Review exercises (Solutions)

Exercise 1

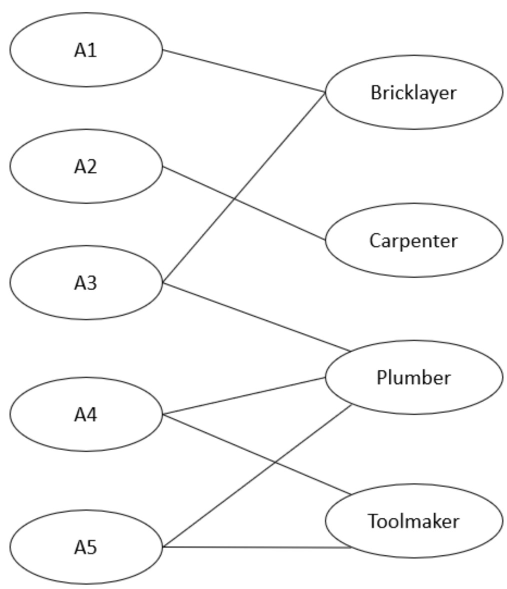
A building contractor advertises for a bricklayer, a carpenter, a plumber and a toolmaker and receives five applicants - one for the job of bricklayer, one for carpenter, one for bricklayer and plumber, and two for plumber and toolmaker.

1- Draw the corresponding bipartite graph.

2- Can all of the jobs be filled by qualified people? Which condition we must check?

Solution

1-



2- In order to know if all of the jobs can be filled by qualified people, we are talking about matching problem and we must check the condition of the hall's theorem (stable marriage condition):

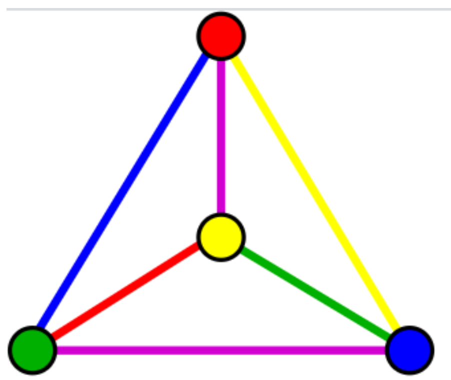
A complete matching from V1 to V2 exists if and only if $|\boldsymbol{A}| \leq |\boldsymbol{\varphi}(\boldsymbol{A})|$ for each subset A of V1.

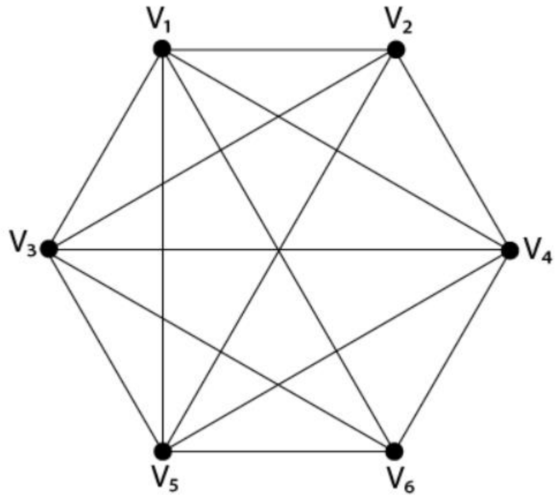
In our graph the set V1 is composed of 5 vertices and the set V2 is composed of 4 vertices. The marriage condition doesn't hold in this case.

Exercise 2

Sketch the graphs of K_4 and K_6. Are they planar?

For which value of n do you think K_n is planar?





