

Stellar Opacity

R. Prouty

January 14, 2021

1	Geometry	2
1.0.1	Metric Tensor	2
1.0.2	Cartesian	3
1.0.3	Spherical	4
2	Radiation Field	8
2.1	Specific Intensity	8
2.1.1	Invariance	8
2.2	Flux-Vector	8
2.3	Radiative Flux	9
2.4	Observational Significance	9
3		10

Chapter 1

Geometry

To my knowledge, Cartesian coordinates are not fundamental in any objective way, they were just the first we formalized and so we consider them easy and pretend to have a robust understanding of their properties.

For example, Cartesian coordinate representations of any vector is unique or non-degenerate. This is nice, but only true for some coordinate systems! This is also very important, we don't want to be able to represent systems in non-unique ways, any objective measure would be rendered useless. Equivalently, perhaps arithmetic based on degenerate coordinate systems might not have led us to such a good language to interpret the natural world. So how do we identify these non-degenerate coordinate systems?

Well, we say a coordinate system is non-degenerate (like Cartesian) if we can demonstrate that the other coordinate system is isomorphic to the Cartesian Coordinate System. By 'isomorphic', I mean that there is a mapping from one space to another that can be exactly reversed by an inverse mapping. In the language of coordinate systems, we use metric tensors to figure this out.

1.0.1 Metric Tensor

Super broadly, a metric tensor is a function that appears in differential geometry. The metric tensor gives definition to generalized mathematical spaces (called manifolds) by relating real-number scalars to distances and angles (in the case of positive-definite metric tensors).

One way to think of a metric tensor is as a rank-2 matrix, g . For a Cartesian coordinate system, $g_{ij} = \delta_{ij}$ where δ is the Kronecker Delta function. Without proof (again, sorry), I claim that if you can characterize a coordinate system A with metric tensor g^A , you can show that it is isomorphic to the Cartesian Coordinate System by demonstrating that the metric tensor is diagonalizable. So, if a metric tensor is diagonalizable, there is an isomorphism between that coordinate system and the Cartesian Coordinate system¹. If a coordinate system is isomorphic to Cartesian, then it must also be non-degenerate.

I'd like to drive home the importance of this in terms of mathematical representations of physical systems. Again, isomorphisms are great! They mean that we can move back and forth between representations without losing information.

For example, you are probably aware that the 2-D map of the surface of the (spherical) Earth *cannot* be perfectly mapped to a 2-D Cartesian representation. However! We can make arbitrarily accurate maps of small patches of the Earth. This is because spherical coordinates define a representation that is only locally diagonalizable in terms of its metric tensor.

Scale Factors

WHY IN THE 3-D CARTESIAN WORLD AM I BOTHERING WITH THIS?

Well, the various indices of a metric are related to what are known as the scale factors. Scale factors (in the context of metric tensors) tell you how space transforms with respect to the coordinate parameters. This idea is super important in general relativity when we introduce the additional coordinate parameter t and see that it depends on (e.g.) the rate of change of the other coordinate parameters.

Happily, I am not going to foray into GR, but I will show you where else scale factors play an important role that has almost certainly been glossed-over in your introductory physics and vector calculus courses.

For a diagonal (or even locally diagonal) metric tensor, $g_{ij} = g_{ii}\delta_{ij}$ with the coordinate parameterization

¹If the metric is only locally diagonalizable, the manifold is only locally isomorphic to the Cartesian Metric.

$$\begin{aligned}
x_1 &= f_1(q_1, q_2, \dots, q_n) \\
x_2 &= f_2(q_1, q_2, \dots, q_n) \\
&\dots \\
x_n &= f_n(q_1, q_2, \dots, q_n)
\end{aligned}$$

The scale factor(h_i) is simply the square root of the diagonal metric element!
That is,

$$h_i = \sqrt{g_{ii}} = \sqrt{\sum_{k=1}^n \left(\frac{\partial x_k}{\partial q_i} \right)^2} \quad (1.1)$$

Here's the fun part (for me ...)

These scale factors allow us to transform any coordinate system differential elements. Of particular use for this study is the line element, the area element, and the volume element.

