Cribsheet

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1 Transport equations

- **Material derivative:**
$$\frac{D\Phi}{Dt} = \frac{\partial \Phi}{\partial t} + v_i \frac{\partial \Phi}{\partial x_i}$$
With v_i (in Einstein notation form) the speed of displacement of the Lagrangian frame, following a streamline.

1.1 Flow

- **Mass balance:** *(in units of $kg.m^{-3}.s^{-1}$)*

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0}$$

- **Momentum balance:** *(in units of $kg.m^{-2}.s^{-2}$)*

$$\boxed{\rho \frac{Dv_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}}$$

$$\underbrace{\rho \frac{Dv_i}{Dt}}_{\text{Amount of acceleration}/m^3} = \underbrace{\rho g_i}_{\text{Volume forces}/m^3} - \underbrace{\frac{\partial p}{\partial x_i}}_{\text{Pressure forces}/m^3} + \underbrace{\frac{\partial \tau_{ij}}{\partial x_j}}_{\text{Viscous forces}/m^3}$$

  - If the fluid is **Eulerian** (or ideal), viscous effects are neglected:
$$\rho \frac{Dv_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i}$$
  - If the fluid is **Newtonian** (linear *and* isotropic relationship between viscous stress and the strain rate):
$$\tau_{ij} = 2\mu d_{ij} - \underbrace{\frac{2}{3}\mu(\nabla.\mathbf{v})\delta_{ij}}_{=0 \text{ if incompressible}}$$
  - If the fluid is **Newtonian** *and* **incompressible**, the viscous stress tensor can be further simplified:
$$\underbrace{\tau_{ij}}_{\text{Viscous stress tensor}} = \underbrace{2\mu}_{\text{Viscosity}} . \underbrace{d_{ij}}_{\text{Strain rate tensor}}$$

with the strain rate tensor defined as: *(i*j matrix, 3*3 in 3D)*

$$d_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$$

- **Navier-Stokes equations:** Momentum balance equations for *Newtonian* fluids:
  - <u>If $\mu = cst$:</u>

$$\underbrace{\frac{\partial \tau_{ij}}{\partial x_j}}_{\text{Viscous forces}/m^3} = \mu \frac{\partial^2 u_i}{\partial x_j^2} + \frac{1}{3}\mu \frac{\partial}{\partial x_i}(\frac{\partial u_k}{\partial x_k})$$

    with

$$\frac{\partial}{\partial x_j}(\nabla.\mathbf{v}\,\delta_{ij}) = \frac{\partial}{\partial x_j}\frac{\partial v_k}{\partial x_k}\,\delta_{ij} = \frac{\partial}{\partial x_i}\frac{\partial v_k}{\partial x_k}$$

  - <u>If $\mu = cst$ **and** incompressible:</u>

$$\underbrace{\frac{\partial \tau_{ij}}{\partial x_j}}_{\text{Viscous forces}/m^3} = \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

1.2 Mass transfer

- **Mass (or species) transport equation**

$$\frac{\partial}{\partial t}(\rho Y_k) + \nabla.\mathcal{J}_\mathbf{k} = \dot{w}_k$$

$$\boxed{\underbrace{\frac{\partial}{\partial t}(\rho Y_k)}_{non-stationary} + \underbrace{\nabla.(\rho Y_k \mathbf{v})}_{convection} = \underbrace{-\nabla.\mathbf{J_k}}_{diffusion} + \underbrace{\dot{w}_k}_{production}}$$

  where ρ is the density of the fluid, Y_k is the mass fraction of species k in the fluid, $\mathbf{v}$ the velocity vector of the fluid, $\mathcal{J}_\mathbf{k}$ the mass flux vector of species k, $\mathbf{J_k}$ the <u>diffusive</u> mass flux vector of species k, $\dot{w}_k$ the production rate of species k.

- **Fick's law:** (mass diffusion)

$$\mathbf{J_k} = -D_k \nabla C_k = -\rho D_k \nabla Y_k$$

  We can then further develop the previous equation by introducing Fick's law and Einstein convention:

$$\rho(\frac{\partial Y_k}{\partial t} + \frac{\partial Y_k v_j}{\partial x_j}) = \rho D_k \frac{\partial^2 Y_k}{\partial x_j \partial x_j} + \dot{w}_k$$

2 Dimensionless numbers

- **Prandtl's number:** (with D_T: thermal diffusivity).

$$Pr = \frac{\nu}{D_T} = \frac{momentum\, transport}{thermal\, transport}$$

- **Schmidt's number:**

$$Sc = \frac{\nu}{D_J} = \frac{momentum\, transport}{species\, or\, mass\, transport}$$

- **Lewis' number:** Thermo-diffusive effects.

$$Le = \frac{D_T}{D_J} = \frac{Sc}{Pr} = \frac{diffusive\, mass\, transfer}{heat\, transfer}$$

- **Progress variable:** (with c=0 in unburnt gases and c=1 in burnt gases)

$$c = \frac{T - T_1}{T_2 - T_1} = \frac{Y_F - Y_{F_1}}{Y_{F_2} - Y_{F_1}}$$

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