Stellar Opacity

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January 29, 2021

Chapter 1

Geometry

1.1 General

To my knowledge, Cartesian coordinates are not fundamental in any objective way, they were just the first we formalized and so we consider them easy and pretend to have a robust understanding of their properties.

For example, Cartesian coordinate representations of any vector is unique or non-degenerate. This is nice, but only true for some coordinate systems! This is also very important, we don't want to be able to represent systems in non-unique ways, any objective measure would be rendered useless. Equivalently, perhaps arithmetic based on degenerate coordinate systems might not have led us to such a good language to interpret the natural world. So how do we identify these non-degenerate coordiante systems?

Well, we say a coordinate system is non-degenerate (like Cartesian) if we can demonstrate that the other coordinate system is isomorphic to the Cartesian Coordinate System. By 'isomorphic', I mean that there is a mapping from one space to another that can be exactly reversed by an inverse mapping. In the language of coordinate systems, we use metric tensors to figure this out.

1.1.1 Metric Tensor

Super broadly, a metric tensor is a function that appears in differential geometry. The metric tensor gives definition to generalized mathematical spaces (called manifolds) by relating real-number scalars to distances and angles (in the case of positive-definite metric tensors).

One way to think of a metric tensor is as a rank-2 matrix, g. For a Cartesian coordinate system, $g_{ij} = \delta_{ij}$ where δ is the Kroenecker Delta function. Without proof (again, sorry), I claim that if you can characterize a coordinate system A with metric tensor g^A, you can show that it is isomorphic to the Cartesian Coordinate System by demonstrating that the metric tensor is diagonalizable. So, if a metric tensor is diagonalizable, there is an isomorphism between that coordinate system and the Cartesian Coordinate system[1]. If a coordinate system is isomorphic to Cartesian, then it must also be non-degenerate.

I'd like to drive home the importance of this in terms of mathematical representations of physical systems. Again, isomorphisms are great! They mean that we can move back and forth between representations without losing information.

For example, you are probably aware that the 2-D map of the surface of the (spherical) Earth *cannot* be perfectly mapped to a 2-D Cartesian representation. However! We can make arbitrarily accurate maps of small patches of the Earth. This is because spherical coordinates define a representation that is only locally diagonalizable in terms of its metric tensor.

1.1.2 Scale Factors

WHY IN THE 3-D CARTESIAN WORLD AM I BOTHERING WITH THIS?

Well, the various indeces of a metric are related to what are known as the scale factors. Scale factors (in the context of metric tensors) tell you how space transforms with respect to the coordinate parameters. This idea is super important in general relativity when we introduce the additional coordinate paramter t and see that it depends on (e.g.) the rate of change of the other coordinate parameters.

Happily, I am not going to forray into GR, but I will show you where else scale factors play an important roles that has almost certainly between glossed-over in your introductory physics and vector calculus courses.

For a diagonal (or even locally diagonal) metric tensor, $g_{ij} = g_{ii}\delta_{ij}$ with the coordinate parameterization

[1] If the metric is only locally diagonalizable, the manifold is only locally isomorphic to the Cartesian Metric.

$$
\begin{aligned}
x_1 &= f_1(q_1, q_2, \cdots, q_n) \\
x_2 &= f_2(q_1, q_2, \cdots, q_n) \\
&\cdots \\
x_n &= f_n(q_1, q_2, \cdots, q_n)
\end{aligned}
$$

The scale factor(h_i) is simply the square root of the diagonal metric element!

That is,

$$h_i = \sqrt{g_{ii}} = \sqrt{\sum_{k=1}^{n} \left(\frac{\partial x_k}{\partial q_i}\right)^2} \tag{1.1}$$

Here's the fun part (for me ...)

These scale factors allow us to transform any coordinate system differential elements. Of particular use for this study is the line element, the area element, and the volume element.

Taking each in turn ...

Line Element The line element is given in terms of scale factors and the above coordinate parameterization as

$$d\vec{l} = h_1\, dq_1 \hat{q_1} + h_2\, dq_2 \hat{q_2} + \cdots + h_n\, dq_n \hat{q_n} \tag{1.2}$$

Area Element The area element is given in terms of scale factors and the above coordinate parameterization as

$$d^2\vec{s}_{ab} = h_a h_b\, dq_a\, dq_b \cdot (\hat{q_a} \times \hat{q_b}) \tag{1.3}$$

Where the cross product appears in order to orient the area element with the generalized cross-product.

Volume Element The volume element is given in terms of scale factors and the above coordinate parameterization as

$$d^3V = h_1 h_2 h_3\, dq_1\, dq_2\, dq_3 \tag{1.4}$$

1.2 Cartesian

Let's quickly make use of these to define the differential elements of our favorite simple coordinate system.

