Affine Jump-Diffusions: Stochastic Stability and Limit Theorems

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Outline

Affine Jump-Diffusions

- Widely used in finance and econometrics
  - Vasicek (1977)
  - Cox et al. (1985)
  - Heston (1993)
  - Bates (2000), Duffie et al. (2000), Barndorff-Nielsen and Shephard (2001), Cheridito et al. (2007), Errais et al. (2010), etc.

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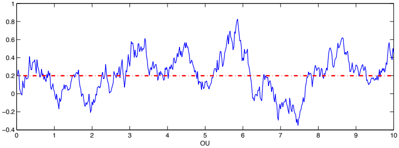
- Computationally tractable
  - Fourier transform is easy to compute (by solving ODEs)

- Defining SDE has "affine" structure
  - drift
  - variance
  - jump intensity

  are all affine in the state variable

AJD Examples

- Ornstein-Uhlenbeck (OU) process in Vasicek model
  - $\mathrm{d}X(t) = (b - \beta X(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)$



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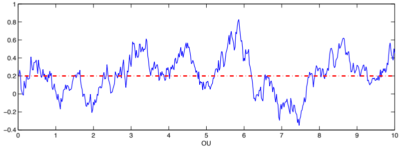

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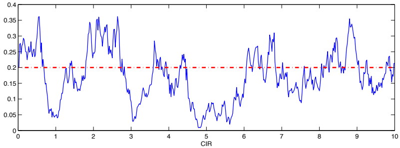

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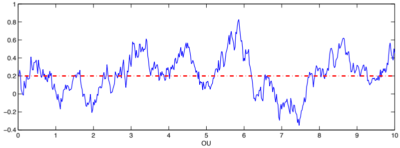

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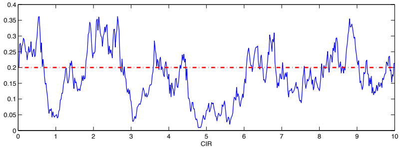

- Mean-reversion: $\beta > 0$

Stochastic Stability

- Long-term behavior
  - Does there exist an equilibrium?
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- Parameter estimation based on large-time asymptotics
  - X is observed at times $0, \Delta, \ldots, n\Delta$
  - Estimating equation for the unknown parameter Θ

$$\frac{1}{n}\sum_{k=1}^{n} h(X(\Delta_{k-1}), X(\Delta_k); \hat{\Theta}_n) = 0$$

  - Maximum likelihood, (generalized) method of moments, least squares, etc.
  - LLN $\Rightarrow$ consistency of $\hat{\Theta}$
  - CLT $\Rightarrow$ asymptotic normality of $\hat{\Theta}$

Related Work

- Sato and Yamazato (1984), Masuda (2004): Lèvy-driven OU process

$$\mathrm{d}X(t) = -\beta X(t)\,\mathrm{d}t + \mathrm{d}J(t)$$

- Glasserman and Kim (2010), Jena et al. (2012): affine diffusions (without jumps)
- Keller-Ressel (2011): a one-dimensional AJD
- Barczy et al. (2014): a two-dimensional AJD with Lèvy-type jumps

1-D Case

- Consider the following 1-dimensional process

$$dX(t) = (b - \beta X(t))\, dt + \sigma \sqrt{X(t)}\, dW(t) + dJ(t)$$

- $J(t) = \sum_{i=1}^{N(t)} Z_i$, Z_i's are iid with $\mathbb{E}|Z_1|^p < \infty$ for some $p > 0$
- $N(t)$ is a counting process with intensity $\kappa + \lambda X(t)$

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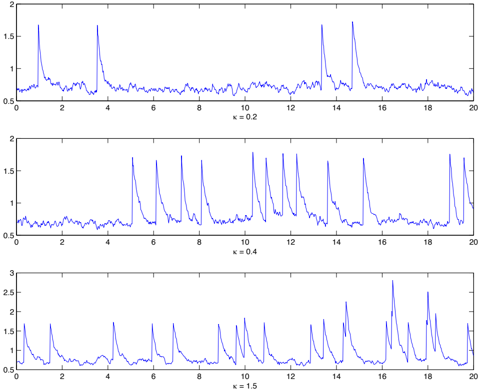
Takeaway: If $\beta > \lambda\mathbb{E}(Z_1)$, then X is "stable".

