On Almost Interpolation*

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Abstract

We obtain some characterizations of almost interpolation configurations of points with respect to finite-dimensional functional spaces. Particularly, a Schoenberg-Whitney type characterization which is valid for any multivariate spline space relative to an arbitrary partition of a domain $A \subset \mathbb{R}^m$ is presented. As a closely related problem we investigate sectional structure of finite-dimensional spaces of real functions on a topological space A. It is shown that under some reasonable restrictions on A any space of this sort may be considered as piecewise almost Chebyshev.

1 Introduction

Let U be a finite-dimensional space of multivariate splines with respect to a partition of a domain $A \subset \mathbb{R}^m$, $m \geq 2$. The problem of describing those finite configurations of points $t_1, \ldots, t_n \in A$, $n = \dim U$, which admit unique Lagrange interpolation from U has attracted considerable interest in recent years. Several methods of constructing such configurations (we call them *interpolation sets*) have been developed (see [1, 2, 3, 10] and references therein). However, in contrast to the univariate case $A \subset \mathbb{R}$, when *all* interpolation configurations with respect to a spline space can be characterized through

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the well-known Schoenberg-Whitney condition [11], the multivariate setting seems to admit only constructing *special* interpolation sets (see [3, p. 136]).

On the other hand, it is quite clear (see, *e.g.*, [2, p. 58]), that for multivariate polynomial interpolation at points not regularly distributed in $\mathbb{R}^m$, the probability of encountering a non-interpolation set is zero. This leads to a concept of *almost interpolation* introduced recently by Sommer & Strauß [13, 14]. A finite set $M = \{t_1, \ldots, t_n\} \subset A$ is called an almost interpolation set (AI-set) with respect to U if for any system of neighborhoods B_i of t_i, $i = 1, \ldots, n$, there exist points $t_i' \in B_i$ such that $M' = \{t_1', \ldots, t_n'\}$ is an interpolation set. It is shown in [13] that almost interpolation sets can be characterized by a condition of Schoenberg-Whitney type for a class of generalized multivariate spline spaces with respect to polyhedral partitions.

In this paper we offer several characterizations of almost interpolation sets. Theorems 3.3 and 3.10 give a "local" characterization which is valid for any finite-dimensional space U of real functions on an arbitrary topological space A. As an application of Theorem 3.3 we obtain a general algorithm of transforming a given AI-set M into an interpolation set in any arbitrarily small neighborhood of M. Theorem 4.21 provides a Schoenberg-Whitney type condition characterizing almost interpolation sets with respect to a class of finite-dimensional spaces which we call *generalized almost Chebyshev spline spaces* (Definition 4.19). This class includes generalized multivariate splines introduced in [13] as well as any space of continuous piecewise polynomial functions with respect to an arbitrary partition of a domain $A \subset \mathbb{R}^m$.

Another topic of the paper is concerned with the properties of restrictions of finite-dimensional functional spaces to some subsets of A (see Section 4).

It is well-known that the most important finite-dimensional spaces in the univariate approximation theory are Chebyshev (*i.e.*, algebraic polynomials) or at least piecewise Chebyshev (*i.e.*, splines, generalized splines). Recall that a space U of real functions on a set A, with $\dim U = n$, is said to be a *Chebyshev space* (*T-space*) if every nonzero function $u \in U$ has at most $n-1$ zeros. It is an important feature of T-spaces that they are as good for interpolation as possible: any set $M = \{t_1, \ldots, t_n\} \subset A$ is an interpolation set with respect to U. However, by the well-known theorem of Mairhuber [9], there exist no T-spaces of dimension $n \geq 2$ in the case that A is compact and is *not* homeomorphic to a subset of the circle. Hence, as we want to deal with multivariate functions, we have to replace T-spaces by a wider class. In this connection so-called *almost Chebyshev spaces* are of interest. According to a definition given by Stechkin [15], a subset Y in a normed space X is called almost Chebyshev if the set of elements $x \in X$ for which there exists a unique best approximation to x from Y, is residual in X. Garkavi [4, 5] investigated almost Chebyshev subspaces of Banach spaces. Particularly, he showed that there exist almost Chebyshev subspaces of arbitrary finite dimensions in any separable Banach space. The following interpolation property of finite-dimensional almost Chebyshev subspaces in $C(A)$, the space of continuous functions on a metric compact A, was also mentioned by Garkavi [5]: $U \subset C(A)$ is an almost Chebyshev subspace if and only if the systems of points $t_1, \dots, t_n$, $n = \dim U$, at which Lagrange interpolation may not always be possible form only a closed nowhere dense subset of A^n. In other words, any system $t_1, \dots, t_n$ is an almost interpolation set with respect to U. The latter seems very close to the above-mentioned characteristic property of T-spaces, and we take this property as a definition of almost Chebyshev spaces of functions on an arbitrary topological space A (see Definition 4.8).

It seems to be a surprising result of Section 4 that under some reasonable limitations on A any finite-dimensional functional space U necessarily possesses a piecewise almost Chebyshev structure (see Theorem 4.13, Corollaries 4.14, 4.16). Notice that a notion of *local dimension* introduced in Section 3 (Definition 3.2) plays an important role in revealing this structure.

2 Algebraic Lemmas

In this section we give a series of lemmas of algebraic nature. Throughout the section, let U be a finite-dimensional linear subspace of $F(A)$, the space of all real functions on a given set A. For any $u \in F(A)$ we set $\operatorname{supp} u \stackrel{\text{def}}{=} \{t \in A : \; u(t) \neq 0\}$. For any $B \subset A$ and any linear subspace $U \subset F(A)$, let $U_{|B}$ denote the restriction of U to the set B, *i.e.*, $U_{|B} \stackrel{\text{def}}{=} \{u_{|B} : \; u \in U\}$. Furthermore, let

$$\begin{aligned} U(B) &\stackrel{\text{def}}{=} \{u \in U : \; \forall t \in B \;\; u(t) = 0\}, \quad B \subset A, \\ Z(U) &\stackrel{\text{def}}{=} \{t \in A : \; \forall u \in U \;\; u(t) = 0\}. \end{aligned}$$

Although some proofs in the section are well-known, we present them here for completeness.