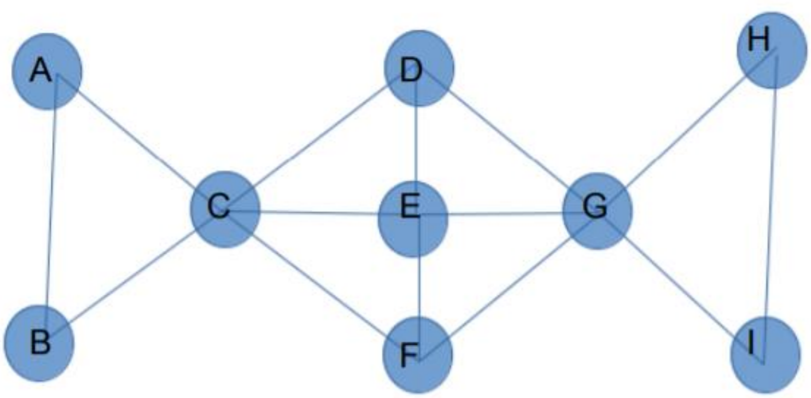
K4 K6

K4 is planar (shown in the figure) and K6 is not planar because if we remove the edges (V1,V4),(V3,V4)and (V5,V4) the graph K6,becomes homeomorphic to K5.Hence it is non-planar.

K_n is planar for n $\leq$ 4.

Exercise 3

What is the chromatic number of this graph?



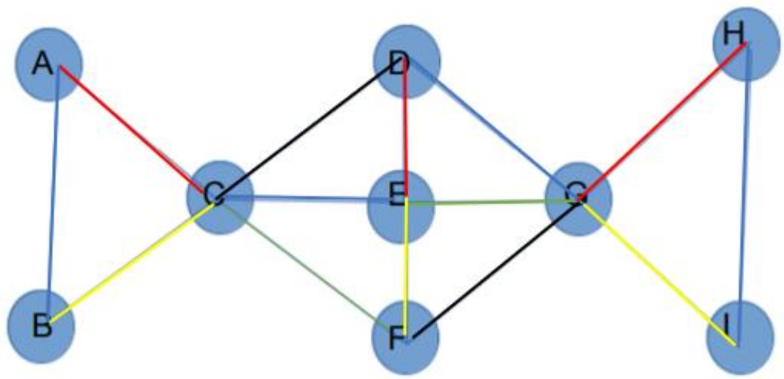
Solution

For the chromatic number, we have to colour the vertices of the graph. We apply the welsh and Powell algorithm

Vertex	C	G	E	D	F	A	B	H	I
Degree	5	5	4	3	3	2	2	2	2
Color	R	R	B	G	G	B	G	B	G

So, the chromatic number $\chi(G) = 3$

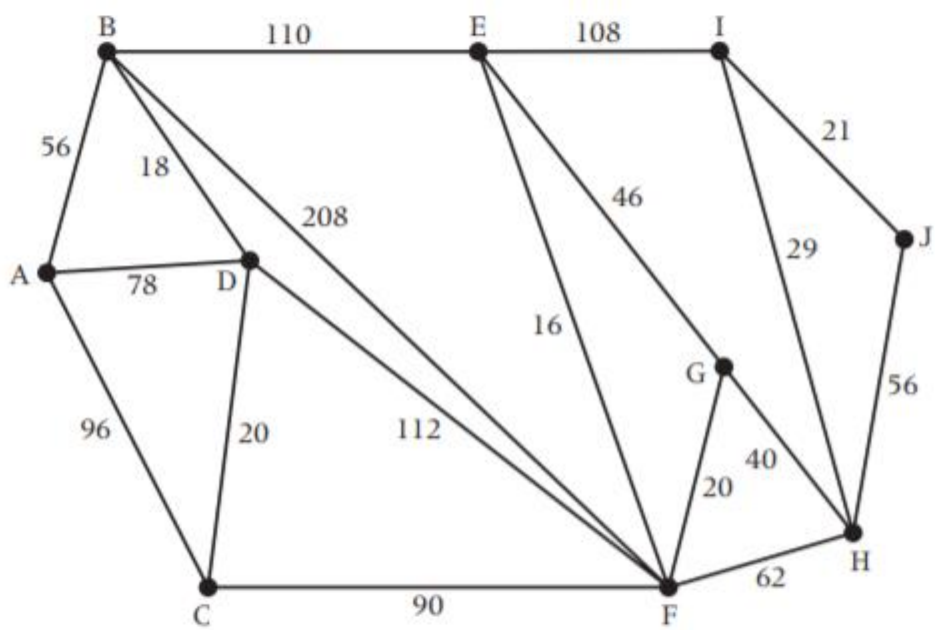
For the chromatic index, we have to colour the edges of the graph. Since the maximal degree is equal to 5 so the chromatic index should be 5 or 6.



