Interpolation by Weak Chebyshev Spaces

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Abstract. We present two characterizations of Lagrange interpolation sets for weak Chebyshev spaces. The first of them is valid for an arbitrary weak Chebyshev space U and is based on an analysis of the structure of zero sets of functions in U extending Stockenberg's theorem. The second one holds for all weak Chebyshev spaces that possess a locally linearly independent basis.

§1. Introduction

Let U denote a finite-dimensional subspace of real valued functions defined on a totally ordered set K, for example, an arbitrary subset of $\mathbb{R}$.

A finite subset $T = \{t_1, \ldots, t_n\}$ of K, where $n = \dim U$, is called an *interpolation set (I-set)* w.r.t. U if for any given data $\{y_1, \ldots, y_n\}$ there exists a unique function $u \in U$ such that

$$u(t_i) = y_i, \quad i = 1, \ldots, n.$$

It is easy to see that T is an I-set w.r.t. U if and only if

$$\dim U_{|T} = n,$$

where $U_{|T} := \{u_{|T} : \; u \in U\}$. For a set of s points, $T = \{t_1, \ldots, t_s\} \subset K$, with $s < n$, we say that T is an I-set if $\dim U_{|T} = s$.

We are interested in describing I-sets w.r.t. U in the case when U is a *weak Chebyshev space (WT-space), i.e.*, every $u \in U$ has at most $n-1$ sign changes. The primary example of a WT-space is the space of univariate polynomial splines, in which case all interpolation sets can be characterized by well-known Schoenberg-Whitney condition (see e.g. [12]). Extensions of Schoenberg-Whitney theorem to some classes of generalized spline spaces were proposed in [11,13,14].

Recently, some characterizations of I-sets w.r.t. weak Chebyshev spaces without any *a priori* assumption about "piecewise Haar" spline-like structure have been found. In [4] it was proved that Schoenberg-Whitney characterization in its "dimension form" holds true for a WT-space U if and only if $U_{|K'}$ is also a WT-space for all $K' \subset K$. This last property is satisfied, for example, if U is a weak Descartes space. In [2] the "support form" of Schoenberg-Whitney theorem has been shown to hold true for every WT-space that possesses a locally linearly independent weak Descartes basis. (See [7] for a review of various forms of Schoenberg-Whitney condition, especially in regard to their extendibility to multivariate splines.)

The purpose of this paper is twofold. In Section 2 we present a characterization of I-sets w.r.t. arbitrary weak Chebyshev spaces (Theorem 2.1), which does not involve any structural properties of U. Instead, the unions of the intervals $[t_i, t_{i+1}]$

between interpolation points are considered. This result relies on an extension of Stockenberg's theorem about zeros of functions in a WT-space (Theorem 2.4), which seems to be of independent interest. A (rather lengthy) proof of Theorem 2.4 is given in Section 4.

In Section 3 we generalize the above mentioned theorem of [2] and show that Schoenberg-Whitney characterization holds true for all weak Chebyshev spaces with a locally linearly independent basis. The proof involves an analysis of the relationship between I-sets and so-called *strong almost interpolation sets* as well as our previous results on almost interpolation [5,9] and Theorem 2.1.

§2. Interpolation by Arbitrary Weak Chebyshev Spaces

We denote by $F(K)$ the linear space of all real valued functions defined on K and by $C(K)$ its subspace consisting of all continuous functions. For any $f \in F(K)$ and any subspace U of $F(K)$, let

$$Z(f) := \{t \in K : \ f(t) = 0\}, \qquad Z(U) := \bigcap_{f \in U} Z(f).$$

We need the following (somewhat unusual in the case $\beta < \alpha$) definition of the *closed interval* with endpoints $\alpha, \beta \in K$,

$$[\alpha, \beta] := \begin{cases} \{t \in K : \alpha \le t \le \beta\} & \text{if } \alpha \le \beta \\ \{t \in K : t \ge \alpha \text{ or } t \le \beta\} & \text{if } \beta < \alpha. \end{cases}$$

In the same way we define *open* and *halfopen intervals.*

The main result of this section reads as follows.

Theorem 2.1. *Let U be an n-dimensional weak Chebyshev subspace of $F(K)$, and let $T = \{t_1, \dots, t_n\} \subset K \setminus Z(U)$ such that $t_1 < \dots < t_n$, and $t_{n+1} := t_1$. The following conditions are equivalent:*

1) *T is an I-set w.r.t. U.*
2) *For all $P \subset \{1, \dots, n\}$,*

$$\operatorname{card}(T \cap \bigcup_{i \in P} [t_i, t_{i+1}]) \le \dim U_{|\bigcup_{i \in P} [t_i, t_{i+1}]}. \tag{2.1}$$

A simple example shows that this characterization is no longer valid if one omits the assumption that U is a weak Chebyshev space. Moreover, we conjecture that only for WT-spaces every set T satisfying condition 2) is an I-set.

Example 2.2. Let $K = [0,3] \subset \mathbb{R}$ and assume that $U = \operatorname{span}\{u_1, u_2\}$ where $u_1 = 1$ on K and

$$u_2(t) = \begin{cases} 1-t & \text{if } 0 \le t \le 1 \\ 0 & \text{if } 1 < t < 2 \\ t-2 & \text{if } 2 \le t \le 3. \end{cases}$$

Set $\tilde{u} = 1/2u_1 - u_2$. Then it is obvious that $\tilde{u}$ has two sign changes at $t_1 = 1/2$ and $t_2 = 5/2$, respectively. This implies that U fails to be a weak Chebyshev space and, in particular, that $T = \{t_1, t_2\}$ fails to be an I-set w.r.t. U. On the other hand, T satisfies condition 2) of Theorem 2.1.

We will prove Theorem 2.1 at the end of this section as a consequence of a result about location of zeros of functions in WT-spaces. The following generalized notion of separation of zeros will be particular important for our analysis.

Suppose that $u \in U$ and $\tilde{Z} \subset Z(u)$ are given. We say that zeros $x_1, \ldots, x_m \in Z(u) \setminus \tilde{Z}$, where $x_1 < \ldots < x_m$, are *cyclically separated with respect to* $\tilde{Z}$ if for every $i \in \{1, \ldots, m\}$ there exists a subinterval

$$[y_{2i-1}, y_{2i}] \subset (x_i, x_{i+1}), \quad y_{2i-1}, y_{2i} \in K, \tag{2.2}$$

(we set $x_{m+1} := x_1$ if $i = m$) such that

$$u(y_j) \neq 0, \quad j = 1, \ldots, 2m, \tag{2.3}$$

and

$$(x_i, x_{i+1}) \cap \tilde{Z} \subset (y_{2i-1}, y_{2i}). \tag{2.4}$$

Note that $y_{2i-1} \le y_{2i}$, $i = 1, \ldots, m-1$. For $i = m$ the following three cases can occur: 1) $x_m < y_{2m-1} \le y_{2m}$, 2) $y_{2m-1} \le y_{2m} < x_1$, and 3) $y_{2m} < x_1 < x_m < y_{2m-1}$.

