $\mathbb{R}^3$. Find a subset of the set $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for $\mathbb{R}^3$.

Problem 10

The vectors $u_1 = (1, 1, 1, 1)$, $u_2 = (0, 1, 1, 1)$, $u_3 = (0, 0, 1, 1)$, and $u_4 = (0, 0, 0, 1)$ form a basis for $\mathbb{R}^4$. Find the unique representation of an arbitrary vector $v = (a_1, a_2, a_3, a_4)$ $\mathbb{R}^4$ as a linear combination of u_1, u_2, u_3, u_4.

Problem 11

In each part, use the Lagrange interpolation formula to construct the polynomial of smallest degree whose graph contains the following points. $(-2, 3), (-1, -6), (1, 0), (3, -2)$.

Problem 12

The set of all upper triangular $n \times n$ matrices is a subspace of W of $M_{n \times n}(\mathbb{F})$. Find a basis for W. What is the dimension of W?