Lemma 2.1 *Let $B \subset A$. Then*

$$\dim U = \dim U_{|_B} + \dim U(B)\,.$$

Proof. Lemma follows from the simple observation that $U(B)$ is the kernel of the mapping $u \mapsto u_{|_B}$. ■

Lemma 2.2 *Let $B, C \subset A$, $B \cap C \neq \emptyset$. Then*

$$\dim U_{|_{B\cup C}} \leq \dim U_{|_B} + \dim U_{|_C} - \dim U_{|_{B\cap C}}\,. \tag{2.1}$$

Proof. By Lemma 2.1, inequality (2.1) can be equivalently expressed in the form

$$\dim U(B \cup C) \geq \dim U(B) + \dim U(C) - \dim U(B \cap C)\,. \tag{2.2}$$

It is easily seen that

$$U(B) \cap U(C) = U(B \cup C)\,, \tag{2.3}$$

$$U(B) + U(C) \subset U(B \cap C)\,. \tag{2.4}$$

Therefore,

$$\begin{aligned} \dim U(B \cap C) &\geq \dim\,(U(B) + U(C)) \\ &= \dim U(B) + \dim U(C) - \dim\,(U(B) \cap U(C)) \\ &= \dim U(B) + \dim U(C) - \dim U(B \cup C)\,, \end{aligned}$$

which gives (2.2). ■

Lemma 2.3 *Let $B \subset A$, $C' \subset C \subset A$. If*

$$\dim U_{|_{C'}} = \dim U_{|_C}\,, \tag{2.5}$$

then

$$\dim U_{|_{B\cup C'}} = \dim U_{|_{B\cup C}}\,. \tag{2.6}$$

Proof. By (2.5) and Lemma 2.1, $\dim U(C') = \dim U(C)$. Since $U(C) \subset U(C')$, it follows that

$$U(C') = U(C)\,,$$

and, by (2.3), we obtain

$$U(B \cup C') = U(B) \cap U(C') = U(B) \cap U(C) = U(B \cup C)\,.$$

Hence,

$$\dim U(B \cup C') = \dim U(B \cup C)\,.$$

Applying Lemma 2.1 once again, we have

$$\dim U_{|_{B\cup C'}} = \dim U_{|_{B\cup C}}\,,$$

and (2.6) is proved. ■

Suppose $\dim U = n$. A finite set $M = \{t_1, \ldots, t_p\} \subset A$, $p \leq n$, is said to be an *interpolation set* (*I-set*) with respect to $U \subset F(A)$ if $\dim U_{|_M} = p$. It is not hard to see that M is an I-set if and only if it admits Lagrange interpolation from U, *i.e.*, for any given data $y_1, \ldots, y_p \in \mathbb{R}$ there exists a function $u \in U$ such that $u(t_i) = y_i$, $i = 1, \ldots, p$.

Lemma 2.4 *For any integer $p \leq n$ there exists an I-set $M = \{t_1, \ldots, t_p\}$ with respect to U.*

Proof. Let $u_1, \ldots, u_n$ be a basis for U. Since u_1 is not the zero function, there exists a point $t_1 \in A$ for which $u_1(t_1) \neq 0$ and $\{t_1\}$ is an I-set with respect to U. Suppose that there exist $t_1, \ldots, t_{p-1}$ such that

$$\det\{u_i(t_j)\}_{i,j=1}^{p-1} \neq 0\,.$$

Set

$$\varphi(t) \stackrel{\text{def}}{=} \det\{u_i(t_1) \cdots u_i(t_{p-1})\, u_i(t)\}_{i=1}^{p}\,.$$

Because $u_1, \ldots, u_p$ are linearly independent, there exists $t_p \in A$ such that $\varphi(t_p) \neq 0$. Hence $\{t_1, \ldots, t_p\}$ is the required I-set. ■

Lemma 2.5 *Suppose that $T = \{t_1, \ldots, t_{p-1}\}$ and $X = \{x_1, \ldots, x_p\}$ are two I-sets with respect to U. Then there exists $s \in \{1, \ldots, p\}$ such that $x_s \notin T$ and $\{t_1, \ldots, t_{p-1}, x_s\}$ is also an I-set.*

Proof. Let $u_1, \ldots, u_n$ be a basis for U. Since X is an I-set, the vectors

$$(u_1(x_i), \ldots, u_n(x_i)) \in \mathbb{R}^n, \quad i = 1, \ldots, p,$$

are linearly independent. They all can not depend on $p-1$ vectors

$$(u_1(t_j), \ldots, u_n(t_j)) \in \mathbb{R}^n, \quad j = 1, \ldots, p-1.$$

Therefore, there exists $s \in \{1, \ldots, p\}$ such that the system

$$\{(u_1(t_j), \ldots, u_n(t_j)) \in \mathbb{R}^n : \ j = 1, \ldots, p-1\} \cup \{(u_1(x_s), \ldots, u_n(x_s))\}$$

is independent. Then evidently $x_s \notin T$ and $\{t_1, \ldots, t_{p-1}, x_s\}$ is an I-set. ■

Lemma 2.6 *Let $\{A_i : \ i = 1, \ldots, p\}$, $p \leq \dim U$, be a system of subsets of A such that for any nonempty $I \subset \{1, \ldots, p\}$,*

$$\dim U_{|_{\cup_{i \in I} A_i}} \geq \operatorname{card} I.$$

Then there exists an I-set $T = \{t_1, \ldots, t_p\}$ with respect to U such that $t_i \in A_i$, $i = 1, \ldots, p$.