Taking each in turn ...

Line Element The line element is given in terms of scale factors and the above coordinate parameterization as

$$d\vec{l} = h_1 dq_1 \hat{q}_1 + h_2 dq_2 \hat{q}_2 + \dots + h_n dq_n \hat{q}_n \quad (1.2)$$

Area Element The area element is given in terms of scale factors and the above coordinate parameterization as

$$d^2 \vec{s}_{ab} = h_a h_b dq_a dq_b \cdot (\hat{q}_a \times \hat{q}_b) \quad (1.3)$$

Where the cross product appears in order to orient the area element with the generalized cross-product.

Volume Element The volume element is given in terms of scale factors and the above coordinate parameterization as

$$d^3 V = h_1 h_2 h_3 dq_1 dq_2 dq_3 \quad (1.4)$$

1.0.2 Cartesian

Let's quickly make use of these to define the differential elements of our favorite simple coordinate system.

The Cartesian Metric Tensor is trivially diagonal. Therefore, $g_{ij} = g_{ii} \delta_{ij}$ with the coordinate parameterization

$$\begin{aligned}
x_1 &= x \\
x_2 &= y \\
x_3 &= z
\end{aligned}$$

Recall equation 1.1

$$\begin{aligned}
h_i &= \sqrt{\sum_{k=1}^n \left(\frac{\partial x_k}{\partial q_i} \right)^2} \\
h_x &= \sqrt{\sum_{k=1}^3 \left(\frac{\partial x_k}{\partial q_i} \right)^2} \\
h_x &= \sqrt{\left(\frac{\partial x}{\partial x} \right)^2 + \left(\frac{\partial y}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial x} \right)^2} \\
h_x &= \sqrt{(1)^2 + (0)^2 + (0)^2} \\
h_x &= 1
\end{aligned}$$

The rest are easily seen to be 1 as well!

Cartesian Scale Factors $h_x = 1; h_y = 1; h_z = 1$

Now, recalling equations 1.2, 1.3, 1.4 ...

$$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z} \quad (1.5)$$

$$\begin{aligned} d^2\vec{s}_{xy} &= dx\,dy \cdot (\hat{x} \times \hat{y}) \\ d^2\vec{s}_{xy} &= dx\,dy \cdot \hat{z} \end{aligned} \quad (1.6)$$

Note: Other combinations of the coordinates would generate other area elements!

$$d^3V = dx\,dy\,dz \quad (1.7)$$

Easy!

1.0.3 Spherical

The Spherical Metric Tensor is only *locally* diagonal. In spite of this, we can still make use of $g_{ij} = g_{ii}\delta_{ij}$ *locally* with regard to differential paramters. And define the parameterization of the Cartesian Coordinates in terms of the Spherical parameters: the radial distance, the polar angular distance, and the azimuthal angular distance (r, θ, ϕ) .

Figures 1.1 and 1.2 give a schematic for how Cartesian and Spherical Coordinate Systems are related.

The Cartesian Coordinate x can be seen to be the radial distance from the origin scaled and projected onto the $x - y$ plane. That is: $x = r \sin \theta \cos \phi$.

Similarly, $y = r \sin \theta \sin \phi$ and $z = r \cos \theta$.

In this parameterization below, note that $(x_1, x_2, x_3) \rightarrow (x, y, z)$.

$$\begin{aligned} x_1 &= r \sin \theta \cos \phi \\ x_2 &= r \sin \theta \sin \phi \\ x_3 &= r \cos \theta \end{aligned}$$

