MATE 5150: Asignacion #3

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Problem 1

Prove that T is a linear transformation, and find bases for both $N(T)$ and $R(T)$. Then compute the nulity and rank of T, and verify the dimension theorem. Finally use the appropriate theorems in this section to determine wheather T is one-to-one or onto.

$$T : R^2 \to R^3 \text{ defined by } T(a_1, a_2) = (a_1 + 2a_2, 0, 2a_1 - a_2)$$

Problem 2

Prove that T is a linear transformation, and find bases for both $N(T)$ and $R(T)$. Then compute the nulity and rank of T, and verify the dimension theorem. Finally use the appropriate theorems in this section to determine wheather T is one-to-one or onto.

$$T : M_{2x3}(F) \to M_{2x2}(F) \text{ defined by } T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$$

Problem 3

In this exercise, $T : R^2 \to R^2$ is a function. State why T is not linear.

$$T(a_1, a_2) = (a_1 + 1, a_2)$$

Problem 4

Is there a linear transformation $T : R^3 \to R^2$ such that $T(1, 0, 3) = (1, 1)$ and $T(-2, 0, -6) = (2, 1)$?

Problem 5

Let V and W be vector spaces, let $T : V \to W$ be a linear, and let $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent set of k vectors from $R(T)$. Prove that if $S = \{v_1, v_2, \ldots, v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, \ldots, k$, then S is linearly independent. Visit goo.gl/kmaQS2 for a solution.

Problem 6

Let β and γ be the standard ordered bases for R^n and R^m, respectively. For each linear transformation $T : R^n \to R^m$, compute $[T]_\beta^\gamma$.

1. $T : R^3 \to R$ defined by $T(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$
2. $T : R^n \to R^n$ defined by $T(a_1, a_2, \ldots, a_n) = (a_n, a_{n-1}, \ldots, a_1)$

Problem 7

Define

$$T : M_{2\times 2}(R) \to P_2 \text{ by } T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2$$

Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \text{ and } \gamma = \{1, x, x^2\}$$

Compute $[T]_\beta^\gamma$.

Problem 8

Let

$$\beta = \{1, x, x^2\} \text{ and } \gamma = \{1\}$$

Define $T : P_2(R) \to R^2$ by $T(f(x)) = f(2)$. Compute $[T]_\beta^\gamma$.

Problem 9

Let V and W be vector spaces, and let T and U be nonzero linear transformations form V to W. If $R(T) \cap R(U) = \{0\}$, prove that $\{T, U\}$ is linearly independent subset of $L(V, W)$.

Proof. To show that $\{T, U\}$ is linearly independent, assume $aT + bU = 0$ for scalars a and b.
Since T and U are linear transformations, for any $v \in V$:

$$aT(v) + bU(v) = 0.$$

This means the combined transformation sends all vectors in V to the zero vector in W.
Let $w \in R(T)$. Then there exists $v_1 \in V$ such that:

$$w = T(v_1).$$

Thus:

$$aT(v_1) + bU(v_1) = 0 \implies aw + bU(v_1) = 0.$$

□

Problem 10

Calculate the compossition of

$$[T(A)]_\alpha, \text{ where } A = \begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix}$$

$$alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Problem 11

Calculate the compossition of

$$[T(A)]_\gamma, \text{ where } A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$\gamma = \{1\}$$

Problem 12

Find linear transformations $U, T : F^2 \to F^2$ such that $UT = T_0$ (the zero transformation) but $TU \neq T_0$. Use your answer to find matrices A and B such that $AB = 0$ but $BA \neq 0$.

Problem 13

Let A and B be $n \times n$ matrices. Recall that the trace of A is defined by

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

Prove that $tr(AB) = tr(BA)$ and $tr(A) = tr(A^T)$.

Problem 14

For the definition of *projection* and related facts, see pages 76-77. Let V be a vector space and $T : V \to V$ be a linear transformation. Prove that $T = T^2$ if and only if T is a projection on $W_1 = \{y : T(y) = y\}$ along $N(T)$.