MATE 5150: Determinants

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1 Determinants of Order 2

2 Determinants of Order n

3 Properties of Determinants

3.1 Facts about elementary Matrices

1. If E is an elementary matrix obtained by interchanging any two rows of I_n, then $\det(E) = -1$.
2. If E is an elementary matrix obtained by multiplying some row of I_n by the nonzero scalar K, then $\det(E) = K$.
3. If E is an elementary matrix obtained by adding a multiple of some row of I_n to another row, the $\det(E) = 1$.

Theorem 1. *For any $A, B \in M_{n\times n}(\mathbb{R})$, $\det(AB) = \det(A)\cdot\det(B)$.*

Proof.

$$\begin{aligned}\det(AB) &= \det(E_m \cdots E_2E_1B)\\ &= \det(E_m)\cdot\det(E_{m-1}\cdots E_2E_1B)\\ &= \det(E_m\cdots E_2E_1)\cdot\det(B)\\ &= \det(A)\cdot\det(B)\end{aligned}$$

Therefore, $\det(AB) = \det(A)\det(B)$. □

Corollary 1. *A Matrix $A \in M_{n\times n}(F)$ is invertible if and only if $\det(A) \neq 0$. Furthermore, if A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.*

Proof.

If A is invertible, then $rank(A) < n$

So by the Correlation 4.6, $\det(A) = 0$

If A is not invertible, then

$$\det(A) \cdot \det(A^{-1}) = \det(AA^{-1}) = \det(I_n) = 1$$

Therefore, $\det(A^{-1}) = \frac{1}{\det(A)}$ if A is invertible and $\det(A) \neq 0$. $\square$

Theorem 2. *For any $A \in M_{n\times n}(F)$, $\det(A^T) = \det(A)$.*

Proof.

$$\begin{aligned}
\det(A^T) &= \det(E_1^T \cdot E_2^T \cdots E_m^T) \\
&= \det(E_1^T) \cdot \det(E_2^T) \cdots \det(E_m^T) \\
&= \det(E_1) \cdot \det(E_2) \cdots \det(E_m) \\
&= \det(E_m \cdots E_2 E_1) \\
&= \det(A)
\end{aligned}$$

Therefore, $\det(A^T) = \det(A)$. $\square$

Theorem 3 (Cramer's Rule)**.** *Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns. Where $x = (x_1, x_2, \ldots, x_n)^T$. If $\det(A) \neq 0$, then this system has a unique solution, and for each k $(k = 1, 2, \ldots, n)$,*

$$x_k = \frac{\det(M_k)}{\det(A)}$$

where M_k is the $n \times n$ matrix obtained from A by replacing column k of A by b.

Proof. Let X_k be the matrix obtained from A by replacing the k-th column of A with the vector $\mathbf{b}$. Then we can write the system as:

$$AX_k = M_k$$

where M_k is the vector formed by the coefficients of the system in the new configuration.

To evaluate $\det(X_k)$ using cofactor expansion along the k-th column, we have:

$$\det(X_k) = x_k \det(I_{n-1})$$

Since $\det(I_{n-1}) = 1$, it follows that:

$$\det(X_k) = x_k$$

By applying the property of determinants, we obtain:

$$\det(M_k) = \det(AX_k) = \det(A) \cdot \det(X_k)$$

Substituting $\det(X_k)$ into this equation gives:

$$\det(M_k) = \det(A) \cdot x_k$$

Therefore, we can rearrange this to find x_k:

$$x_k = \frac{\det(M_k)}{\det(A)}$$

This proves Cramer's Rule for the variable x_k. $\square$

4 Summary - Important Facts about Determinants