If $\tilde{Z} \cap (x_i, x_{i+1}) = \emptyset$, then it is sufficient to have one point $y_{2i-1} = y_{2i}$, with $u(y_{2i-1}) \neq 0$. Otherwise, we need two different points y_{2i-1} and y_{2i} satisfying (2.4). If $\tilde{Z} = \emptyset$, then we simply say that $x_1, \ldots, x_m$ are *cyclically separated zeros* of u. Note that this definition requires that x_m and x_1 are also separated from each other by some points y_{2m-1} and y_{2m} (possibly equal) that lie outside $[x_1, x_m]$, which is the reason for the word "cyclically". If $\tilde{Z} = \emptyset$ and (2.2) and (2.3) are only satisfied for $i = 1, \ldots, m-1$ and $j = 1, \ldots, 2m-2$, respectively, then the zeros $x_1, \ldots, x_m$ are *separated* in usual sense.

We say that $x \in Z(u)$ is an *essential zero* of $u \in U$ if $x \notin Z(U)$.

A relationship between the number of separated essential zeros of functions $u \in U$ and the dimension n of U was found by Stockenberg [15]. We recall his theorem which has played an important role in characterizing continuous selections for metric projections (see e.g. [10]).

Theorem 2.3. [15] *Suppose that U is an n-dimensional weak Chebyshev space.*

1) *If there exists $u \in U$ with n separated essential zeros $x_1, \ldots, x_n$ such that $x_1 < \ldots < x_n$, then $u(t) = 0$ for all $t \in [x_n, x_1]$.*

2) *No $u \in U$ has more than n separated essential zeros.*

Note that both statements of Theorem 2.3 are obviously contained in the following formulation: *no $u \in U$ has more than $n-1$ cyclically separated essential zeros.*

We generalize Theorem 2.3 as follows.

Theorem 2.4. *Suppose that U is an n-dimensional weak Chebyshev subspace of $F(K)$, and let $\tilde{Z} \subset K$. If there exists $u \in U$ such that $\tilde{Z} \subset Z(u)$, and u has m essential zeros $x_1 < \ldots < x_m$ that are cyclically separated w.r.t. $\tilde{Z}$, then*

$$\dim U_{|\tilde{Z}} \leq n - m - 1. \tag{2.5}$$

The proof of Theorem 2.4 will be given in Section 4.

Let us see that Theorem 2.4 contains Theorem 2.3 as a special case. Indeed, if we take $\tilde{Z} = \emptyset$, then (2.5) gives the bound $m \leq n-1$ on the number m of cyclically separated essential zeros of u, which yields Theorem 2.3.

Moreover, in the situation of Theorem 2.3, 1), *i.e.*, when $u \in U$ has n separated essential zeros $x_1 < \ldots < x_n$, we can deduce from (2.5) a slightly stronger statement by setting $\tilde{Z} := [x_n, x_1]$. Since $x_1, \ldots, x_n$ are separated in $[x_1, x_n]$, it clearly follows that $x_2, \ldots, x_{n-1}$ are cyclically separated w.r.t. $\tilde{Z}$. Thus, by (2.5), and in view of the assumption that $x_1 \notin Z(U)$, we have $\dim U_{|[x_n, x_1]} = 1$. Therefore, *under the hypotheses of Theorem 2.3, 1), not only u itself, but also every function $v \in U$ such that $v(x_1) = 0$ or $v(x_n) = 0$ necessarily satisfies $v(t) = 0$ for all $t \in [x_n, x_1]$.* Particularly, in the important special case when K contains its minimal and maximal elements, $a = \min K$ and $b = \max K$, we have the following corollary: *if $\dim U_{|\{a,b\}} = 2$, then no $u \in U$ has more than $n-1$ separated essential zeros.* This applies specifically to spline spaces and recovers their well-known property (see [12]).

We will see now that Theorem 2.1 immediately follows from Theorem 2.4.

Proof of Theorem 2.1: If T is an I-set, then $\operatorname{card}(T \cap K') \leq \dim U_{|K'}$ for *every* subset $K' \subset K$. Therefore, we only have to show that 2) implies 1). On the contrary, suppose that $T = \{t_1, \ldots, t_n\}$ satisfies 2), but fails to be an I-set. Hence, there exists a function $u \in U \setminus \{0\}$ such that

$$u(t_i) = 0, \qquad i = 1, \ldots, n.$$

We set

$$\tilde{P} = \{i : \ u_{|[t_i, t_{i+1}]} = 0\}, \qquad \tilde{Z} = \bigcup_{i \in \tilde{P}} [t_i, t_{i+1}].$$

Let

$$T \setminus \tilde{Z} = \{x_1, \ldots, x_m\}.$$

Then it is easy to see that $x_1, \ldots, x_m$ are essential zeros of u that are cyclically separated w.r.t. $\tilde{Z}$. Therefore, by (2.5), $\dim U_{|\tilde{Z}} \leq n - m - 1$. On the other hand, (2.1), with $P = \tilde{P}$, implies $\dim U_{|\tilde{Z}} \geq n - m$, a contradiction. $\square$

§3. WT-spaces with a Locally Linearly Independent Basis

Throughout this section we assume that K is endowed with a topology consistent with the ordering. We denote by $\overline{M}$ and $\operatorname{int} M$ the closure and the interior, respectively, of any subset $M \subset K$. For every function $f \in F(K)$, we set

$$\operatorname{supp} f := \overline{\{x \in K : \ f(x) \neq 0\}}.$$

Let $\{u_1, \dots, u_n\}$ be a system of functions in $F(K)$. The following notion of a locally linearly independent system which generalizes a well-known property of univariate B-splines has proven to be important in the problems of interpolation.

Definition 3.1. We say that $\{u_1, \dots, u_n\}$ is a *locally linearly independent system (LI-system)* if for any $t \in K$ and any neighborhood $B(t)$ of t there exists an open set B', with $t \in B' \subset B(t)$, such that the subsystem

$$\{u_i : B' \cap \operatorname{supp} u_i \neq \emptyset\}$$

is linearly independent on B'.