Proof. We will prove the lemma by induction on p. The statement is trivial when $p = 1$. Let $\{A_i : \ i = 1, \ldots, p\}$ be as above. By the induction hypothesis, the lemma is valid for $p-1$. Therefore, there exist $y_i \in A_i$, $i = 1, \ldots, p-1$, such that $Y = \{y_1, \ldots, y_{p-1}\}$ is an I-set. Furthermore, Lemma 2.4 shows that there exists an I-set $Z = \{z_1, \ldots, z_p\} \subset \bigcup_{i=1}^{p} A_i$. By Lemma 2.5 we can find $s \in \{1, \ldots, p\}$ such that $X = \{y_1, \ldots, y_{p-1}, z_s\}$ is also an I-set. If $z_s \in A_p$, then $T = X$ has required properties. Suppose $z_s \in A_{m_0}$ for some $m_0 \in \{1, \ldots, p-1\}$. Set $y'_{m_0} \stackrel{\text{def}}{=} z_s$, so that

$$X = \{y_1, \ldots, y_{m_0}, y'_{m_0}, y_{m_0+1}, \ldots, y_{p-1}\}, \quad y_i \in A_i, \quad i = 1, \ldots, p-1, \quad y'_{m_0} \in A_{m_0}.$$

By the induction hypothesis, we can find an I-set

$$X_0 = \{x_1, \ldots, x_{m_0-1}, x_{m_0+1}, \ldots, x_p\}, \quad x_i \in A_i, \quad i = 1, \ldots, m_0-1, m_0+1, \ldots, p.$$

Applying Lemma 2.5 to X and X_0, we conclude that at least one of the following sets

$$X_0 \cup \{y_i\}, \quad i = 1, \ldots, p-1,$$

$$X_0 \cup \{y'_{m_0}\}$$

is an I-set. We can take $T = X_0 \cup \{y_{m_0}\}$ or $X_0 \cup \{y'_{m_0}\}$ if one of them is an I-set. Otherwise, there exists $m_1 \in \{1, \ldots, p-1\} \setminus \{m_0\}$ such that $X_0 \cup \{y_{m_1}\}$ is an I-set. In this case, set

$$X_1 = (X_0 \cup \{y_{m_1}\}) \setminus \{x_{m_1}\}$$

and apply Lemma 2.5 to the pair X, X_1. The same argument as above shows that either at least one of $X_1 \cup \{y_{m_0}\}$ and $X_1 \cup \{y'_{m_0}\}$ can be taken for T, or there exists $m_2 \in \{1, \ldots, p-1\} \setminus \{m_0, m_1\}$ such that $X_1 \cup \{y_{m_2}\}$ is an I-set. (Note that $m_2 \neq m_1$ because $y_{m_1} \in X_1$.) In the latter case we set

$$X_2 = (X_1 \cup \{y_{m_2}\}) \setminus \{x_{m_2}\}$$

and apply the same argument repeatedly. After several steps we construct a set

$$X_l = (X_1 \cup \{y_{m_1}, \ldots, y_{m_l}\}) \setminus \{x_{m_1}, \ldots, x_{m_l}\}, \quad l \in \{2, \ldots, p-1\},$$

such that either $X_l \cup \{y_{m_0}\}$ or $X_l \cup \{y'_{m_0}\}$ is an I-set, which completes the proof. ■

3 Almost Interpolation Sets

Throughout this section A denotes a topological space and, as previously, U is assumed to be a finite-dimensional linear subspace of $F(A)$, $\dim U = n$. For any $B \subset A$, denote by $\overline{B}$, $\operatorname{int} B$ and $\operatorname{bd} B$ the closure, the interior and the boundary set of B respectively.

Definition 3.1 A finite set $M = \{t_1, \ldots, t_p\} \subset A$ is said to be an *almost interpolation set* (*AI-set*) with respect to $U \subset F(A)$ if for any neighborhoods $B(t_1), \ldots, B(t_p)$ there exist points $t'_i \in B(t_i)$, $i = 1, \ldots, p$, such that $\{t'_1, \ldots, t'_p\}$ is an I-set with respect to U.

Note that almost interpolation sets were introduced in two recent papers by Sommer & Strauß [13, 14].

Our first purpose is to give a "local" characterization of AI-sets through the "neighborhood dimension" which is defined as follows.

Definition 3.2 Let $M = \{t_1, \ldots, t_p\}$ be any finite subset of A. By the *neighborhood dimension* of U on M we mean the quantity

$$\text{n–dim}_M U \stackrel{\text{def}}{=} \inf \{\dim U_{|_B} : \ B \supset M\,, \ B \text{ is open}\}\,.$$

In the particular case when $M = \{t\}$ we say about the *local dimension* of U at the point $t \in A$,

$$\text{l–dim}_t U \stackrel{\text{def}}{=} \text{n–dim}_{\{t\}} U = \inf \{\dim U_{|_B} : \ B \text{ is a neighborhood of } t\}\,.$$

Theorem 3.3 *A finite set $M = \{t_1, \ldots, t_p\} \subset A$ is an AI-set with respect to U if and only if*

$$\text{n–dim}_N U \geq \text{card}\, N \tag{3.1}$$

for any $N \subset M$, $N \neq \emptyset$.

Proof. *Necessity.* Since M is an AI-set, $N = \{t_{i_1}, \ldots, t_{i_k}\} \subset M$ is also an AI-set. Therefore, for any open set $B \supset N$ (which is a neighborhood of each element t_{i_j}) there exist points $t'_{i_j} \in B$ such that $N' = \{t'_{i_1}, \ldots, t'_{i_k}\}$ is an I-set. Because of this,

$$\dim U_{|_B} \geq \dim U_{|_{N'}} = k = \text{card}\, N\,,$$

and (3.1) holds.

Sufficiency. Let $B(t_1), \ldots, B(t_p)$ be arbitrary neighborhoods of the points $t_1, \ldots, t_p$. It follows from (3.1) that

$$\dim U_{|_{\cup_{t\in N} B(t)}} \geq \text{n–dim}_N U \geq \text{card}\, N$$

for any $N \subset M$, $N \neq \emptyset$. Then by Lemma 2.6 there exist points $t'_i \in B(t_i)$ such that $\{t'_1, \ldots, t'_p\}$ is an I-set, and the proof is complete. ■

We now turn to the problem on how to find efficiently an I-set in a neighborhood of a given AI-set. The following assertion will be of use.