The chromatic index $\chi'(G) = 5$

Exercise 4

The following network shows the time, in minutes, between some points.



1. Find the minimum time to travel from A to J and state the route.

2. A new road is to be constructed connecting B to G. Find the time needed for travelling this section of road if the overall minimum journey time to travel from A to J is reduced by 10 minutes. State the new route.

Solution

1. We apply Dijkstra Algorithm with the starting point A.

Initialization

Q={A, B, C, D, E, F, G, H, I, J}

S={}

Dist=(0, ∞, ∞, ∞, ∞, ∞, ∞, ∞, ∞, ∞)

P=(Null, Null, Null, Null, Null, Null, Null, Null, Null, Null)

Iteration 1

From Q we choose v=A (Dist(A)=0)

Q={B, C, D, E, F, G, H, I, J}

S={A}

The neighbors of A from Q are: B, D and C

Dist(B) = min(∞, 56)= 56 ; P(B)= A

Dist(C) = min(∞, 96)= 96 ; P(C)= A

Dist(D) = min(∞, 78)= 78 ; P(D)= A

Dist=(0, 56, 96, 78, ∞, ∞, ∞, ∞, ∞, ∞)

P=(Null, A, A, A, Null, Null, Null, Null, Null, Null)

Iteration 2

From Q we choose v=B (Dist(B)=56)

Q={C, D, E, F, G, H, I, J}

S={A, B}

The neighbors of B from Q are: D, E and F

Dist(D) = min(78, 74)= 74 ; P(D)= B

Dist(E) = min(∞, 166)= 166 ; P(E)= B

Dist(F) = min(∞, 264)= 264 ; P(F)= B

Dist=(0, 56, 96, 74, 166, 264, ∞, ∞, ∞, ∞)

P=(Null, A, A, B, B, B, Null, Null, Null, Null)

Iteration 3

From Q we choose v=D (Dist(D)=74)

Q={C, E, F, G, H, I, J}

S={A, B, D}

The neighbors of D from Q are: C and F

Dist(C) = min(96, 94)= 94 ; P(C)= D

Dist(F) = min(264, 186)= 186 ; P(F)= D

Dist=(0, 56, 94, 74, 166, 186, ∞, ∞, ∞, ∞)

P=(Null, A, D, B, B, D, Null, Null, Null, Null)

Iteration 4

From Q we choose v=C (Dist(C)=94)

Q={E, F, G, H, I, J}

S={A, B, D, C}

The neighbors of C from Q is F

Dist(F) = min(186, 184)= 184 ; P(F)= C

Dist=(0, 56, 94, 74, 166, 184, ∞, ∞, ∞, ∞)

P=(Null, A, D, B, B, C, Null, Null, Null, Null)

Iteration 5

From Q we choose v=E (Dist(E)=166)

Q={F, G, H, I, J}

S={A, B, D, C, E}

The neighbors of E from Q are F, G and I

Dist(F) = min(184, 182)= 182 ; P(F)= E

Dist(G) = min(∞, 212)= 212 ; P(G)= E

Dist(I) = min(∞, 274)= 274 ; P(I)= E

Dist=(0, 56, 94, 74, 166, 182, 212, ∞, 274, ∞)

P=(Null, A, D, B, B, E, E, Null, E, Null)

Iteration 6

From Q we choose v=F (Dist(F)=182)

Q={G, H, I, J}

S={A, B, D, C, E, F}

The neighbors of F from Q are G and H

Dist(G) = min(212, 202)= 182 ; P(G)= F

Dist(H) = min(∞, 244)= 244 ; P(H)= F

Dist=(0, 56, 94, 74, 166, 182, 202, 244, 274, ∞)