It has been shown in [9] that the above definition is equivalent to the standard definition of local linear independence by de Boor and Höllig [1], so that $\{u_1, \dots, u_n\}$ is an LI-system if and only if for every open $B \subset K$, the condition

$$\sum_{i=1}^{n} a_i u_i(x) = 0, \qquad x \in B,$$

implies $a_i = 0$ for all i such that $B \cap \operatorname{supp} u_i \neq \emptyset$.

An important feature of an LI-system $\{u_1, \dots, u_n\}$ is that it forms a least supported basis for its span (see Carnicer and Peña [3]).

Carnicer and Peña [2] have also shown that for a space of continuous functions on the real interval spanned by an LI-system satisfying weak Descartes property, interpolation sets can be characterized by Schoenberg-Whitney condition in support form. In order to formulate this result, we need the definition of a weak Descartes system. Recall that a matrix is said to be *totally positive* if all its minors are nonnegative.

Definition 3.2. We say that a system of functions $\{u_1, \dots, u_n\}$ in $F(K)$ is a *weak Descartes system (WD-system)* if the matrix $(u_i(t_j))_{i,j=1}^{n}$ is totally positive for all choices of $t_1, \dots, t_n \in K$ such that $t_1 < \dots < t_n$.

Theorem 3.3. [2] *Suppose that $K = [a, b]$ is an interval of the real line $\mathbb{R}$, and $\{u_1, \dots, u_n\} \subset C(K)$ is simultaneously an LI-system and WD-system. Let $T = \{t_1, \dots, t_n\} \subset K$ such that $t_1 < \dots < t_n$. The following conditions are equivalent:*

1) *T is an I-set w.r.t. $U = \operatorname{span}\{u_1, \dots, u_n\}$.*
2) *$t_i \in \{x \in K : \ u_i(x) \neq 0\}$, $i = 1, \dots n$.*

The main objective of this section is to provide a generalization of Theorem 3.3 in two directions. First, we relax the condition that $\{u_1, \ldots, u_n\}$ is a WD-system and show that the theorem essentially holds for every weak Chebyshev space possessing an LI-basis. Second, we allow K to be a general totally ordered set. Our main tools are Theorem 2.1 and some results of our previous research [8,9] on almost interpolation by spaces with locally linearly independent bases.

Definition 3.4. Let U be a finite-dimensional subspace of $F(K)$, $\dim U = n$. A set $T = \{t_1, \ldots, t_s\} \subset K$, $s \le n$ is called an *almost interpolation set (AI-set)* w.r.t. U if for any system of neighborhoods B_i of t_i, $i = 1, \ldots, s$, there exist points $t_i' \in B_i$ such that $T' = \{t_1', \ldots, t_s'\}$ is an I-set w.r.t. U.

Next two theorems are valid for any topological space K.

Theorem 3.5. [5] *Let U be a finite-dimensional subspace of $F(K)$, $\dim U = n$, and let $T = \{t_1, \ldots, t_s\} \subset K$, $s \le n$. Then T is an AI–set w.r.t. U if and only if*

$$\dim U_{|B(T')} \ge \operatorname{card} T', \qquad \text{all open } B(T') \supset T', \tag{3.1}$$

for every choice of a nonempty subset $T' \subset T$.

We note that condition (3.1) can be also written as

$$\text{l-dim}_{T'} U \ge \operatorname{card} T',$$

where $\text{l-dim}_{T'} U$ denotes the *local dimension of U on T', i.e.*,

$$\text{l-dim}_{T'} U := \inf \{\dim U_{|B} : \ T' \subset B, \ B \text{ open}\}.$$

Theorem 3.6. [9] *Let $\{u_1, \ldots, u_n\} \subset F(K)$ be a locally linearly independent system and $U = \operatorname{span}\{u_1, \ldots, u_n\}$. A finite set $T = \{t_1, \ldots, t_s\} \subset K$, $s \le n$, is an AI-set w.r.t. U if and only if there exists some permutation σ of $\{1, \ldots, n\}$ such that*

$$t_i \in \operatorname{supp} u_{\sigma(i)}, \quad i = 1, \ldots, s.$$

Generally, for example if K is a domain in $\mathbb{R}^d$, $d > 1$, many almost interpolation sets fail to be I-sets. However, in our case of a totally ordered K the situation is much better. In fact, it is often enough to strengthen the condition of almost interpolation in the following obvious way, in order to get a characterization of I-sets.

Definition 3.7. A set $T = \{t_1, \ldots, t_s\} \subset K$, $s \le n$ is called a *strong AI-set* w.r.t. U if there exist neighborhoods B_i of t_i, $i = 1, \ldots, s$, such that $T' = \{t_1', \ldots, t_s'\}$ is an AI-set w.r.t. U as soon as $t_i' \in B_i$, $i = 1, \ldots, s$.

If U includes only continuous functions on K, *i.e.*, $U \subset C(K)$, then every I-set w.r.t. U is easily seen to be a strong AI-set.

Let now K be again a totally ordered set. We say that a point $t \in K$ has *V-property* if either t is an isolated point of K, or

$$t = \sup\{x \in K : \ x < t\} = \inf\{x \in K : \ x > t\}.$$

(The latter means, in particular, that both sets in the last display are nonempty.)

Lemma 3.8. *Let U be a finite-dimensional subspace of $F(K)$, $\dim U = n$, and let $T = \{t_1, \ldots, t_s\} \subset K$, $s \leq n$, such that $t_1 < \ldots < t_s$, and $t_{s+1} := t_1$. Suppose that every point t_i, $i = 1, \ldots, s$, satisfies V-property. If T is a strong AI-set w.r.t. U, then for every $P \subset \{1, \ldots, s\}$,*

$$\operatorname{card}(T \cap \bigcup_{i \in P} [t_i, t_{i+1}]) \leq \dim U_{|\bigcup_{i \in P}[t_i, t_{i+1}]}.$$

Proof: On the contrary, suppose that

$$\dim U_{|R_{\tilde{P}}} < \operatorname{card}(T \cap R_{\tilde{P}}) \tag{3.2}$$

for some $\tilde{P} \subset \{1, \ldots, s\}$, where $R_{\tilde{P}} = \bigcup_{j \in \tilde{P}} [t_j, t_{j+1}]$. Then we also have

$$\dim U_{|\operatorname{int} R_{\tilde{P}}} < \operatorname{card}(T \cap R_{\tilde{P}}). \tag{3.3}$$

However, since T is a strong AI-set and every point t_i, $i = 0, \ldots, s$, satisfies V-property, for each $t_j \in R_{\tilde{P}}$ we can find a point $t'_j \in \operatorname{int} R_{\tilde{P}}$ such that

$$T' = \{t'_1, \ldots, t'_s\}$$

is an AI-set (we set $t'_j = t_j$ when $t_j \notin R_{\tilde{P}}$). Thus,

$$\operatorname{card}(T' \cap \operatorname{int} R_{\tilde{P}}) = \operatorname{card}(T \cap R_{\tilde{P}}), \tag{3.4}$$

and, because of (3.3),

$$\dim U_{|\operatorname{int} R_{\tilde{P}}} < \operatorname{card}(T' \cap \operatorname{int} R_{\tilde{P}}),$$

which is impossible in view of Theorem 3.5. $\square$

¿From this lemma and Theorem 2.1 we immediately get the following result describing relationship between I-sets and strong AI-sets w.r.t. a weak Chebyshev space.