The Cartesian Metric Tensor is trivially diagonal. Therefore, $g_{ij} = g_{ii}\delta_{ij}$ with the coordinate parameterization

$$
\begin{aligned}
x_1 &= x \\
x_2 &= y \\
x_3 &= z
\end{aligned}
$$

Recall equation 1.1

$$h_i = \sqrt{\sum_{k=1}^{n} \left(\frac{\partial x_k}{\partial q_i}\right)^2}$$

$$
\begin{aligned}
h_x &= \sqrt{\sum_{k=1}^{3} \left(\frac{\partial x_k}{\partial q_i}\right)^2} \\
h_x &= \sqrt{\left(\frac{\partial x}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2} \\
h_x &= \sqrt{\left(1\right)^2 + \left(0\right)^2 + \left(0\right)^2} \\
h_x &= 1
\end{aligned}
$$

The rest are easily seen to be 1 as well!

Cartesian Scale Factors $h_x = 1; h_y = 1; h_z = 1$

Now, recalling equations 1.2, 1.3, 1.4 ...

$$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z} \tag{1.5}$$

$$\begin{aligned} d^2\vec{s}_{xy} &= dx\,dy \cdot (\hat{x} \times \hat{y}) \\ d^2\vec{s}_{xy} &= dx\,dy \cdot \hat{z} \end{aligned} \tag{1.6}$$

Note: Other combinations of the coordinates would generate other area elements! This will be sometimes noted as an oriented area element or a vector area element to distinguish it from uses of its magnitude.

$$d^3V = dx\,dy\,dz \tag{1.7}$$

Easy!

1.3 Spherical

The Spherical Metric Tensor is only *locally* diagonal. In spite of this, we can still make use of $g_{ij} = g_{ii}\delta_{ij}$ *locally* with regard to differential paramters. And define the parameterization of the Cartesian Coordinates in terms of the Spherical parameters: the radial distance, the polar angular distance, and the azimuthal angular distance (r, θ, ϕ).

Figures 1.1 and 1.2 give a schematic for how Cartesian and Spherical Coordinate Systems are related.

The Cartesian Coordinate x can be seen to be the radial distance from the origin scaled and projected onto the $x - y$ plane. That is: $x = r\sin\theta\cos\phi$.

Similarly, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$.

In this parameterization below, note that $(x_1, x_2, x_3) \to (x, y, z)$.

$$\begin{aligned} x_1 &= r\sin\theta\cos\phi \\ x_2 &= r\sin\theta\sin\phi \\ x_3 &= r\cos\theta \end{aligned}$$

Recall equation 1.1

$$h_i = \sqrt{\sum_{k=1}^{n}\Big(\frac{\partial x_k}{\partial q_i}\Big)^2}$$

With the

$$\begin{aligned}
h_r &= \sqrt{\sum_{k=1}^{3}\Big(\frac{\partial x_k}{\partial r}\Big)^2} \\
h_r &= \sqrt{\Big(\frac{\partial x_1}{\partial r}\Big)^2 + \Big(\frac{\partial x_2}{\partial r}\Big)^2 + \Big(\frac{\partial x_3}{\partial r}\Big)^2} \\
h_r &= \sqrt{\Big(\frac{\partial}{\partial r} r\sin\theta\cos\phi\Big)^2 + \Big(\frac{\partial}{\partial r} r\sin\theta\sin\phi\Big)^2 + \Big(\frac{\partial}{\partial r} r\cos\theta\Big)^2} \\
h_r &= \sqrt{\sin^2\theta\cos^2\phi\Big(\frac{\partial}{\partial r} r\Big)^2 + \sin^2\theta\sin^2\phi\Big(\frac{\partial}{\partial r} r\Big)^2 + \cos^2\theta\Big(\frac{\partial}{\partial r} r\Big)^2} \\
h_r &= \sqrt{\sin^2\theta\cos^2\phi\Big(1\Big) + \sin^2\theta\sin^2\phi\Big(1\Big) + \cos^2\theta\Big(1\Big)} \\
h_r &= \sqrt{\sin^2\theta(\underbrace{\cos^2\phi + \sin^2\phi}_{1;PythagoreanIdentity}) + \cos^2\theta} \\
h_r &= \sqrt{\sin^2\theta + \cos^2\theta} = \sqrt{1^2} = 1
\end{aligned}$$