Special Case: Compound Poisson Jumps

- Suppose $N(t)$ is independent of $X(t)$, i.e. $\lambda = 0$
  - $J(t) = \sum_{i=1}^{N(t)} Z_i$ is a compound Poisson process
- $X(t)$ is stable if $\beta > 0$



Multidimensional Case

- Follow the formulation in Duffie et al. (2003)
- $X(t) \in \mathcal{X} = \mathbb{R}_+^m \times \mathbb{R}^{d-m}$ satisfies

$$\begin{cases} \mathrm{d}X(t) = \mu(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t) + \mathrm{d}J(t) \\ J(t) = \sum_{i=1}^{N(t)} Z_i \end{cases}$$

  $N(t)$ is a counting process with intensity $\Lambda(X(t))$, where

$$\mu(x) = b - \beta x, \qquad b \in \mathbb{R}^d,\ \beta \in \mathbb{R}^{d\times d}$$

$$\sigma(x)\sigma(x)^\intercal = a + \sum_{i=1}^{d} x_j \alpha_i, \qquad a \in \mathbb{R}^{d\times d},\ \alpha_i \in \mathbb{R}^{d\times d},\ i = 1,\dots,d$$

$$\Lambda(x) = \kappa + \lambda^\intercal x, \qquad \kappa \in \mathbb{R},\ \lambda \in \mathbb{R}^d$$

- Assume $\mathbb{E}\|Z_1\|^p < \infty$ for some $p > 0$

Insights from 1-D Case

1. Mean-reversion, i.e. β is "positive"
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- ODE analog: $X'(t) = b - \beta X(t)$
- X is asymptotically stable if and only if β is **positive stable**
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How to represent "dominance" for matrices?

- $\beta - (\mathbb{E}Z_1)\lambda^\intercal$ is positive stable
  - **strong mean-reversion** condition

Main Result

Theorem

If $\beta - \mathbb{E}(Z_1)\lambda^\intercal$ *is positive stable, then* X *is* **exponentially ergodic**, *i.e.,*

$$\|\mathbb{P}_x(X(t) \in \cdot) - \pi(\cdot)\| \le c(x)e^{-\rho t}, \quad t \ge 0,$$

for some positive function $c(\cdot)$ *and some positive constant* ρ, *where* π *is the unique stationary distribution of* X.

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- Proof relies on the "Lyapunov approach" (Meyn and Tweedie, 1993)

Lyapunov Approach

- Originated in the study of stability of ODE's
- Extended to stochastic stability of Markov processes in 1970's (Meyn and Tweedie, 2009)
- Establish a Lyapunov inequality: find constants $c > 0$, $d < \infty$ and a norm-like function $V \geq 0$, for which

$$\mathscr{A}V(x) \leq -cV(x) + d, \quad \text{for all } \|x\| \text{ large enough,}$$

where $\mathscr{A}$ is the infinitesimal generator of X

$$\begin{aligned}
\mathscr{A}g(x) &\triangleq \mathscr{G}g(x) + \mathscr{L}g(x) \\
\mathscr{G}g(x) &\triangleq \nabla g(x) \cdot (b + \beta x) + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 g(x)}{\partial x_i \partial x_j} \left(a_{i,j} + \sum_{k=1}^{d} \alpha_{k,ij} x_k \right) \\
\mathscr{L}g(x) &\triangleq (\lambda + \kappa^{\mathsf{T}} x) \int_{\mathcal{X}} (g(x+z) - g(x))\, \nu(\mathrm{d}z)
\end{aligned}$$

Lyapunov Function

- If $B = \beta - (\mathbb{E}Z_1)\lambda^\intercal$ is positive stable, then there exists a positive definite matrix H, denoted as $H \succ 0$, such that

$$HB + B^\intercal H \succ 0$$

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- Set $V(x) = \|x\|_H^p$, where $\|x\|_H = (x^\intercal Hx)^{\frac{1}{2}}$, and show

$$\mathscr{A}V(x) = pV(x)\left(-\frac{x^\intercal(HB + B^\intercal H)x}{2\|x\|_H^2} + o(1)\right)$$

as $\|x\| \to \infty$

Remark on Strong Mean-Reversion

- Positive stability of $\beta - \mathbb{E}(Z_1)\lambda^\intercal$ cannot be relaxed in general

Remark on Strong Mean-Reversion

- Positive stability of $\beta - \mathbb{E}(Z_1)\lambda^\intercal$ cannot be relaxed in general

- Revisit the following 1-d process

$$\mathrm{d}X(t) = (b - \beta X(t))\,\mathrm{d}t + \sigma\sqrt{X(t)}\,\mathrm{d}W(t) + \mathrm{d}J(t)$$

