

MATE 5150: Asignacion #7

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Problem 1

For each of the following linear operators T on a vector space V and ordered basis β , compute $[T]_\beta$ and determine whether β is a basis consisting of eigenvectors of T .

$$V = \mathbb{R}^3, T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a + 2b - 2c \\ -4a - 3b + 2c \\ -c \end{pmatrix}, \text{ and } \beta = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

Compute $[T]_\beta$:

$$\begin{aligned} \begin{pmatrix} 3(0) + 2(1) - 2(1) \\ -4(0) - 3(1) + 2(1) \\ -(1) \end{pmatrix} &= \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \\ a \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} &= \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \\ v_1 &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\ v_2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ v_3 &= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \\ [T]_\beta &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

Finding the eigenvalues of T :

$$\begin{aligned} \det([T]_\beta - \lambda I) &= \det \left(\begin{pmatrix} -1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{pmatrix} \right) \\ &= (1-\lambda)(1+\lambda)^2 = 0 \\ \lambda_1, \lambda_2 &= 1, -1 \end{aligned}$$

Calculating the eigenvectors for $\lambda_1 = 1$:

$$\begin{aligned} \begin{pmatrix} -1-1 & 0 & 0 \\ 0 & 1-1 & 0 \\ 0 & 0 & -1-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ v_1 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

Calculating the eigenvectors for $\lambda_2 = -1$:

$$\begin{pmatrix} -1+1 & 0 & 0 \\ 0 & 1+1 & 0 \\ 0 & 0 & -1+1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Problem 2

For each of the following matrices $A \in M_{n \times n}(F)$,

1. Determine all the eigenvalues of A .
2. For each eigenvalue λ of A , find the set of eigenvectors corresponding to λ .
3. If possible, find a basis for F^n consisting of eigenvectors of A .
4. If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

$$A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} \text{ for } F = \mathbb{R}$$

Determine all the eigenvalues of A :

$$\begin{aligned} \det(A - \lambda I) &= \det \left(\begin{pmatrix} 0-\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2 & 2 & 5-\lambda \end{pmatrix} \right) \\ &= (3-\lambda)(2-\lambda)(1-\lambda) = 0 \\ \lambda_1, \lambda_2, \lambda_3 &= 3, 2, 1 \end{aligned}$$

Calculating the eigenvectors for $\lambda_1 = 3$:

$$\begin{pmatrix} 0-3 & -2 & -3 \\ -1 & 1-3 & -1 \\ 2 & 2 & 5-3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & -2 & -3 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Calculating the eigenvectors for $\lambda_2 = 2$:

$$\begin{pmatrix} 0-2 & -2 & -3 \\ -1 & 1-2 & -1 \\ 2 & 2 & 5-2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Calculating the eigenvectors for $\lambda_3 = 1$:

$$\begin{pmatrix} 0-1 & -2 & -3 \\ -1 & 1-1 & -1 \\ 2 & 2 & 5-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

The basis of eigenvectors for A is $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

$$Q = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$Q^{-1} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Problem 3

For each linear operator T on V , find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

$$V = M_{2 \times 2}(\mathbb{R}), \text{ and } T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\beta = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$[T]_\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Calculating the eigenvalues of T :

$$\begin{aligned} \det(T - \lambda I) &= \det \left(\begin{pmatrix} 0 - \lambda & 0 & 1 & 0 \\ 0 & 0 - \lambda & 0 & 1 \\ 1 & 0 & 0 - \lambda & 0 \\ 0 & 1 & 0 & 0 - \lambda \end{pmatrix} \right) \\ &= (1 - \lambda)^2 (\lambda - 1)^2 = 0 \end{aligned}$$

Calculating the eigenvectors for $\lambda_1 = 1$:

$$\begin{pmatrix} 0-1 & 0 & 1 & 0 \\ 0 & 0-1 & 0 & 1 \\ 1 & 0 & 0-1 & 0 \\ 0 & 1 & 0 & 0-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v = a \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Calculating the eigenvectors for $\lambda_2 = 1$:

$$\begin{pmatrix} 0-1 & 0 & 1 & 0 \\ 0 & 0-1 & 0 & 1 \\ 1 & 0 & 0-1 & 0 \\ 0 & 1 & 0 & 0-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v = a \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

V is a basis of eigenvectors for T . $V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Problem 4

Let V be a finite-dimensional vector space, and let λ be any scalar.

1. For any ordered basis β for V , prove that $[\lambda|_V]_\beta = \lambda I$.
2. Compute the characteristic polynomial of $\lambda|_V$.
3. Show that $\lambda|_V$ is diagonalizable and has only one eigenvalue.

