Lecture 11: February 16

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11.1 Simplex Algorithm (Continued)

11.1.1 Assumptions

So far we have made the following assumptions:

1. The LP is in the standard form i.e.

$$\begin{aligned} \min c^T x \\ \text{s.t } Ax = b, \\ x \geq 0 \end{aligned}$$

   where $C, x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $rank(A) = m$

2. Every Basic Feasible Solution i.e. BFS is non-degenerate

3. BFS is in the form

$$\left[I \mid Y\right]\left[x\right] = y_0 \tag{11.1}$$

   where I is $m \times m$ Identity Matrix, $x = \begin{bmatrix} x_B \\ x_{NB} \end{bmatrix}$, $x_B \in R^n$ and $x_{NB} \in R^{n-m}$

11.2 Basic Steps of Algorithm

- Generic step of the algorithm is to swap a basic variable with a non basic variable. For now assume that we have selected basic variable x_p and non-basic variable x_q to swap

- x_p can be swapped with x_q if and only if $Y_{pq} \neq 0$ because if Y_{pq} is equal to 0 then column vector Y_q can be represented as linear combination of m − 1 basis vectors i.e.

$$Y_q = \sum_{i=1\ i\neq p}^{m} y_{iq} * I_i$$

  and hence Y_q cannot be included in basic solution

- Now make q^{th} column as $\begin{bmatrix} 0 & ... & 0 & 1_p & 0 & ... & 0 \end{bmatrix}$ where 1_p signifies 1 at p^{th} position. For that dividep_{th}row of matrix$\begin{bmatrix} I & Y \end{bmatrix}$ and matrix $\begin{bmatrix} Y(0) \end{bmatrix}$ by Y_{pq} and apply the row operation $R_i \Rightarrow R_i - Y_{iq} * R_p$

11.3 Determining the Leaving Variable p

- While applying row transformation of $\begin{bmatrix} I & Y \end{bmatrix}$ rows of $\begin{bmatrix} I \end{bmatrix}$ also changes and are given by

$$Y_{i0}' = Y_{i0} - Y_{iq} * Y_{p0}/Y_{i0}$$

  Condition $Y_{i0}' \geq 0$ must satisfy otherwise x_q would not be a BFS.

- So choose p such that

$$p \in S = \underset{i}{argmin}\{Y_{i0}/Y_{iq} | Y_{iq} \geq 0\}$$

- If number of elements in S is > 1 then the would become degenerate. Since non-degeneracy is assumed

$$p = \underset{i}{argmin}\{Y_{i0}/Y_{iq} | Y_{iq} \geq 0\}$$

11.4 Determining the Entering Variable q

- We Know that

$$\begin{bmatrix} I & Y \end{bmatrix} \begin{bmatrix} x_B \\ x_{NB} \end{bmatrix} = \begin{bmatrix} Y(0) \end{bmatrix}$$

$$x_B = Y_0 - Y x_{NB}$$

$$Where \begin{bmatrix} x_B \\ x_{NB} \end{bmatrix} \geq 0$$

- Initial Cost:

$$c^T \begin{bmatrix} x_B \\ x_{NB} \end{bmatrix} = c_B^T x_B + c_{NB}^T x_{NB}$$

$$= c_B^T x_B$$

$$= c_B^T Y_0$$

$$since \;\; x_{NB} = 0 \;\; and \;\; I * x_B + Y * x_{NB} = Y_0$$

- Now cost is

$$c^T \begin{bmatrix} x_B \\ x_{NB} \end{bmatrix} = c_B^T x_B + c_{NB}^T x_{NB}$$

$$= c_B^T (Y_0 - Y x_{NB}) + c_{NB}^T x_{NB}$$

$$= c_B^T Y_0 + (c_{NB - Y^T c_B})^T x_{NB}$$

- Now we can choose q such for which $(c_{NB} - Y^T c_B)_q < 0$

- Formalizing the above concept

$$\begin{bmatrix} 1 & 0 & ... & 0 & Y_{1,m+1} & Y_{1,m+2} & ... & Y_{1,n} \\ 0 & 1 & ... & 0 & Y_{2,m+1} & Y_{2,m+2} & ... & Y_{2,n} \\ . & . & . & . & . & . & ... & . \\ . & . & . & . & . & . & ... & . \\ . & . & . & . & . & . & ... & . \\ 0 & 0 & ... & 1 & Y_{m,m+1} & Y_{m,m+2} & ... & Y_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_m \\ . \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} y_{10} \\ . \\ . \\ . \\ y_{m0} \\ . \\ . \\ . \\ y_{n0} \end{bmatrix}$$

and

$$(c_{NB} - Y^T c_B)^T x_{NB} = \sum_{j=m+1}^{n} (c_j - Z_j) * x_j$$

$$Where \;\; Z_j = \sum_{i=1}^{m} (Y_{i,j} * c_i)$$

- To determine the entering variable choose j such that $(c_j - Z_j) < 0$

11.4.1 Theorem 8.1.

Given a non-degenerate Basic Feasible Solution with objective value Z' . Suppose $c_j - Z_j{}' < 0$ for some j there is a feasible solution with objective value $< Z'$. Also if variable x_j can be substituted for a variable in the basis for a new BFS, we get new BFS with value $Z_0 < 0$. If this cannot be done then the solution is unbounded.

11.5 Optimality condition

The Basic Feasible Solution is optimal if

$$\forall, \quad c_j - Z_j \geq 0$$

11.6 Some Points to Ponder

- f there does not exist p to replace then we have founded the recession direction and the cost can be reduced to $-\infty$

- In the worst case the Simplex Algorithm might visit all the extreme points. Example - Klee Minty cube

11.7 Duality

Every linear programming problem, referred to as a primal problem, can be converted into a dual problem, which provides an upper bound to the optimal value of the primal problem. The primal problem is:

$$\underset{x}{min}\ c^T x$$

$$Ax = b$$

$$x \geq 0$$

$$Where\ \ A \in R^{m \times n},\ Rank(A) = m$$

with the corresponding symmetric dual problem,

$$\underset{y}{max}\ b^T y$$

$$A^T y \leq c$$

$$Where\ \ A \in R^{m \times n},\ Rank(A) = m$$