18.102 Assignment 3

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Problem 1

(a)

Proof. We want to show that $u \in M'$. First, we show that u is linear.

Let $a, b \in M$ and let $\lambda \in \mathbb{C}$. Then

$$u(\lambda a) = \lim_{k \to \infty} (\lambda a_k) = \lambda \cdot \lim_{k \to \infty} a_k = \lambda u(a)$$
, and (1)

$$u(a+b) = \lim_{k \to \infty} (a_k + b_k) = \lim_{k \to \infty} a_k + \lim_{k \to \infty} b_k = u(a) + u(b).$$
 (2)

So u is linear on M. Next, we show that u is bounded.

Let $a \in M$, i.e. $\lim_{k \to \infty} a_k$ exists. Then a is bounded, so $\exists B \ge 0$ such that $\forall k \in \mathbb{N}, |a_k| \le B$. Then by continuity of the norm,

$$||u|| \le |u(a)| \tag{3}$$

$$= \left| \lim_{k \to \infty} a_k \right| \tag{4}$$

$$=\lim_{k\to\infty}|a_k|\tag{5}$$

$$\leq B,$$
 (6)

so u is bounded.

Then we conclude that u is a bounded linear functional on M.

(b)

Proof. (By contradiction). Suppose instead that $\exists b \in \ell^1$ such that $\forall a \in \ell^\infty$,

$$v(a) = \sum_{k=1}^{\infty} a_k b_k. \tag{7}$$

Define $e_n := \{\delta_{kn}\}_k \in \ell^{\infty}$, for fixed $n \in \mathbb{N}$. Then $\lim_{k \to \infty} \delta_{nk} = 0$, so $e_n \in M$ as well. By equation (7), we have

$$v(e_n) = \sum_{k=1}^{\infty} \delta_{kn} b_k = b_n.$$
 (8)

By the Hahn-Banach theorem, $v|_M=u$. But $u(e_n)=\lim_{k\to\infty}\delta_{kn}=0$, and since $e_n\in M$, we have

$$b_n = v(e_n) = u(e_n) = 0.$$
 (9)

This must hold for any $n \in \mathbb{N}$, so $b_n = 0 \ \forall n \in \mathbb{N}$. Then $b = \{b_k\}_k = (0, 0, ...)$, so v = 0 by definition. But

$$0 = v(1, 1, \dots) = u(1, 1, \dots) = 1, \quad (\Rightarrow \Leftarrow)$$
 (10)

so we arrive at a contradiction to the initial assumption.

Therefore
$$\nexists b \in \ell^1$$
 such that $\forall a \in \ell^\infty$, $v(a) = \sum_k a_k b_k$.

Problem 2

(a)

Proof. First we show that $||T^{\dagger}|| \leq ||T||$. We have

$$||T^{\dagger}|| = \sup_{||f||=1} ||T^{\dagger}f||$$
 (11)

$$= \sup_{\|f\|=1} \|f \circ T\| \tag{12}$$

$$\leq \sup_{\|f\|=1} \|f\| \|T\| \tag{13}$$

$$\leq ||T||. \tag{14}$$

So, $T^{\dagger}: W' \to V'$ is bounded, i.e. $T^{\dagger} \in \mathcal{B}(W', V')$.

Next we show that $||T^{\dagger}|| \ge ||T||$.

Let $x \in V$ with ||x|| = 1. Since W is a normed space, then by the theorem from lecture 6 (corollary to Hahn-Banach Thm.), $\exists f \in W'$ such that ||f|| = 1 and $f(w) = ||w|| \ \forall w \in W \setminus \{0\}$.

Since $T: V \to W$, then w = Tx for $x \in V$, so f(Tx) = ||Tx||. Then we have

$$||T^{\dagger}|| = \sup_{\|f\|=1} ||T^{\dagger}f|| \tag{15}$$

$$= \sup_{\|f\|=1} \sup_{\|x\|=1} |(f \circ T)(x)| \tag{16}$$

$$= \sup_{\|f\|=1} \sup_{\|x\|=1} |f(Tx)| \tag{17}$$

$$\geq \sup_{\|x\|=1} \|Tx\| \tag{18}$$

$$= ||T||, \tag{19}$$

as desired.

Thus
$$T^{\dagger} \in \mathcal{B}(W', V')$$
 and $||T^{\dagger}|| = ||T||$.

(b)

Proof. Let $a \in \ell^p$, and $b_k = Ra_k$. We want to show that $b = \{b_k\}_k \in \ell^p$. Since $a \in \ell^p$, then a is bounded, which implies that $\exists B \geq 0$ such that $\forall k \in \mathbb{N}$, $|a_k| \leq B$. Then $b_k = Ra_k := a_{k-1}$, with $b_1 = a_0 := 0$.

Thus $\forall k \in \mathbb{N}$, $|b_k| = |a_{k-1}| \leq B$, so b is also bounded. We have

$$||b||_{\infty} = \sup_{k} |b_k| = \sup_{k} |a_{k-1}|.$$
 (20)

So, $R: \ell^p \to \ell^p$. Next we compute the operator norm of R:

$$||R|| = \sup_{||a||=1} ||Ra|| \tag{21}$$

$$= \sup_{\|a\|=1} \sup_{k} |Ra_k| \tag{22}$$

$$= \sup_{\|a\|=1} \sup_{k} |a_{k-1}| \tag{23}$$

$$= \sup_{\|a\|=1} \sup_{k} \{0, |a_1|, |a_2|, \dots\}$$
 (24)

$$= \sup_{\|a\|=1} \sup_{k} |a_k| \tag{25}$$

$$= \sup_{\|a\|=1} \|a\| \tag{26}$$

$$=1. (27)$$

Therefore, $R \in \mathcal{B}(\ell^p, \ell^p)$ with ||R|| = 1.

