Domain Recovery From Stochastic Inverse Problems

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Overview

Example

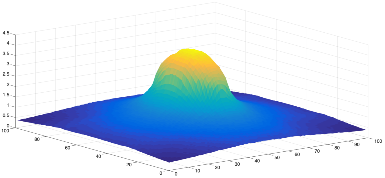
Model 1

$$y = Q_x(a, b) = ax + b, \tag{1}$$

where a, b are parameters of interest and x is an input variable in map Q_x. Some graphs of solutions of the inverse problem (1) on different domains are shown on the next page.

- $\{a, b\}$ are random physical parameters with compact support Λ.
- In statistics, x is referred to as the predictor variable.
- Given measured data on the output of Q_x, we wish to determine information on $\{a, b\}$.
- Problem: Q_x is a measurable map but is not $1 - 1$.

Example



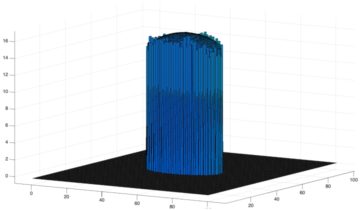


Figure: When estimating density on the true domain of (a, b), it is much closer to true density, a truncated normal.

Stochastic forward problem (SFP)

Consider a general system

$$y = Q_{\boldsymbol{x}}(\boldsymbol{a}),$$

where $Q_{\boldsymbol{x}} : (\Lambda, \mathcal{B}_\Lambda, P_\Lambda) \to (\mathcal{D}_{\boldsymbol{x}}, \mathcal{B}_{\mathcal{D}_{\boldsymbol{x}}}, P_{\mathcal{D}_{\boldsymbol{x}}})$ is an invertible measurable map between the n dimensional domain Λ and the 1 dimensional range $\mathcal{D}_{\boldsymbol{x}}$ and P_Λ induces a probability distribution $P_{\mathcal{D}_{\boldsymbol{x}}}$ on $\mathcal{D}_{\boldsymbol{x}}$.

- P_Λ is called the *Original Generating Distribution* (OGD) or *Natural Generating Distribution* (NGD), $\boldsymbol{a}$ is the physical parameter, $\boldsymbol{x}$ is the observable predictor.
- $P_{\mathcal{D}_{\boldsymbol{x}}}$ is an induced probability distribution which depends on the deterministic value of $\boldsymbol{x}$.
- The output of the map $Q_{\boldsymbol{x}}$ is observed after the value of $\boldsymbol{x}$ is given.

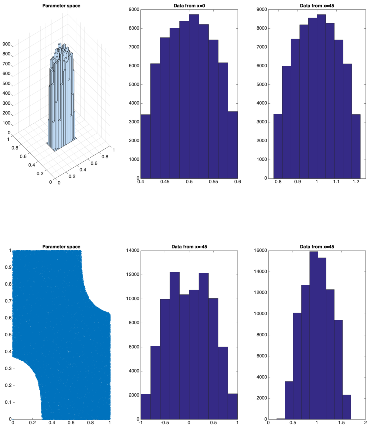


Figure: Induced probability distribution on the range in the stochastic forward problem. Λ is given as a regular ellipse and P_Λ is a truncated normal (Top); Λ is given as a non-convex set and and P_Λ is uniform (Bottom)

Stochastic inverse problem (SIP)

This is the direct inverse of forward problem. Given $(\mathcal{D}_{\boldsymbol{x}}, \mathcal{B}_{\mathcal{D}_{\boldsymbol{x}}}, P_{\mathcal{D}_{\boldsymbol{x}}})$, what can we say about probability distribution P_Λ?

Solution of SIP

This requires inverting a map that is from n dim to 1 dim.

- Inverse of map $Q_{\boldsymbol{x}}$ at a point is called a generalized contour or an equivalence class, $Q_{\boldsymbol{x}}^{-1}(y), y \in \mathcal{D}_{\boldsymbol{x}}$.
- Decomposition of $\Lambda =$ a union of generalized contours indexed by points in $\mathcal{D}_{\boldsymbol{x}}$, $\bigcup_{y \in \mathcal{D}_{\boldsymbol{x}}} Q_{\boldsymbol{x}}^{-1}(y)$, which are set-valued solutions.
- Disintegration: the distribution $P_\Lambda =$ iterated integral of a "conditional" distribution on each generalized contour $\&$ probability distribution on the set of equivalence classes.
- Ansatz: assume a uniform distribution on each generalized contour.