Proposition 3.4 *Suppose that $M = \{t_1, \ldots, t_p\} \subset A$ is an almost interpolation set with respect to $U \subset F(A)$. Let any neighborhoods B_i of t_i, $i = 1, \ldots, p$, be given. If $A_i \subset B_i$, $i = 1, \ldots, p$, satisfy*

$$\dim U_{|_{A_i}} = \dim U_{|_{B_i}}\,, \qquad i = 1, \ldots, p\,, \tag{3.2}$$

then there exist points $t'_i \in A_i$, $i = 1, \ldots, p$, such that $\{t'_1, \ldots, t'_p\}$ is an I-set with respect to U.

Proof. It is sufficient to show that the system $\{A_1, \ldots, A_p\}$ satisfies the hypotheses of Lemma 2.6. Let $I \subset \{1, \ldots, p\}$, $I \neq \emptyset$. Set $N = \{t_i : i \in I\}$. By Lemma 2.3 and Theorem 3.3,

$$\dim U_{|_{\cup_{i\in I} A_i}} = \dim U_{|_{\cup_{i\in I} B_i}} \geq \text{n–}\dim_N U \geq \operatorname{card} N = \operatorname{card} I\,. \quad \blacksquare$$

We describe an **Algorithm** which, for a given AI-set $M = \{t_1, \ldots, t_p\}$ and a system of neighborhoods $B_i \ni t_i$, $i = 1, \ldots, p$, produces an I-set $M' = \{t'_1, \ldots, t'_p\}$ such that $t'_i \in B_i$.

Step 1. Find, according to Lemma 2.4, I-sets $A_i = \{\tau_{i,1}, \ldots, \tau_{i,k_i}\} \subset B_i$, $i = 1, \ldots, p$, where $k_i = \dim U_{|_{B_i}}$. It is significant that in some important cases such interpolation points are easily obtainable in any neighborhood. For example, when $B_i \subset \mathbb{R}^m$ and $U_{|_{B_i}}$ is a space of multivariate polynomials, they can be constructed as intersections of some hyperplanes (Chung–Yao interpolation; see [1, p. 207]).

Step 2. Find points $t'_i \in A_i$ such that $M' = \{t'_1, \ldots, t'_p\}$ is an I-set. The existence of such an M' follows from Proposition 3.4. Since A_i's are finite, the points t'_i can be chosen by simple exhaustion of a finite number of possibilities. However, the proof of Lemma 2.6 in fact provide another constructive procedure of choosing t'_i's, which may be much faster.

Note that a completely different algorithm of changing an AI-set to an I-set was offered by Sommer & Strauß [14].

In view of Theorem 3.3, the evaluation of neighborhood dimensions of arbitrary finite sets is of particular importance for characterizing AI-sets. The rest of the section is devoted to identifying the largest neighborhood on which neighborhood dimension is attained.

Definition 3.5 Let $t \in A$ and let B be a neighborhood of t. We say that B is an *L-neighborhood* of t if

$$\dim U_{|_B} = \text{l–}\dim_t U\,. \tag{3.3}$$

Proposition 3.6 *Let M be any finite subset of A and for each $t \in M$ let $B(t)$ be an L-neighborhood of t. Then*

$$\text{n–}\dim_M U = \dim U_{|_{\cup_{t\in M} B(t)}}\,.$$

Proof. It is evident by the definition of neighborhood dimension that

$$\text{n–dim}_M U \leq \dim U_{|_{\cup_{t\in M} B(t)}} .$$

Thus, it remains to prove the opposite inequality. Let B be any open set such that $M \subset B$. By (3.3),

$$\dim U_{|_{B(t)\cap B}} = \dim U_{|_{B(t)}} .$$

Therefore, in view of Lemma 2.3,

$$\dim U_{|_B} \geq \dim U_{|_{\cup_{t\in M}(B(t)\cap B)}} = \dim U_{|_{\cup_{t\in M} B(t)}} ,$$

which establishes the desired inequality. ■

Definition 3.7 Let $t \in A$. By the *principal neighborhood* of t, denoted by $PN(t)$, we mean the union of all L-neighborhoods of t.

Proposition 3.8 *Let $B \subset A$ be any L-neighborhood of t. Then $PN(t) = Z(U(B))$.*

Proof. Suppose B is an L-neighborhood of t. Since $U(Z(U(B))) = U(B)$, it follows from Lemma 2.1 and (3.3) that

$$\dim U_{|_{Z(U(B))}} = \dim U_{|_B} = \text{l–dim}_t U , \tag{3.4}$$

i.e., the set $Z(U(B))$ is also an L-neighborhood of t. Hence,

$$Z(U(B)) \subset PN(t) .$$

On the other hand, suppose that B' is another L-neighborhood of t. Then, by the definition of local dimension,

$$\dim U_{|_{B\cap B'}} = \dim U_{|_B} = \dim U_{|_{B'}} = \text{l–dim}_t U .$$

Lemma 2.1 now yields

$$\dim U(B \cap B') = \dim U(B) = \dim U(B') ,$$

which, together with the simple observation that

$$U(B) \subset U(B \cap B'), \qquad U(B') \subset U(B \cap B'),$$

leads to the conclusion that $U(B) = U(B')$, hence that $Z(U(B)) = Z(U(B'))$, and finally that $B' \subset Z(U(B))$. Therefore,

$$PN(t) \subset Z(U(B)),$$

which completes the proof. ■

The following proposition is a consequence of Proposition 3.8 and equality (3.4).

Proposition 3.9 *Both $PN(t)$ and $\operatorname{int} PN(t)$ are L-neighborhoods of t.*

Propositions 3.6 and 3.9 enable us to reformulate Theorem 3.3 as follows.

Theorem 3.10 *A finite set $M = \{t_1, \dots, t_p\} \subset A$ is an AI-set with respect to U if and only if*

$$\dim U_{|_{\cup_{t \in N} PN(t)}} \geq \operatorname{card} N \tag{3.5}$$

for any $N \subset M$, $N \neq \emptyset$.

It is easily seen that Theorem 3.10 remains valid if we replace $PN(t)$ by any L-neighborhood of t, for example, $\operatorname{int} PN(t)$.

