# Introduction to numerical optimisation techniques

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# Overview

# Plan

# Convexity

For a convex function $f$, $0 \leq \theta \leq 1$

- $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$
- $-1 * f$ must be concave
- a convex function has a global minima

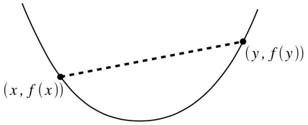


Figure 1: convex

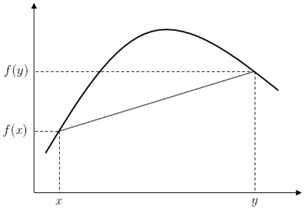


Figure 2: concave

# Conditions to be convex

For a function $f$, its Hessian matrix $Hf(x)$ must be positive definite

- $f_1(x) = x_1^2 + 2x_1 + 2x_2^2$
- $f_2(x) = x_1^2 + 2x_1 - 6x_2^2$

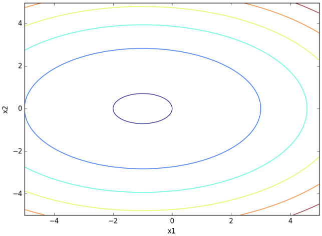


Figure 3: $f_1(x)$

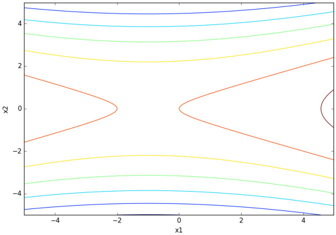


Figure 4: $f_2(x)$

# Plan

# Optimisation with constraints: Quadratic programming

$$
\begin{aligned}
\text{minimize} \quad & f(x) = \frac{1}{2}x^\top P x + c^\top x + d \\
\text{subject to} \quad & h(x) = 0 \\
& g(x) \leq 0
\end{aligned} \tag{1}
$$

- strictly convex or concave
- the easiest form of NLP
- other complex problems converted into QP

# KKT conditions

The Lagrangian of the QP:

$$\begin{aligned} \text{minimize} \quad & L(x, \lambda, \mu) = f(x) + \mu h(x) + \lambda g(x) \\ \text{subject to} \quad & h(x) = 0 \\ & g(x) \leq 0 \end{aligned} \tag{2}$$

To optimise the Lagrangian, following conditions:

- $g(x) \leq 0, h(x) = 0$
- $\lambda > 0$
- $\lambda g(x) = 0$
- $\nabla_x L(x, \lambda, \mu) = 0$

# Using KKT conditions to solve QP problems

For a QP problem:

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{1}{2}x^\top P x + c^\top x + d \\ \text{subject to} \quad & h(x) = 0 \\ & g(x) \leq 0 \end{aligned} \tag{3}$$

First ignore the inequality constraint and solve

$$\begin{aligned} \nabla_x f(x) + \nabla_x \mu g(x) &= 0 \\ g(x) &= 0 \end{aligned} \tag{4}$$

If the inequality constraints violated, activate it and solve:

$$\begin{aligned} \nabla_x f(x) + \nabla_x \mu g(x) + \nabla_x \lambda h(x) &= 0 \\ g(x) &= 0 \\ h(x) &= 0 \end{aligned} \tag{5}$$

If $\lambda > 0$, done !

# Examples

Problem 1 with only equality constraint

$$\begin{aligned} \text{minimize} \quad & f(x) = x_1^2 - 2x_1 + 3x_2^2 \\ \text{subject to} \quad & x_1 + x_2 = 3 \end{aligned} \tag{6}$$

Problem 2 with both equality and inequality constraints

$$\begin{aligned} \text{minimize} \quad & f(x) = x_1^2 - 2x_1 + 3x_2^2 \\ \text{subject to} \quad & x_1 + x_2 = 3 \\ & x_2 \geq 1 \end{aligned} \tag{7}$$

# More

Besides, there also exist the following two popular, effective methods:

- Active Set
- Interior point

KKT conditions server as the fundamental block in them !

# Plan

# Sequential Quadratic programming

Core idea: decomposing a complex NLP problem into a series of Quadratic programming problem by its local approximation :

$$f(x) \approx f(x_*) + \nabla f(x_*)^\top (x - x_*) + \frac{1}{2}(x - x_*)^\top Hf(x)(x - x_*) \tag{8}$$

The procedure of SQP

- initialise $x$ with random values $x_*$
- solve its local approximation at $x_*$
- use the new $x$ to form its new local approximation
- continue the above steps until convergent

Off-shelf software package:matlab fmincon,scipy.optimize

# Plan

- Welcome the new members !