

18.102 Assignment 1

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Problem 1

(a)

[Hölder's Inequality]

Proof. Let $A, B > 0$ and $t \in (0, 1)$. We claim that

$$A^t B^{1-t} \leq tA + (1-t)B. \quad (1)$$

For $x > 0$, define

$$f(x) := tx + (1-t)B - x^t B^{1-t}.$$

Computing the first and second derivatives of f , we find

$$f'(x) = t - tx^{t-1} B^{1-t}, \quad \text{and}$$

$$f''(x) = -t(t-1)x^{t-2} B^{1-t}.$$

Then at $x = B$, we have $f'(B) = 0$ and $f''(B) = -t(t-1)\frac{1}{B} > 0$ for $0 < t < 1$. Hence, $f(x)$ has a minimum at $x = B$ by the second derivative test. Since $f(B) = 0$, we conclude that f attains a minimum value of 0 at $x = B$. If $A \neq B$, then it follows that

$$f(A) \geq f(B) = 0 \quad (2)$$

$$\implies tA + (1-t)B - A^t B^{1-t} \geq 0 \quad (3)$$

$$\implies A^t B^{1-t} \leq tA + (1-t)B, \quad (4)$$

and the claim is proven.

Now let $A = \frac{\sum_{k=1}^n |a_k|^p}{\sum_{k=1}^n |a_k|^p}$ and $B = \frac{\sum_{k=1}^n |b_k|^p}{\sum_{k=1}^n |b_k|^p}$ for $n \in \mathbb{N}$. Note that these choices satisfy the positivity conditions required in the previous claim. Then by (1), letting $t = \frac{1}{p}$, we have

$$A^{\frac{1}{p}} B^{\frac{1}{q}} \leq \frac{A}{p} + \frac{B}{q} \quad (5)$$

Substituting the expressions for A and B gives

$$\frac{|a_k||b_k|}{(\sum_{k=1}^n |a_k|^p)^{\frac{1}{p}} (\sum_{k=1}^n |b_k|^q)^{\frac{1}{q}}} \leq \frac{|a_k|^p}{p \sum_{k=1}^n |a_k|^p} + \frac{|b_k|^q}{q \sum_{k=1}^n |b_k|^q}. \quad (6)$$

Summing from $k = 1$ to n on both sides of the inequality, we find

$$\sum_{k=1}^n \frac{|a_k||b_k|}{(\sum_{k=1}^n |a_k|^p)^{\frac{1}{p}} (\sum_{k=1}^n |b_k|^q)^{\frac{1}{q}}} \leq \frac{1}{p} \sum_{k=1}^n \frac{|a_k|^p}{\sum_{k=1}^n |a_k|^p} + \frac{1}{q} \sum_{k=1}^n \frac{|b_k|^q}{\sum_{k=1}^n |b_k|^q} \quad (7)$$

$$= \frac{1}{p} + \frac{1}{q} = 1 \quad (8)$$

$$\Rightarrow \sum_{k=1}^n \frac{|a_k||b_k|}{(\sum_{k=1}^n |a_k|^p)^{\frac{1}{p}} (\sum_{k=1}^n |b_k|^q)^{\frac{1}{q}}} \leq 1. \quad (9)$$

Multiplying both sides by the product $(\sum_{k=1}^n |a_k|^p)^{\frac{1}{p}} (\sum_{k=1}^n |b_k|^q)^{\frac{1}{q}}$, we obtain the desired result,

$$\sum_{k=1}^n |a_k b_k| \leq \left[\sum_{k=1}^n |a_k|^p \right]^{\frac{1}{p}} \left[\sum_{k=1}^n |b_k|^q \right]^{\frac{1}{q}}. \quad (10)$$

□

(b)

[Minkowski's Inequality]

Proof. By the triangle inequality, we have

$$\sum_{k=1}^n |a_k + b_k|^p = \sum_{k=1}^n |a_k + b_k| |a_k + b_k|^{p-1} \quad (11)$$

$$\leq \sum_{k=1}^n |a_k| |a_k + b_k|^{p-1} + \sum_{k=1}^n |b_k| |a_k + b_k|^{p-1}. \quad (12)$$