  - If $\beta - \lambda\mathbb{E}(Z_1) < 0$, then this process is *transient*
  - Proof also relies on the Lyapunov approach

Limit Theorems

Theorem (SLLN)

If $\beta - \mathbb{E}(Z_1)\lambda^\intercal$ *is positive stable and* $|h(x)| \leq C\|x\|^p$*, then*

$$\mathbb{P}_x\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} h(X(i\Delta)) = \pi(h)\right) = 1, \quad x \in \mathcal{X}.$$

Theorem (FCLT)

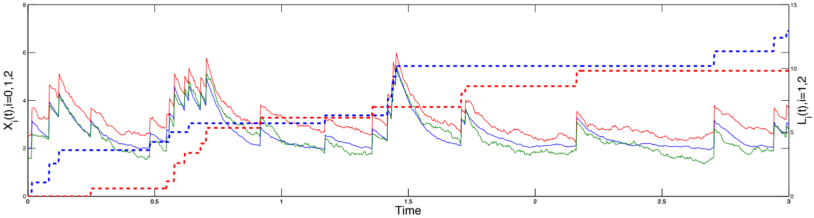
If $\beta - \mathbb{E}(Z_1)\lambda^\intercal$ *is positive stable and* $|h(x)|^{2+\epsilon} \leq C\|x\|^p$*, then*

$$n^{1/2}\left(\frac{1}{n}\sum_{i=1}^{\lfloor n\cdot\rfloor} h(X(i\Delta)) - \pi(h)\right) \Rightarrow \gamma_h W(\cdot)$$

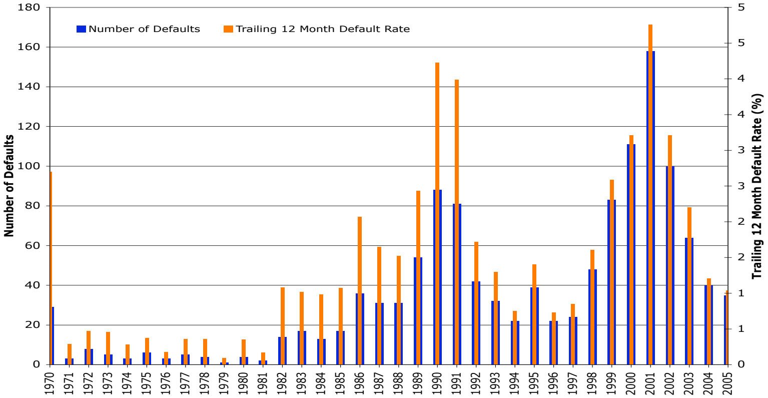
as $n \to \infty$ $\mathbb{P}_x$*-weakly in* $\mathcal{D}[0,1]$ *for each* $x \in \mathcal{X}$*.*

Affine Point Processes

- The jump process in an AJD
- Hawkes process is a popular example (limit order book, credit default)
- Jump intensity is state-dependent, capturing the clustering of jumps

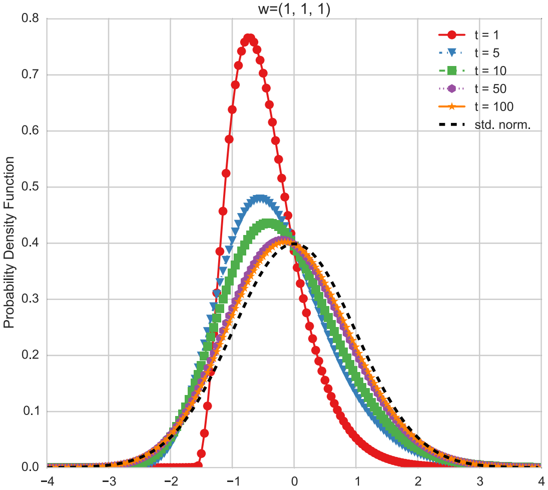


Annual Defaults of Moodys rated U.S. Firms



CLT for Affine Point Processes

- $J(t) \overset{\mathcal{D}}{\approx} rt + \mathcal{N}(0, \eta^2 t)$ for large t



Conclusions

- Proved exponential ergodicity of AJDs under a simple condition
  - Strong mean-reversion cannot be relaxed in general
- Proved SLLN and FCLT
  - provide theoretical support for many estimation methods for AJDs

Thanks!

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