18.102 Assignment 7

Octavio Vega

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Problem 1

(a)

Proof. Let $E = [a, b]$ and let $f : E \to \mathbb{C}$. Suppose $f \in L^p([a, b])$. Then

$$\Rightarrow \int_E |f|^p < \infty \tag{1}$$

$$\Rightarrow \left(\int_E |f|^p\right)^{\frac{1}{p}} < \infty \tag{2}$$

$$\iff ||f||_{L^p(E)} < \infty. \tag{3}$$

Now let $1 \leq q \leq p$. Then by Hölder's inequality, since $1 : E \to \mathbb{C}$ is measurable, then

$$\begin{aligned}
||f||^q_{L^q(E)} &= \int_E |f|^q && (4)\\
&= |||f|^q||_{L^1(E)} && (5)\\
&= ||1 \cdot |f|^q||_{L^1(E)} && (6)\\
&\leq ||1||_{L^{\frac{p}{p-q}}(E)} |||f|^q||_{L^{\frac{p}{q}}(E)} && (7)\\
&= \left(\int_E 1\right)^{\frac{p-q}{p}} \left(\int_E \left||f|^q\right|^{\frac{p}{q}}\right)^{\frac{q}{p}} && (8)\\
&= (b-a)^{\frac{p-q}{p}} \left(\int_E |f|^p\right)^{\frac{q}{p}} && (9)\\
&= (b-a)^{\frac{p-q}{p}} ||f||^q_{L^p(E)} && (10)\\
&< \infty. && (11)
\end{aligned}$$

Hence, $f \in L^p([a, b]) \Rightarrow f \in L^q([a, b])$.

Therefore $L^p([a, b]) \subset L^q([a, b])$. □

(b)

Proof. Let $f \in L^p([a,b])$ and $\epsilon > 0$. Choose N such that

$$||f - f\chi_{[-f^{-1}(N), f^{-1}(N)]}||_p < \frac{\epsilon}{2}. \tag{12}$$

Let $f_n = f\chi_{[-f^{-1}(n), f^{-1}(n)]}$. From PS6.2, Littlewood's third principle tells us that "every measurable function is nearly continuous." This gives us a closed set F such that

$$m([a,b] \backslash F) < \left(\frac{\epsilon}{4N}\right)^p, \tag{13}$$

and the restriction $f_n\big|_F$ is continuous with $f_N(a) = f_N(b) = 0$. $\square$