Principle of Ansatz

Since the probability distribution on each generalized contour is unknown in the solution of SIP, we assume it has certain known probability distribution which is called the Ansatz, e.g. a uniform distribution or a truncated normal. Generally, without any information about the NGD, we choose the uniform ansatz meaning that events are equally likely to occur on the generalized contour. Then, the inverse distribution using the uniform ansatz

- is the maximum entropy solution of all continuous inverse distributions
- Maximum Entropy Principle implies it is least-informative among all choices
- is a default when we solve the SIP
- non-uniqueness of the solution
- the NGD is one solution of the SIP.

Inverse Support

The solution of SIP is called the inverse distribution and inverse support is the support of inverse distribution which is defined as

$$\Lambda_{\boldsymbol{x}} = \{\lambda \in \Lambda : Q_{\boldsymbol{x}}(\lambda) \in Q_{\boldsymbol{x}}(\mathcal{K}_0)\}, \tag{2}$$

where $\mathcal{K}_0 \subset \Lambda$ is the true support of physical parameters. The support defined in (2) is the maximal element $\mathscr{C}_{\boldsymbol{x}}$ in the equivalence class of all feasible support defined as

$$B_{\boldsymbol{x}}^{-1} := \{c \in \mathcal{B}_\Lambda : Q_{\boldsymbol{x}}(c) = B(= Q_{\boldsymbol{x}}(\mathcal{K}_0)) \in \mathcal{B}_{\mathcal{D}_{\boldsymbol{x}}}\}, \tag{3}$$

where $B_{\boldsymbol{x}}^{-1}$ contains all the support that yields the support of $P_{\mathcal{D}_{\boldsymbol{x}}}$. The uniform distribution on $\Lambda_{\boldsymbol{x}}$ has the maximum entropy compared to any other continuous distribution on any other support $S \in B_{\boldsymbol{x}}^{-1}$.

Inverse Support

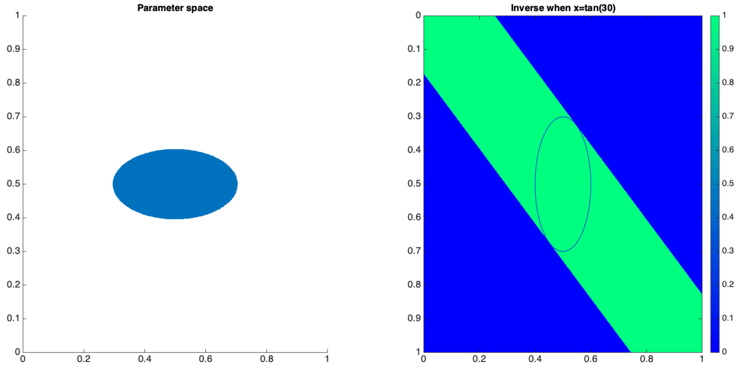


Figure: P_Λ *in Model 1 is "smeared out" into a larger domain. The inverse support is much larger than the true support* $\Lambda_x \supset \mathcal{K}_0$. *Inefficient to draw samples in the larger domain and lose the its physical meanings.*

Inverse Support

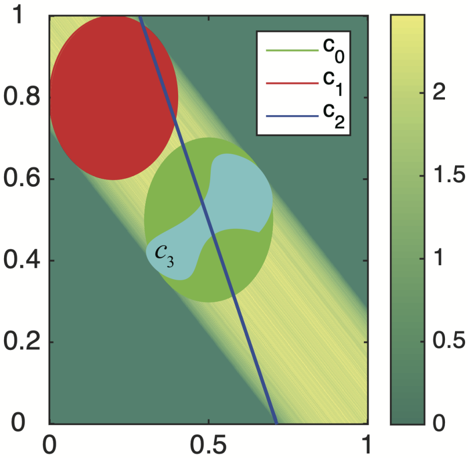


Figure: Different elements $c_0, c_1, c_2, c_3, \mathscr{C}_{\boldsymbol{x}}$ *in* $B_{\boldsymbol{x}}^{-1}$ *are shown.*

Role of $\boldsymbol{x}$

$\Lambda_{\boldsymbol{x}}$ contains the true support $\mathcal{K}_0$ for all possible values of $\boldsymbol{x} \in \mathcal{X}$ by definition. Each $\Lambda_{\boldsymbol{x}}$ gives information of the true support. Therefore, given a family of $\{Q_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathcal{X}}$ and marginal distributions $\{P_{\mathcal{D}_{\boldsymbol{x}}}\}_{\boldsymbol{x}\in\mathcal{X}}$, we compute the intersection