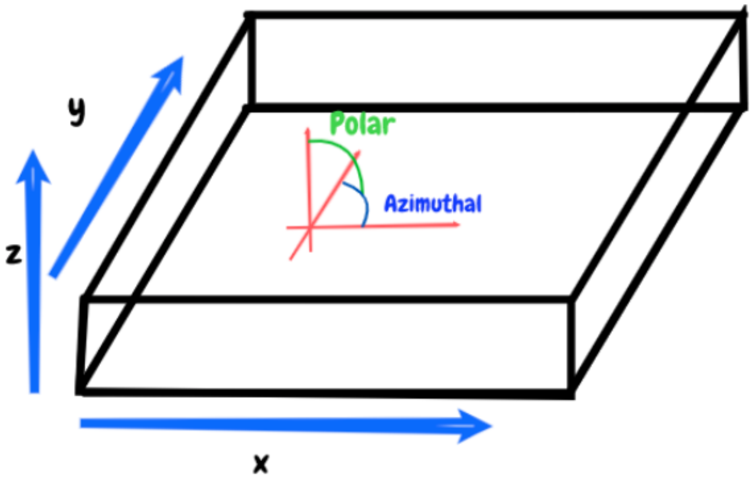

Figure 1.1: Along with

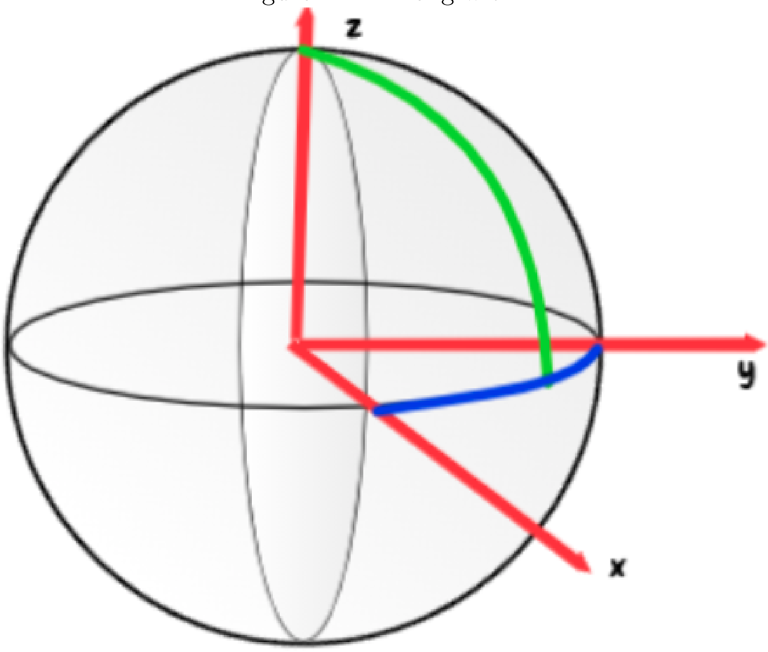

Figure 1.2: Along with

$$h_\theta = \sqrt{\sum_{k=1}^{3}\Big(\frac{\partial x_k}{\partial r}\Big)^2}$$

$$h_\theta = \sqrt{\Big(\frac{\partial x_1}{\partial \theta}\Big)^2 + \Big(\frac{\partial x_2}{\partial \theta}\Big)^2 + \Big(\frac{\partial x_3}{\partial \theta}\Big)^2}$$

$$h_\theta = \sqrt{\Big(\frac{\partial}{\partial \theta} r\sin\theta\cos\phi\Big)^2 + \Big(\frac{\partial}{\partial \theta} r\sin\theta\sin\phi\Big)^2 + \Big(\frac{\partial}{\partial \theta} r\cos\theta\Big)^2}$$

$$h_\theta = \sqrt{r^2\cos^2\phi\Big(\frac{\partial}{\partial \theta}\sin\theta\Big)^2 + r^2\sin^2\phi\Big(\frac{\partial}{\partial \theta}\sin\theta\Big)^2 + r^2\Big(\frac{\partial}{\partial \theta}\cos\theta\Big)^2}$$

$$h_\theta = r\sqrt{\Big(\cos^2\phi\Big(\cos\theta\Big)^2 + \sin^2\phi\Big(\cos\theta\Big)^2 + \Big(-\sin\theta\Big)^2}$$

$$h_\theta = r\sqrt{\cos^2\phi\cos^2\theta + \sin^2\phi\cos^2\theta + \sin^2\theta}$$

$$h_\theta = r\sqrt{\cos^2\theta(\underbrace{\cos^2\phi + \sin^2\phi}_{1;PythagoreanIdentity}) + \sin^2\theta}$$

$$h_\theta = r\sqrt{\cos^2\theta + \sin^2\theta} = r\sqrt{1^2}$$

Do the ϕ one! You should get $r\sin\theta$.

