MATH 103A - FINAL EXAM

Date: July 31, 2022.
Instructor: Be'eri Greenfeld

- You have 2.5 hours (unless you have an approved time extension). This includes submitting your solutions.
- This is an open book exam. You may use notes and any material from the course, but you may not search for material online.
- Solve all of the questions (27 points each, except for the bonus). The maximum possible grade is 100.
- Justify every step in your solutions. You may quote any result presented in the lectures, including practice exercises given therein.
- The time window is 12pm–10pm Pacific time, so the whole 2.5 hours period of your exam must be within it. For example, if you start taking the exam only at 9pm, you will have only one hour.
- Submit your solution through Gradescope only. Solutions through email will not be accepted.

Good luck!

(1) (a) (20 points) Let $f\colon G \to H$ be a group homomorphism. Let $K = \{(g, f(g)) | g \in G\} \subseteq G \times H$. Prove that K is a subgroup of $G \times H$.
(b) (7 points) Prove that $K \trianglelefteq G \times H$ if and only if $\mathrm{Im}(f) \subseteq Z(H)$ (recall that $Z(H)$ is the center of H).

(2) (a) (22 points) Let N be a normal subgroup of a group G. Suppose that $[G : N] = 4$. Prove that for every $x, y \in G$, we have that $xyx^{-1}y^{-1} \in N$.
(b) (5 points) Give a counterexample to the assertion of the first part of this problem if N is not normal (but still $[G : N] = 4$).

(3) (27 points) Let $\pi = (1\ 2\ 3\ \cdots\ n-1\ n) \in S_n$. Let $\sigma_1, \dots, \sigma_n \in S_n$ be arbitrary (any) permutations. For each $1 \leq i \leq n$, define $\pi_i = \sigma_i \pi \sigma_i^{-1}$. Find the sign of the permutation $\pi_1 \pi_2 \cdots \pi_n$.

(4) (a) (9 points) Suppose that a group G of order p, which is a prime number, acts on a set X. Prove that for every $x \in X$ we have either $\mathrm{Stab}_G(x) = \{e\}$ or $\mathrm{Stab}_G(x) = G$.
(b) (9 points) Let $p > 2$ be prime and let $G = \{id, \sigma, \cdots \sigma^{p-1}\}$ be the group of rotations of a regular polygon with p sides (here σ is a rotation to the right by $\frac{360}{p}$ degrees). How many colorings with k colors of the vertices of that polygon are fixed by *all* of the elements of G? (Namely how many colorings have $\mathrm{Stab}_G(x) = G$?)
(c) (9 points) In how many different ways can one color the vertices of a regular polygon with $p > 2$ (p is prime) sides, using k colors, where we identify colorings that can be obtained from each other by rotations?[1]

Bonus (10 points): Let G be an abelian group acting on a set X. Suppose that there is only one orbit under this action. Assume that $|G| \neq |X|$. Prove[2] that there exists $e \neq g \in G$ such that $\mathrm{Fix}(g) = X$.

[1]Hint: Use that the number of orbits is equal to $\frac{1}{|G|} \sum_{x \in X} |\mathrm{Stab}_G(x)|$, as well as the first two parts of this problem.

[2]Recall that $(g) = \{x \in X | g \star x = x\}$.