MATE 5150: Asignacion #3

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Problem 1

Prove that T is a linear transformation, and find bases for both $N(T)$ and $R(T)$. Then compute the nulity and rank of T, and verify the dimension theorem. Finally use the appropriate theorems in this section to determine wheather T is one-to-one or onto.

$$T : R^2 \to R^3 \text{ defined by } T(a_1, a_2) = (a_1 + 2a_2, 0, 2a_1 - a_2)$$

$$\begin{aligned}
T(x+y) &= T(x) + T(y) \\
T(b_1 + d_1, b_2 + d_2) &= T(b_1, b_2) + T(d_1, d_2) \\
(b_1 + b_2 + d_1 + d_2, 0, 2b_1 - b_2 + 2d_1 - d_2) &= (b_1 + 2b_2, 0, 2b_1 - b_2) + (d_1 + 2d_2, 0, 2d_1 - d_2) \\
T(cx) &= cT(x) \\
T(c(b_1, b_2)) &= cT(b_1, b_2) \\
(cb_1 + 2cb_2, 0, 2cb_1 - cb_2) &= c(b_1 + 2b_2, 0, 2b_1 - b_2) \\
(cb_1 + 2cb_2, 0, 2cb_1 - cb_2) &= (cb_1 + 2cb_2, 0, 2cb_1 - cb_2)
\end{aligned}$$

Thus T is a linear transformation. The nulity of T is 0, and the rank of T is 2.

Problem 2

Prove that T is a linear transformation, and find bases for both $N(T)$ and $R(T)$. Then compute the nulity and rank of T, and verify the dimension theorem. Finally use the appropriate theorems in this section to determine wheather T is one-to-one or onto.

$$T : M_{2x3}(F) \to M_{2x2}(F) \text{ defined by } T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$$

$$\begin{aligned}
T(x+y) &= T(x) + T(y) \\
T(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}) &= T(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}) + T(\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}) \\
T(\begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{pmatrix}) &= \begin{pmatrix} 2(a_{11} + b_{11}) - (a_{12} + b_{12}) & a_{13} + 2(a_{12} + b_{12}) \\ 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 2(a_{11} + b_{11}) - (a_{12} + b_{12}) & a_{13} + 2(a_{12} + b_{12}) \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 2(a_{11} + b_{11}) - (a_{12} + b_{12}) & a_{13} + 2(a_{12} + b_{12}) \\ 0 & 0 \end{pmatrix} \\
T(cx) &= cT(x) \\
T(c\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}) &= cT(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}) \\
\begin{pmatrix} 2ca_{11} - ca_{12} & ca_{13} + 2ca_{12} \\ 0 & 0 \end{pmatrix} &= c\begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 2ca_{11} - ca_{12} & ca_{13} + 2ca_{12} \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 2ca_{11} - ca_{12} & ca_{13} + 2ca_{12} \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

Thus T is a linear transformation. The nulity of T is 0.

Problem 3

In this exercise, $T : R^2 \to R^2$ is a function. State why T is not linear.

$$T(a_1, a_2) = (a_1 + 1, a_2)$$

Problem 3 continued on next page. . .

$$\begin{aligned} T(2(1,0)) &= T(2,0) = (2+1,0) = (3,0) \\ 2T(1,0) &= 2(1+1,0) = 2(2,0) = (4,0) \\ T(2(1,0)) &\neq 2T(1,0) \end{aligned}$$

Thus T is not linear.

Problem 4

Is there a linear transformation $T : R^3 \to R^2$ such that $T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$?

$$T((-2,0,-6)) = T(-2(1,0,3)) = -2T(1,0,3) = -2(1,1) = (-2,-2) \neq (2,1)$$

Thus there does not exist a linear transformation

Problem 5

Let V and W be vector spaces, let $T : V \to W$ be a linear, and let $\{w_1, w_2, \dots, w_k\}$ be a linearly independent set of k vectors from $R(T)$. Prove that if $S = \{v_1, v_2, \dots, v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, \dots, k$, then S is linearly independent. Visit goo.gl/kmaQS2 for a solution.

Proof.

$$\text{Asume } \sum_{i=1}^{k} c_i v_i = 0$$

$$\text{Then } T\left(\sum_{i=1}^{k} c_i v_i\right) = T(0)$$

$$\text{Since } T \text{ is linear, } \sum_{i=1}^{k} c_i w_i = 0$$

$$\text{Since } \{w_1, w_2, \dots, w_k\} \text{ is linearly independent, } c_i = 0 \text{ for } i = 1, 2, \dots, k$$

Thus $\{v_1, v_2, \dots, v_k\}$ is linearly independent.

□

Problem 6

Let β and γ be the standard ordered bases for R^n and R^m, respectively. For each linear transformation $T : R^n \to R^m$, compute $[T]_\beta^\gamma$.