Spherical Scale Factors $h_r = 1; h_\theta = r; h_\phi = r\sin\theta$

Now, recalling equations 1.2, 1.3, 1.4 ...

$$\vec{dl} = h_r\, dr\hat{r} + h_\theta\, d\theta\hat{\theta} + h_\phi\, d\phi\hat{\phi}$$

$$\vec{dl} = dr\hat{r} + r\, d\theta\hat{\theta} + r\sin\theta\, d\phi\hat{\phi} \tag{1.8}$$

$$d^2\vec{s}_{\theta\phi} = h_\theta h_\phi\, d\theta\, d\phi \cdot (\hat{\theta} \times \hat{\phi})$$
$$d^2\vec{s}_{\theta\phi} = r \cdot r\sin\theta\, d\theta\, d\phi \cdot \hat{r}$$
$$d^2\vec{s}_{\theta\phi} = r^2\sin\theta\, d\theta\, d\phi\hat{r} \tag{1.9}$$

Note: Other combinations of the coordinates would generate other area elements! Also note that $d^2\vec{s}_{\theta\phi}$ is oriented away from the surface of the sphere, that is: in the radial direction. This will be sometimes noted as an oriented area element or a vector area element to distinguish it from uses of its magnitude

$$d^3V = h_r h_\theta h_\phi\, dr\, d\theta\, d\phi$$
$$d^3V = r^2\sin\theta\, dr\, d\theta\, d\phi \tag{1.10}$$

These definitions will be extemely useful moving forward!

Solid Angle

To start the discussion with solid angles, let us actually begin with a surface integral over the entire area of a sphere. We will use the definition of the spherical area element given in 1.9.

Recall that $d^2\vec{s}_{\theta\phi}$ is a vector quantity. We will just take the magnitude of this oriented area element.

$$\int_{sphere} |\, d^2\vec{s}_{\theta\phi}|$$

In order to cover the entire sphere, θ must vary from 0 (at the top of the sphere in 1.2) to π andϕ must vary from 0 to 2π along the great circle containing the $x-y$ plane.

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} r^2\sin\theta\, d\theta\, d\phi$$

What we're left with is actually fairly easy to solve! First, we can pull r^2 out of the integral, since it does not vary with θ or ϕ. Additionally, we can separate the two integrals fairly cleanly!

$$r^2\int_{\phi=0}^{2\pi} d\phi \cdot \int_{\theta=0}^{\pi} \sin\theta\, d\theta$$

The ϕ integral readily goes to 2π.

$$r^2 \cdot 2\pi \cdot \int_{\theta=0}^{\pi} \sin\theta\, d\theta$$

The θ integral is only *slightly* more complicated, but happily yields 2.

$$r^2 \cdot 2\pi \cdot 2$$

I will repeat that θ integral with a substitution that will be later employed. $\mu = \cos\theta$ and $d\mu = -\sin\theta\, d\theta$

$$r^2 \cdot 2\pi \cdot \int_{\mu(0)=1}^{\mu(\pi)=-1} -\, d\mu$$

Flip the bounds to remove the $-$ sign ...

$$r^2 \cdot 2\pi \cdot \int_{\mu=-1}^{1} d\mu$$

And it's even easier to see that we end up with 2.

Taking the product, we end up with the fact that the surface area of a sphere is $4\pi r^2$. Agreed? GOOD.

Now, that r^2 actually gets us into a lot of trouble. In some weird way, it specifies an otherwise generalizable sphere.

If we strip the r^2 from the $d^2\vec{s}_{\theta\phi}$, we're actually left with an intriguing quantity that depends only on two angular measures, θ and ϕ.

$$\frac{|\, d^2\vec{s}_{\theta\phi}|}{r^2} = \sin\theta \, d\theta \, d\phi = \, d^2\Omega \tag{1.11}$$

It is important to note that the *differential* solid angle element implies a direction, namely $\hat{n}$. This can be made explicit with the inclusion of $\hat{n}$ with the notation: $d^2\Omega\hat{n}$

By comparison to the surface integrals performed above, the integral of the differential solid angle over the entire sphere ($\theta \in [0, \pi], \phi \in [0, 2\pi]$) yields 4π.

$$\int_{sphere} d^2\Omega = 4\pi \tag{1.12}$$

In analogy with the radian for measure of a 1-D angle, the quantity we discuss here is built from a 2-D angle. It therefore is owed a new measure: $rad^2 = sr$–the steradian.

Just as one might claim a circle has 2π radians, a sphere has 4π steradians.

We'll see that defining quantities in terms of the solid angle and the differential element thereof allows for the calculation of important quantities that arise in astrophysics and astrophysical observation.

Chapter 2

Radiation Field

2.1 Specific Intensity

The Specific Intensity is a differential, scalar quantity [1]:

$$\frac{dI}{d\lambda} = I_\lambda(\vec{r}, \hat{n}, \lambda, t) = \frac{\delta E}{d\lambda\, dt\, d^2S \cos\theta\, d^2\Omega}$$

Where δE is the amount of energy transported by radiation belonging to the wavelength range $[\lambda, \lambda + d\lambda]$. Equivalently ...

$$\delta E \approx E_\lambda\, d\lambda = dE$$

In the arguments of I_λ reside the location vector $\vec{r}$ locating the oriented area element d^2S, $\hat{n}$ representing the direction or orientation of the differential solid angle element, λ the spectral location, and time t.

The specific intensity therefore represents the amount of spectral, radiative energy that passes perpendicularly across an oriented area element ($d^2\vec{S}$) in unit time (dt) confined to a differential solid angle element ($d^2\Omega$).