4 Sectional Structure of Finite-Dimensional Spaces

As before, A denotes a topological space and U denotes a finite-dimensional linear subspace of $F(A)$, $\dim U = n$. In this section we sometimes impose some restrictions on A and require that $U \subset C(A)$, the space of continuous real functions on A. However, unless otherwise specified, neither continuity nor any other additional conditions are assumed.

4.1 Properties of Local Dimension

We now consider some properties of local dimension which will be used in the subsequent analysis.

Recall that a real function f on a topological space X is said to be *lower (upper) semicontinuous* if for any $x \in X$ and $r \in \mathbb{R}$ satisfying $f(x) > r$ $(f(x) < r)$, there exists a neighborhood $B(x)$ such that $\forall x' \in B(x) \quad f(x') > r \;\; (f(x') < r)$.

Proposition 4.1 *Local dimension* $\text{l–dim}_t U$ *is upper semicontinuous as a function of* t. *Furthermore, for any* $k = 0, 1, \ldots,$

$$\{t \in A :\ \text{l–dim}_t U \geq k\} \quad \textit{is a closed set}, \tag{4.1}$$

$$\{t \in A :\ \text{l–dim}_t U \leq k\} \quad \textit{is an open set}. \tag{4.2}$$

Proof. Given $x \in A$, suppose that $\text{l–dim}_x U < r$. Then there exists an open neighborhood $B(x)$ such that $\dim U_{|_{B(x)}} < r$. Therefore, for any $t \in B(x)$ we also have $\text{l–dim}_t U < r$. Statements (4.1) and (4.2) follow immediately from the well-known properties of upper semicontinuous functions when it is considered that $\text{l–dim}_t U$ assumes only integer values. ■

Proposition 4.2 *Let* F_U *be the set of all points of discontinuity of* $\text{l–dim}_t U$. *Then* F_U *is a closed nowhere dense subset of* A.

Proof. We first prove that $G_U = A \setminus F_U$ is an open set. Let $x \in G_U$. Then $\text{l–dim}_t U$ is continuous at the point x. Hence, there exists an open neighborhood $B(x)$ such that $\forall t \in B(x) \quad |\,\text{l–dim}_t U - \text{l–dim}_x U| < 1$, *i.e.*,

$$\text{l–dim}_t U = \text{l–dim}_x U\,, \quad t \in B(x)\,.$$

Thus, $\text{l–dim}_t U$ is constant in $B(x)$ and therefore $B(x) \subset G_U$. This proves that G_U is open and F_U is closed.

We now prove that F_U is nowhere dense. Let $B \subset A$ be an open set and let $m = \min\{\text{l–dim}_t U :\ t \in B\}$. Consider the set

$$B' \stackrel{\text{def}}{=} \{t \in B :\ \text{l–dim}_t U \leq m\}\,.$$

By (4.2) B' is open and it is evidently nonempty. However,

$$\text{l–dim}_t\, U = m\,, \quad t \in B'\,.$$

Hence, the local dimension is continuous inside B' and therefore $B' \cap F_U = \emptyset$. ■

The following example shows that F_U can be of a rather complicated nature.

Example 4.3 Let $U = \operatorname{span}\{u_1, u_2\} \subset C[0,1]$, where $u_1(t) \equiv 1$ and u_2 is the Cantor continuous nondecreasing function which is constant on each component of the complement of a Cantor discontinuum F_α of measure $\alpha < 1$ (see, for example, [6, Chapter 8]). It is not difficult to see that

$$\text{l–dim}_t\, U = \begin{cases} 2, & t \in F_\alpha, \\ 1, & t \notin F_\alpha, \end{cases}$$

hence that $F_U = F_\alpha$.

As an application of Proposition 4.2 we obtain an analogue of Proposition 1.6 in [13].

Proposition 4.4 *Assume that $U \subset C(A)$. Let $M = \{t_1, \dots, t_p\} \subset A$ be an AI-set with respect to U and B_i be a neighborhood of t_i, $i = 1, \dots, p$. Then there exists an I-set $M' = \{t'_1, \dots, t'_p\}$ with respect to U such that $t'_i \in B_i \cap G_U$, $i = 1, \dots, p$.*

Indeed, Proposition 4.4 is a consequence of Proposition 3.4 in which we take $A_i = B_i \cap G_U$, so that condition (3.2) follows from Proposition 4.2 and the next simple lemma.

Lemma 4.5 *Let A' be a dense subset of A, $U \subset C(A)$. Then*

$$\dim U_{|_{A'}} = \dim U\,.$$

Proof. By Lemma 2.4 there exists an I-set $M = \{x_1, \dots, x_n\}$ with respect to U. Let $u_1, \dots, u_n$ be a basis of U, so that

$$\det\{u_i(x_j)\}_{i,j=1}^n \neq 0\,.$$

By the continuity of u_i's, there exist open sets $B_j \ni x_j$, $j = 1, \ldots, n$, such that

$$\det\{u_i(x'_j)\}_{i,j=1}^n \neq 0\,,$$

for any $x'_j \in B_j$. Because A' is a dense subset of A, we can choose $x'_j \in A' \cap B_j$. Hence $\{x'_1, \ldots, x'_n\} \subset A'$ is an I-set and $\dim U_{|_{A'}} \geq n$. ■

Proposition 4.6 *Let C be a connected subset of A. If local dimension is constant in C,*

$$\text{l–dim}_t\, U = m\,, \quad t \in C\,, \tag{4.3}$$

then

$$\dim U_{|_C} \leq m\,.$$

To prove this, we need the following topological result (see [8, p. 136]).