Theorem 3.9. *Let U be an n-dimensional weak Chebyshev subspace of $F(K)$, and let $T = \{t_1, \ldots, t_n\} \subset K \setminus Z(U)$. Suppose that every point t_i, $i = 1, \ldots, n$, satisfies V-property.*

1) *If T is a strong AI-set w.r.t. U, then T is an I-set.*
2) *Moreover, if $U \subset C(K)$, then the following conditions are equivalent:*
   - *T is an I-set w.r.t. U.*
   - *T is a strong AI-set w.r.t. U.*

The following example shows that Theorem 3.9 is not true in general if the points of T do not satisfy V-property.

Example 3.10. Let $K = [-1,1] \cup \{-2,2\} \subset \mathbb{R}$ and assume that $U = \text{span}\{u_1, u_2\}$ where $u_1(t) = t,\ t \in K$, and

$$u_2(t) = \begin{cases} 1 - t^2 & \text{if } t \in [-1,1] \\ 0 & \text{if } t \in \{-2,2\}. \end{cases}$$

It then follows that U is a weak Chebyshev space. Set $T = \{t_1, t_2\}$ where $t_1 = -1$ and $t_2 = 1$. Then in view of Theorem 3.5, T is an AI-set w.r.t. U. Moreover, T is a strong AI-set, since $T' = \{t_1', t_2'\}$ is an AI-set for all $t_1 \le t_1' < t_2' \le t_2$. However, T fails to be an I-set w.r.t. U since $T \subset Z(u_2)$. It is also easy to see that both t_1 and t_2 fail to have V-property.

We now turn to the main subject of this section: characterization of I-sets for weak Chebyshev spaces with LI-basis.

Theorem 3.11. *Let U be an n-dimensional weak Chebyshev subspace of $C(K)$, such that $U = \text{span}\{u_1, \ldots, u_n\}$, where $\{u_1, \ldots, u_n\}$ is an LI-system, and let $T = \{t_1, \ldots, t_n\} \subset K \setminus Z(U)$. Suppose that every point t_i, $i = 1, \ldots, n$, satisfies V-property. Then T is an I-set w.r.t. U if and only if there exists some permutation σ of $\{1, \ldots, n\}$ such that*

$$t_i \in \text{int supp}\, u_{\sigma(i)}, \quad i = 1, \ldots, n. \tag{3.5}$$

Proof: Let us first assume that (3.5) holds. We show that T is a strong AI-set w.r.t. U. It follows from Theorem 3.6 that T is an AI-set. Let $V_i :=$ int supp $u_{\sigma(i)}$, $i = 1, \ldots, n$. Then V_i is an open neighborhood of t_i, $i = 1, \ldots, n$, and again in view of Theorem 3.6, $T' = \{t_1', \ldots, t_n'\}$ is an AI-set w.r.t. U for all $t_i' \in V_i$, $i = 1, \ldots, n$. This shows that T is a strong AI-set. It then follows from Theorem 3.9 that T is an I-set w.r.t. U. (This direction is even true without the assumption that $U \subset C(K)$.)

For the converse assume that T is an I-set w.r.t. U. We prove (3.5) by induction on n (where we do not use the hypothesis on U to be a weak Chebyshev space).

Let $n = 1$. Then $U = \text{span}\,\{u_1\}$ and $T = \{t_1\} \subset K \setminus Z(u_1)$. Hence, $u_1(t_1) \ne 0$, and, since $u_1 \in C(K)$, $t_1 \in$ int supp u_1.

Assume now that the statement is true up to $n-1$. Since $T = \{t_1, \ldots, t_n\}$ is an I-set w.r.t. U, it is also an AI-set, which, in view of Theorem 3.6, implies that there exists some permutation σ of $\{1, \ldots, n\}$ such that

$$t_i \in \text{supp}\, u_{\sigma(i)}, \quad i = 1, \ldots, n.$$

Without loss of generality assume that $\sigma(i) = i$, $i = 1, \ldots, n$. Suppose now that $t_1 \notin$ int supp u_1. Then $u_1(t_1) = 0$ since $u_1 \in C(K)$.

Let $M := (u_i(t_j))_{i,j=1}^n$ and let M_{i1} denote the submatrix of M obtained by omitting the i-th row and the first column. Then

$$\det M = \sum_{i=1}^{n} (-1)^{i+1} u_i(t_1) \det M_{i1}.$$

Since T is an I-set, $\det M \neq 0$, which implies that $u_\ell(t_1) \neq 0$ and $\det M_{\ell 1} \neq 0$ for some $\ell \in \{2, \ldots, n\}$.

Hence, $\{t_2, \ldots, t_n\}$ is an I-set w.r.t. $\operatorname{span}\{u_1, \ldots, u_{\ell-1}, u_{\ell+1}, \ldots, u_n\}$. Applying the induction hypothesis to this situation we find a bijection $\tilde{\sigma}$ from $\{2, \ldots, n\}$ to $\{1, \ldots, \ell-1, \ell+1, \ldots, n\}$ such that

$$t_i \in \operatorname{int} \operatorname{supp} u_{\tilde{\sigma}(i)}, \quad i = 2, \ldots, n.$$

Moreover, $u_\ell(t_1) \neq 0$ implies that $t_1 \in \operatorname{int} \operatorname{supp} u_\ell$.

Therefore, setting

$$\sigma(i) := \begin{cases} \tilde{\sigma}(i) & \text{if } i = 2, \ldots, n, \\ \ell & \text{if } i = 1, \end{cases}$$

we obtain the desired statement

$$t_i \in \operatorname{int} \operatorname{supp}\, u_{\sigma(i)},\ i = 1, \ldots, n.$$

This completes the proof of Theorem 3.11. $\square$

Example 3.10 also shows that V-property of points t_i is essential in the formulation of Theorem 3.11. Indeed, it is easy to see that the functions u_1, u_2 of Example 3.10 form an LI-system, and $T = \{-1, 1\}$ fails to be an I-set w.r.t. $U = \operatorname{span}\{u_1, u_2\}$ despite the fact that (3.5) holds.

