

18.102 Assignment 5

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We denote by \mathcal{M} the set of all Lebesgue-measurable subsets of \mathbb{R} .

Problem 1

TODO TODO TODO

Problem 2

TODO TODO TODO

Problem 3

(a)

Proof. (\Rightarrow) Suppose f is measurable.

Let $\alpha \in \mathbb{R}$. We may express the preimage of the set $(\alpha, \infty]$ under the inverse of the restriction of f to E as follows:

$$f^{-1}|_E((\alpha, \infty]) = f^{-1}((\alpha, \infty]) \cap E, \quad (1)$$

and similarly for F :

$$f^{-1}|_F((\alpha, \infty]) = f^{-1}((\alpha, \infty]) \cap F. \quad (2)$$

Since f is measurable, then $f^{-1}((\alpha, \infty]) \in \mathcal{M}$. By assumption, E and F are also measurable. Hence, the intersections in (1) and (2) are also measurable.

Therefore, $f|_E$ and $f|_F$ are measurable.

(\Leftarrow) Suppose $f|_E$ and $f|_F$ are measurable.

Then for ever $\alpha \in \mathbb{R}$, $f^{-1}|_E((\alpha, \infty]) \in \mathcal{M}$ and $f^{-1}|_F((\alpha, \infty]) \in \mathcal{M}$. Since \mathcal{M} is closed under taking finite unions, then the union of each of these sets is also measurable, i.e.

$$f^{-1}|_E((\alpha, \infty]) \cup f^{-1}|_F((\alpha, \infty]) \in \mathcal{M}. \quad (3)$$

We also have that $E, F \in \mathcal{M}$, so $E \cup F \in \mathcal{M}$. Then we have

$$f^{-1}|_E((\alpha, \infty]) \cup f^{-1}|_F((\alpha, \infty]) \quad (4)$$

$$= (f^{-1}((\alpha, \infty]) \cap E) \cup (f^{-1}((\alpha, \infty]) \cap F) \quad (5)$$

$$= f^{-1}((\alpha, \infty]) \cap (E \cup F) \quad (6)$$

where in line (6) we used the fact that $f^{-1}((\alpha, \infty]) \subset (E \cup F)$.

Therefore, as desired, f must be measurable. \square

(b)

Proof. (\Rightarrow) Suppose f is measurable.

We define the indicator function χ_E on E via

$$\chi_E(x) := \begin{cases} 1, & x \in E \\ 0, & x \in E^c. \end{cases} \quad (7)$$

Then we can express g as the product

$$g(x) = f(x) \cdot \chi_E(x). \quad (8)$$

In problem **2a**, we showed that the product of measurable functions is measurable. By assumption, f is measurable. so we need only check that χ_E is measurable.

Let $\alpha \in \mathbb{R}$.

Case 1: $1 \leq \alpha \leq \infty$. Then,

$$\chi_E^{-1}((\alpha, \infty]) = \emptyset \in \mathcal{M}. \quad (9)$$

Case 2: $0 \leq \alpha < 1$. Then,

$$\chi_E^{-1}((\alpha, \infty]) = E \in \mathcal{M}. \quad (10)$$

Case 3: $-\infty \leq \alpha < 0$. Then,

$$\chi_E^{-1}((\alpha, \infty]) = \mathbb{R} \in \mathcal{M}. \quad (11)$$

Hence, χ_E is measurable, so $f \cdot \chi_E$ is also measurable.

Therefore, g is measurable.

(\Leftarrow) Suppose g is measurable. Since $g : E \cup E^c = \mathbb{R} \rightarrow [-\infty, \infty]$ is defined by

$$g(x) := \begin{cases} f(x), & x \in E \\ 0, & x \in E^c, \end{cases} \quad (12)$$

then by restricting g to E we get $g|_E(x) = f(x)$. By part **(a)**, $g|_E$ must be measurable.

Therefore, f is measurable. \square

(c)

Proof. We have already shown in class that sums and products of measurable functions are measurable. So if u and v are measurable, then both u^2 and v^2 are measurable, which implies that $u^2 + v^2$ is measurable.

Define

$$f(x) := u^2(x) + v^2(x). \quad (13)$$

Then we need only check that $f^{\frac{1}{2}}$ is measurable.

Let $g(x) = x^{\frac{1}{2}}$. Then $f^{\frac{1}{2}}(x) = (g \circ f)(x)$, and $f : E \rightarrow [0, \infty]$
 $\implies g : [0, \infty] \rightarrow [0, \infty]$. We use the fact that the composition of measurable functions is measurable, proven in appendix A.1, to show that $f^{\frac{1}{2}}$ is measurable.

Let $\alpha \in \mathbb{R}$.

Case 1: $0 \leq \alpha \leq \infty$. Then,

$$g^{-1}((\alpha, \infty]) = (\alpha^2, \infty) \in \mathcal{M}. \quad (14)$$

Case 2: $-\infty \leq \alpha < 0$. Then,

$$g^{-1}((\alpha, \infty]) = \emptyset \in \mathcal{M}. \quad (15)$$

Hence, g is measurable, so by A.1, $f^{\frac{1}{2}}$ is measurable.

Therefore, $(u^2 + v^2)^{\frac{1}{2}}$ is measurable. \square

Problem 4

TODO TODO TODO

Appendices

A Appendix A

A.1

TODO TODO TODO