Lemma 4.7 *Let X be a connected topological space. For any open cover $\{B_s :\ s \in S\}$ of X and any two points $x, y \in X$ there exists a sequence $s_1, \ldots, s_k \in S$ such that $x \in B_{s_1}$, $y \in B_{s_k}$ and $B_{s_i} \cap B_{s_{i+1}} \neq \emptyset$, $i = 1, \ldots, k-1$.*

Proof of Proposition 4.6. Suppose, to the contrary, that

$$\dim U_{|_C} = n > m\,.$$

Then, by Lemma 2.4, there exist points $x_1, \ldots, x_n \in C$ for which

$$\dim U_{|_{\{x_1, \ldots, x_n\}}} = n\,. \tag{4.4}$$

In view of (4.3), for each $t \in C$ we can find an open neighborhood $B(t) \subset A$ such that

$$\dim U_{|_{B(t)}} = m\,. \tag{4.5}$$

Then $\mathcal{B} = \{B(t) :\ t \in C\}$ is an open cover of C. It follows from Lemma 4.7 that there exist $B_1, \ldots, B_r \in \mathcal{B}$ such that

$$x_i \in B_1 \cup \cdots \cup B_r\,, \quad i = 1, \ldots, n\,, \tag{4.6}$$

and

$$B_j \cap B_{j+1} \cap C \neq \emptyset\,, \quad j = 1, \ldots, r-1\,.$$

Hence, by (4.3) and (4.5) we have

$$\begin{aligned}\dim U_{|_{B_j}} &= m\,, \quad j = 1, \ldots, r\,,\\ \dim U_{|_{B_j \cap B_{j+1}}} &= m\,, \quad j = 1, \ldots, r-1\,.\end{aligned}$$

Now Lemma 2.2 shows that

$$\dim U_{|_{B_1 \cup \cdots \cup B_r}} = m\,,$$

contrary to (4.4) and (4.6). ■

4.2 Almost Chebyshev Spaces

We define almost Chebyshev spaces as follows.

Definition 4.8 Let A be a topological space, $U \subset F(A)$, $\dim U = n < \infty$. We say that U is an *almost Chebyshev space (AT-space)* if any set of n points $\{t_1, \ldots, t_n\} \subset A$ is an almost interpolation set with respect to U.

Next theorem gives a characterization of AT-spaces which, in view of a result by Garkavi [5, Theorem I], shows that in the case when $U \subset C(A)$, with A being a metric compact, our definition is equivalent to one given by Stechkin [15]. (See also Kitahara [7, p. 9] for the case of a locally compact subset of $\mathbb{R}$.)

Theorem 4.9 *$U \subset F(A)$ is an AT-space if and only if for any nonempty open set $B \subset A$,*

$$\dim U_{|_B} = \min\{n,\ \mathrm{card}\, B\}\,. \tag{4.7}$$

Proof. *Necessity.* Set $k = \min\{n,\ \mathrm{card}\, B\}$ and choose any k distinct points $t_1, \ldots, t_k \in B$. Since U is an almost Chebyshev space, $\{t_1, \ldots, t_k\}$ is an AI-set. Therefore, there exist points $t_i' \in B$, $i = 1, \ldots, k$, such that $\{t_1', \ldots, t_k'\}$ is an I-set. Thus, $\dim U_{|_B} \geq k$. On the other hand, it is evident that $\dim U_{|_B} \leq k$.

Sufficiency. Let $M = \{t_1, \ldots, t_n\} \subset A$, let N be a subset of M, $N \neq \emptyset$, and let $B \supset N$ be an open set. By (4.7) we have

$$\dim U_{|_B} = \min\{n,\ \mathrm{card}\, B\} \geq \mathrm{card}\, N\,.$$

Therefore, $\text{n–dim}_N U \geq \operatorname{card} N$ and, by Theorem 3.3, M is an AI-set. ■

Notice that, in view of Lemma 2.1, (4.7) can be written in the form

$$\dim U(B) = \max\{0,\ n - \operatorname{card} B\}\,, \tag{4.8}$$

which leads to the following consequence of Theorem 4.9.

Proposition 4.10 *$U \subset F(A)$ is an AT-space if and only if in A any nonempty open set of cardinality at most n is an I-set with respect to U and the only function $u \in U$ vanishing on an open set $B \subset A$, with $\operatorname{card} B > n$, is the zero function.*

Thus, it is easily seen that the class of AT-spaces is fairly wide. For example, any finite-dimensional space of analytic functions on a set $A \subset \mathbb{R}^m$ containing no isolated points, is an AT-space. In the case $A \subset \mathbb{R}$ the same is true for any subspace of a T-space.

It follows from Theorem 4.9 that if in a topological space A every nonempty open set $B \subset A$ consists of at least n points, then $U \subset F(A)$ is an AT-space if and only if $\text{l–dim}_t U = n$ for any $t \in A$. Next theorem shows that under some additional assumptions of connectedness and separation it is sufficient to require that $\text{l–dim}_t U$ should be constant. Before stating this result, we formulate a standard topological assertion which will be required in the proof.

Lemma 4.11 *Let A be a connected T_1-space such that $\operatorname{card} A \geq 2$. Then any nonempty open set $B \subset A$ is infinite.*

Theorem 4.12 *Let A be a connected topological T_1-space containing at least two distinct points. A finite-dimensional space $U \subset F(A)$ is an AT-space if and only if $\text{l–dim}_t U$ is constant in A.*

Proof. *Necessity* follows immediately from Theorem 4.9.

Sufficiency. Suppose that

$$\text{l–dim}_t U = n\,, \quad t \in A\,.$$

By Proposition 4.6, $\dim U = n$. Let $B \subset A$ be a nonempty open set and let $x \in B$. Then

$$n = \dim U \geq \dim U_{|B} \geq \text{l–dim}_x U = n\,.$$

Hence, $\dim U_{|B} = n$. Because B is infinite, (4.7) is satisfied and an application of Theorem 4.9 completes the proof. ■

4.3 Main Structure Theorem

Let A be a topological T_1-space and let $U \subset F(A)$ be a finite-dimensional space. Denote by G_U the set of all points of continuity of $\mathrm{l\text{--}dim}_t\, U$. According to Proposition 4.2, G_U is an open subset of A and its complement, $F_U = A \setminus G_U$, is nowhere dense in A.

Consider the decomposition of G_U into the union of its connected components,

$$G_U = \bigcup_{s \in S} G_s\,, \tag{4.9}$$

where S is an index set.