P=(Null, A, D, B, B, E, F, F, E, Null)

Iteration 7

From Q we choose v=G (Dist(F)=202)

Q={H, I, J}

S={A, B, D, C, E, F, G}

The neighbors of G from Q is H

Dist(H) = min(244, 242)= 242 ; P(H)= G

Dist=(0, 56, 94, 74, 166, 182, 202, 242, 274, ∞)

P=(Null, A, D, B, B, E, F, G, E, Null)

Iteration 8

From Q we choose v=H (Dist(F)=242)

Q={I, J}

S={A, B, D, C, E, F, G, H}

The neighbors of H from Q are I and J

Dist(I) = min(274, 271)= 271 ; P(I)= H

Dist(J) = min(∞, 298)= 298 ; P(J)= H

Dist=(0, 56, 94, 74, 166, 182, 202, 242, 271, 298)

P=(Null, A, D, B, B, E, F, G, H, H)

Iteration 9

From Q we choose v=I (Dist(I)=271

Q={J}

S={A, B, D, C, E, F, G, H, I}

The neighbors of I from Q is J

Dist(J) = min(298, 292)= 292 ; P(J)= I

Dist=(0, 56, 94, 74, 166, 182, 202, 242, 271, 292)

P=(Null, A, D, B, B, E, F, G, H, I)

Iteration 10

From Q we choose v=J (Dist(J)=292

Q={}

S={A, B, D, C, E, F, G, H, I, J}

Q is empty => End of the algorithm

Dist=(0, 56, 94, 74, 166, 182, 202, 242, 271, 292)

P=(Null, A, D, B, B, E, F, G, H, I)

The shortest path from A to J is : A -B -E -F -G -H -I -J with the value of 292 minutes.

2- A new road is to be constructed connecting B to G. Find the time needed for travelling this section of road if the overall minimum journey time to travel from A to J is reduced by 10 minutes. State the new route.

The new road from B to G will reduce the overall minimum journey time to travel from A to J by 10 minutes: The new route will be A-B-G-H-I-J

Dist(J)= 282

Dist(I)= 261

Dist(H)= 232

Dist(G)= 192

Dist(B)= 56

Dist(A)= 0

So the time needed for travelling the section B-G is 192-56= 136 minutes.

Exercise 5

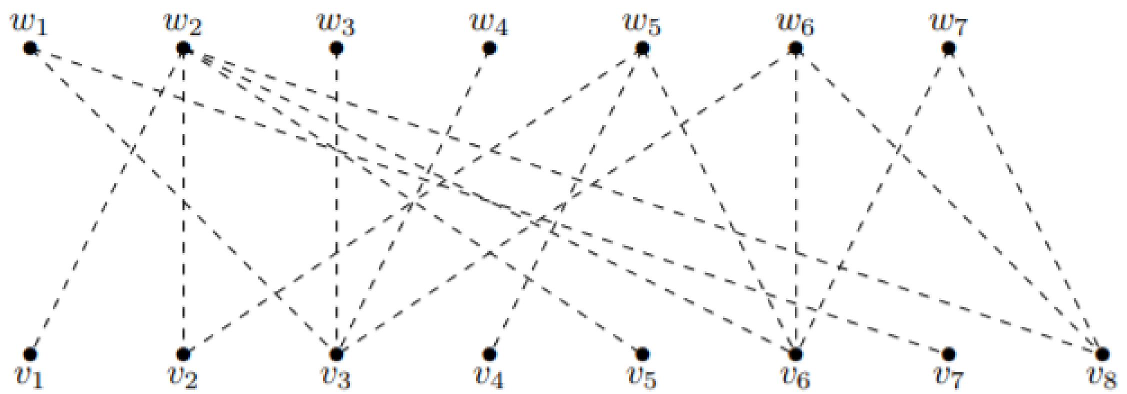
Show that a bipartite planar graph with n vertices has at most $2n - 4$ edges.