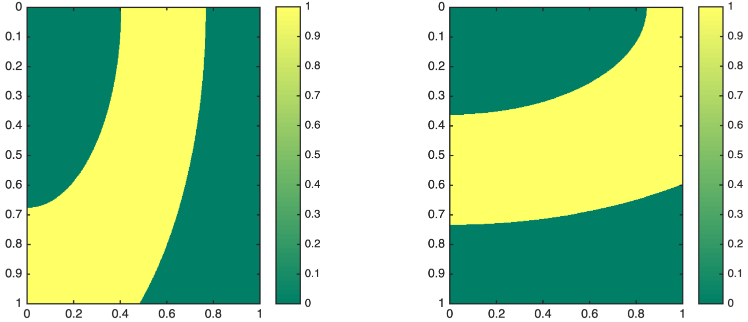
$$\bigcap_{\boldsymbol{x}\in\mathcal{X}} \Lambda_{\boldsymbol{x}},$$

which also contains the true support but gets smaller. Then, we solve the inverse problem on this new parameter space. This helps to

- avoid assigning nonzero probability to events outside the true support
- reduce the loss of information and bias on events that have nonzero probability
- yield a better solution for the SIP meaning that it is “closer” to P_Λ.

Note that this procedure essentially computes the “almost always” set.

Role of x



(a) (b)

Figure: Inverse support from different values of x

Marginal of Inverses

Suppose the inverse distribution is denoted $P_{\boldsymbol{a}|\boldsymbol{x}}$ and the inverse density function is $P'_{\boldsymbol{a}|\boldsymbol{x}}$. Marginal of inverses is defined as:

$$\bar{P}_{\boldsymbol{a}|\Lambda} = \int_{\mathcal{X}} P_{\boldsymbol{a}|\boldsymbol{x},\Lambda} d\boldsymbol{x}, \tag{4}$$

$$\bar{P'}_{\boldsymbol{a}|\Lambda} = \int_{\mathcal{X}} P'_{\boldsymbol{a}|\boldsymbol{x},\Lambda} d\boldsymbol{x}, \tag{5}$$

which

- give the marginal distribution of physical parameters not associated with $\boldsymbol{x}$,
- give the high probability region that describes the original high probability region, which means the high probability region is "closer" to the original by a lower bound property.

(4) and (5) can be approximated by using a quadrature rule or Monte Carlo sampling.

Marginal of Inverses

Suppose (5) is computed through

$$\int_{\boldsymbol{x}\in\mathcal{X}} P'_{\boldsymbol{a}|\boldsymbol{x}} dP_{\mathcal{X}}(\boldsymbol{x}). \tag{6}$$

The marginal (6) is a lower bound

$$\sup_{\boldsymbol{x}\in\mathcal{X}} \int_{\Lambda} |P'_{\Lambda} - P'_{\boldsymbol{a}|\boldsymbol{x},\Lambda}| d\mu_{Leb}(\lambda) \geq \int_{\Lambda} |P'_{\Lambda} - \int_{\mathcal{X}} P'_{\boldsymbol{a}|\boldsymbol{x},\Lambda} dP_{\mathcal{X}}(\boldsymbol{x})| d\mu_{Leb}(\lambda),$$

where P'_{Λ} is the density function of the NGD. This also shows

- (6) is a lower bound under Wasserstein metric on a quotient space with respect to the kernel $|\cdot|$,
- what "closer" means.

Marginal of Inverses

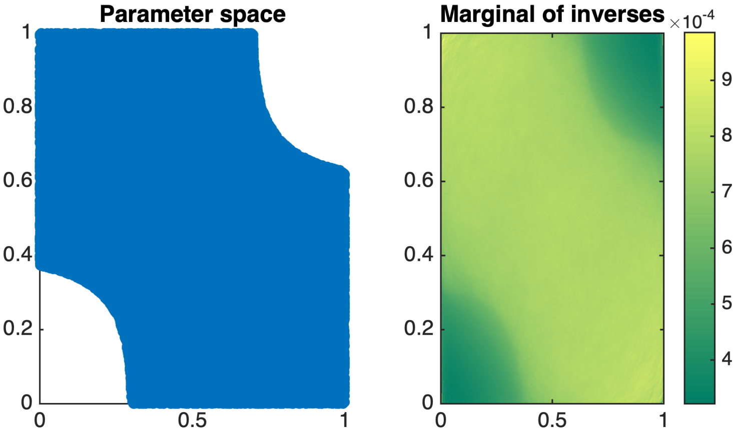


Figure: Actual support on the parameter space $\Lambda = [0, 1] \times [0, 1]$ (left); Integration over a family of all inverses on the original parameter space Λ (right). The marginal shows the high probability region.