The units of the specific intensity are $\frac{W}{m^2 \mathring{A} sr}$.

Confining the energy to the differential solid angle implicates a propogation direction for this specific intensity, namely $\hat{n}$ as discussed in section 1.3.

The $\cos\theta$ term in the denominator owes to the projection of the differential area element onto the direction of energy propogation. That is, the general oriented area element can be written as $d^2\vec{S} = d^2S \cdot \hat{s}$ with $\hat{s}$ representing the unit normal to the area element. We discussed this notation in spherical coordinates (i.e., 1.9). For reasons we will discuss below, it is generally convienient to align the direction of propogation to the polar axis (or z-axis). Therefore, $d^2\vec{S}\, d^2\Omega$ is equivalent to $d^2S\, d^2\Omega \hat{s} \cdot \hat{n}$ and finally $d^2S\, d^2\Omega \cos\theta$. With θ defining the angle between the propogation direction ($\hat{n}$) and the orientation of the oriented area element ($\hat{s}$).

2.1.1 Invariance

Importantly, the use of the solid angle in the development of the specific intensity enables the quantity to be independent of the distance between the source and the observer[2].

As an illustrative example without illustration, take $\delta E = I_\lambda(\vec{r}, \hat{n}, \lambda, t)\, d\lambda\, dt\, d^2S \cos\theta\, d^2\Omega$ as the amount of spectral, radiative energy that passes perpendicularly across an oriented area element $d^2\vec{S}$ in unit time confined to a solid angle element $d^2\Omega$.

Some distance r away from $d^2\vec{S}$ is another oriented area element $d^2\vec{S}'$. As seen from $d^2\vec{S}'$, $d^2\vec{S}$ subtends a solid angle element $d^2\Omega$ and from $d^2\vec{S}$, $d^2\vec{S}'$ subtends a solid angle element $d^2\Omega'$.

That is, $d^2\Omega = \frac{d^2\vec{S}'}{r^2}$ and $d^2\Omega' = \frac{d^2\vec{S}}{r^2}$, both by our definition of solid angle 1.11.

So, by conservation of energy, $\delta E = I_\lambda(\vec{r}, \hat{n}, \lambda, t)\, d\lambda\, dt\, d^2S \cos\theta\, d^2\Omega = I'_\lambda(\vec{r}, \hat{n}, \lambda, t)\, d\lambda\, dt\, d^2S' \cos\theta'\, d^2\Omega'$.

And $I_\lambda = I'_\lambda$.

Therefore, the specific intesity is spatially invariant.

2.2 Flux-Vector

Flux is a vector-valued quantity representing the amount of spectral, radiative energy that passes perpendicularly across an oriented area element in unit time.

[1] In radiometry, this quantity is called the spectral radiance.

[2] Provided there are no sources or sinks of radiant energy along $\hat{n}$

The flux is measured in units of $\frac{W}{m^2 \mathring{A}}$.

$$\vec{\mathcal{F}}_\lambda(\vec{r},\lambda,t) = \int_\Omega I_\lambda(\vec{r},\lambda,t,\hat{n})\, d^2\Omega\hat{n} \tag{2.1}$$

In the above, it is the implied propogation direction $\hat{n}$ associated with $d^2\Omega$ that yields a vector-valued quantity. With the $\vec{\mathcal{F}}_\lambda(\vec{r},\lambda,t)$ representable as a Cartesian vector:

$$\vec{\mathcal{F}}_\lambda = \mathcal{F}_{\lambda,x}\hat{x} + \mathcal{F}_{\lambda,y}\hat{y} + \mathcal{F}_{\lambda,z}\hat{z}$$

Dropping the spectral (λ) subscript for ease only for the moment...

$$\vec{\mathcal{F}} = \mathcal{F}_x\hat{x} + \mathcal{F}_y\hat{y} + \mathcal{F}_z\hat{z}$$

$$\vec{\mathcal{F}} = \int_\Omega I_\lambda\, d^2\Omega\hat{n}\cdot\hat{x} + \int_\Omega I_\lambda\, d^2\Omega\hat{n}\cdot\hat{y} + \int_\Omega I_\lambda\, d^2\Omega\hat{n}\cdot\hat{z}$$

At this point it is useful to make use of the substitution $\mu = \cos\theta$ and $d\mu = -\sin\theta\, d\theta$. Therefore,

$$\begin{aligned}
\hat{n}\cdot\hat{x} &= \sqrt{(1-\mu^2)}\cos\phi \\
\hat{n}\cdot\hat{y} &= \sqrt{(1-\mu^2)}\sin\phi \\
\hat{n}\cdot\hat{z} &= \mu
\end{aligned}$$

Substitute these relations into the previous flux vector integrals to write ...

