# MATE 5150: Determinants

Alejandro Ouslan

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# 2 Determinants of Order $n$

# 3 Properties of Determinants

## 3.1 Facts about elementary Matrices

1. If $E$ is an elementary matrix obtained by interchanging any two rows of $I_n$, then $\det(E) = -1$.

2. If $E$ is an elementary matrix obtained by multiplying some row of $I_n$ by the nonzero scalar $K$, then $\det(E) = K$.

3. If $E$ is an elementary matrix obtained by adding a multiple of some row of $I_n$ to another row, the $\det(E) = 1$.

**Theorem 1.** *For any $A, B \in M_{n \times n}(\mathbb{R})$, $\det(AB) = \det(A) \cdot \det(B)$.*

*Proof.*

$$\begin{aligned}
\det(AB) &= \det(E_m \cdots E_2 E_1 B) \\
&= \det(E_m) \cdot \det(E_{m-1} \cdots E_2 E_1 B) \\
&= \det(E_m \cdots E_2 E_1) \cdot \det(B) \\
&= \det(A) \cdot \det(B)
\end{aligned}$$

Therefore, $\det(AB) = \det(A) \det(B)$. $\qquad \square$

**Corollary 1.** *A Matrix $A \in M_{n \times n}(F)$ is invertible if and only if $\det(A) \neq 0$. Furthermore, if $A$ is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.*

*Proof.*

If $A$ is invertible, then $rank(A) < n$

So by the Correlation 4.6, $\det(A) = 0$

If $A$ is not invertible, then

$$\det(A) \cdot \det(A^{-1}) = \det(AA^{-1}) = \det(I_n) = 1$$

Therefore, $\det(A^{-1}) = \frac{1}{\det(A)}$ if $A$ is invertible and $\det(A) \neq 0$. $\square$

**Theorem 2.** *For any $A \in M_{n \times n}(F)$, $\det(A^T) = \det(A)$.*

*Proof.*

$$\begin{aligned}
\det(A^T) &= \det(E_1^T \cdot E_2^T \cdots E_m^T) \\
&= \det(E_1^T) \cdot \det(E_2^T) \cdots \det(E_m^T) \\
&= \det(E_1) \cdot \det(E_2) \cdots \det(E_m) \\
&= \det(E_m \cdots E_2 E_1) \\
&= \det(A)
\end{aligned}$$

Therefore, $\det(A^T) = \det(A)$. $\square$

**Theorem 3** (Cramer's Rule)**.** *Let $Ax = b$ be the matrix form of a system of $n$ linear equations in $n$ unknowns. Where $x = (x_1, x_2, \ldots, x_n)^T$. If $\det(A) \neq 0$, then this system has a unique solution, and for each $k$ $(k = 1, 2, \ldots, n)$,*

$$x_k = \frac{\det(M_k)}{\det(A)}$$

*where $M_k$ is the $n \times n$ matrix obtained from $A$ by replacing column $k$ of $A$ by $b$.*

*Proof.* Let $X_k$ be the matrix obtained from $A$ by replacing the $k$-th column of $A$ with the vector $\mathbf{b}$. Then we can write the system as:

$$AX_k = M_k$$

Where $M_k$ is the vector formed by the coefficients of the system in the new configuration.
To evaluate $\det(X_k)$ using cofactor expansion along the $k$-th column, we have:

$$\det(X_k) = x_k \det(I_{n-1})$$

Since $\det(I_{n-1}) = 1$, it follows that:

$$\det(X_k) = x_k$$

By applying the property of determinants, we obtain:

$$\det(M_k) = \det(AX_k) = \det(A) \cdot \det(X_k)$$

Substituting $\det(X_k)$ into this equation gives:

$$\det(M_k) = \det(A) \cdot x_k$$

Therefore, we can rearrange this to find $x_k$:

$$x_k = \frac{\det(M_k)}{\det(A)}$$

This proves Cramer's Rule for the variable $x_k$. $\square$

# 4 Summary - Important Facts about Determinants