General Linear System

Model 2

$$y = Q_{\boldsymbol{x}}(\boldsymbol{a}) = a_1 x_1 + a_2 x_2 \cdots + a_d x_d, \tag{7}$$

where $\boldsymbol{a} = (a_1, a_2, \ldots, a_d)^T \in \Lambda, \boldsymbol{x} = (x_1, x_2, \ldots, x_d)^T \in \mathcal{X}$. Λ is compact and has non-empty interior and measure zero boundary. $\mathcal{K}_0$ is the true support of $\boldsymbol{a}$ which has non-empty interior and measure zero boundary.

Theorem 1

Consider (7), the inverse support $\Lambda_{\boldsymbol{x}}$ can be uniquely expressed as

$$\Lambda_{\boldsymbol{x}} = \Lambda \cap S_{c_1,c_2}, \tag{8}$$

where c_1, c_2 are some constants, and $S_{c_1,c_2} = \{\boldsymbol{a} \in \mathbb{R}^d : \boldsymbol{a} \cdot \boldsymbol{x} \in [c_1, c_2]\}$ is a parallel slab in $\mathbb{R}^d$.

We call $\Lambda_{\boldsymbol{x}}$ a generalized parallel slab.

General Linear System

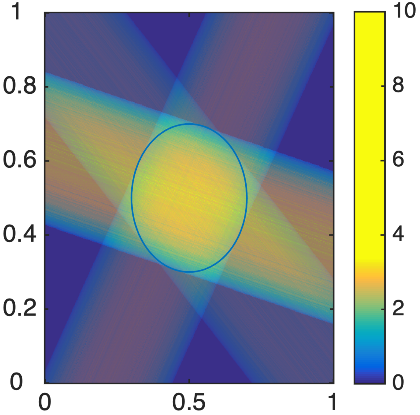


Figure: Display of generalized parallel slabs.

General Linear System

- $\boldsymbol{a} \cdot \boldsymbol{x} = c_1$ or $\boldsymbol{a} \cdot \boldsymbol{x} = c_2$ characterize the boundary of the true support $\mathcal{K}_0$, and are referred to as the supporting hyperplanes.
- supporting hyperplane:
  - the set is contained in one of the two closed half-spaces characterized by the hyperplane;
  - the set has at least a boundary point on the hyperplane.
- $\mathcal{K}_0$ is contained in $\Lambda_{\boldsymbol{x}}$.
- $\bigcap_{\boldsymbol{x} \in \mathbb{R}^d} \Lambda_{\boldsymbol{x}}$ is a convex hull of $\mathcal{K}_0$.

Theorem 2

Suppose there is a collection $\{\boldsymbol{x}_i\}_{i=1}^{\infty}$, which is a dense subset of $\mathbb{R}^d$.

Then, $\mathcal{K}_{conv} = \bigcap\limits_{i=1}^{\infty} \Lambda_{\boldsymbol{x}_i}$ is the convex hull of $\mathcal{K}_0$.

General Linear System

Inverse distribution on the convex hull is better since

$$g_{conv} = \arg\inf_{g_c \in \mathcal{H}_c} \int_{\mathcal{L}_{\boldsymbol{x}'}} \int_{\boldsymbol{a} \in \pi^{-1}_{\mathcal{L}_{\boldsymbol{x}'}}(\ell) \cap \mathcal{K}_{conv}} (g_c(\boldsymbol{a}) - g_0(\boldsymbol{a})) d\mu_{Leb}(\boldsymbol{a}) d\mu_{\mathcal{L}_{\boldsymbol{x}'}}(\ell)$$

if g_0 is a general density function and

$$g_{conv} = \arg\inf_{g_c \in \mathcal{H}_c} \int_{\mathcal{L}_{\boldsymbol{x}'}} \sup_{\pi^{-1}_{\mathcal{L}_{\boldsymbol{x}'}}(\ell) \cap \Lambda} (g_c - g_0) d\mu_{\mathcal{L}_{\boldsymbol{x}'}}(\ell),$$

if g_0 is unimodal. $\boldsymbol{x}'$ a new value. g_{conv} is the inverse density function on $\mathcal{K}_{conv}$ and g_c is the inverse density function on certain convex support that contains $\mathcal{K}_0$ and $\mathcal{H}_c$ denote the class of such g_c.

