MATE 5150: Asignacion #3

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Problem 1

Determine whether T is invertible and justify your answer.

$$T : \mathbb{R}^2 \to \mathbb{R}^3, T(a_1, a_2) = (3a_1 - a_2, a_2, 4a_1)$$

Since the $Rank(\mathbb{R}^3) \neq Dim(\mathbb{R}^2)$, then T is not invertible.

Problem 2

Determine whether T is invertible and justify your answer.

$$T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R}), T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b & a \\ c & c+d \end{bmatrix}$$

Problem 2 continued on next page. . .

Check if T is linear:

$$\begin{aligned}
T(A+B) &= T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) \\
&= \begin{bmatrix} (a_1+a_2)+(b_1+b_2) & (a_1+a_2) \\ (c_1+c_2) & (c_1+c_2)+(d_1+d_2) \end{bmatrix} \\
&= \begin{bmatrix} a_1+b_1+a_2+b_2 & a_1+a_2 \\ c_1+c_2 & c_1+c_2+d_1+d_2 \end{bmatrix} \\
T(A)+T(B) &= \begin{bmatrix} (a_1+b_1)+(a_2+b_2) & a_1+a_2 \\ c_1+c_2 & (c_1+d_1)+(c_2+d_2) \end{bmatrix} \\
&= \begin{bmatrix} a_1+b_1+a_2+b_2 & a_1+a_2 \\ c_1+c_2 & c_1+c_2+d_1+d_2 \end{bmatrix} \\
&\Rightarrow T(A+B) = T(A)+T(B). \\
cT(A) :&= c\begin{bmatrix} a_1+b_1 & a_1 \\ c_1 & c_1+d_1 \end{bmatrix} \\
&= \begin{bmatrix} c(a_1+b_1) & ca_1 \\ cc_1 & c(c_1+d_1) \end{bmatrix} \\
T(cA) &= T\left(\begin{bmatrix} ca_1 & cb_1 \\ cc_1 & cd_1 \end{bmatrix}\right) \\
&= \begin{bmatrix} ca_1+cb_1 & ca_1 \\ cc_1 & cc_1+cd_1 \end{bmatrix} \\
&\Rightarrow T(cA) = cT(A).
\end{aligned}$$

Find the kernel of T :

$$\begin{aligned}
\ker(T) &= \{X \in M_{2\times 2}(\mathbb{R}) : T(X) = 0\} \\
T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&\Rightarrow a+b=0, a=0, c=0, c+d=0 \\
&\Rightarrow a=b=c=d=0 \\
&\Rightarrow \ker(T) = \{0\}.
\end{aligned}$$

Determine the range of T :

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b & a \\ c & c+d \end{bmatrix}.$$

Let $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$ be in the range.

$$\begin{aligned}
x_1 = a+b, x_2 = a, y_1 = c, y_2 = c+d. \\
\Rightarrow a = x_2, b = x_1 - x_2, c = y_1, d = y_2 - y_1.
\end{aligned}$$

Therefore, T is invertible.

Problem 3

Is the following pairs of vector spaces are isomorphic? Justify your answer.

$$V = \{A \in M_{2\times 2}(\mathbb{R}) : \operatorname{tr}(A) = 0\} \text{ and } \mathbb{R}^4$$

Problem 3 continued on next page. . .

Dimension of $\mathbb{R}^4$: The vector space $\mathbb{R}^4$ has dimension 4.

Dimension of V : The space $M_{2\times 2}(\mathbb{R})$ has dimension 4.

The condition $\operatorname{tr}(A) = 0$ implies $a + d = 0$.

Thus, a general element in V can be written as:

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

The parameters a, b, c can be chosen freely, leading to a dimension of 3.

Comparing dimensions: Dimension of V is 3.

Dimension of $\mathbb{R}^4$ is 4.

Since the dimensions are different, V cannot be isomorphic to $\mathbb{R}^4$.

Conclusion: The vector spaces V and $\mathbb{R}^4$ are not isomorphic.

Problem 4

Let

$$V = \left\{ \begin{bmatrix} a & a+b \\ 0 & c \end{bmatrix} : a, b, c \in F \right\}$$

Construct an isomorphism from V to F^3.

$$T : V \to F^3$$

$$T\left(\begin{bmatrix} a & a+b \\ 0 & c \end{bmatrix}\right) = (a, b, c)$$

To show T is linear:

Additivity:

$$T\,(A + B) = T\left(\begin{bmatrix} a_1 + a_2 & (a_1 + b_1) + (a_2 + b_2) \\ 0 & c_1 + c_2 \end{bmatrix}\right) = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$$

$$= T(A) + T(B)$$

Scalar multiplication:

$$T(kA) = T\left(\begin{bmatrix} ka & k(a+b) \\ 0 & kc \end{bmatrix}\right) = (ka, kb, kc) = kT(A)$$

Injectivity: $T(A) = T(B) \implies A = B$

Surjectivity: $\forall (a, b, c) \in F^3, \exists \begin{bmatrix} a & a+b \\ 0 & c \end{bmatrix} \in V$

Conclusion: T is an isomorphism.