If we now take $K = [a, b] \subset \mathbb{R}$, then V-property is satisfied for every $t \in K$ except $t = a$ or b. Therefore, the hypotheses of Theorem 3.11 do not allow T to include the endpoints of the interval $[a, b]$. In fact, some extra conditions at these points have to be imposed.

Theorem 3.12. *Let U be an n-dimensional weak Chebyshev subspace of $F[a, b]$, such that $U = \operatorname{span}\{u_1, \ldots, u_n\}$, where $\{u_1, \ldots, u_n\}$ is an LI-system, and let $T = \{t_1, \ldots, t_n\} \subset K \backslash Z(U)$. Suppose that there exists some permutation σ of $\{1, \ldots, n\}$ such that*

1) $t_i \in \operatorname{int} \operatorname{supp} u_{\sigma(i)}, \quad i = 1, \ldots, n,$
2) $u_{\sigma(i)}(t_i) \neq 0 \quad$ *if* $\quad t_i \in \{a, b\}, \quad$ *and*
3) $u_{\sigma(i)}(t_i) u_{\sigma(j)}(t_j) - u_{\sigma(i)}(t_j) u_{\sigma(j)}(t_i) \neq 0 \quad$ *if* $\quad t_i, t_j \in \{a, b\},\ t_i \neq t_j.$

Then T is an I-set w.r.t. U.

Proof: Let us first consider the case $n = 1$. If $t_1 \in (a, b)$, then t_1 obviously satisfies V-property, and hence $T = \{t_1\}$ is an I-set w.r.t. U by Theorem 3.11. Otherwise, $u_1(t_1) \neq 0$ by 2), and T is an I-set again.

Suppose $n \geq 2$. If $T \subset (a, b)$, then every point t_i has V-property, and the statement follows from Theorem 3.11. However, in the case $T \cap \{a, b\} \neq \emptyset$ Theorem 3.11 is not applicable. Therefore, we argue as follows.

We first extend the interval $[a,b]$ to the open interval $\tilde{K} := (a-\epsilon, b+\epsilon)$ for some $\epsilon > 0$. Then every $t_i \in T$, $i = 1, \ldots, n$, obviously satisfies V-property (w.r.t. $\tilde{K}$). Moreover, extend each function u_i, $i = 1, \ldots, n$, to a function $\tilde{u}_i \in F(\tilde{K})$ by

$$\tilde{u}_i(t) = \begin{cases} u_i(t) & \text{if } t \in [a,b], \\ u_i(a) & \text{if } t \in (a-\epsilon, a), \\ u_i(b) & \text{if } t \in (b, b+\epsilon). \end{cases}$$

Then $\tilde{U} := \operatorname{span}\{\tilde{u}_1, \ldots, \tilde{u}_n\}$ is again a weak Chebyshev space (while $\{\tilde{u}_1, \ldots, \tilde{u}_n\}$ is no longer an LI-system).

If T is a strong AI-set w.r.t. $\tilde{U}$, then, by Theorem 3.9, it is also an I-set w.r.t. $\tilde{U}$ and, in particular, w.r.t. U (since $U = \tilde{U}_{|[a,b]}$ and $T \subset [a,b]$). Thus, it suffices to show that T is a strong AI-set w.r.t. $\tilde{U}$.

To this end we consider sufficiently small open neighbourhoods V_i of t_i's, such that

$$\begin{aligned} & V_i \subset \operatorname{int} \operatorname{supp} \tilde{u}_{\sigma(i)}, \\ & V_i \cap V_j = \emptyset \ \text{ if } \ i \neq j, \\ & V_i \subset (a,b) \ \text{ if } \ t_i \in (a,b), \end{aligned}$$

and take arbitrary points $\tilde{t}_i \in V_i$, $i = 1, \ldots, n$. We have to check that $\tilde{T} := \{\tilde{t}_1, \ldots, \tilde{t}_n\}$ is an AI-set w.r.t. $\tilde{U}$. In view of Theorem 3.5 this will follow if we prove that

$$\text{l-dim}_{T'} \tilde{U} \geq \operatorname{card} T', \tag{3.6}$$

for every nonempty $T' \subset \tilde{T}$.

Suppose without loss of generality that $\tilde{t}_1 < \tilde{t}_2 < \cdots < \tilde{t}_n$. Let $T' = \{\tilde{t}_{i_1}, \ldots, \tilde{t}_{i_r}\} \subset \tilde{T}$. If $T' \subset (a,b)$, then Theorem 3.6 ensures that T' is an AI-set w.r.t. U since $\tilde{t}_{i_j} \in V_{i_j} \subset \operatorname{supp} u_{\sigma(i_j)}$, $j = 1, \ldots, r$. Therefore,

$$\text{l-dim}_{T'} \tilde{U} = \text{l-dim}_{T'} U \geq r = \operatorname{card} T'$$

by Theorem 3.5, and (3.6) holds.

Assume that $T' \setminus (a,b) \neq \emptyset$. Obviously, at most two points in T' may lie outside (a,b). We consider only the worst case $T' \setminus (a,b) = \{\tilde{t}_{i_1}, \tilde{t}_{i_r}\} = \{\tilde{t}_1, \tilde{t}_n\}$. (The other cases can be handled analogously.) Then necessarily $t_1 = a$, $t_n = b$. Set $\hat{T} := T' \setminus \{\tilde{t}_1, \tilde{t}_n\}$. Since $\hat{T} \subset (a,b)$, we have, as in the above,

$$\text{l-dim}_{\hat{T}} U \geq \operatorname{card} \hat{T} = r - 2.$$

If $\text{l-dim}_{\hat{T}} U \geq r$, then

$$\text{l-dim}_{T'} \tilde{U} \geq \text{l-dim}_{\hat{T}} \tilde{U} = \text{l-dim}_{\hat{T}} U \geq r = \operatorname{card} T',$$

and (3.6) follows. Otherwise, recall that by the definition of $\tilde{u}_i$ we have

$$\begin{aligned} \tilde{u}_{\sigma(1)}(\tilde{t}_1) &= u_{\sigma(1)}(t_1), \quad & \tilde{u}_{\sigma(n)}(\tilde{t}_n) &= u_{\sigma(n)}(t_n), \\ \tilde{u}_{\sigma(1)}(\tilde{t}_n) &= u_{\sigma(1)}(t_n), \quad & \tilde{u}_{\sigma(n)}(\tilde{t}_1) &= u_{\sigma(n)}(t_1), \end{aligned}$$

and hence conditions 2) and 3) ensure that

$$\tilde{u}_{\sigma(1)}(\tilde{t}_1) \neq 0, \quad \tilde{u}_{\sigma(n)}(\tilde{t}_n) \neq 0, \quad \det \begin{pmatrix} \tilde{u}_{\sigma(1)}(\tilde{t}_1) & \tilde{u}_{\sigma(1)}(\tilde{t}_n) \\ \tilde{u}_{\sigma(n)}(\tilde{t}_1) & \tilde{u}_{\sigma(n)}(\tilde{t}_n) \end{pmatrix} \neq 0. \tag{3.7}$$