General Linear System

Since the true support of the OGD is unknown, we replace it by its convex hull in practice. Accordingly, we assume $\mathcal{K}_0$ is convex. The convergence $\bigcap_{i=1}^{n} \Lambda_{\boldsymbol{x}_i} \to \mathcal{K}_0$ can be measured in several metrics

- the Nikodym metric: $d_N(U,V) = \mu_{Leb}(U \triangle V)$ where $U, V \in \mathcal{C}^d :=$ {the set of convex bodies (i.e., of compact convex subsets with non-empty interior) in $\mathbb{R}^d$};
- the Hausdorff metric: $d_H(U,V) = \inf\{\epsilon > 0 : U \subset B(V,\epsilon)$ and $V \subset B(U,\epsilon)\}$, where $U, V \subset \mathbb{R}^d$ and $B(U,\epsilon) := \{y \in \mathbb{R}^d : B(y,U) \leq \epsilon\}$, $B(x,\epsilon) := B(\{x\},\epsilon)$.
- In $\mathcal{C}^d$, Nikodym metric and Hausdorff metric are topologically equivalent.

General Linear System

Corollary 3

Suppose a collection $\{\boldsymbol{x}_i\}_{i=1}^{\infty}$ is drawn from a dense subset of $\mathcal{X}$ such that $F_n = \frac{1}{n}\sum_{i=1}^{n} I(\boldsymbol{x}_i \leq \boldsymbol{x}) \to F(\boldsymbol{x})$ where I is the indicator function and F has a positive continuous density w.r.t μ_{Leb}. Then $d_N\left(\bigcap_{i=1}^{n} \Lambda_{\boldsymbol{x}_i}, \mathcal{K}_0\right) \to 0$ as $n \to \infty$,

which is a direct result from $\Lambda_n = \bigcap_{i=1}^{n} \Lambda_{\boldsymbol{x}_i} \to \mathcal{K}_0$.

Based on $\Lambda_n \to \mathcal{K}_0$, we consider the convergence of inverse distributions on the corresponding support.

Theorem 4

Suppose $P_{\mathcal{L}_{\boldsymbol{x}}|\mathcal{K}_0}$ is absolutely continuous with respect to the Lebesgue measure $\mu_{\mathcal{L}_{\boldsymbol{x}}}$. Then, $P_{\boldsymbol{a}|\boldsymbol{x},\Lambda_n} \xrightarrow{w.} P_{\boldsymbol{a}|\boldsymbol{x},\mathcal{K}_0}$.

General Linear System

$P_{\boldsymbol{a}|\boldsymbol{x},\Lambda_n} \xrightarrow{w.} P_{\boldsymbol{a}|\boldsymbol{x},\mathcal{K}_0}$:

- $P_{\boldsymbol{a}|\boldsymbol{x},\Lambda_n}$ is the inverse distribution computed on the parameter space Λ_n,
- $P_{\boldsymbol{a}|\boldsymbol{x},\mathcal{K}_0}$ is the inverse distribution computed on the parameter space $\mathcal{K}_0$.

We also have some results on the properties of the marginal of $\{P_{\boldsymbol{a}|\boldsymbol{x}_i,\Lambda}\}_{i=1}^{\infty}$. Note that $\bar{P}_{\boldsymbol{a}|\Lambda} = \int_{\mathcal{X}} P_{\boldsymbol{a}|\boldsymbol{x},\Lambda} d\boldsymbol{x}$ can be approximated by

$$\mathcal{M}_{\boldsymbol{X}}[P_{\boldsymbol{a}|\boldsymbol{X},\Lambda}](A) = \int_{\mathcal{X}=\mathbb{R}^d} P_{\boldsymbol{a}|\boldsymbol{x},\Lambda}(A) f_{\boldsymbol{X}}(\boldsymbol{x}) d\mu_{Leb}(\boldsymbol{x}),$$

where $A \in \mathcal{B}_\Lambda$ and $f_{\boldsymbol{X}}$ is the density function. In this case, the marginal has some good properties.

Theorem 5

$\mathcal{M}_{\boldsymbol{X}}[P_{\boldsymbol{a}|\boldsymbol{X},\Lambda}]$ is a probability measure on $(\Lambda, \mathcal{B}_\Lambda)$.

Theorem 6

$\mathcal{M}_{\boldsymbol{X}}[P_{\boldsymbol{a}|\boldsymbol{X},\Lambda_n}] \xrightarrow{w.} \mathcal{M}_{\boldsymbol{X}}[P_{\boldsymbol{a}|\boldsymbol{X},\mathcal{K}_0}]$.

