MATE 5150: Asignacion #3

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Problem 1

For each of the following homogeneous systems of linear equations, find the dimensions of and a basis for the solution space.

$$x_1 + 2x_2 + x_3 + x_4 = 0$$
$$x_2 - x_3 + x_3 = 0$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{bmatrix}$$

$$x_2 = x_3 - x_4$$

$$x_1 = -x_3 - x_4$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} x_4$$

The rank of the matrix is 2, and dimension of the solution space is 2. A basis for the solution space is $\{(1, 1, 1, 0), (1, -1, 0, 1)\}$.

Problem 2

Using the results of Exercise 2, find all solutions to the following system of linear equations.

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 \quad &= 0 \\ x_2 - x_3 + x_3 \qquad &= 0 \end{aligned}$$

$$K = \left\{ t_1 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} : t_1, t_2 \in \mathbb{R} \right\}$$

Problem 3

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(a, b, c) = (a + b, 2a - c)$. Determine $T^{-1}(1, 11)$.

$$T(a, b, c) = (a + b, 2a - c)$$
$$1 = a + b \quad 11 = 2a - c$$
$$b = 1 - a \quad c = 2a - 11$$
$$(a, b, c) = (a, 1 - a, 2a - 11)$$
$$T^{-1}(1, 11) = \{(a, 1 - a, 2a - 11) : a \in \mathbb{R}\}$$

Problem 4

Determine which of the following systems of liner equations has a solution.

$$\begin{aligned} x_1 + x_2 + 3x_3 - x_3 \quad &= 0 \\ x_1 + x_2 + x_3 + x_4 \quad &= 1 \\ x_1 - 2x_2 + x_3 - x_4 \quad &= 1 \\ 4x_1 + x_2 + 8x_3 - x_4 \quad &= 0 \end{aligned}$$

Problem 4 continued on next page. . .

$$A = \begin{bmatrix} 1 & 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & 1 \\ 4 & 1 & 8 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{25}{6} \\ 0 & 1 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & 0 & -\frac{5}{2} \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

$$s = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{25}{6} \\ \frac{4}{3} \\ -\frac{5}{2} \\ -2 \end{bmatrix}$$

Problem 5

u A certain economy consists of two sectors: goods and services. Suppose that 60% of all goods and 30% of all services are used in the production of goods. What proportion of the total economic output is used in the production of goods?

$$P = \frac{.6G + .3S}{G + S} = \frac{.6G + .3S}{T_2}$$
$$P = \frac{.9G}{T_2} = .45, \text{ where } G = 1, S = 1, T_2 = 2$$

Problem 6

Use Gaussian elimination to solve the following systems of linear equations.

$$\begin{aligned} 2x_1 - 2x_2 - x_3 + 6x_4 - 2x_5 &= 1 \\ x_1 - x_2 + x_3 + 2x_4 - x_5 &= 2 \\ 4x_1 - 4x_2 + 5x_3 + 7x_4 - x_5 &= 6 \end{aligned}$$

$$A = \begin{bmatrix} 2 & -2 & -1 & 6 & -2 & 1 \\ 1 & -1 & 1 & 2 & -1 & 2 \\ 4 & -4 & 5 & 7 & -1 & 6 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 0 & 0 & 23 & -23 \\ 0 & 0 & 1 & 0 & -6 & 7 \\ 0 & 0 & 0 & 1 & -9 & 9 \end{bmatrix}$$
$$S = \begin{bmatrix} x_2 - 23x_5 - 23 \\ x_2 \\ 6x_5 + 7 \\ 9x_5 + 9 \\ x_5 \end{bmatrix}$$

Problem 7

Let the reduce row echolon form of A be

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

Problem 7 continued on next page. . .

Determine A if the first, second , and forth columns of A are

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

Problem 8

Let W be the subspace of $M_{2\times 2}(\mathbb{R})$ consisting of the symmetric 2×2 matrices. Then set

$$S = \left\{ \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 9 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix} \right\}$$

generates W. Find a subset of S that is a basis for W.

$$A_1 = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 2 & 1 \\ 1 & 9 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}.$$

$$M = \begin{bmatrix} 0 & 1 & 2 & 1 & -1 \\ -1 & 2 & 1 & -2 & -2 \\ 1 & 3 & 9 & 4 & -1 \end{bmatrix}.$$

$$1.R_2 \leftarrow R_2 - 2R_1, \quad R_3 \leftarrow R_3 - 1R_1$$

$$\Rightarrow M \sim \begin{bmatrix} 0 & 1 & 2 & 1 & -1 \\ -1 & 0 & -3 & -4 & 1 \\ 1 & 3 & 9 & 4 & -1 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & -3 & -4 & 1 \\ 0 & 1 & 2 & 1 & -1 \\ 1 & 3 & 9 & 4 & -1 \end{bmatrix}.$$

$$3.R_1 \leftarrow -R_1 \text{ and then } R_3 \leftarrow R_3 + R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 & 4 & -1 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 3 & 12 & 8 & 0 \end{bmatrix}.$$

$$4.R_3 \leftarrow R_3 - 3R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 & 4 & -1 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 6 & 5 & 3 \end{bmatrix}.$$

This shows there are 3 non-zero rows, hence dimension of span = 3.

Thus, a basis for W is $\{A_1, A_2, A_3\}$.

Problem 9

Let V be a in Exercise 12

$$\begin{aligned} x_1 - x_2 + 2x_4 - 3x_5 + x_6 &= 0 \\ 2x_1 - x_2 - x_3 + 3x_4 - 4x_5 + 4x_6 &= 0 \end{aligned}$$

1. Show that $S = \{(0,1,0,1,1,0),(0,2,1,1,0,0)\}$ is a linerly independent subset of V.

2. Extend S to a basis for V.