Moreover, by [9, Theorem 3.4], since $\{u_1, \ldots, u_n\}$ is an LI-system, we have

$$\text{l-dim}_{\hat{T}} U = \operatorname{card} \{i = 1, \ldots, n : \hat{T} \cap \operatorname{supp} u_i \neq \emptyset\}. \tag{3.8}$$

If $\text{l-dim}_{\hat{T}} U = r - 2$, then (3.8) implies that

$$\tilde{t} \notin \operatorname{supp} u_{\sigma(1)} \cup \operatorname{supp} u_{\sigma(n)}, \quad \text{all } \tilde{t} \in \hat{T}.$$

Combining this with (3.7), we see that $\text{l-dim}_{T'} \tilde{U} \geq r$, and (3.6) holds. If $\text{l-dim}_{\hat{T}} U = r - 1$, then by (3.8),

$$\tilde{t} \notin \operatorname{supp} u_{\sigma(i)}, \quad \text{all } \tilde{t} \in \hat{T},$$

for at least one $i \in \{1, n\}$. Since $\tilde{u}_{\sigma(i)}(\tilde{t}_i) \neq 0$, $i \in \{1, n\}$, we again have $\text{l-dim}_{T'} \tilde{U} \geq r$, which completes the proof. □

It is easy to see that condition 3) of Theorem 3.12 is superfluous if $\{u_1, \ldots, u_n\} \subset C[a, b]$ is simultaneously an LI-system and WD-system, *i.e.*, in the setting of Theorem 3.3. Indeed, in this case 3) is a consequence of 2) in view of the following lemma due to Carnicer and Peña.

Lemma 3.13. [2] *Let $\{u_1, u_2\} \subset C[a, b]$ be simultaneously an LI-system and WD-system. If $u_1(a) \neq 0$ and $u_2(b) \neq 0$, then*

$$\det \begin{pmatrix} u_1(a) & u_1(b) \\ u_2(a) & u_2(b) \end{pmatrix} \neq 0.$$

Moreover, conditions 1) and 2) are now equivalent to

$$t_i \in \{x \in K : u_{\sigma(i)}(x) \neq 0\}, \qquad i = 1, \ldots, n,$$

which shows that Theorem 3.3 follows from Theorem 3.12.

§4. Proof of Theorem 2.4

On the contrary, suppose that

$$\dim U_{|\tilde{Z}} \geq n - m.$$

Then there exists $T = \{t_1, \ldots, t_{n-m}\} \subset \tilde{Z}$ such that

$$\dim U_{|T} = n - m,$$

and, since $x_1, \ldots, x_m$ are cyclically separated w.r.t. $\tilde{Z}$,

$$T \subset \bigcup_{i=1}^{m} (y_{2i-1}, y_{2i})$$

where $\{y_j\}_{j=1}^{2m}$ satisfy (2.2) – (2.4). We set

$$X := \{x_1, \ldots, x_m\}, \quad Y := \{y_1, \ldots, y_{2m}\},$$
$$x_{m+1} := x_1, \quad y_{2m+1} := y_1.$$

Let $j^* \in \{2m-2, 2m-1, 2m\}$ be a unique index such that

$$y_{j^*} > y_{j^*+1}.$$

We set

$$n_j := \text{card}\,([y_j, y_{j+1}] \cap (X \cup T)),\ j = 1, \ldots, 2m.$$

It is obvious that

$$\sum_{j=1}^{2m} n_j = n. \tag{4.1}$$

We now construct a function $v \in U$ such that $(X \cup T) \cap Z(v) = \emptyset$. Since $\dim U_{|T} = n - m = \text{card}\, T$, we interpolate on T as follows. Let $j \neq j^*$ and $T \cap [y_j, y_{j+1}] \neq \emptyset$. Then in view of (2.4), j is an odd number, which implies that $[y_j, y_{j+1}] \cap X = \emptyset$. Thus we have

$$T \cap [y_j, y_{j+1}] = \{t_{k_j} < \ldots < t_{k_j+n_j-1}\},\ n_j \geq 1.$$

We require

$$\text{sign}\ v(t_{k_j+s}) = (-1)^s \text{sign}\ u(y_j),\ s = 0, \ldots, n_j - 1. \tag{4.2}$$

Consider now the index j^* and assume that $T \cap [y_{j^*}, y_{j^*+1}] \neq \emptyset$. In view of (2.4), it is quite clear that only the case $j^* = 2m - 1$; *i.e.*, $y_{2m} < x_1, x_m < y_{2m-1}$ can occur. Moreover, it then follows that $n_{j^*} = \text{card}\,(T \cap [y_{j^*}, y_{j^*+1}]) \geq 1$ and there exists $p \in \{0, \ldots, n_{j^*}\}$ such that

$$T \cap [y_{j^*}, y_{j^*+1}] = \{t_{k_{j^*}}, \ldots, t_{k_{j^*}+n_{j^*}-1}\}$$

and

$$t_{k_{j^*}+p} < \ldots < t_{k_{j^*}+n_{j^*}-1} < y_{2m} < x_1 <$$
$$\ldots < x_m < y_{2m-1} < t_{k_{j^*}} < \ldots < t_{k_{j^*}+p-1}.$$

This means that all points lie to the left of y_{2m} if $p = 0$ and to the right of y_{2m-1} if $p = n_{j^*}$. If $p \neq 0$, we require

$$\text{sign}\ v(t_{k_{j^*}+s}) = \begin{cases} (-1)^s \text{sign}\ u(y_{2m-1}) & \text{if } s = 0, \ldots, p-1 \\ (-1)^{s+n-1} \text{sign}\ u(y_{2m-1}) & \text{if } s = p, \ldots, n_{j^*} - 1. \end{cases} \tag{4.3}$$

If $p = 0$, we require

$$\text{sign } v(t_{k_{j^*}+s}) = (-1)^{s+n_{j^*}-1}\text{sign } u(y_{2m}), \; s = 0, \ldots, n_{j^*} - 1. \tag{4.4}$$

Since $x_1, \ldots, x_m$ are essential zeros of u, we can apply [15, Lemma 2] and require

$$v(x_i) \neq 0, \quad i = 1, \ldots, m. \tag{4.5}$$

Thus, a function $v \in U$ with properties (4.2) – (4.5) must exist. In view of (2.3), we can find $\epsilon > 0$ such that

$$|\epsilon v(y_j)| < |u(y_j)|, \; j = 1, \ldots, 2m. \tag{4.6}$$

We now show that at least one of the functions $u - \epsilon v$, $u + \epsilon v \in U$ has n sign changes on K contradicting the weak Chebyshev property of U.