Equivalently, $\mathcal{M}_{\boldsymbol{X}}[P_{\boldsymbol{a}|\boldsymbol{X},\Lambda_n}](A) \to \mathcal{M}_{\boldsymbol{X}}[P_{\boldsymbol{a}|\boldsymbol{X},\mathcal{K}_0}](A)$ for any $A \in \mathcal{B}_{\mathcal{K}_0}$ with $P_{\boldsymbol{a}|\boldsymbol{x},\mathcal{K}_0}(\partial A) = 0$ for any $\boldsymbol{x} \in \mathbb{R}^d$.

- $\Lambda_n = \bigcap_{i=1}^{n} \Lambda_{\boldsymbol{X}_i}$
- $\boldsymbol{X}$ is a new sample from $\{\boldsymbol{X}_i\}$.

General Linear System

There are several ways to predict an inverse distribution from a new map $Q_{\boldsymbol{x}_0}$ by using a collection of computed inverse distributions $\{P_{\boldsymbol{a}|\boldsymbol{x}_i,\Lambda}\}_{\boldsymbol{x}_i \in \mathbb{R}^d}$:

- use the marginal $\mathcal{M}_{\boldsymbol{X}}[P_{\boldsymbol{a}|\boldsymbol{X},\Lambda_n}]$ as an approximation of the NGD to calculate the $P_{\boldsymbol{a}|\boldsymbol{x}_0,\Lambda_n}$,
- use Monte Carlo estimation and kernel functions.

Suppose $K_h(\boldsymbol{x}_i, \boldsymbol{x}_0) = e^{-\frac{||\boldsymbol{x}_i - \boldsymbol{x}_0||_2^2}{2h^2}}$ which is the Radial Basis Function (RBF) kernel with the Euclidean distance.

General Linear System

Theorem 7

For any $A \in \mathcal{B}_\Lambda$ and a new value $\boldsymbol{x}_0 \in \mathbb{R}^d$

$$\frac{1}{n}\sum_{i=1}^{n} P_{\boldsymbol{a}|\boldsymbol{X}_i,\Lambda}(A) K_h(\boldsymbol{X}_i, \boldsymbol{x}_0) \Big/ \frac{1}{n}\sum_{i=1}^{n} K_h(\boldsymbol{X}_i, \boldsymbol{x}_0) \xrightarrow[h\to 0]{n\to\infty} P_{\boldsymbol{a}|\boldsymbol{x}_0,\Lambda}(A),$$

if $P_{\boldsymbol{a}|\boldsymbol{x},\Lambda}(A)$ is continuous with respect to $\boldsymbol{x}$.

- $\{\boldsymbol{X}_i\}_{i=1}^{\infty}$ is a collection of i.i.d. random samples in $\mathbb{R}^d$ with positive bounded continuous density function $f_{\boldsymbol{X}}$
- $P_{\boldsymbol{a}|\boldsymbol{X}_i,\Lambda}$ is the inverse distribution for random sample $\boldsymbol{X}_i$.
- the continuity of $P_{\boldsymbol{a}|\boldsymbol{x},\Lambda}(A)$ is determined by the continuity of $\pi_{\mathcal{L}_{\boldsymbol{x}},\Lambda}$, $Q_{\boldsymbol{x}}^{-1}$, $P_{\mathcal{D}_{\boldsymbol{x}}}$ and the boundary of Λ.

General Linear System

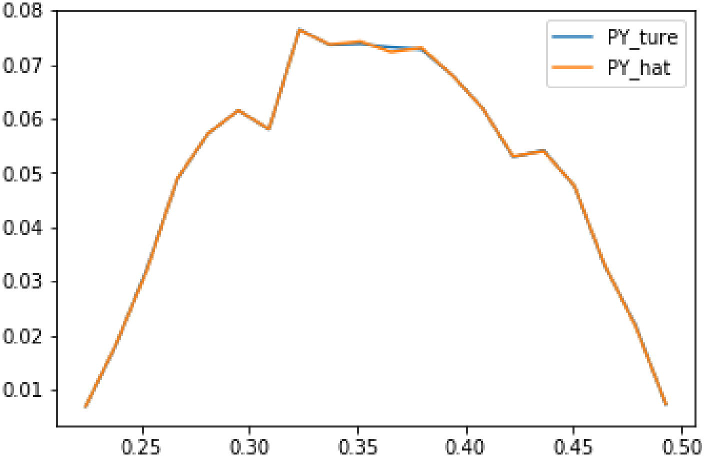


Figure: Prediction of a new distribution $P_{\mathcal{D}_{\boldsymbol{x}_0}}$.

