MATE 5150: Asignacion #7

Due on Noviembre 14, 2024

Dr. Pedro Vasquez

Alejandro Ouslan

Problem 1

For each of the following linear operators T on a vector space V and ordered basis β, compute $[T]_\beta$ and determine whether β is a basis consisting of eigenvectors of T.

$$V = \mathbb{R}^3, T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a + 2b - 2c \\ -4a - 3b + 2c \\ -c \end{pmatrix}, \text{ and } \beta = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

Problem 2

For each of the following matrices $A \in M_{n\times n}(F)$,

1. Determine all the eigenvalues of A.
2. For each eigenvalue λ of A, find the set of eigenvecotrs corresponding to λ.
3. If possible, find a basis for F^n consisting of eigenvectors of A.
4. If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

$$A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} \text{ for } F = \mathbb{R}$$

Problem 3

For each linear operator T on V, find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

$$V = M_{2\times 2}(\mathbb{R}), \text{ and } T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

Problem 4

Let V be a finite-dimensional vector space, and let λ be any scalar.

1. For any ordered basis β for V, prove that $[\lambda|_V]_\beta = \lambda I$.
2. Compute the characteristic polynomial of $\lambda|_V$.
3. Show that $\lambda|_V$ is diagonalizable and has only one eigenvalue.

Problem 5

Let T be a linear operator on a vector space V over the field F, and let $g(t)$ be a polynomial with coefficients form F. Prove that if x is an eigenvector of T with corresponding eigenvalue λ, then $g(T)(x) = g(\lambda)x$. That is, x is an eigenvector of $g(T)$ with corresponding eigenvalue $g(\lambda)$.

Problem 6

For each of the following matrices $A \in M_{n\times n}(\mathbb{R})$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

Problem 7

For

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2\times 2}(\mathbb{R}),$$

find a expression for A^n, where n is an arbitrary positive integer.

Problem 8

Let T be a linear operator on a finite-dimensional vector space V, and suppose there exists an ordered basis β for V such that $[T]_\beta$ is an upper triangular matrix.

1. Prove that the caracteristic polynomial for T splits.
2. State and prove an analogous result for matrices.

The converse of (a) is treated in Exercise 12(b).

Problem 9

For each of the following linear operators T on the vector space V, determine whether the given subspace W is a T-invariant subspace of V.

$$V = C([0,1]), T(f(t)) = \left[\int_0^1 f(x)dt\right] t, \text{ and } W = \{f \in V : f(t) = at + b \text{ for some } a, b\}$$

Problem 10

For each linear operator T on the vector space V, find an ordered basis for the T-cyclic subspace generated by the given vector z

$$V = M_{2\times 2}(\mathbb{R}), T(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A, \text{ and } z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Problem 11

For each linear operator in Exercise 6, find the characteristic polynomial of $f(t)$ of T, and verify that the characteristic polynomial of T_W (computed in Exercise 6) divides $f(t)$.

Problem 12

Let T be a linear operator on a vector space V, let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. Prove that

1. W is T-invariant.
2. Any T-invariant subspace of V containing v also contains W.

Problem 13

In $C([0,1])$, let $f(t) = t$ and $g(t) = e^t$. Compute $\langle f, g \rangle$. (as defined in Example 3), $\|f\|$, $\|g\|$, and $\|f+g\|$. Then verify both the Cauchy-Schwarz inequality and the triangle inequality.

Problem 14

In C^2, show that $\langle x, y \rangle = xAy^\star$ is an inner product, where

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$$

Compute $\langle x, y \rangle$ for $x = (1-i, 2+3i)$ and $y = (2+i, 3-2i)$.

Problem 15

Provide reasons why each of the following is not an inner product on the given vector space.

$$\langle A, B \rangle + tr(A+B) \text{ on } M_{2\times 2}(\mathbb{R})$$

Problem 16

Suppose that $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are two inner products on a vector space V. Prove that $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 + \langle \cdot, \cdot \rangle_2$ is another inner product on V.

Problem 17

Let T be a linear operator on an inner product space V, and suppose $\|T(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

Problem 18

Prove that the following are norms on the given vector space V.

$$V = C([0,1]), \|f\|_V = \int_0^1 |f(t)| dt \text{ for all } f \in V$$

Problem 19

In each part, apply the Gram-Schmidt process to the given subset S of the inner product space V to obtain an orthogonal basis for span(S). Then normalize the vectors in this basis to obtain an orthonormal basis β for span(S), and compute the Fourier coefficients of the given vector relative to β. Finally, use Theorem 6.5 to verify your results.

$$V = P_2(\mathbb{R}), \text{ with the inner product } \langle f(x), g(x)\rangle = \int_0^1 f(t)g(t)dt, S = \{1, x, x^2\}, \text{ and } h(x) = 1 + x$$

Problem 20

Same as the previous problem

$$V + span(S) \text{ with the inner product } \langle f(x), g(x)\rangle = \int_0^\pi f(t)g(t)dt, S = \{\sin(t), \cos(t), 1, t\}, \text{ and } h(x) = 2t+1$$

Problem 21

Let $S = \{(1, 0, i), (1, 2, 1)\}$ in C^3. Compute $S^\perp$.

Problem 22

Let $W = span(\{(i, 0, 1)\})$ in C^3. Find orthogonal bases for W and $W^\perp$.

Problem 23

In each of the following parts, find the orthogonal projection of the given vector on the given subspace W of the inner product space V.

$$V = R^2, u = (2, 6), \text{ and } W = \{(x, y) : y = 4x\}$$