Recall equation 1.1

$$h_i = \sqrt{\sum_{k=1}^n \left(\frac{\partial x_k}{\partial q_i} \right)^2}$$

With the

$$\begin{aligned} h_r &= \sqrt{\sum_{k=1}^3 \left(\frac{\partial x_k}{\partial r} \right)^2} \\ h_r &= \sqrt{\left(\frac{\partial x_1}{\partial r} \right)^2 + \left(\frac{\partial x_2}{\partial r} \right)^2 + \left(\frac{\partial x_3}{\partial r} \right)^2} \\ h_r &= \sqrt{\left(\frac{\partial}{\partial r} r \sin \theta \cos \phi \right)^2 + \left(\frac{\partial}{\partial r} r \sin \theta \sin \phi \right)^2 + \left(\frac{\partial}{\partial r} r \cos \theta \right)^2} \\ h_r &= \sqrt{\sin^2 \theta \cos^2 \phi \left(\frac{\partial}{\partial r} r \right)^2 + \sin^2 \theta \sin^2 \phi \left(\frac{\partial}{\partial r} r \right)^2 + \cos^2 \theta \left(\frac{\partial}{\partial r} r \right)^2} \\ h_r &= \sqrt{\sin^2 \theta \cos^2 \phi (1) + \sin^2 \theta \sin^2 \phi (1) + \cos^2 \theta (1)} \\ h_r &= \sqrt{\sin^2 \theta (\underbrace{\cos^2 \phi + \sin^2 \phi}_{1; Pythagorean Identity}) + \cos^2 \theta} \\ h_r &= \sqrt{\sin^2 \theta + \cos^2 \theta} = \sqrt{1^2} = 1 \end{aligned}$$