Theorem 4.13 *Suppose that A is a T_1-space and $U \subset F(A)$ is a finite-dimensional space. Let G_s be any connected component of G_U. If G_s contains at least two distinct points and*

$$G_s \subset \overline{\mathrm{int}\, G_s}\,, \tag{4.10}$$

then $U_{|G_s}$ is an almost Chebyshev space.

Proof. The function $\varphi(t) = \mathrm{l\text{--}dim}_t\, U$ is continuous when $t \in G_s$. By [8, p. 128], $\varphi(t)$ has Darboux property, *i.e.*, it passes from one value to another through all intermediate values. Because $\varphi(t)$ takes only integer values, it has to be constant in G_s. Let us show that

$$\varphi(t) = \mathrm{l\text{--}dim}_t\, U_{|G_s}\,, \qquad t \in G_s\,.$$

Suppose $t \in G_s$. It follows from the definition of local dimension that $\varphi(t) \geq \mathrm{l\text{--}dim}_t\, U_{|G_s}$. Conversely, let B be an open neighborhood of t such that

$$\dim U_{|B \cap G_s} = \mathrm{l\text{--}dim}_t\, U_{|G_s}\,.$$

In view of (4.10), $B \cap \mathrm{int}\, G_s \neq \emptyset$. Let $x \in B \cap \mathrm{int}\, G_s$. Let us find an open set B' such that $x \in B' \subset B \cap \mathrm{int}\, G_s$ and

$$\dim U_{|B'} = \mathrm{l\text{--}dim}_x\, U = \varphi(x)\,.$$

Since φ is constant in G_s, we have

$$\mathrm{l\text{--}dim}_t\, U_{|G_s} = \dim U_{|B \cap G_s} \geq \dim U_{|B'} = \varphi(x) = \varphi(t)\,.$$

Thus, $1\text{–}\dim_t U_{|_{G_s}}$ is constant in G_s. It remains to apply Theorem 4.12. ■

We consider some consequences of this main structure theorem.

First, in the case $A \subset \mathbb{R}$ condition (4.10) always holds because any connected set $G \subset \mathbb{R}$ containing at least two points is an (finite or infinite) open, closed or half-open interval.

Corollary 4.14 *Let A be any subset of $\mathbb{R}$ and let $U \subset F(A)$ be a finite-dimensional space. If a connected component G_s of G_U contains at least two distinct points, then $U_{|_{G_s}}$ is an almost Chebyshev space.*

On the other hand, the following example shows that the assumption (4.10) cannot be omitted in general.

Example 4.15 Let $A \subset \mathbb{R}^2$ be defined as follows: $A = A_1 \cup A_2$, where

$$A_1 = [-1,1] \times \{0\}\,,$$
$$A_2 = \{(x,y) \in \mathbb{R}^2 :\ 0 < y \le 1\,, \quad \sin\tfrac{\pi}{2y} - \varepsilon \le (1+\varepsilon)x \le \sin\tfrac{\pi}{2y} + \varepsilon\}\,,$$

for some sufficiently small $\varepsilon > 0$. Next, let $U = \operatorname{span}\{f_1, f_2, f_3\} \subset C(A)$, where $f_i = \tilde{f}_{i|_A}$, $i = 1,2,3$,

$$\begin{aligned}
\tilde{f}_1(x,y) &\equiv 1\,,\\
\tilde{f}_2(x,y) &= \begin{cases} xy, & \text{if } x \ge 0,\\ -x, & \text{if } x < 0,\end{cases}\\
\tilde{f}_3(x,y) &= \tilde{f}_2(-x,y)\,.
\end{aligned}$$

It is easily checked that

$$1\text{–}\dim_{(x,y)} U = 3 = \dim U\,, \qquad \text{for any} \quad (x,y) \in A\,.$$

Therefore, U is an AT-space, $G_U = A$. The connected components of G_U are A_1 and A_2. However, $U_{|_{A_1}}$ is the space of univariate continuous piecewise linear functions with a knot at the point $(0,0)$, which evidently fails to be almost Chebyshev.

Notice that, in view of Example 4.15, none of the following assumptions ensure condition (4.10) for all components G_s:

- *A is compact*;
- *A is connected* (this requires a slight modification of the example);
- *U is an almost Chebyshev space*;
- *$U \subset C(A)$*;
- *the number of connected components of G_U is finite.*

However, if A is *locally connected*, then any component of each open subset of A is open (see [8]), so that (4.10) is necessarily satisfied. This leads to another consequence of Theorem 4.13.

Corollary 4.16 *Suppose that A is a locally connected T_1-space and $U \subset F(A)$ is a finite-dimensional space. Let G_s be any connected component of G_U. If G_s contains at least two distinct points, then $U_{|G_s}$ is an almost Chebyshev space.*

Thus, let U be any finite-dimensional space of real functions on a locally connected T_1-space A which, say, contains no isolated points. Corollary 4.16 shows that U is in some sense composed of AT-spaces defined on disjoint connected open sets $G_s \subset A$ whose union is residual in A. In other words, U may be thought of as "piecewise almost Chebyshev".

The following results throw light on the structure of principal neighborhoods: components G_s play an important role in this question as well.

Proposition 4.17 *Under the hypotheses of Theorem 4.13, let $t \in A$ and let G_s be a connected component of G_U such that $G_s \subset \overline{\operatorname{int} G_s}$.*

(a) If $G_s \cap \operatorname{int} PN(t) \neq \emptyset$, then $G_s \subset PN(t)$.

(b) If $t \in \overline{G_s}$, then $G_s \subset PN(t)$.

(c) If $t \in \operatorname{int} G_s$, then G_s is an L-neighborhood of t.

Proof. It is clear that (b) and (c) follow immediately from (a). The latter is evident when G_s is a singleton. Suppose that G_s contains at least two distinct points. Then, by Theorem 4.13, $U_{|G_s}$ is an AT-space. Set $B = \operatorname{int} PN(t)$. Let u be any function in $U(B)$. Obviously, $u \in U(B \cap \operatorname{int} G_s)$. $B \cap \operatorname{int} G_s$

is infinite by Lemma 4.11. Then Proposition 4.10 shows that $u \in U(G_s)$. Therefore, by Proposition 3.8, $G_s \subset Z(U(B)) = PN(t)$. ■

Corollary 4.18 *If $U \subset C(A)$, where A is a locally connected T_1-space, then for any $t \in A$ there exists an L-neighborhood $B(t)$ such that*

$$B(t) = \overline{\bigcup_{s \in S'} G_s}\,,$$

for some subset $S' \subset S$.