Problem 5

Let T be a linear operator on a vector space V over the field F , and let $g(t)$ be a polynomial with coefficients from F . Prove that if x is an eigenvector of T with corresponding eigenvalue λ , then $g(T)(x) = g(\lambda)x$. That is, x is an eigenvector of $g(T)$ with corresponding eigenvalue $g(\lambda)$.

Problem 6

For each of the following matrices $A \in M_{n \times n}(\mathbb{R})$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

Problem 7

For

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2 \times 2}(\mathbb{R}),$$

find an expression for A^n , where n is an arbitrary positive integer.

Problem 8

Let T be a linear operator on a finite-dimensional vector space V , and suppose there exists an ordered basis β for V such that $[T]_\beta$ is an upper triangular matrix.

1. Prove that the characteristic polynomial for T splits.
2. State and prove an analogous result for matrices.

The converse of (a) is treated in Exercise 12(b).

Problem 9

For each of the following linear operators T on the vector space V , determine whether the given subspace W is a T -invariant subspace of V .

$$V = C([0, 1]), T(f(t)) = \left[\int_0^1 f(x) dx \right] t, \text{ and } W = \{f \in V : f(t) = at + b \text{ for some } a, b\}$$

Problem 10

For each linear operator T on the vector space V , find an ordered basis for the T -cyclic subspace generated by the given vector z

$$V = M_{2 \times 2}(\mathbb{R}), T(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A, \text{ and } z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Problem 11

For each linear operator in Exercise 6, find the characteristic polynomial of $f(t)$ of T , and verify that the characteristic polynomial of T_W (computed in Exercise 6) divides $f(t)$.

Problem 12

Let T be a linear operator on a vector space V , let v be a nonzero vector in V , and let W be the T -cyclic subspace of V generated by v . Prove that

1. W is T -invariant.
2. Any T -invariant subspace of V containing v also contains W .

Problem 13

In $C([0, 1])$, let $f(t) = t$ and $g(t) = e^t$. Compute $\langle f, g \rangle$. (as defined in Example 3), $\|f\|$, $\|g\|$, and $\|f + g\|$. Then verify both the Cauchy-Schwarz inequality and the triangle inequality.

Problem 14

In C^2 , show that $\langle x, y \rangle = xAy^*$ is an inner product, where

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$$

Compute $\langle x, y \rangle$ for $x = (1 - i, 2 + 3i)$ and $y = (2 + i, 3 - 2i)$.

Problem 15

Provide reasons why each of the following is not an inner product on the given vector space.

$$\langle A, B \rangle + \text{tr}(A + B) \text{ on } M_{2 \times 2}(\mathbb{R})$$

Problem 16

Suppose that $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are two inner products on a vector space V . Prove that $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 + \langle \cdot, \cdot \rangle_2$ is another inner product on V .

Problem 17

Let T be a linear operator on an inner product space V , and suppose $\|T(x)\| = \|x\|$ for all x . Prove that T is one-to-one.

Problem 18

Prove that the following are norms on the given vector space V .

$$V = C([0, 1]), \|f\|_V = \int_0^1 |f(t)| dt \text{ for all } f \in V$$

Problem 19

In each part, apply the Gram-Schmidt process to the given subset S of the inner product space V to obtain an orthogonal basis for $\text{span}(S)$. Then normalize the vectors in this basis to obtain an orthonormal basis β for $\text{span}(S)$, and compute the Fourier coefficients of the given vector relative to β . Finally, use Theorem 6.5 to verify your results.

$$V = P_2(\mathbb{R}), \text{ with the inner product } \langle f(x), g(x) \rangle = \int_0^1 f(t)g(t)dt, S = \{1, x, x^2\}, \text{ and } h(x) = 1 + x$$

Problem 20

Same as the previous problem

$$V = \text{span}(S) \text{ with the inner product } \langle f(x), g(x) \rangle = \int_0^\pi f(t)g(t)dt, S = \{\sin(t), \cos(t), 1, t\}, \text{ and } h(x) = 2t + 1$$

Problem 21

Let $S = \{(1, 0, i), (1, 2, 1)\}$ in C^3 . Compute S^\perp .

Problem 22

Let $W = \text{span}(\{(i, 0, 1)\})$ in C^3 . Find orthogonal bases for W and W^\perp .

Problem 23

In each of the following parts, find the orthogonal projection of the given vector on the given subspace W of the inner product space V .

$$V = \mathbb{R}^2, u = (2, 6), \text{ and } W = \{(x, y) : y = 4x\}$$