$$\vec{\mathcal{F}} = \int_\Omega I_\lambda \cos\phi\sqrt{(1-\mu^2)}\, d^2\Omega + \int_\Omega I_\lambda \sin\phi\sqrt{(1-\mu^2)}\, d^2\Omega + \int_\Omega I_\lambda\mu\, d^2\Omega$$

Evaluating each component over the entire 4π steradians of a sphere ...

$$\begin{aligned}
\vec{\mathcal{F}}\cdot\hat{x} &= \int_{\phi=0}^{2\pi}\int_{\mu=-1}^{1} I_\lambda(\vec{r},\lambda,t,\mu,\phi)\cos\phi\sqrt{(1-\mu^2)}\, d\mu\, d\phi \\
\vec{\mathcal{F}}\cdot\hat{y} &= \int_{\phi=0}^{2\pi}\int_{\mu=-1}^{1} I_\lambda(\vec{r},\lambda,t,\mu,\phi)\sin\phi\sqrt{(1-\mu^2)}\, d\mu\, d\phi \\
\vec{\mathcal{F}}\cdot\hat{z} &= \int_{\phi=0}^{2\pi}\int_{\mu=-1}^{1} I_\lambda(\vec{r},\lambda,t,\mu,\phi)\mu\, d\mu\, d\phi
\end{aligned}$$

This is as far as we can go before giving more definite form to the specific intensity of a particular system.

2.3 Radiative Flux

In the context of small portions of [stellar] atmospheres, we can make a single simpliying assumption. Consider a plane-parallel atmosphere perhaps not unlike the volume element shown schematically in 1.1. If the planar atmosphere is homogenous in the $x-y$ plane, only $\vec{\mathcal{F}}\cdot\hat{z}$ can be non-zero. Since the scalar radiation field ($I_\lambda(\vec{r},\lambda,t,\mu,\phi)$) would be symmetric with respect to the z axis, there will be ray-by-ray cancellation in the net flux across a surface perpendicular to that axis of symmetry (the $x-y$ plane). It is here that we decide to align the propogation direction with this axis of symmetry and slightly modify the specific intensity.

$$I_\lambda(\vec{r},\lambda,t,\mu,\phi) \rightarrow I_\lambda(\vec{r},\lambda,t,\mu)$$

In terms of the integrals above, we take advantage of this simplification as follows:

$$\vec{\mathcal{F}} \cdot \hat{x} = \underbrace{\int_{\phi=0}^{2\pi} \cos\phi\, d\phi}_{0} \int_{\mu=-1}^{1} I_\lambda(x, \lambda, t, \mu)\sqrt{(1-\mu^2)}\, d\mu$$

$$\vec{\mathcal{F}} \cdot \hat{y} = \underbrace{\int_{\phi=0}^{2\pi} \sin\phi\, d\phi}_{0} \int_{\mu=-1}^{1} I_\lambda(y, \lambda, t, \mu)\sqrt{(1-\mu^2)}\, d\mu$$

$$\vec{\mathcal{F}} \cdot \hat{z} = \underbrace{\int_{\phi=0}^{2\pi} d\phi}_{2\pi} \int_{\mu=-1}^{1} I_\lambda(z, \lambda, t, \mu)\mu\, d\mu$$

So only the $\vec{\mathcal{F}} \cdot \hat{z}$ term may be non-zero.

We will therefore define this scalar quantity as the *total* [spectral] radiative flux in the limit of plane-parallel atmospheres with symmetry about the radial axis[3].

$$F_{\lambda,tot} = 2\pi \int_{\mu=-1}^{1} I_\lambda(r, \lambda, t, \mu)\mu\, d\mu$$

We can also define the portion of the flux that passes through the surface of a [stellar] atmosphere in the positive radial ($+r$) direction as the surface flux. Check where that factor of 2 goes!

$$F_{\lambda,surf} = \pi \int_{\mu=-1}^{1} I_\lambda(r, \lambda, t, \mu)\mu\, d\mu \tag{2.2}$$

2.4 Observational Significance

Imagine a system where an observer measures the energy received by a star a distance, D away. Orient the polar/radial/z axis to be along the line-of-sight. Also assume that the radius of the star, r, is much smaller than D ($r << D$). In this way, all of the observed rays of light from the star can be taken to be parallel.

Any ray of specific intensity originating from the a surface element (d^2S) of this star is projected to be parallel to the line-of-sight by a factor, μ. Where μ is the cosine of the angle between the line-of-sight and the local normal where the specific intensity originated.

In this way, we completely ignore specific intensities originating perpendicularly from the top and bottom of the star (schematically given in 2.1) and from the far-side (left-side in figure) of the star. Also, the specific intensity originating from the center of the disk of the star as seen by the observer (the point on the far right-side of the figure) is not affected by the factor μ whereas those originating from an angle θ off of this line-of-sight are reduced by μ.

The d^2S from where the specific intensity originates subtends a differential solid angle element $d^2\Omega = \frac{d^2S}{D^2}$ as seen by the distant observer.

