Ray-Sphere Intersection

Definitions:

Parametric representation of a line in 3D: $f(t) = \vec{p} + t\vec{d}$

Implicit representation of a sphere centered at the point $\vec{c}$ and with radius r: $(\vec{x} - \vec{c}) \cdot (\vec{x} - \vec{c}) - r^2 = 0$

Intersection:

Substitute $f(t) = \vec{p} + t\vec{d}$ as the point $\vec{x}$ in the sphere equation:

$$(\vec{p} + t\vec{d} - \vec{c}) \cdot (\vec{p} + t\vec{d} - \vec{c}) - r^2 = 0$$

This substitution produces a quadratic equation of the form:

$$At^2 + Bt + C = 0$$

where the coefficients are

$$A = (\vec{d} \cdot \vec{d})$$

$$B = 2\vec{d} \cdot (\vec{p} - \vec{c})$$

$$C = (\vec{p} - \vec{c}) \cdot (\vec{p} - \vec{c}) - r^2$$

Solve for the parameter t using the quadratic formula. If the discriminant is less than 0.0, then the given line does not intersect the sphere. The smaller positive solution for t is the intersection point. That is, t_0 is the $-$ solution and t_1 is the $+$ solution.

Full Derivation:

$(\vec{p} + t\vec{d} - \vec{c}) \cdot (\vec{p} + t\vec{d} - \vec{c}) - r^2 = 0$

Let $\vec{u} = ((\vec{p} + t\vec{d}) - \vec{c})$. Rewrite the equation as

$$\vec{u} \cdot ((\vec{p} + t\vec{d}) - \vec{c}) - r^2 = 0.$$

Since the dot product is distributive, this form is equal to

$$\vec{u} \cdot (\vec{p} + t\vec{d}) - (\vec{u} \cdot \vec{c}) - r^2 = (\vec{u} \cdot \vec{p}) + (\vec{u} \cdot t\vec{d}) - (\vec{u} \cdot \vec{c}) - r^2 = 0.$$

Undo the substitution for $\vec{u}$:

$$(\vec{p} \cdot ((\vec{p} + t\vec{d}) - \vec{c})) + (t\vec{d} \cdot ((\vec{p} + t\vec{d}) - \vec{c})) - (\vec{c} \cdot ((\vec{p} + t\vec{d}) - \vec{c})) - r^2 = 0$$

Distribute each term then rearrange into standard quadratic form:

$$(\vec{d} \cdot \vec{d})t^2 + 2((\vec{p} \cdot \vec{d}) - (\vec{d} \cdot \vec{c}))t + ((\vec{p} \cdot \vec{p}) - 2(\vec{p} \cdot \vec{c}) + (\vec{c} \cdot \vec{c}) - r^2) = 0$$

To simplify the form of B, note that $\vec{d}$ appears in both terms of $B = 2((\vec{p} \cdot \vec{d}) - (\vec{d} \cdot \vec{c}))$. $\vec{d}$ can be factored out using the distributive property of dot products.

For C, we have something of the form $(a+b)^2 = a^2 + 2ab + b^2$ written using dot product notation, which can be factored into:

$$(\vec{p} \cdot \vec{p}) - 2(\vec{p} \cdot \vec{c}) + (\vec{c} \cdot \vec{c}) = (\vec{p} - \vec{c}) \cdot (\vec{p} - \vec{c}).$$

Also see: https://education.siggraph.org/static/HyperGraph/raytrace/rtinter1.htm