To this end we determine a subset D of $\{1, \ldots, 2m\}$ as follows. We say that $j \in D$ if both $n_j \neq 0$ and

$$\text{sign } u(y_j)u(y_{j+1}) = \begin{cases} (-1)^{n_j+1} & \text{if } j \neq j^* \\ (-1)^{n_j+n} & \text{if } j = j^*. \end{cases} \tag{4.7}$$

We now divide D into two subsets P and N as follows. Let $j \in D$. We say that $j \in P$ if either j is odd, or $j = 2i$ and

$$\text{sign } u(y_{2i})v(x_{i+1}) = \begin{cases} 1 & \text{if } y_{2i} < x_{i+1} \\ (-1)^{n+1} & \text{if } y_{2i} > x_{i+1}. \end{cases} \tag{4.8}$$

Note that $y_{2i} > x_{i+1}$ can happen only when $2i = 2m = j^*$. In this case $n_{j^*} = 1$ and, in view of (4.7), (4.8) is equivalent to

$$\text{sign } u(y_1)v(x_1) = 1.$$

We set $N = D \setminus P$ and suppose, without loss of generality, that

$$\text{card } P \geq \text{card } N. \tag{4.9}$$

We shall show that $u - \epsilon v$ has at least n sign changes on K contradicting the assumption on U to be a weak Chebyshev space. (If card $P <$ card N, then similar argumentation shows that $u + \epsilon v$ has at least n sign changes.)

We first prove the following statement.

Lemma. *The function $u - \epsilon v$ has at least*

$$n - n_{j^*} + \text{card } (P \setminus \{j^*\}) - \text{card } (N \setminus \{j^*\}) \tag{4.10}$$

sign changes in the interval $[y_{j^*+1}, y_{j^*}]$.

Proof: Suppose that $j \neq j^*$ and $n_j \geq 1$. Let

$$[y_j, y_{j+1}] \cap (T \cup X) = \{\zeta_1, \dots, \zeta_{n_j}\}$$

such that $y_j < \zeta_1 < \dots < \zeta_{n_j} < y_{j+1}$. Since $u(\zeta_i) = 0,\ i = 1, \dots, n_j$, it follows from (4.2) that $u - \epsilon v$ has at least $n_j - 1$ sign changes in $[\zeta_1, \zeta_{n_j}]$. Moreover, if $j \notin N$, we can find some additional sign changes of $u - \epsilon v$ in $[y_j, y_{j+1}]$.

Indeed, if $j \in \{1, \dots, 2m\} \setminus D$, then by the definition of D,

$$\text{sign } u(y_j)u(y_{j+1}) = (-1)^{n_j}.$$

Therefore, in view of (4.6), we obtain

$$\text{sign } (u - \epsilon v)(y_j) = (-1)^{n_j} \text{sign } (u - \epsilon v)(y_{j+1})$$

which would be impossible if $u - \epsilon v$ had exactly $n_j - 1$ sign changes in $[y_j, y_{j+1}]$. Thus, $u - \epsilon v$ has at least n_j sign changes there when $j \in \{1, \dots, 2m\} \setminus D$.

We next consider the case when $j \in P$. Then, if j is an odd number, it follows from (4.2) that

$$\text{sign } u(y_j) = \text{sign } v(\zeta_1).$$

(Note that in this case $X \cap [y_j, y_{j+1}] = \emptyset$ and $t_{k_j} = \zeta_1$.) Otherwise, if j is even, it follows from (2.4) and the fact that $T \subset \tilde{Z}$ that $[y_j, y_{j+1}] \cap (T \cup X) = \{x_{i+1}\}$ where $j = 2i$. Thus $\zeta_1 = x_{i+1}$ and by (4.8), again

$$\text{sign } u(y_j) = \text{sign } v(\zeta_1).$$

Summarizing both cases, and by (4.6), we obtain

$$\text{sign } (u - \epsilon v)(y_j) = -\text{sign } (u - \epsilon v)(\zeta_1).$$

Therefore, $u - \epsilon v$ has at least n_j sign changes in $[y_j, \zeta_{n_j}]$ if $j \in P$. Moreover, it follows from (4.7) that

$$\text{sign } (u - \epsilon v)(y_j) = (-1)^{n_j+1} \text{sign } (u - \epsilon v)(y_{j+1}).$$

Hence, $u - \epsilon v$ cannot have exactly n_j sign changes in $[y_j, y_{j+1}]$. By the above arguments, it has at least $n_j + 1$ sign changes there when $j \in P$.

Thus we have shown that $u - \epsilon v$ has at least $n_j - 1$, n_j or n_{j+1} sign changes in $[y_j, y_{j+1}]$ if $j \in N$, $j \in \{1, \dots, 2m\} \setminus D$ or $j \in P$, respectively. Taking into consideration that we had supposed that $j \neq j^*$ we conclude that $u - \epsilon v$ has at least

$$\sum_{j \in N \setminus \{j^*\}} (n_j - 1) + \sum_{j \notin D \cup \{j^*\}} n_j + \sum_{j \in P \setminus \{j^*\}} (n_j + 1)$$

sign changes in $[y_{j^*+1}, y_{j^*}] = \bigcup_{\substack{j=1 \\ j \neq j^*}}^{2m} [y_j, y_{j+1}]$, which, in view of (4.1), implies (4.10) and completes the proof of the lemma. $\square$

To finish the proof of Theorem 2.4 we have to show that $u - \epsilon v$ has additional sign changes in the interval $[y_{j^*}, y_{j^*+1}]$ if necessary. To this end we consider several cases. In each case we show that $u-\epsilon v$ has at least n sign changes on K contradicting the assumption on U to be a weak Chebyshev space.

Case 1. Assume that $n_{j^*} = 0$. Then $j^* \notin D$ which implies that $P \setminus \{j^*\} = P$, $N \setminus \{j^*\} = N$ and, in view of (4.9), the lemma immediately yields that $u - \epsilon v$ has at least n sign changes.

Case 2. Assume that $n_{j^*} = 1$ and $j^* \in N$. Then

$$\text{card}\,(P \setminus \{j^*\}) - \text{card}\,(N \setminus \{j^*\}) \geq 1,$$

and, hence,

$$n - n_{j^*} + \text{card}\,(P \setminus \{j^*\}) - \text{card}\,(N \setminus \{j^*\}) \geq n$$

which implies that $u - \epsilon v$ has at least n sign changes.