Therefore, differential spectral radiative flux observed is

$$dF_{\lambda,obs} = I_\lambda\, d^2\Omega$$

On the surface of the star, the differential area element is $d^2S = 2\pi r'\, dr'$. The radius that locates the annulus, r', is related to the radius of the star by $r' = r\sin\theta$, so then $dr' = r\cos\theta\, d\theta$.

So then, the solid angle subtended by this surface area element is ...

$$\begin{aligned} d^2\Omega &= \frac{2\pi r'\, dr'}{D^2} \\ &= \frac{2\pi r^2 \sin\theta\cos\theta\, d\theta}{D^2} \\ &= \frac{2\pi r^2 \mu\, d\mu}{D^2} \end{aligned}$$

$$d^2\Omega = 2\pi \frac{r^2}{D^2}\mu\, d\mu$$

So the actual spectral flux observed would be equal to ...

[3]Check your geometry and make sure you agree that I can call the radial axis the z axis!

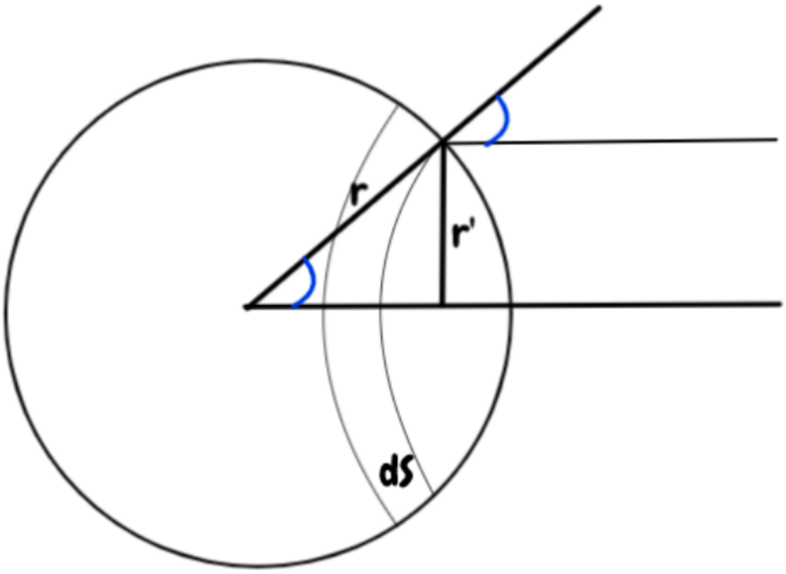

Figure 2.1: Consider the annulus of differential area $d^2S_{\theta\phi}$

$$F_{\lambda,obs} = 2\pi \frac{r^2}{D^2} \int_0^1 I_\lambda(r,\lambda,t,\mu)\mu \, d\mu$$

In the limit of small θ (equivalent to $r << D$) $\frac{r}{R} \approx \theta$ and $\frac{r}{R} \approx \frac{\alpha}{2}$. Here, α is the angular stellar diameter. So the observed spectral flux can be re-written as ...

$$F_{\lambda,obs} = \frac{r^2}{D^2} F_{\lambda,surf} = \frac{\alpha^2}{2^2} F_{\lambda,surf}$$

This enables observers to relate a known distance and radius to and of a distant star to the surface spectral flux of that star. Equivalently, if the star's angular diamter is known.

Above, the limits on μ go from $0 \rightarrow 1$ as opposed to the definition in 2.2. This is accounted for in an assumption that we get 0 specific intensity from the far-side of the star–the side where $\mu \in [-1, 0)$.

$$F_{\lambda,obs}(D,\lambda,t) = \frac{\alpha^2}{4} F_{\lambda,surf}(r,\lambda,t) \tag{2.3}$$

Chapter 3

Interaction of Radiation and Matter

3.1 Absorption & Emission

We use the radiation field as a description of the movement of energy through space via electromagnetic radiation. The movement of energy between particles or molecules or bulk matter is handled by the *thermodynamics* of the material.

These two modes of energy transport can interact with each other. Through interactions with matter, energy may be removed from or delivered into the radiation field. Absorption interactions remove energy from the radiation field and deliver it to the thermal properties of the material–the *thermal pool*. Emission interactions deliver energy to the radiation field derived from this thermal pool.

Up until now, the radiation field has been agnostic of the particle or wave picture of light. We'll now adopt the notion of the photon as the quantization of radiative energy and therefore of the field. We know that Planck's Constant (h) is given definition by the formula $E_\gamma = \frac{hc}{\lambda}$ where the spectral location or wavelength λ is related to the speed of light c by $c = \lambda\nu$, where ν is another marker of spectral location, called the frequency. The energy a photon carries with it may leave the radiation field and therefore diminish the specific intensity of the radiation field through interactions with matter (absorption). Photons may also be generated by the interaction of matter with the radiation field (emission).

The radiative energy carried by a photon may also be redirected in scattering processes.