Problem 5

For each of the following pairs of ordered bases β and β' for $\mathbb{R}^2$, find the change of coordinates matrix that changes β-coordinates into β-coordinate

$$\beta = \{(-1, 3), (2, -1)\} \text{ and } \beta' = \{(0, 10), (5, 0)\}$$

$$\text{Let } \beta = \{\mathbf{v_1} = (-1,3), \mathbf{v_2} = (2,-1)\}$$
$$\text{Let } \beta' = \{\mathbf{u_1} = (0,10), \mathbf{u_2} = (5,0)\}$$
$$\text{Solve } \mathbf{v_1} = a_1\mathbf{u_1} + b_1\mathbf{u_2}$$
$$(-1,3) = a_1(0,10) + b_1(5,0)$$
$$5b_1 = -1 \quad \Rightarrow \quad b_1 = -\frac{1}{5}$$
$$10a_1 = 3 \quad \Rightarrow \quad a_1 = \frac{3}{10}$$
$$\text{Solve } \mathbf{v_2} = a_2\mathbf{u_1} + b_2\mathbf{u_2}$$
$$(2,-1) = a_2(0,10) + b_2(5,0)$$
$$5b_2 = 2 \quad \Rightarrow \quad b_2 = \frac{2}{5}$$
$$10a_2 = -1 \quad \Rightarrow \quad a_2 = -\frac{1}{10}$$
$$\text{Change of coordinates matrix } P = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{10} & -\frac{1}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

Problem 6

For each of the following pairs of ordered bases β and β' for $P_2(\mathbb{R})$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

$$\beta = \{1, x, x^2\} \text{ and } \beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$$

$$\text{Let } \beta' = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$$
$$\text{Express } \mathbf{u_1} = a_2x^2 + a_1x + a_0 \text{ in terms of } \beta:$$
$$\mathbf{u_1} = a_2x^2 + a_1x + a_0 = c_1(1) + c_2(x) + c_3(x^2)$$
$$\text{Coefficients: } \begin{cases} c_1 = a_0 \\ c_2 = a_1 \\ c_3 = a_2 \end{cases}$$
$$\text{Express } \mathbf{u_2} = b_2x^2 + b_1x + b_0 \text{ in terms of } \beta:$$
$$\mathbf{u_2} = b_2x^2 + b_1x + b_0 = c_1(1) + c_2(x) + c_3(x^2)$$
$$\text{Coefficients: } \begin{cases} c_1 = b_0 \\ c_2 = b_1 \\ c_3 = b_2 \end{cases}$$
$$\text{Express } \mathbf{u_3} = c_2x^2 + c_1x + c_0 \text{ in terms of } \beta:$$
$$\mathbf{u_3} = c_2x^2 + c_1x + c_0 = c_1(1) + c_2(x) + c_3(x^2)$$
$$\text{Coefficients: } \begin{cases} c_1 = c_0 \\ c_2 = c_1 \\ c_3 = c_2 \end{cases}$$
$$\text{Change of coordinates matrix } P = \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

Problem 7

Let T be the linear operator on $\mathbb{R}^2$ defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a+b \\ a-3b \end{pmatrix}$$

let β be the standard ordered basis for $\mathbb{R}^2$, and let

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

Use Theorem 2.23 and the fact that

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

to find $[T]_{\beta'}$.

$$[T]_\beta = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$$

$$\text{Let } P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \text{ then } P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\text{Use the formula } [T]_{\beta'} = P^{-1}[T]_\beta P :$$

$$[T]_{\beta'} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\text{Calculate } [T]_\beta P :$$

$$[T]_\beta P = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -2 & -5 \end{pmatrix}$$

$$\text{Calculate } P^{-1}[T]_\beta P :$$

$$[T]_{\beta'} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -2 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 2\cdot 3 + (-1)(-2) & 2\cdot 4 + (-1)(-5) \\ -1\cdot 3 + 1\cdot(-2) & -1\cdot 4 + 1\cdot(-5) \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix}$$

$$\text{Thus, } [T]_{\beta'} = \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix}.$$

Problem 8

For each matrix A and ordered basis β, find $[L_A]_\beta$. Also, find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \text{ and } \beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

Problem 8 continued on next page. . .

$$\text{Let } \mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$\text{Compute } A\mathbf{v_1} \text{ and } A\mathbf{v_2}:$$

$$A\mathbf{v_1} = A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\mathbf{v_1},$$

$$A\mathbf{v_2} = A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-2 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\mathbf{v_2}.$$

$$\text{Thus, } [L_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\text{To find } Q, \text{ use } Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$\text{Now, compute } Q^{-1}:$$

$$Q^{-1} = \frac{1}{\det(Q)}\,\text{adj}(Q) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

$$\text{Confirm that } [L_A]_\beta = Q^{-1}AQ:$$

$$Q^{-1}AQ = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$\text{Compute } AQ:$$

$$AQ = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}.$$

$$\text{Then compute } Q^{-1}AQ:$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\text{Thus, } [L_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$