MATE 5150: Determinants

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3.1 Facts about elementary Matrices

1. If E is an elementary matrix obtained by interchanging any two rows of I_n, then $\det(E) = -1$.
2. If E is an elementary matrix obtained by multiplying some row of I_n by the nonzero scalar K, then $\det(E) = K$.
3. If E is an elementary matrix obtained by adding a multiple of some row of I_n to another row, the $\det(E) = 1$.

Theorem 1. *For any* $A, B \in M_{n\times n}(\mathbb{R})$, $\det(AB) = \det(A)\cdot\det(B)$.

Proof.

$$\begin{aligned}\det(AB) &= \det(E_m \cdots E_2E_1B)\\ &= \det(E_m)\cdot\det(E_{m-1}\cdots E_2E_1B)\\ &= \det(E_m\cdots E_2E_1)\cdot\det(B)\\ &= \det(A)\cdot\det(B)\end{aligned}$$

Therefore, $\det(AB) = \det(A)\det(B)$. □

Corollary 1. *A Matrix* $A \in M_{n\times n}(F)$ *is invertible if and only if* $\det(A) \neq 0$. *Furthermore, if* A *is invertible, then* $\det(A^{-1}) = \frac{1}{\det(A)}$.

Proof.

If A is invertible, then $rank(A) < n$

So by the Correlation 4.6, $\det(A) = 0$

If A is not invertible, then

$$\det(A) \cdot \det(A^{-1}) = \det(AA^{-1}) = \det(I_n) = 1$$

Therefore, $\det(A^{-1}) = \frac{1}{\det(A)}$ if A is invertible and $\det(A) \neq 0$. $\square$

Theorem 2. *For any $A \in M_{n\times n}(F)$, $\det(A^T) = \det(A)$.*

Proof.

$$\begin{aligned}
\det(A^T) &= \det(E_1^T \cdot E_2^T \cdots E_m^T) \\
&= \det(E_1^T) \cdot \det(E_2^T) \cdots \det(E_m^T) \\
&= \det(E_1) \cdot \det(E_2) \cdots \det(E_m) \\
&= \det(E_m \cdots E_2 E_1) \\
&= \det(A)
\end{aligned}$$

Therefore, $\det(A^T) = \det(A)$. $\square$

Theorem 3. *Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns. Where $x = (x_1, x_2, \ldots, x_n)^T$. If $\det(A) \neq 0$, then this system has a unique solution, and for each k ($k = 1, 2, \ldots, n$),*

$$x_k = \frac{\det(M_k)}{\det(A)}$$

where M_k is the $n \times n$ matrix obtained from A by replacing column k of A by b.

4 Summary - Important Facts about Determinants