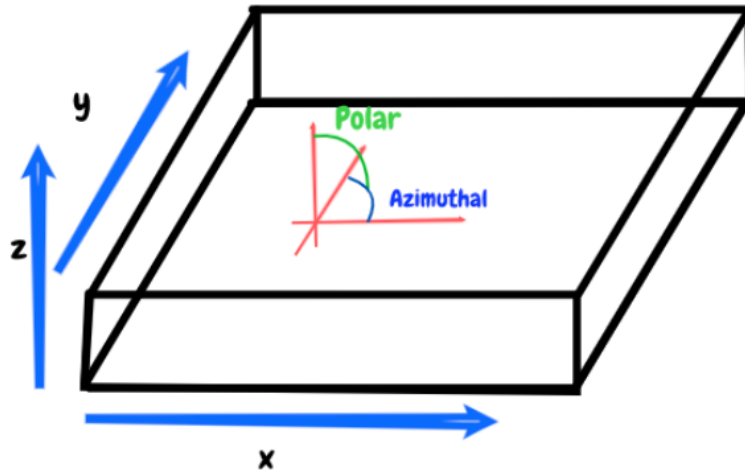


Figure 1.1: Along with

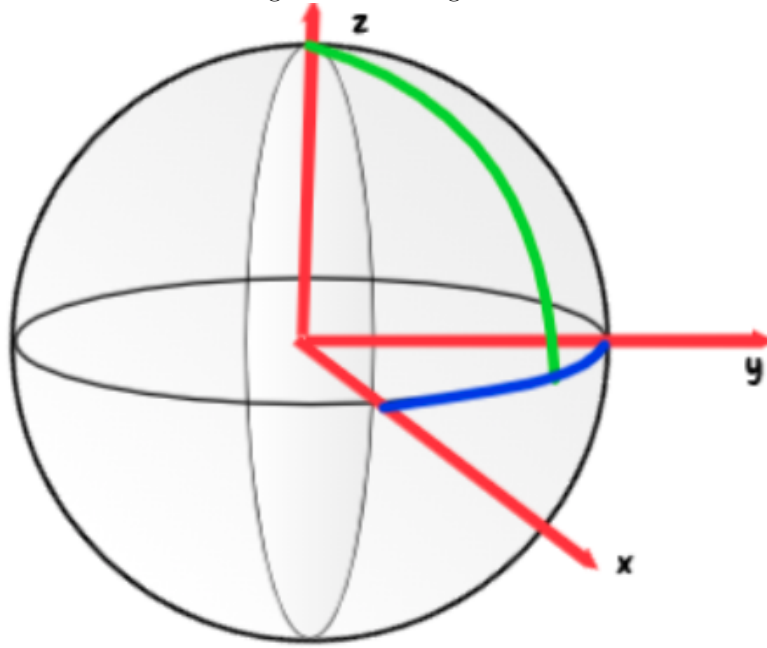


Figure 1.2: Along with

$$\begin{aligned}
 h_\theta &= \sqrt{\sum_{k=1}^3 \left(\frac{\partial x_k}{\partial r} \right)^2} \\
 h_\theta &= \sqrt{\left(\frac{\partial x_1}{\partial \theta} \right)^2 + \left(\frac{\partial x_2}{\partial \theta} \right)^2 + \left(\frac{\partial x_3}{\partial \theta} \right)^2} \\
 h_\theta &= \sqrt{\left(\frac{\partial}{\partial \theta} r \sin \theta \cos \phi \right)^2 + \left(\frac{\partial}{\partial \theta} r \sin \theta \sin \phi \right)^2 + \left(\frac{\partial}{\partial \theta} r \cos \theta \right)^2} \\
 h_\theta &= \sqrt{r^2 \cos^2 \phi \left(\frac{\partial}{\partial \theta} \sin \theta \right)^2 + r^2 \sin^2 \phi \left(\frac{\partial}{\partial \theta} \sin \theta \right)^2 + r^2 \left(\frac{\partial}{\partial \theta} \cos \theta \right)^2} \\
 h_\theta &= r \sqrt{\left(\cos^2 \phi (\cos \theta)^2 + \sin^2 \phi (\cos \theta)^2 + (-\sin \theta)^2 \right)} \\
 h_\theta &= r \sqrt{\cos^2 \phi \cos^2 \theta + \sin^2 \phi \cos^2 \theta + \sin^2 \theta} \\
 h_\theta &= r \sqrt{\cos^2 \theta \left(\underbrace{\cos^2 \phi + \sin^2 \phi}_{1; Pythagorean Identity} \right) + \sin^2 \theta} \\
 h_\theta &= r \sqrt{\cos^2 \theta + \sin^2 \theta} = r \sqrt{1^2}
 \end{aligned}$$

Do the ϕ one! You should get $r \sin \theta$.

Spherical Scale Factors $h_r = 1; h_\theta = r; h_\phi = r \sin \theta$

Now, recalling equations 1.2, 1.3, 1.4 ...

$$\begin{aligned} d\vec{l} &= h_r dr \hat{r} + h_\theta d\theta \hat{\theta} + h_\phi d\phi \hat{\phi} \\ d\vec{l} &= dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi} \end{aligned} \quad (1.8)$$

$$\begin{aligned} d^2 \vec{s}_{\theta\phi} &= h_\theta h_\phi d\theta d\phi \cdot (\hat{\theta} \times \hat{\phi}) \\ d^2 \vec{s}_{\theta\phi} &= r \cdot r \sin \theta d\theta d\phi \cdot \hat{r} \\ d^2 \vec{s}_{\theta\phi} &= r^2 \sin \theta d\theta d\phi \hat{r} \end{aligned} \quad (1.9)$$

Note: Other combinations of the coordinates would generate other area elements! Also note that $d^2 \vec{s}_{\theta\phi}$ is oriented away from the surface of the sphere, that is: in the radial direction.

$$\begin{aligned} d^3 V &= h_r h_\theta h_\phi dr d\theta d\phi \\ d^3 V &= r^2 \sin \theta dr d\theta d\phi \end{aligned} \quad (1.10)$$

These definitions will be extremely useful moving forward!

Solid Angle

To start the discussion with solid angles, let us actually begin with a surface integral over the entire area of a sphere. We will use the definition of the spherical area element given in 1.9.

Recall that $d^2 \vec{s}_{\theta\phi}$ is a vector quantity. The direction is the unit normal to the area element. This will be sometimes noted as an oriented area element or a vector area element to distinguish it from uses of its magnitude.

$$\oint_{\text{sphere}} d^2 \vec{s}_{\theta\phi}$$

We know that the area element along the coordinates θ and ϕ will have a unit normal pointing along the radial direction away from the sphere.