Solution

By Euler's formula, we know that a in a planar graph we have that $n+f-e = 2$, where n represents the number of vertices, e the number of edges and f the number of faces. In a bipartite graph, every face has at least 4 edges. On the other hand, each edge is incident to exactly 2 faces, so by double counting the face-edge pairs such that the edge is incident to the face, we obtain that $f \geq e/2$, and therefore $n + e/2 - e \geq 2$. This implies that $e \leq 2n - 4$, which completes the proof.

Exercise 6

We consider this bipartite graph.



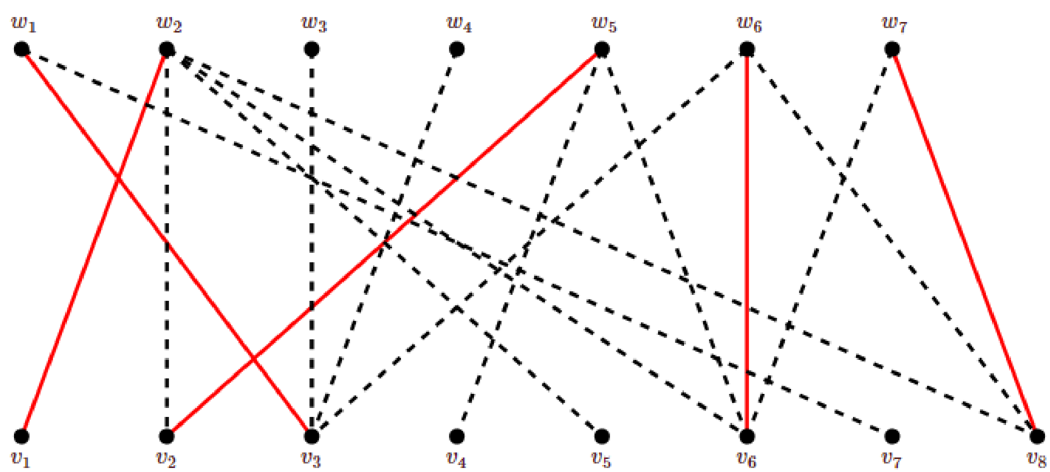
1- Is it possible to have a perfect matching? If yes give a perfect matching, otherwise give a maximum matching if we consider as initial matching: (w1, v3); (w2, v1); (w5, v2); (w6, v6); (w7, v8). Explain your approach

Solution

It is impossible to have a perfect matching because the subset A={v1, v5, v6, v7, v8} has a neighbors $\boldsymbol{\varphi(A)}$**= {w2, w1, w7};** $|\boldsymbol{A}| > |\boldsymbol{\varphi(A)}|$ so the marriage condition doesn't hold.

In order to find a maximum matching, we apply the augmenting alternating algorithm

We start with the initial matching



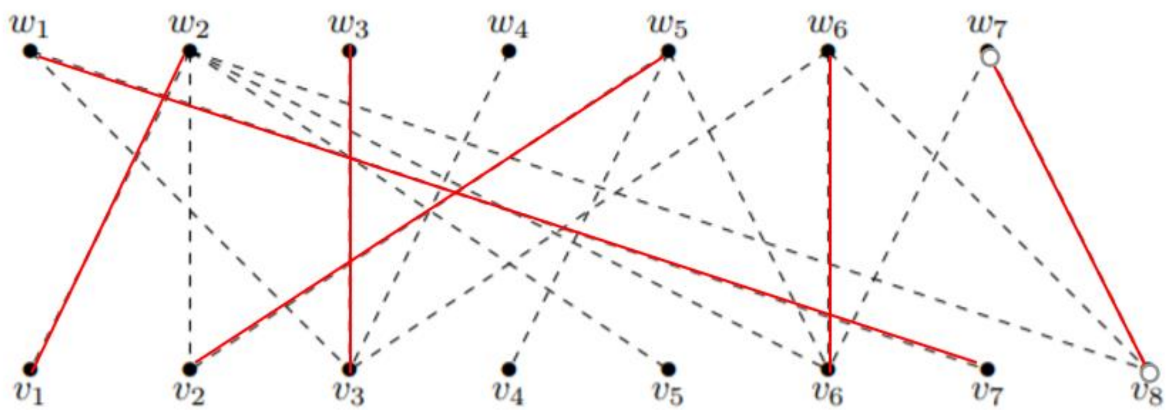
The vertices without matching are v4, v5, v7, w3, w4

