18.102 Assignment 3

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Problem 1

(a)

Proof. We want to show that $u \in M'$. First, we show that u is linear.

Let $a, b \in M$ and let $\lambda \in \mathbb{C}$. Then

$$u(\lambda a) = \lim_{k\to\infty} (\lambda a_k) = \lambda \cdot \lim_{k\to\infty} a_k = \lambda u(a) \text{ , and} \tag{1}$$

$$u(a+b) = \lim_{k\to\infty} (a_k + b_k) = \lim_{k\to\infty} a_k + \lim_{k\to\infty} b_k = u(a) + u(b). \tag{2}$$

So u is linear on M. Next, we show that u is bounded.

Let $a \in M$, i.e. $\lim_{k\to\infty} a_k$ exists. Then a is bounded, so $\exists B \geq 0$ such that $\forall k \in \mathbb{N}$, $|a_k| \leq B$. Then by continuity of the norm,

$$\begin{aligned} ||u|| &\leq |u(a)| && (3)\\ &= \left|\lim_{k\to\infty} a_k\right| && (4)\\ &= \lim_{k\to\infty} |a_k| && (5)\\ &\leq B, && (6) \end{aligned}$$

so u is bounded.

Then we conclude that u is a bounded linear functional on M. $\square$