In order to cover the entire sphere, θ must vary from 0 (at the top of the sphere in 1.2) to π and ϕ must then vary from 0 to 2π along the great circle containing the $x - y$ plane.

$$\int_0^{2\pi} \int_0^\pi r^2 \sin \theta d\theta d\phi \hat{r} \cdot \hat{r}$$

A quick note on that $\hat{r} \cdot \hat{r}$, since the closed surface integral also incurs a normal component, the two appear here. However, they're the same! So they result in 1, by definition of a unit normal vector.

What we're left with is actually fairly easy to solve! First, we can pull r^2 out of the integral, since it does not vary with θ or ϕ . Additionally, we can separate the two integrals fairly cleanly!

$$r^2 \int_0^{2\pi} d\phi \cdot \int_0^\pi \sin \theta d\theta$$

The ϕ integral readily goes to 2π .

$$r^2 \cdot 2\pi \cdot \int_0^\pi \sin \theta d\theta$$

The θ integral is a only *slightly* more complicated, but happily yields 2.

$$r^2 \cdot 2\pi \cdot 2$$

I will repeat that θ integral after substituting $\mu = \cos \theta$ and $d\mu = -\sin \theta d\theta$

$$r^2 \cdot 2\pi \cdot \int_{\mu(0)=1}^{\mu(\pi)=-1} -d\mu$$

Flip the bounds to remove the $-$ sign ...

$$r^2 \cdot 2\pi \cdot \int_{-1}^1 d\mu$$

And it's even easier to see that we end up with 2.

Taking the product, we end up with the fact that the surface area of a sphere is $4\pi r^2$. Agreed? GOOD.

Now, that r^2 actually gets us into a lot of trouble. In some weird way, it specifies an otherwise generalizable sphere.

If we strip the r^2 from the $d^2\vec{s}_{\theta\phi}$, we're actually left with an intriguing quantity that depends only on two angular measures, θ and ϕ .

$$\frac{d^2\vec{s}_{\theta\phi}}{r^2} = \sin\theta d\theta d\phi \hat{n} = d^2\Omega \quad (1.11)$$

Where I change what has been \hat{r} to a more general \hat{n} for generality, that is, differential solid angles can exist anywhere in space, not necessarily tied to the distance from an origin, r .

It is important to note that the differential solid angle element implies a direction, namely \hat{n} .

By comparison to the surface integrals performed above, the integral of the differential solid angle over the entire sphere ($\theta \in [0, \pi], \phi \in [0, 2\pi]$) yields 4π .

$$\oint_{sphere} d^2\Omega = 4\pi \quad (1.12)$$

In analogy with the radian for measure of a 1-D angle, the quantity we discuss here is built from a 2-D angle. It therefore is owed a new measure: $rad^2 = sr$ —the steradian.

Just as one might claim a circle has 2π radians, a sphere has 4π steradians.

We'll see that defining quantities in terms of the solid angle and the differential element thereof allows for the calculation of important quantities that arise in astrophysics and astrophysical observation.

Chapter 2

Radiation Field

2.1 Specific Intensity

The Specific Intensity is a differential, scalar quantity ¹:

$$\frac{dI}{d\lambda} = I_\lambda(\vec{r}, \hat{n}, \lambda, t) = \frac{\delta E}{d\lambda dt d^2S \cos \theta d^2\Omega}$$

Where δE is the amount of energy transported by radiation belonging to the wavelength range $[\lambda, \lambda + d\lambda]$. That is,

$$\delta E \approx E_\lambda d\lambda = dE$$

The specific intensity therefore represents this amount of spectral, radiative energy that passes perpendicularly across an oriented area element ($d^2\vec{S}$) in unit time (dt) confined to a solid angle element ($d^2\Omega$). In the arguments of I_λ is the location vector \vec{r} locating the oriented area element d^2S , \hat{n} representing the direction or orientation of the differential solid angle element, λ the spectral location, and t the time.