We search an augmenting path from v4: v4-w5-v2-w2-v1. V1 is not an unmatched vertex so we drop this path.

We search an augmenting path from v5: v5-w2-v1. Again, we drop this path.

We search an augmenting path from v7: v7-w1-v3-w3 (we can also choose v7-w1-v3-w3)

We delete the matching w1v3 and we add the new matchings: v7w1 and v3w3. So we obtain the maximum matching:



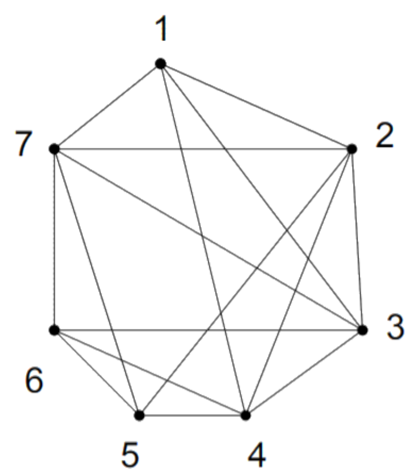
Exercise 7

Suppose there are 7 finals to be scheduled. Suppose the courses are numbered 1 through 7. Suppose that the following pairs of courses have common students : 1 and 2, 1 and 3, 1 and 4, 1 and 7, 2 and 3, 2 and 4, 2 and 5, 2 and 7, 3 and 4, 3 and 6, 3 and 7, 4 and 5, 4 and 6, 5 and 6, 5 and 7, 6 and 7.

How can the final exams at a university be scheduled so that no student has two exams at the same time?

Solution

If we model this situation, we find the following graph:



In order to avoid conflicts, we should colour the vertices of the graph and apply the welsh and powell algorithm.

Vertex	2	3	4	7	1	5	6
Degree	5	5	5	5	4	4	4
Color	R	B	G	G	Y	B	R

Chromatic number $\chi(G) = 4$

Thus, 4 time slots needed:

Slot 1: Courses 2 and 6

Slot 2: Courses 3 and 5

Slot 3: Courses 4 and 7

Slot 4 : Course 1

Exercise 8

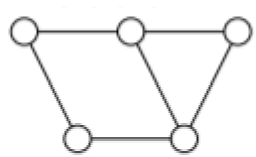
Draw an example graph for each of these.

(a) A planar graph has 5 vertices and 3 faces. How many edges does it have?

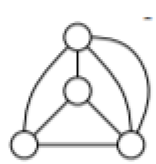
(b) A planar graph has 7 edges and 5 faces. How many vertices does it have?

Solution

We use Euler's formula: V + F = E + 2. (a) There are E = V + F − 2 = 6 edges. Here's an example:

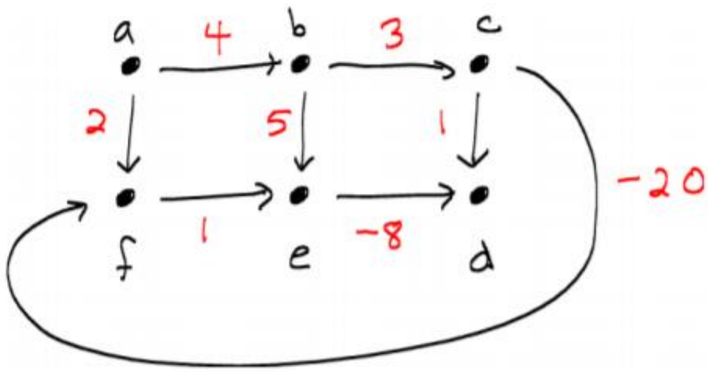


(b) There should be V = E − F + 2 = 4 vertices. However, this is not possible without creating duplicate edges. With duplicate edges, it is possible, and the formula gives the correct answer if we count the space between two duplicate edges also as a face. Here's an example:



Exercise 9

We consider the following graph:



Find the shortest path between a and d. Explain the choice of the algorithm.