 (\mathbf{c})

Suppose $1 \leq p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. From assignment 1, we identify $(\ell^p)'$ with ℓ^q via the pairing: $f \in (\ell^p)' \iff \exists b \in \ell^q$ such that $\forall a \in \ell^p$,

$$f(a) = \sum_{k=1}^{\infty} a_k b_k, \tag{28}$$

and $||f|| = ||b||_q$.

For example, let $b = \{b_k\}_k = e_1$ defined by $e_1 := \{\delta_{1k}\}_k = \{1, 0, 0, ...\} \in \ell^q$. Then $\forall a \in ell^p$,

$$(R^{\dagger}e_1)(a) := \sum_{k=1}^{\infty} Ra_k \delta_{1k}$$
(29)

$$= Ra_1 \tag{30}$$

$$= a_0 \tag{31}$$

$$=0 (32)$$

$$=\sum_{k=1}^{\infty}a_k\cdot 0. \tag{33}$$

Thus $R^{\dagger}e_1 = 0 \in \ell^q$.

Now let $a \in \ell^p$. Then

$$(R^{\dagger}b)(a) = \sum_{k=1}^{\infty} (Ra)_k b_k \tag{34}$$

$$=\sum_{k=1}^{\infty}a_{k-1}b_k\tag{35}$$

$$= 0 \cdot b_k + \sum_{k=2}^{\infty} a_{k-1} b_k \tag{36}$$

$$= \sum_{k=1}^{\infty} a_k b_{k+1}. \tag{37}$$

Hence, $\{(R^{\dagger}b)_k\}_k = \{b_{k+1}\}_k$. Therefore, where R was the right-shift operator, we can identify R^{\dagger} as a left-shift operator.

Problem 3

Proof. To show that $m^*(E+x) = m^*(E)$, we will show both that $m^*(E+x) \ge m^*(E)$ and $m^*(E+x) \le m^*(E)$.

Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of open intervals that covers E. Then $\{I_n+x\}_n$ covers E+x. Since interval length is invariant under translation, we have

$$m^*(E+x) \le \sum_n \ell(I_n + x) = \sum_n \ell(I_n).$$
 (38)

So for every such sequence of intervals $\{I_n\}_n$, we have $m^*(E+x) \leq \sum_n \ell(I_n)$. Thus,

$$m^*(E+x) \le m^*(E).$$
 (39)

Now let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of open intervals that covers E+x. Then $\{I_n-x\}_n$ covers E. Once again, by the translation-invariance of interval length, we have

$$m^*(E) \le \sum_n \ell(I_n - x) = \sum_n \ell(I_n).$$
 (40)

Since this is true for every such sequence of intervals $\{I_n\}$, then

$$m^*(E) \le m^*(E+x).$$
 (41)

From equations (39) and (41), we conclude that $m^*(E+x) = m^*(E)$.

Therefore, the outer measure m^* is translation-invariant.

Problem 4

(a)

Proof. Let $x \in U$. Then $a_x \leq b_x$, and both $(a_x, x] \subset U$ and $[x, b_x) \subset U$. Thus,

$$(a_x, b_x) = (a_x, x] \cup [x, b_x) \subset U, \tag{42}$$

as desired. \Box

(b)

Proof. Let $x, y \in U$ and suppose $y \in (a_x, b_x)$.

Case 1: $y \in (a_x, x]$. Then since y < x,

$$a_y = \inf\{a \in \mathbb{R} \mid (a, y] \subset U\} \tag{43}$$

$$=\inf\{a\in\mathbb{R}\mid (a,y]\cup[y,x]\subset U\}\tag{44}$$

$$=\inf\{a\in\mathbb{R}\mid (a,x]\subset U\}\tag{45}$$

$$= a_x. (46)$$

Similarly,

$$b_y = \sup\{b \in \mathbb{R} \mid [y, b) \subset U\} \tag{47}$$

$$= \sup\{b \in \mathbb{R} \mid [x, y] \cup [y, b) \subset U\} \tag{48}$$

$$= \sup\{b \in \mathbb{R} \mid [x, b) \subset U\} \tag{49}$$

$$=b_x. (50)$$

Case 2: $y \in [x, b_x)$. Then since y > x,

$$a_y = \inf\{a \in \mathbb{R} \mid (a, y] \subset U\} \tag{51}$$

$$=\inf\{a\in\mathbb{R}\mid (a,x]\cup[x,y]\subset U\}\tag{52}$$

$$=\inf\{a\in\mathbb{R}\mid (a,x]\subset U\}\tag{53}$$

$$= a_x. (54)$$

Similarly,

$$b_y = \sup\{b \in \mathbb{R} \mid [y, b) \subset U\}$$
 (55)

$$= \sup\{b \in \mathbb{R} \mid [x, y] \cup [y, b) \subset U\} \tag{56}$$

$$= \sup\{b \in \mathbb{R} \mid [x, b) \subset U\} \tag{57}$$

$$=b_x. (58)$$

So we conclude that $(a_x, b_x) = (a_y, b_y)$