Proof. Let B be any open L-neighborhood of t. Set

$$S' = \{s \in S : \; G_s \cap B \neq \emptyset\}\,.$$

By Proposition 4.17(a), $G_s \subset PN(t)$, $s \in S'$. Therefore,

$$\bigcup_{s \in S'} G_s \subset PN(t)\,.$$

In view of Lemma 4.5, we obtain

$$\dim U_{|B(t)} = \dim U_{|\bigcup_{s \in S'} G_s} \leq \dim U_{|PN(t)} = \text{l–}\dim_t U\,.$$

On the other hand, by Proposition 4.2, $B \setminus \bigcup_{s \in S'} G_s = F_U \cap B$ is nowhere dense in B. Hence $B \subset B(t)$, so that $B(t)$ is an neighborhood of t, which completes the proof. ■

Note that there evidently exist spaces U such that every L-neighborhood of a point $t \in F_U$ contains an infinite number of components G_s (see Example 4.3 above).

4.4 Generalized Almost Chebyshev Splines

We now impose some conditions which seem to suffice for U to have the right to be called a generalized spline space.

Definition 4.19 Suppose that A is a locally connected T_1-space and $U \subset C(A)$ is a finite-dimensional space. We say that U is a space of *generalized almost Chebyshev (or: AT-) splines* if the number of connected components of G_U is finite.

Thus, if U is a space of generalized AT-splines, then the decomposition (4.9) of G_U into the union of its connected components determines a *partition* of A,

$$A = \bigcup_{s \in S} \overline{G_s}\,, \tag{4.11}$$

where S is a finite index set. *Cells* G_s of this partition are disjoint connected open subsets of A such that $U_{|G_s}$ is an almost Chebyshev space for each $s \in S$. The set of all *edge points* (points of discontinuity of local dimension) of the partition,

$$F_U = A \setminus G_U = \bigcup_{s \in S} \overline{G_s} \setminus G_s = \bigcup_{s \in S} \operatorname{bd} G_s\,,$$

is a closed nowhere dense subset of A.

Note that any space of multivariate polynomial splines with respect to an arbitrary partition of a bounded domain $A \subset \mathbb{R}^m$, is a space of generalized AT-splines. The same is evidently true for a class of generalized multivariate spline spaces introduced by Sommer & Strauß [13, Remark 2.1(iii)].

Given a space of generalized AT-splines U, set

$$S(t) \overset{\text{def}}{=} \{s \in S :\ t \in \overline{G_s}\}\,, \tag{4.12}$$

$$G(t) \overset{\text{def}}{=} \bigcup_{s \in S(t)} \overline{G_s}\,, \quad t \in A\,. \tag{4.13}$$

The following theorem gives an useful formula for the neighborhood dimension of an arbitrary finite set $M \subset A$. It rests on the fact that $G(t)$ is an L-neighborhood of t.

Theorem 4.20 *Let $U \subset C(A)$ be a space of generalized almost Chebyshev splines, and let $M \subset A$ be a finite set. Then*

$$\text{n–dim}_M\, U = \dim U_{|\cup_{t \in M} G(t)}\,. \tag{4.14}$$

Proof. Let B be an open L-neighborhood of t such that $B \cap G_s = \emptyset$ for any $s \notin S(t)$. Then

$$S(t) = \{s \in S :\ G_s \cap B \neq \emptyset\}\,.$$

Arguing exactly as in the proof of Corollary 4.18 we obtain that

$$\overline{\bigcup_{s \in S(t)} G_s} \quad \left(= G(t)\right)$$

is also an L-neighborhood of t. It remains to apply Proposition 3.6. ■

We can now give a characterization of almost interpolation sets with respect to an arbitrary space of generalized almost Chebyshev splines through a condition of Schoenberg-Whitney type. Our theorem generalizes the main result in [13].

For any $S' \subset S$, $S' \neq \emptyset$, set

$$G_{S'} \stackrel{\text{def}}{=} \bigcup_{s \in S'} \overline{G_s}\,.$$

Theorem 4.21 *Let $U \subset C(A)$ be a space of generalized almost Chebyshev splines, and let $M \subset A$ be a finite set. Then M is an almost interpolation set if and only if*

$$\dim U_{|G_{S'}} \geq \operatorname{card}\,(M \cap \operatorname{int} G_{S'}) \tag{4.15}$$

for any $S' \subset S$, $S' \neq \emptyset$.

Proof. *Necessity.* Suppose, to the contrary, that (4.15) does not hold for some S' although M is an AI-set. Set $N = M \cap \operatorname{int} G_{S'}$. Since $\operatorname{int} G_{S'}$ is a dense subset of $G_{S'}$, it follows from Lemma 4.5 that

$$\dim U_{|G_{S'}} = \dim U_{|\operatorname{int} G_{S'}}\,.$$

Therefore,

$$\text{n–}\dim_N U \leq \dim U_{|\operatorname{int} G_{S'}} < \operatorname{card} N\,,$$

which, by Theorem 3.3, shows that M is not an AI-set, a contradiction.

Sufficiency. Suppose that (4.15) is valid. Let N be any subset of M, $N \neq \emptyset$. Set $S' = \bigcup_{t \in N} S(t)$. Then

$$G_{S'} = \bigcup_{t \in N} G(t)\,.$$

Since $G(t)$ is a neighborhood of t, $N \subset \operatorname{int} G_{S'}$. Therefore,

$$\operatorname{card}\,(M \cap \operatorname{int} G_{S'}) \geq \operatorname{card} N\,.$$

Thus, by (4.14) and (4.15),

$$\text{n–}\dim_N U = \dim U_{|G_{S'}} \geq \operatorname{card} N\,,$$

and it follows from Theorem 3.3 that M is an AI-set. ■

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