Solution

We apply the bellman-Ford Algorithm because we have negative weights.

Bellman Ford Algorithm

Initialization

Vertex	a	b	c	d	e	f
Dist	0	∞	∞	∞	∞	∞
P	Null	Null	Null	Null	Null	Null

Iteration 1

Vertex	a	b	c	d	e	f
Dist	0	4	7	1	-12	-13
P	Null	a	b	e	f	c

Iteration 2

Vertex	a	b	c	d	e	f
Dist	0	4	7	-20	-12	-13
P	Null	a	b	e	f	c

Iteration 3

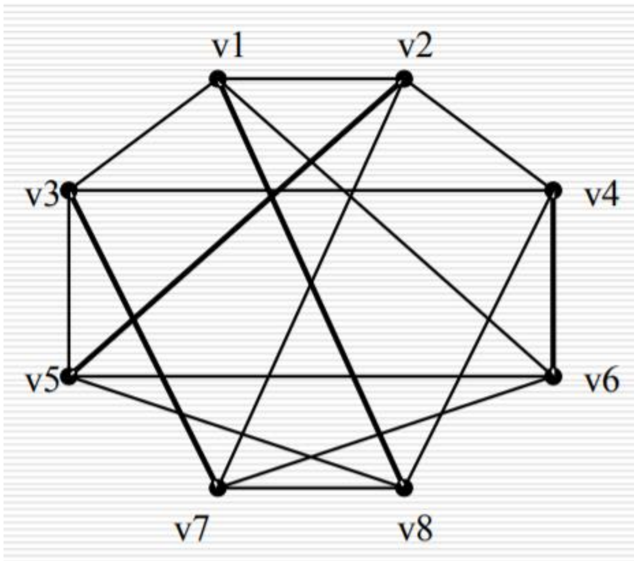
Vertex	a	b	c	d	e	f
Dist	0	4	7	-20	-12	-13
P	Null	a	b	e	f	c

There is no negative cycle in the graph.

The shortest path from a to d is: a-b-c-f-e-d with the value of -20.

Exercise 10

The Graph below is a bipartite graph. Determine two subsets of vertices for this graph.



Solution

In order to determine the two subsets, we will colour the vertices of the graph.

Vertex	V1	V2	V3	V4	V5	V6	V7	V8
Degree	4	4	4	4	4	4	4	4
Color	R	B	B	R	R	B	R	B

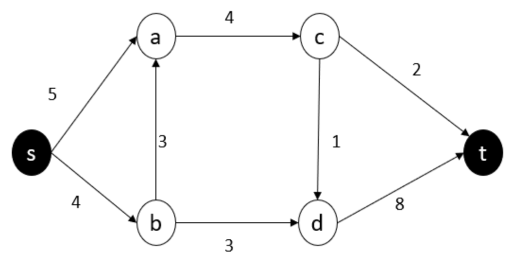
The two sets of the bipartite graph are:

Set 1: V1, V4, V5, V7

Set 2: V2, V3, V6, V8

Exercise 11

Consider the following network :



1- List all the cuts in this network, and find a minimum cut.

2- Find a maximum flow, and verify the max-flow min-cut theorem.