Rate of Convergence

We first consider the rate of convergence in the plane according to the model:

Model 3

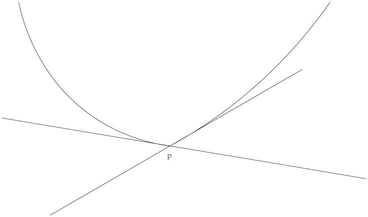
$$y = a \tan \theta + b,$$

where (a, b) is the physical parameter and θ has the same role as $\boldsymbol{x}$. Suppose we have a collection of i.i.d random samples $\{\Theta_i\}_{i=1}^{\infty}$ from $(I_0 = (-\frac{\pi}{2}, \frac{\pi}{2}), \mathcal{B}_{I_0}, P_\Theta)$ and the corresponding computed inverse support $\{\Lambda_{\Theta_i}\}_{i=1}^{\infty}$.

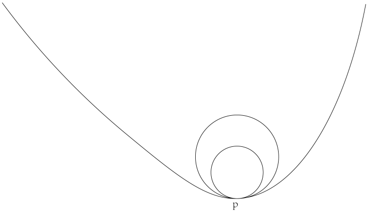
- P_Θ has a positive continuous density f with respect to the Lebesgue measure μ_{Leb}.
- this model is equivalent to the model 1.

Different boundary conditions:

- (PL) is the boundary of certain Lipschitz class with a countable many of discontinuity points of the first or the second derivative.
- (S2) is the boundary of class C^2 and has positive curvature everywhere.



(a) Discontinuities in the first derivatives



(b) Discontinuities in the second derivatives

Figure: Smoothness of boundary condition of (PL)

Rate of Convergence

When P_Θ is uniform:

Theorem 8

$$d_N\left(\bigcap_{i=1}^{n}\Lambda_{\Theta_i},\mathcal{K}_0\right)=o(1/n^{\beta}),\beta<2 \text{ a.s. under (PL), or equivalently}$$
$$d_N\left(\bigcap_{i=1}^{n}\Lambda_{\Theta_i},\mathcal{K}_0\right)=o_p((\log(n))^{\gamma}/n^{2}),\gamma\geq 3 \text{ under (PL).}$$

When f is positive continuous:

Theorem 9

$$\lim_{n\to\infty} n^2 d_N\left(\bigcap_{i=1}^{n}\Lambda_{\Theta_i},\mathcal{K}_0\right)=\frac{1}{4}\int_{(-\frac{\pi}{2},\frac{\pi}{2})}\frac{r^2_{\mathcal{K}_0}(\theta+\frac{\pi}{2})+r^2_{\mathcal{K}_0}(\theta+\frac{3\pi}{2})}{f^2(\theta)}d\theta \text{ a.s.}$$

under (S2).

- β,γ depends on the geometry and the set of discontinuities,
- $r_{\mathcal{K}_0}$ is the radius curvature of the boundary of $\mathcal{K}_0$.

Rate of Convergence

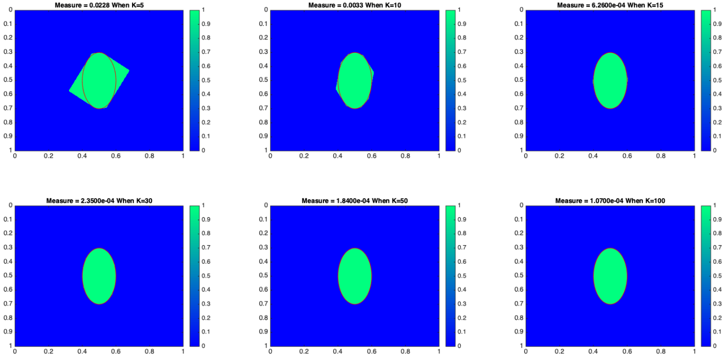


Figure: Steps of convergence.

Next we consider the rate of convergence in $\mathbb{R}^d$ according the general linear system in model 2. Note that

- $\boldsymbol{x} = \nabla_{\boldsymbol{a}} y = (x_1, x_2, \ldots, x_d)^T$ is a nonzero vector that coincides with the the normal vector of the hyperplane at y.
- $3a_1 + 4a_2 + 6a_3 = y(= 12)$ and $-3a_1 - 4a_2 - 6a_3 = y(= -12)$ are distinct hyperplanes indexed by distinct normal vectors since they have distinct output distributions.
- define $\boldsymbol{s} := \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \in \partial B(0, 1)$ is a point on the boundary of a unit ball. $|\cdot|$ is the Euclidean norm.

