MATE 5150: Elementary Matrix Operations and Systrems of Linear Equations

Alejandro Ouslan

Academic Year 2024-2025

Contents

1 Elementary Matrix Operations and Elementary Matrices

In this section, we difine the elementary operations that are used throughout the chapter. In subsequent sections, we use these operations to obtrain simiple computational methods for determining the rank of a linear transformation and the solution of a system of linear equations. There are two types of elementary operations -row operations and column operations. As we will see, the row operations are more useful. They arise from the three operations that can be used to eliminate variables in a system of linear equations.

1.1 Elementary Operations

Definition 1. *Let A be an $m \times n$ matrix. Any of the following three operations on the rows [columns] of A is called an* ***elementary row [column] operation****:*

1. *interchanging any two rows [columns] of A;*
2. *multiplying any row [column] of A by a nonzero scalar;*
3. *adding any scalar multiple of a row [column] of A to another row [column] of A.*

Any of these three operations is called an **elementary operation**. Elementary operations are of **type 1**, **type 2**, or **type 3** depending on wheather they are obtained by (1), (2), or (3) above.

Definition 2. *An $n \times n$* ***elementary matrix*** *is a matrix obtained by performing an elementary operations on I_n. The elemetnary matrix is said to be of* ***type 1****,* ***type 2****, or* ***type 3*** *acording to wheather the elementary operation performed on I_n is of type 1, type 2, or type 3.*

1.1.1 Example Elementary Matrices

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \quad , B = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -2 & 1 \\ 1 & -3 & 1 \end{bmatrix} \quad \text{and} \quad , C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{bmatrix}$$

Find an elementary operation that transforms A into B and an elementary operation that transforms B into C. By means of several additional operations, transform C into I_3.

$$AE = B \quad \text{where} \quad E = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$EB = C \quad \text{where} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1E_2E_3E_4 = I_3 \quad \text{where} \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad , \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \quad , E_4 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

1.2 Properties of Elementary Matrices

Theorem 1. *Elementary matrices are invertible, and the inverse of an elementary matrix is an elementary matrix of the same type.*

Proof.

Let E be an elementary matrix $n \times n$. The E is defided by an elementary operation on I_n.

□

1.2.1 Example

Use the proog in Theroem 1 to obtain the inverse of each of the folowing elementary matrices:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Finding the inverse of each of the elementary matrices in the example above, we have:

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{, Therefore} \quad A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{, Therefore} \quad B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{, Therefore} \quad C^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

1.3 Matrix Multiplication and Elementary Matrices

Let A be an $m \times n$ matrix. Prove that if E can be otained from A by an elementary row [column] operation, then B^T can be obtained from A^T by the corresponding elementary column [row] operation.

Proof.

$$(E_R B)^T = (A)^T$$
$$B^T E_R^T = A^T$$

Therefore, B^T can be obtained from A^T by the corresponding elementary column operation. □