An absorption process is said to thermalize the involved photon–satisfying the interaction between the radiation field and thermal pool. Absorption processes feed energy directly into the thermal kinetic energy of the matter and must therefore couple strongly with thermodynamic properties.

An emission process generates a photon from the energy of the thermal pool–also satisfying this interaction between the radiation field and thermal pool.

Absorption and emission processes tend to introduce local equilibrium with respect to energy exchange between the thermal pool (matter) and the radiation field.

In contrast, scattering processes allow for photons to move through matter without coupling to local thermodynamic properties. Scattering processes therefore tend to delocalize the balance of the matter-radiation equilibrium process and introduce the presence of global properties to any bulk material (i.e., boundaries).

Detailed understanding of absorption and emission processes require a quantum mechanical interpretation of specific energy landscapes and transitions. This will be the microscopic view of these processes.

For now, we will focus on the macroscopic analogs of these processes.

3.2 Extinction Coefficient

The macroscopic coefficient that describes the removal of energy from the radiation field by matter is named the 'extinction coefficient' ($\chi(r, \lambda, t, \mu)$) since radiative energy is 'extincted' from the the ray of light.

It is defined such that an element of material of cross-section d^2S and length ds removes from a ray with specific intensity $I_\lambda(r, \lambda, t, \mu)$, incident normal to d^2S and propogating into solid angle element $d^2\Omega$, an amount of energy

$$\delta E = \chi(r, \lambda, t, \mu) I_\lambda(r, \lambda, t, \mu)\, d^2S\, ds\, d\lambda\, dt\, d^2\Omega \tag{3.1}$$

in spectral region λ of width $d\lambda$ in a time dt. The extinction coefficient is the product of an atomic absorption cross-section (m^2) and the number density of absorbers (m^{-3}) summed over all of the states that can interact with photons of wavelength λ. The dimensions of χ are therefore m^{-1} with the inverse of χ giving the distance a photon can propogate before it is removed from the ray. This χ^{-1} is also called the *mean free path* of the photon.

The extinction properties of a material are generally isotropic. Interaction with moving materials introduces angular dependences due to Doppler Shifts in various frames. Said another way, the absorption or scattering of radiation by matter is generally agnostic of the direction of incidence for the specific intensity.

We say 'extinction' instead of 'absorption' to include scattering processes. We assume outright that absorption and scattering processes occur independently and add linearly[1].

$$\chi(r,\lambda,t) = \underbrace{\kappa(r,\lambda,t)}_{Absorption} + \underbrace{\sigma(r,\lambda,t)}_{Scattering}$$

3.3 Emission Coefficient

The macroscopic emission coefficient is defined such that an element of matter of cross-section d^2S and length ds contributes an amount of energy δE into solid angle $d^2\Omega$ within wavelength band $d\lambda$ in direction $\hat{n}$ in time dt.

$$\delta E = \eta(r,\hat{n},\lambda,t)\, d^2S\, ds\, d^2\Omega\, d\lambda\, dt$$

3.4 LTE

Great! We've now written down a formulation for the extinction and emission coefficients of specific intensity by matter! This is a macroscopic description of the rate at which matter removes or adds radiant energy from or to a beam of radiation (the specific intensity). Uhhhh ... KINDA.

Let me explain. These coefficients described above depend on the microscopic states that the material finds itself in. For example, material sufficiently excited to be partially ionized will extinct and emit light very differently than that same material in a state of full ionization or even no ionization. I don't mean to be focusing on ionization, either. The material's elections could enjoy a relative abundance of excited electrons. This changes the microscopic energy landscape of the material and affects the radiative processes to varying degrees.

So these coefficients that describe the interaction between matter and the radiation field actually very strongly couple to the radiation field itself! Materials enjoying relative abundance of excited electrons may have seen these electrons elevated by past interactions with the radiation field!

More precisely, the rate of energy removal from the radiation field or emission into the radiation field is determined by the radiation field via photoexcitiations, photoionizations, radiative emission, radiative recombination, and related processes. The reality of the situation is therefore that the interaction between the radiation field and matter is nonlinear. Ugh.

What to do ... WELL, let's happily ignore it! Let's pretend the interactions between the radiation field and matter is strictly linear.

How can we get away with this? We've just described the processes that govern the exchange of energy between the radiation field and the thermal pool also determine the microscopic thermal energy states of the material. These microscopic thermal energy states of the material in turn govern the rate of energy exchange! If we assume that the ambient temperature and density are stable in a region of material, we are at the same time saying that this stability is driven by an equilibrium with respect to this energy exchange. So if we assume *Local Thermodynamic Equilibrium* (LTE), then we can treat the extinction coefficient as determined for the region for which LTE applies.

Now the question becomes, what is LTE? And how appropriate is this assumption? Let's address that later and for now play pretend!

[1] How good of an assumption is this? Does it depend on thermodynamic properties of the material? E.g., temperature and density?