18.102 Assignment 3

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February 17, 2023

Problem 1

(a)

Proof. We want to show that $u \in M'$. First, we show that u is linear.

Let $a, b \in M$ and let $\lambda \in \mathbb{C}$. Then

$$u(\lambda a) = \lim_{k\to\infty} (\lambda a_k) = \lambda \cdot \lim_{k\to\infty} a_k = \lambda u(a) \text{ , and} \tag{1}$$

$$u(a+b) = \lim_{k\to\infty} (a_k + b_k) = \lim_{k\to\infty} a_k + \lim_{k\to\infty} b_k = u(a) + u(b). \tag{2}$$

So u is linear on M. Next, we show that u is bounded.

Let $a \in M$, i.e. $\lim_{k\to\infty} a_k$ exists. Then a is bounded, so $\exists B \geq 0$ such that $\forall k \in \mathbb{N}$, $|a_k| \leq B$. Then by continuity of the norm,

$$\begin{aligned} ||u|| &\leq |u(a)| & (3) \\ &= \left| \lim_{k\to\infty} a_k \right| & (4) \\ &= \lim_{k\to\infty} |a_k| & (5) \\ &\leq B, & (6) \end{aligned}$$

so u is bounded.

Then we conclude that u is a bounded linear functional on M. □

(b)

Proof. (By contradiction). Suppose instead that $\exists b \in \ell^1$ such that $\forall a \in \ell^\infty$,

$$v(a) = \sum_{k=1}^{\infty} a_k b_k. \tag{7}$$

Define $e_n := \{\delta_{kn}\}_k \in \ell^\infty$, for fixed $n \in \mathbb{N}$. Then $\lim_{k\to\infty} \delta_{nk} = 0$, so $e_n \in M$ as well. By equation (7), we have

$$v(e_n) = \sum_{k=1}^{\infty} \delta_{kn} b_k = b_n. \tag{8}$$

By the Hahn-Banach theorem, $v|_M = u$. But $u(e_n) = \lim_{k\to\infty} \delta_{kn} = 0$, and since $e_n \in M$, we have

$$b_n = v(e_n) = u(e_n) = 0. \tag{9}$$

This must hold for any $n \in \mathbb{N}$, so $b_n = 0\ \forall n \in \mathbb{N}$. Then $b = \{b_k\}_k = (0, 0, ...)$, so $v = 0$ by definition. But

$$0 = v(1, 1, ...) = u(1, 1, ...) = 1, \quad (\Rightarrow\Leftarrow) \tag{10}$$

so we arrive at a contradiction to the initial assumption.

Therefore $\nexists b \in \ell^1$ such that $\forall a \in \ell^\infty$, $v(a) = \sum_k a_k b_k$. $\square$

Problem 2

(a)

Proof. First we show that $\|T^\dagger\| \leq \|T\|$. We have

$$\begin{aligned} \|T^\dagger\| &= \sup_{||f||=1} \|T^\dagger f\| && (11)\\ &= \sup_{\|f\|=1} \|f \circ T\| && (12)\\ &\leq \sup_{\|f\|=1} \|f\|\|T\| && (13)\\ &\leq \|T\|. && (14) \end{aligned}$$

So, $T^\dagger : W' \to V'$ is bounded, i.e. $T^\dagger \in \mathcal{B}(W', V')$.

Next we show that $\|T^\dagger\| \geq \|T\|$.

Let $x \in V$ with $\|x\| = 1$. Since W is a normed space, then by the theorem from lecture 6 (corollary to Hahn-Banach Thm.), $\exists f \in W'$ such that $\|f\| = 1$ and $f(w) = \|w\|\ \forall w \in W \backslash \{0\}$.

Since $T : V \to W$, then $w = Tx$ for $x \in V$, so $f(Tx) = \|Tx\|$. Then we have

$$\begin{aligned}
\|T^\dagger\| &= \sup_{\|f\|=1} \|T^\dagger f\| && (15)\\
&= \sup_{\|f\|=1} \sup_{\|x\|=1} |(f \circ T)(x)| && (16)\\
&= \sup_{\|f\|=1} \sup_{\|x\|=1} |f(Tx)| && (17)\\
&\geq \sup_{\|x\|=1} \|Tx\| && (18)\\
&= \|T\|, && (19)
\end{aligned}$$

as desired.

Thus $T^\dagger \in \mathcal{B}(W', V')$ and $\|T^\dagger\| = \|T\|$. $\square$