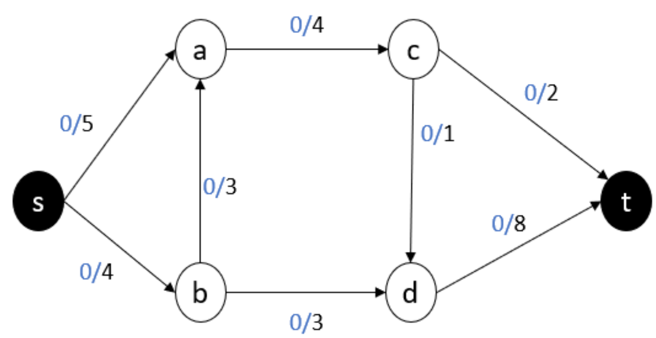
Solution

1- The cuts are {sa, sb}, {sa, ba, bd}, {sa, ba, cd, dt}, {sb, ba, ac] {sb, ba, cd, ct}, {ac, bd}, {ac, cd, dt}, {bd, cd, ct} and {ct, dt}.

The unique minimum cut is {bd, cd, ct} with capacity 6.

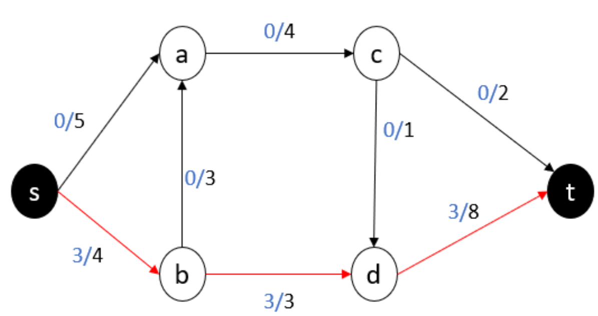
2- In order to find maximum flow, we apply Ford-Fulkerson Algorithm

Step 0



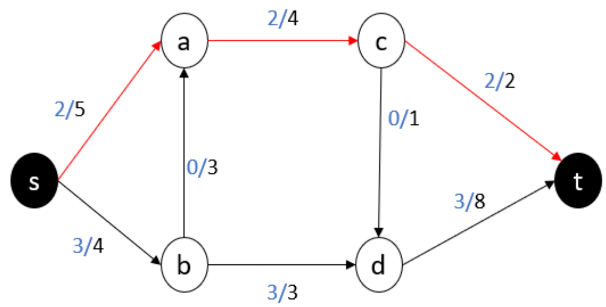
Step 1

We choose the augmenting path s-b-d-t (Remember that this choice is totally arbitrary!!!)



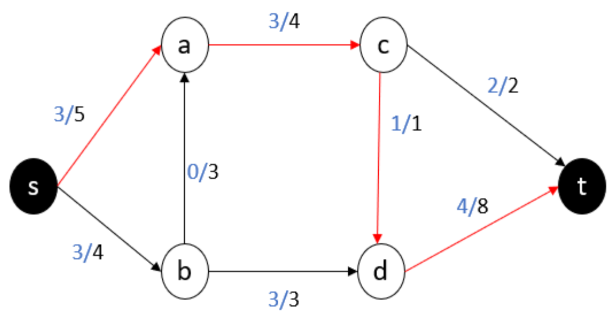
Step 2

We choose the augmenting path s-a-c-t



Step 3

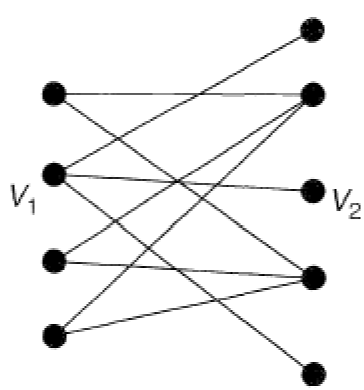
We still can improve and we choose s-a-c-d-t



Now we have our maximum flow and it is equal to 6 = equal to the minimum cut (question 1)

Exercise 12

Let’s consider the following graph:



It is possible to have a perfect matching in this graph. Explain.

Solution

The first, third and fourth vertices in V1 are collectively joined to only two vertices of V2, and hence the marriage condition fails. (The marriage condition: A complete matching from V1 to V2 exists if and only if $|\boldsymbol{A}| \leq |\boldsymbol{\varphi}(\boldsymbol{A})|$ for each subset A of V1.)