MATE 5150: Determinants

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Contents

1 Diagonalizability

In Example 6 of Sectoin5.1, we obtained a basiss of eigenvectors by choosing one eigenvector corresponding to each eigenvalue. In general, such a procedure does not yeild a basis, but the following theorem shows that any set constructed in this manner is linearly independen.

Theorem 1. *Let T be a linear operator on a vector space V, and let $\lambda_1, \ldots, \lambda_m$ be distict eigencalues of T. If $v_1, \ldots, v_m$, then $\{v_1, \ldots, v_m\}$ is linearly independent*

Corollary 1. *Let T be a linear operator on an n-dimensional vector space V. If T has n distict eigenalues, then T is diagonalizable.*

Proof. Supongamos que T tiene n valores propios distintos $\lambda_1, \ldots, \lambda_n$ y sea $v_1, \ldots, v_n$ vectores propios de T correspondientes a $\lambda_1, \ldots, \lambda_n$. para $1 \leq i \leq n$. por el terorema 5.5 son linealmente independientes Dado que $dim(V)$ entonces el candidato para ser una base de V es $\{v_1, \ldots, v_n\}$. Por el terorema 5.1 entonce. T es diagonalizable. □

Definition 1. *A polynomial $f(x)$ is $P(F)$ **splits over** F if ther are scalars $c, a_1, \ldots, a_n$ (not necessarlily distinct) in F such that*

$$f(x) = c(t - a_1) \cdots (t - a_n)$$

Theorem 2. *The characteristic polynomial of any diagonalizable linear operator splits.*

Proof. Supongamos T es diagonalizable y $\exists$ una base β para v tal que $[T]_\beta = D$, donde D es diagonal y supongamos que

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Sea $f(t)$ el polinomio caracteristico de T. Entonces

$$f(t) = det(D - tI_n)$$

$$det \begin{bmatrix} \lambda_1 - t & 0 & \cdots & 0 \\ 0 & \lambda_2 - t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n - t \end{bmatrix} = (\lambda_1 - t) \cdots (\lambda_n - t)$$

□

Definition 2. *Let λ be an eigenvalue of linear operator or matrix wiht characteristic polynomial $f(t)$. The **algebraic multiplicity** of λ is the largest positive integer k for which $(t - \lambda)^k$ is a factor of $f(t)$.*