The appearance of the $\cos \theta$ term in the denominator owes to the deconstruction of the oriented area element into its magnitude and direction. That is, the general oriented area element can be written as $d^2\vec{S} = d^2S \cdot \hat{s}$ with \hat{s} representing the unit normal to the area element. We discussed this notation in spherical coordinates (i.e., 1.9), but the potential orientation of an area element extends to all coordiante systems. Since the specific intensity is confined to a differential solid angle ($d^2\Omega$), the direction of the radiation is given by \hat{n} . For reasons we will discuss below, it is generally convenient to align the direction of propogation to the polar axis (or z -axis). Therefore, $d^2\vec{S} d^2\Omega$ is equivalent to $d^2S d^2\Omega \hat{s} \cdot \hat{n}$ and finally $d^2S d^2\Omega \cos \theta$. With θ defining the angle between the propogation direction (\hat{n}) and the orientation of the oriented area element (\hat{s}).

2.1.1 Invariance

Importantly, the use of the solid angle in the development of the specific intensity enables the quantity to be independent of the distance between the source and the observer².

As an illustrative example without illustration, take $\delta E = I_\lambda(\vec{r}, \hat{n}, \lambda, t) d\lambda dt d^2S \cos \theta d^2\Omega$ as the amount of spectral, radiative energy that passes perpendicularly across an oriented area element $d^2\vec{S}$ in unit time confined to a solid angle element $d^2\Omega$.

Some distance r away from $d^2\vec{S}$ is another oriented area element $d^2\vec{S}'$. As seen from $d^2\vec{S}'$, $d^2\vec{S}$ subtends a solid angle element $d^2\Omega$ and from $d^2\vec{S}$, $d^2\vec{S}'$ subtends a solid angle element $d^2\Omega'$.

That is, $d^2\Omega = \frac{d^2\vec{S}'}{r^2}$ and $d^2\Omega' = \frac{d^2\vec{S}}{r^2}$, both by 1.11.

So, by conservation of energy, $\delta E = I_\lambda(\vec{r}, \hat{n}, \lambda, t) d\lambda dt d^2S = I'_\lambda(\vec{r}, \hat{n}, \lambda, t) d\lambda dt d^2S' \cos \theta' d^2\Omega'$. And $I_\lambda = I'_\lambda$.

2.2 Flux-Vector

Flux is a vector-valued quantity representing the amount of spectral, radiative energy that passes perpendicularly across an oriented area element in unit time.

$$\vec{\mathcal{F}}_\lambda(\vec{r}, \lambda, t) = \oint_\Omega I_\lambda(\vec{r}, \lambda, t, \hat{n}) d^2\Omega \quad (2.1)$$

¹In radiometry, this quantity is called the spectral radiance

²Provided there are no sources or sinks of radiant energy along \hat{n}

In the above, it is again the ‘hidden’ \hat{n} within $d^2\Omega$ that yields a vector-valued quantity. With the $\vec{\mathcal{F}}_\lambda(\vec{r}, \lambda, t)$ representable as a Cartesian vector:

$$\vec{\mathcal{F}}_\lambda = \mathcal{F}_{\lambda,x}\hat{x} + \mathcal{F}_{\lambda,y}\hat{y} + \mathcal{F}_{\lambda,z}\hat{z}$$

Dropping the spectral (λ) subscript for ease only for the moment...

$$\vec{\mathcal{F}} = \mathcal{F}_x\hat{x} + \mathcal{F}_y\hat{y} + \mathcal{F}_z\hat{z}$$

$$\vec{\mathcal{F}} = \oint_{\Omega} I_\lambda d^2\Omega \hat{x} + \oint_{\Omega} I_\lambda d^2\Omega \hat{y} + \oint_{\Omega} I_\lambda d^2\Omega \hat{z}$$

Expanding the differential solid angle reveals the \hat{n} .

$$d^2\Omega = \sin\theta d\theta d\phi \hat{n}$$

At this point it is useful to make use of the substitution $\mu = \cos\theta$ and $d\mu = -\sin\theta d\theta$.

$$d^2\Omega = -d\mu d\phi \hat{n}$$

Therefore,

$$\hat{n} \cdot \hat{x} = \sqrt{(1 - \mu^2)} \cos\phi$$

$$\hat{n} \cdot \hat{y} = \sqrt{(1 - \mu^2)} \sin\phi$$

$$\hat{n} \cdot \hat{z} = \mu$$

2.3 Radiative Flux

2.4 Observational Significance

Chapter 3