

# **MATE 5150: Asignacion #3**

Due on Septiembre 26, 2024

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## Problem 1

Prove that  $T$  is a linear transformation, and find bases for both  $N(T)$  and  $R(T)$ . Then compute the nullity and rank of  $T$ , and verify the dimension theorem. Finally use the appropriate theorems in this section to determine wheather  $T$  is one-to-one or onto.

$$T : R^2 \rightarrow R^3 \text{ defined by } T(a_1, a_2) = (a_1 + 2a_2, 0, 2a_1 - a_2)$$

$$T(x + y) = T(x) + T(y)$$

$$T(b_1 + d_1, b_2 + d_2) = T(b_1, b_2) + T(d_1, d_2)$$

$$(b_1 + b_2 + d_1 + d_2, 0, 2b_1 - b_2 + 2d_1 - d_2) = (b_1 + 2b_2, 0, 2b_1 - b_2) + (d_1 + 2d_2, 0, 2d_1 - d_2)$$

$$T(cx) = cT(x)$$

$$T(c(b_1, b_2)) = cT(b_1, b_2)$$

$$(cb_1 + 2cb_2, 0, 2cb_1 - cb_2) = c(b_1 + 2b_2, 0, 2b_1 - b_2)$$

$$(cb_1 + 2cb_2, 0, 2cb_1 - cb_2) = (cb_1 + 2cb_2, 0, 2cb_1 - cb_2)$$

Thus  $T$  is a linear transformation. The nullity of  $T$  is 0, and the rank of  $T$  is 2.

## Problem 2

Prove that  $T$  is a linear transformation, and find bases for both  $N(T)$  and  $R(T)$ . Then compute the nullity and rank of  $T$ , and verify the dimension theorem. Finally use the appropriate theorems in this section to determine wheather  $T$  is one-to-one or onto.

$$T : M_{2 \times 3}(F) \rightarrow M_{2 \times 2}(F) \text{ defined by } T \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$$

$$T(x + y) = T(x) + T(y)$$

$$T \left( \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} \right) = T \left( \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \right) + T \left( \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} \right)$$

$$T \left( \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{pmatrix} \right) = \begin{pmatrix} 2(a_{11} + b_{11}) - (a_{12} + b_{12}) & a_{13} + 2(a_{12} + b_{12}) \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2(a_{11} + b_{11}) - (a_{12} + b_{12}) & a_{13} + 2(a_{12} + b_{12}) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2(a_{11} + b_{11}) - (a_{12} + b_{12}) & a_{13} + 2(a_{12} + b_{12}) \\ 0 & 0 \end{pmatrix}$$

$$T(cx) = cT(x)$$

$$T \left( c \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \right) = cT \left( \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \right)$$

$$\begin{pmatrix} 2ca_{11} - ca_{12} & ca_{13} + 2ca_{12} \\ 0 & 0 \end{pmatrix} = c \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2ca_{11} - ca_{12} & ca_{13} + 2ca_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2ca_{11} - ca_{12} & ca_{13} + 2ca_{12} \\ 0 & 0 \end{pmatrix}$$

Thus  $T$  is a linear transformation. The nullity of  $T$  is 0.

## Problem 3

In this exercise,  $T : R^2 \rightarrow R^2$  is a function. State why  $T$  is not linear.

$$T(a_1, a_2) = (a_1 + 1, a_2)$$

$$\begin{aligned}
T(2(1, 0)) &= T(2, 0) = (2 + 1, 0) = (3, 0) \\
2T(1, 0) &= 2(1 + 1, 0) = 2(2, 0) = (4, 0) \\
T(2(1, 0)) &\neq 2T(1, 0)
\end{aligned}$$

Thus  $T$  is not linear.

## Problem 4

Is there a linear transformation  $T : R^3 \rightarrow R^2$  such that  $T(1, 0, 3) = (1, 1)$  and  $T(-2, 0, -6) = (2, 1)$ ?

$$T((-2, 0, -6)) = T(-2(1, 0, 3)) = -2T(1, 0, 3) = -2(1, 1) = (-2, -2) \neq (2, 1)$$

Thus there does not exist a linear transformation

## Problem 5

Let  $V$  and  $W$  be vector spaces, let  $T : V \rightarrow W$  be a linear, and let  $\{w_1, w_2, \dots, w_k\}$  be a linearly independent set of  $k$  vectors from  $R(T)$ . Prove that if  $S = \{v_1, v_2, \dots, v_k\}$  is chosen so that  $T(v_i) = w_i$  for  $i = 1, 2, \dots, k$ , then  $S$  is linearly independent. Visit [goo.gl/kmaQS2](http://goo.gl/kmaQS2) for a solution.