Case 3. Assume that $n_{j^*} = 1$ and $j^* \notin D$. By the lemma, $u - \epsilon v$ has at least $n-1$ sign changes in $[y_{j^*+1}, y_{j^*}]$. Since $n_{j^*} \neq 0$, it follows from the definition of D (see (4.7)), that

$$\text{sign}\, u(y_{j^*}) u(y_{j^*+1}) = (-1)^{n_{j^*}+n+1}.$$

Hence, by (4.6) we obtain

$$\text{sign}\,(u - \epsilon v)(y_{j^*+1}) = (-1)^n \text{sign}\,(u - \epsilon v)(y_{j^*}).$$

Therefore, $u - \epsilon v$ has at least n sign changes in $[y_{j^*+1}, y_{j^*}]$.

Case 4. Assume that $n_{j^*} = 1$ and $j^* \in P$. It follows from the lemma that $u - \epsilon v$ has at least $n - 2$ sign changes in $[y_{j^*+1}, y_{j^*}]$. By (4.6) and (4.7) we obtain that

$$\text{sign}\,(u - \epsilon v)(y_{j^*}) = (-1)^{n+1} \text{sign}\,(u - \epsilon v)(y_{j^*+1}).$$

Therefore, $u-\epsilon v$ must have at least $n-1$ sign changes in $[y_{j^*+1}, y_{j^*}]$. Since $n_{j^*} = 1$, we have

$$(y_{j^*}, y_{j^*+1}) \cap (X \cup T) = \{\zeta\}.$$

It follows from (4.3), (4.4), (4.7) and (4.8) that

$$\text{sign}\, v(\zeta) = \begin{cases} \text{sign}\, u(y_{j^*}) & \text{if } \zeta > y_{j^*} \\ \text{sign}\, u(y_{j^*+1}) & \text{if } \zeta < y_{j^*+1}. \end{cases}$$

Then, since $u = 0$ on $X \cup T$, the function $u - \epsilon v$ has a sign change in (y_{j^*}, ζ) if $\zeta > y_{j^*}$ and in (ζ, y_{j^*+1}) if $\zeta < y_{j^*+1}$, respectively. Anyway, $u - \epsilon v$ has at least one sign change outside $[y_{j^*+1}, y_{j^*}]$.

Again, the total number of sign changes is at least n.

Case 5. Assume that $n_{j^*} \geq 2$. Then j^* must be odd which implies that $j^* = 2m-1$. It then follows that

$$T \cap [y_{2m-1}, y_{2m}] = \{t_{k_{2m-1}}, \ldots, t_{k_{2m-1}+n_{2m-1}-1}\},$$

and, for some $p \in \{0, \ldots, n_{2m-1}\}$,

$$\begin{aligned} &t_{k_{2m-1}+p} < \ldots < t_{k_{2m-1}+n_{2m-1}-1} < y_{2m} < x_1 < \\ &\ldots < x_m < y_{2m-1} < t_{k_{2m-1}} < \ldots < t_{k_{2m-1}+p-1}. \end{aligned}$$

We set

$$t_{\min} = \begin{cases} t_{k_{2m-1}+p} & \text{if } p \neq n_{2m-1}, \\ y_{2m} & \text{if } p = n_{2m-1}, \end{cases} \qquad t_{\max} = \begin{cases} t_{k_{2m-1}+p-1} & \text{if } p \neq 0, \\ y_{2m-1} & \text{if } p = 0. \end{cases}$$

In view of (4.3), it is easy to see that $u - \epsilon v$ has at least $n_{2m-1} - p - 1$ sign changes in $[t_{\min}, y_{2m}]$ and at least p sign changes in $[y_{2m-1}, t_{\max}]$ if $p \neq 0$. Moreover, by (4.4), $u - \epsilon v$ has at least n_{2m-1} sign changes in $[t_{\min}, y_{2m}]$ if $p = 0$.

If $j^* \notin D$, then by the lemma, $u - \epsilon v$ has at least $n - n_{2m-1}$ sign changes in $[y_{2m}, y_{2m-1}]$. By the definition of D,

$$\text{sign } u(y_{2m}) = (-1)^{n_{2m-1}+n-1} \text{sign } u(y_{2m-1})$$

and, in view of (4.6),

$$\text{sign } (u - \epsilon v)(y_{2m}) = (-1)^{n_{2m-1}+n-1} \text{sign } (u - \epsilon v)(y_{2m-1}).$$

Therefore, $u - \epsilon v$ must in fact have at least $n - n_{2m-1} + 1$ sign changes in $[y_{2m}, y_{2m-1}]$.

Thus, by the above arguments, $u - \epsilon v$ has at least $(n_{2m-1} - p - 1) + p + (n - n_{2m-1} + 1) = n$ sign changes in K.

Finally, let $j^* \in D$. Then $j^* \in P$ since j^* is odd. By the lemma, the function $u - \epsilon v$ has at least $n - n_{2m-1} - 1$ sign changes in $[y_{2m}, y_{2m-1}]$. As above, we deduce from (4.6) and (4.7) that

$$\text{sign } (u - \epsilon v)(y_{2m}) = (-1)^{n_{2m-1}+n} \text{sign } (u - \epsilon v)(y_{2m-1})$$

which shows that $u - \epsilon v$ must have at least $n - n_{2m-1}$ sign changes in $[y_{2m}, y_{2m-1}]$. If now $p = 0$ or $p = n_{2m-1}$, then by the above arguments, $u - \epsilon v$ has at least $(n - n_{2m-1}) + n_{2m-1} = n$ sign changes in $[t_{\min}, y_{2m-1}]$ or $[y_{2m}, t_{\max}]$, respectively. If $p \in \{1, \ldots, n_{2m-1} - 1\}$, then $u - \epsilon v$ has at least $(n - n_{2m-1}) + (n_{2m-1} - p - 1) + p = n - 1$ sign changes in $[t_{\min}, t_{\max}]$. Moreover, in view of (4.3),

$$\begin{aligned} \text{sign } (u - \epsilon v)(t_{\min}) &= (-1)^{p+n} \text{sign } u(y_{2m-1}), \\ \text{sign } (u - \epsilon v)(t_{\max}) &= (-1)^{p} \text{sign } u(y_{2m-1}), \end{aligned}$$

which implies that

$$\text{sign } (u - \epsilon v)(t_{\min}) = (-1)^{n} \text{sign } (u - \epsilon v)(t_{\max}).$$

Hence, $u - \epsilon v$ must in fact have at least n sign changes in $[t_{\min}, t_{\max}]$. $\square$

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