Then, consider a transformed model:

Model 2.1

$$y = Q_{\boldsymbol{s}}(\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{s},$$

Rate of Convergence

- the inverse support in model 2.1 is $\Lambda_{\boldsymbol{s}} := \{\lambda \in \Lambda : Q_{\boldsymbol{s}}(\lambda) \in Q_{\boldsymbol{s}}(\mathcal{K}_0)\}$ where Λ is sufficiently large,
- Λ is a compact convex set with non-empty interior and $0 \in int(\mathcal{K}_0)$.
- $\Lambda_{\boldsymbol{s}}$ is a generalized parallel slab, which is simply the intersection of two halfspaces characterized by the corresponding support functions.
- the support function $\partial B(0,1) \ni \boldsymbol{s} \to h(\mathcal{K}_0, \boldsymbol{s}) := \sup_{\boldsymbol{x} \in \mathcal{K}_0} < \boldsymbol{x}, \boldsymbol{s} >$, where $< \cdot, \cdot >$ is the standard inner product.
- the supporting halfspaces $H(\mathcal{K}_0, \boldsymbol{s}) := \{\boldsymbol{x} \in \mathbb{R}^d :< \boldsymbol{x}, \boldsymbol{s} >\leq h(\mathcal{K}_0, \boldsymbol{s})\}$.

Rate of Convergence

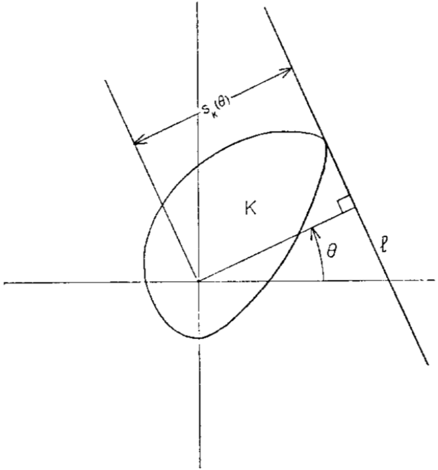


Figure: Support function and supporting halfspaces in $\mathbb{R}^2$.

Rate of Convergence

Theorem 10

For independent, uniformly distributed $\boldsymbol{U}_1, \boldsymbol{U}_2, \ldots, \in \partial B(0,1)$, let $S_n = \{\pm\boldsymbol{U}_1, \pm\boldsymbol{U}_2, \ldots, \pm\boldsymbol{U}_n\}$, where $-\boldsymbol{U}_i$ denotes the opposite component w.r.t. the origin. Then,

$$d_H(\mathcal{K}_0, \bigcap_{\boldsymbol{s}\in S_n} H(\mathcal{K}_0, \boldsymbol{s})) = \begin{cases} O((\log(n)/n)^{1/(d-1)}) \ a.s. \ in \ general, \\ O((\log(n)/n)^{2/(d-1)}) \ a.s. \ under \ (NF). \end{cases}$$

- Smoothness of boundary condition (NF) is having 'no flat parts'.
- $\boldsymbol{U}_i$ is uniformly distributed on $\partial B(0,1)$; that is, $P(\boldsymbol{U}_i \in A) = H_{d-1}(A)/H_{d-1}(\partial B(0,1))$ for any Borel set $A \subset \partial B(0,1)$. H_{d-1} denotes the $(d-1)-$dimensional Hausdorff measure on $\mathbb{R}^d$.
- $\Lambda_{\boldsymbol{U}_i} = H(\Lambda, \boldsymbol{U}_i) \cap H(\Lambda, -\boldsymbol{U}_i)$ is a parallel slab, which is exactly the inverse support of the SIP.

References

- Efim M Bronstein. Approximation of convex sets by polytopes.Journal of Mathematical Sciences,153(6):727–762, 2008.
- T Butler, D Estep, S Tavener, C Dawson, and JJ Westerink. A measure-theoretic computational methodfor inverse sensitivity problems iii: Multiple quantities of interest.SIAM/ASA Journal on UncertaintyQuantification, 2(1):174–202, 2014.
- Lutz D umbgen and G unther Walther. Rates of convergence for random approximations of convex sets.Advances in applied probability, 28(2):384–393, 1996.
- Rolf Schneider. Random approximation of convex sets.Journal of Microscopy, 151(3):211–227, 1988.


Thank You!