1. $T : R^3 \to R$ defined by $T(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$

$$\begin{aligned} T(1,0,0) &= 2(1) + 0 - 3(0) = 2 \\ T(0,1,0) &= 2(0) + 1 - 3(0) = 1 \\ T(0,0,1) &= 2(0) + 0 - 3(1) = -3 \end{aligned}$$

$$[T]_\beta^\gamma = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

2. $T : R^n \to R^n$ defined by $T(a_1, a_2, \ldots, a_n) = (a_n, a_{n-1}, \ldots, a_1)$

$$\begin{gathered} T(1, 0, \ldots, 0) = (0, 0, \ldots, 1) \\ T(0, 1, \ldots, 0) = (0, 0, \ldots, 0) \\ \vdots \\ T(0, 0, \ldots, 1) = (1, 0, \ldots, 0) \end{gathered}$$

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \ldots & 0 \end{pmatrix}$$

Problem 7

Define

$$T : M_{2\times 2}(R) \to P_2 \text{ by } T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + b) + (2d)x + bx^2$$

Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \text{ and } \gamma = \{1, x, x^2\}$$

Compute $[T]_\beta^\gamma$.

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Problem 8

Let

$$\beta = \{1, x, x^2\} \text{ and } \gamma = \{1\}$$

Define $T : P_2(R) \to R^2$ by $T(f(x)) = f(2)$. Compute $[T]_\beta^\gamma$.

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$$

Problem 9

Let V and W be vector spaces, and let T and U be nonzero linear transformations form V to W. If $R(T) \cap R(U) = \{0\}$, prove that $\{T, U\}$ is linearly independent subset of $L(V, W)$.

Proof. To show that $\{T, U\}$ is linearly independent, assume $aT + bU = 0$ for scalars a and b.
Since T and U are linear transformations, for any $v \in V$:

$$aT(v) + bU(v) = 0.$$

This means the combined transformation sends all vectors in V to the zero vector in W.

Problem 9 continued on next page. . .

Let $w \in R(T)$. Then there exists $v_1 \in V$ such that:

$$w = T(v_1).$$

Thus: $aT(v_1) + bU(v_1) = 0 \implies aw + bU(v_1) = 0.$ □

Problem 10

Calculate the compossition of

$$[T(A)]_\alpha, \text{ where } A = \begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix}$$

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$AB_1 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$$

$$AB_2 = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$$

$$AB_3 = \begin{pmatrix} 3 & 0 \\ 4 & 0 \end{pmatrix}$$

$$AB_4 = \begin{pmatrix} 0 & 3 \\ 0 & 4 \end{pmatrix}$$

$$T[A]_\alpha = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix}$$

Problem 11

Calculate the composition of

$$[T(A)]_\gamma, \text{ where } A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$\gamma = \{1\}$$

$$A \times \gamma = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$T[A]_\gamma = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Problem 12

Find linear transformations $U, T : F^2 \to F^2$ such that $UT = T_0$ (the zero transformation) but $TU \neq T_0$. Use your answer to find matrices A and B such that $AB = 0$ but $BA \neq 0$.

Problem 12 continued on next page. . .

Proof.

Let U and T be defined as follows:

$$U(a_1, a_2) = (0, a_1) \text{ and } T(a_1, a_2) = (a_1, 0)$$

Then $UT = T_0$

$$UT(a_1, a_2) = U(T(a_1, a_2)) = U(a_1, 0) = (0, a_1) = T_0$$

But $TU \neq T_0$

$$TU(a_1, a_2) = T(U(a_1, a_2)) = T(0, a_1) = (0, 0) \neq T_0$$

Thus $AB = 0$ but $BA \neq 0$. □

Given $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, verify that $AB = 0$ but $BA \neq 0$.

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Problem 13

Let A and B be $n \times n$ matrices. Recall that the trace of A is defined by

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

Prove that $tr(AB) = tr(BA)$ and $tr(A) = tr(A^T)$.

Proof.

$$\begin{aligned} tr(A) &= \sum_{i=1}^{n} a_{ii} \\ tr(AB) &= \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij} \\ &= \sum_{j=1}^{n} (BA)_{ii} \\ &= tr(BA) \end{aligned}$$

Now let $A = (a_{ii})$ and $A^T = (b_{ii})$

Then $a_{ij} = b_{ii}$

$$tr(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} b_{ii} = tr(A^T)$$

Thus $tr(AB) = tr(BA)$ and $tr(A) = tr(A^T)$. $\square$

Problem 14

For the definition of *projection* and related facts, see pages 76-77. Let V be a vector space and $T : V \to V$ be a linear transformation. Prove that $T = T^2$ if and only if T is a projection on $W_1 = \{y : T(y) = y\}$ along $N(T)$.