*Proof.*

$$\begin{aligned}
&\text{Asume } \sum_{i=1}^k c_i v_i = 0 \\
&\text{Then } T\left(\sum_{i=1}^k c_i v_i\right) = T(0) \\
&\text{Since } T \text{ is linear, } \sum_{i=1}^k c_i w_i = 0
\end{aligned}$$

Since  $\{w_1, w_2, \dots, w_k\}$  is linearly independent,  $c_i = 0$  for  $i = 1, 2, \dots, k$

Thus  $\{v_1, v_2, \dots, v_k\}$  is linearly independent.

□

## Problem 6

Let  $\beta$  and  $\gamma$  be the standard ordered bases for  $R^n$  and  $R^m$ , respectively. For each linear transformation  $T : R^n \rightarrow R^m$ , compute  $[T]_{\beta}^{\gamma}$ .

1.  $T : R^3 \rightarrow R$  defined by  $T(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$

$$\begin{aligned}
T(1, 0, 0) &= 2(1) + 0 - 3(0) = 2 \\
T(0, 1, 0) &= 2(0) + 1 - 3(0) = 1 \\
T(0, 0, 1) &= 2(0) + 0 - 3(1) = -3
\end{aligned}$$

$$[T]_{\beta}^{\gamma} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

2.  $T : R^n \rightarrow R^n$  defined by  $T(a_1, a_2, \dots, a_n) = (a_n, a_{n-1}, \dots, a_1)$

$$T(1, 0, \dots, 0) = (0, 0, \dots, 1)$$

$$T(0, 1, \dots, 0) = (0, 0, \dots, 0)$$

$$\vdots$$

$$T(0, 0, \dots, 1) = (1, 0, \dots, 0)$$

$$[T]_{\beta}^{\gamma} = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

## Problem 7

Define

$$T : M_{2 \times 2}(R) \rightarrow P_2 \text{ by } T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2$$

Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \text{ and } \gamma = \{1, x, x^2\}$$

Compute  $[T]_{\beta}^{\gamma}$ .

$$[T]_{\beta}^{\gamma} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

## Problem 8

Let

$$\beta = \{1, x, x^2\} \text{ and } \gamma = \{1\}$$

Define  $T : P_2(R) \rightarrow R^2$  by  $T(f(x)) = f(2)$ . Compute  $[T]_{\beta}^{\gamma}$ .

$$[T]_{\beta}^{\gamma} = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$$

## Problem 9

Let  $V$  and  $W$  be vector spaces, and let  $T$  and  $U$  be nonzero linear transformations from  $V$  to  $W$ . If  $R(T) \cap R(U) = \{0\}$ , prove that  $\{T, U\}$  is linearly independent subset of  $L(V, W)$ .

*Proof.* To show that  $\{T, U\}$  is linearly independent, assume  $aT + bU = 0$  for scalars  $a$  and  $b$ . Since  $T$  and  $U$  are linear transformations, for any  $v \in V$ :

$$aT(v) + bU(v) = 0.$$

This means the combined transformation sends all vectors in  $V$  to the zero vector in  $W$ .

Let  $w \in R(T)$ . Then there exists  $v_1 \in V$  such that:

$$w = T(v_1).$$

Thus:  $aT(v_1) + bU(v_1) = 0 \implies aw + bU(v_1) = 0$ . □

## Problem 10

Calculate the composition of

$$[T(A)]_\alpha, \text{ where } A = \begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix}$$

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$AB_1 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$$

$$AB_2 = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$$

$$AB_3 = \begin{pmatrix} 3 & 0 \\ 4 & 0 \end{pmatrix}$$

$$AB_4 = \begin{pmatrix} 0 & 3 \\ 0 & 4 \end{pmatrix}$$

$$T[A]_\alpha = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix}$$

## Problem 11

Calculate the composition of

$$[T(A)]_\gamma, \text{ where } A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$\gamma = \{1\}$$

$$T[A]_\gamma = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

## Problem 12

Find linear transformations  $U, T : F^2 \rightarrow F^2$  such that  $UT = T_0$  (the zero transformation) but  $TU \neq T_0$ .  
Use your answer to find matrices  $A$  and  $B$  such that  $AB = 0$  but  $BA \neq 0$ .

**Problem 13**

Let  $A$  and  $B$  be  $n \times n$  matrices. Recall that the trace of  $A$  is defined by

$$\operatorname{tr}(A) = \sum_{i=1}^n a_{ii}$$

Prove that  $\operatorname{tr}(AB) = \operatorname{tr}(BA)$  and  $\operatorname{tr}(A) = \operatorname{tr}(A^T)$ .

**Problem 14**

For the definition of *projection* and related facts, see pages 76-77. Let  $V$  be a vector space and  $T : V \rightarrow V$  be a linear transformation. Prove that  $T = T^2$  if and only if  $T$  is a projection on  $W_1 = \{y : T(y) = y\}$  along  $N(T)$ .