Then by Hölder's inequality [proved in (a)],

$$\sum_{k=1}^n |a_k| |a_k + b_k|^{p-1} \leq \left[\sum_{k=1}^n |a_k|^p \right]^{\frac{1}{p}} \left[\sum_{k=1}^n |a_k + b_k|^p \right]^{\frac{p-1}{p}}, \quad \text{and} \quad (13)$$

$$\sum_{k=1}^n |b_k| |a_k + b_k|^{p-1} \leq \left[\sum_{k=1}^n |b_k|^p \right]^{\frac{1}{p}} \left[\sum_{k=1}^n |a_k + b_k|^p \right]^{\frac{p-1}{p}}, \quad (14)$$

where we have identified $q = \frac{p}{p-1}$. Then

$$\sum_{k=1}^n |a_k + b_k|^p \leq \left(\left[\sum_{k=1}^n |a_k|^p \right]^{\frac{1}{p}} + \left[\sum_{k=1}^n |b_k|^p \right]^{\frac{1}{p}} \right) \left[\sum_{k=1}^n |a_k + b_k|^p \right]^{\frac{p-1}{p}} \quad (15)$$

$$\Rightarrow \left[\sum_{k=1}^n |a_k + b_k|^p \right] \left[\sum_{k=1}^n |a_k + b_k|^p \right]^{\frac{1-p}{p}} \leq \left[\sum_{k=1}^n |a_k|^p \right]^{\frac{1}{p}} + \left[\sum_{k=1}^n |b_k|^p \right]^{\frac{1}{p}}. \quad (16)$$

Combining exponents on the left side, we arrive at

$$\left[\sum_{k=1}^n |a_k + b_k|^p \right]^{\frac{1}{p}} \leq \left[\sum_{k=1}^n |a_k|^p \right]^{\frac{1}{p}} + \left[\sum_{k=1}^n |b_k|^p \right]^{\frac{1}{p}}. \quad (17)$$

□

Problem 2

Proof. We first show that ℓ^p is a normed space.

Let $a = \{a_j\}_{j=1}^\infty$ and $b = \{b_j\}_{j=1}^\infty$ be sequences in ℓ^p . Suppose $\|a\|_p = 0$. Then by Hölder's inequality, letting $b_j = n^{-\frac{1}{p}}$ for $n \in \mathbb{N}$, $\forall j \in \mathbb{N}$, we have

$$0 = \left[\sum_{j=1}^n |a_j|^p \right]^{\frac{1}{p}} = \left[\sum_{j=1}^n |a_j|^p \right]^{\frac{1}{p}} \left[\sum_{j=1}^n \frac{1}{n} \right]^{\frac{1}{p}} \quad (18)$$

$$\geq \sum_{j=1}^n |a_j n^{-\frac{1}{p}}| = n^{-\frac{1}{p}} \sum_{j=1}^n |a_j| \quad (19)$$

$$\geq 0. \quad (20)$$

Thus, we have that

$$0 \leq \sum_{j=1}^n |a_j| \leq 0, \quad (21)$$

but since $|a_j|$ is always nonnegative, this must imply that $a_j = 0 \forall j \in \mathbb{N}$. Going in the opposite direction, suppose $a = 0$ [i.e. $a_j = 0 \forall j \in \mathbb{N}$]. Then

$$\|a\|_p = \left[\sum_{j=1}^n |a_j|^p \right]^{\frac{1}{p}} = \left[\sum_{j=1}^n 0 \right]^{\frac{1}{p}} = 0^{\frac{1}{p}} = 0. \quad (22)$$

Hence, we have shown $\|a\|_p = 0 \iff a = 0$ [definiteness]. Now let $\lambda \in \mathbb{K}$ [an element in a field of scalars, \mathbb{R} or \mathbb{C}]. Then

$$\|\lambda a\|_p = \left[\sum_{j=1}^n |\lambda a_j|^p \right]^{\frac{1}{p}} = \left[|\lambda|^p \sum_{j=1}^n |a_j|^p \right]^{\frac{1}{p}} = |\lambda| \left[\sum_{j=1}^n |a_j|^p \right]^{\frac{1}{p}}. \quad (23)$$

Hence, $\|\lambda a\|_p = |\lambda| \cdot \|a\|_p$ [homogeneity]. Now consider the norm of the sum, $\|a + b\|_p$. By Minkowski's inequality, we have

$$\|a + b\|_p = \left[\sum_{j=1}^n |a_j + b_j|^p \right]^{\frac{1}{p}} \leq \left[\sum_{j=1}^n |a_j|^p \right]^{\frac{1}{p}} + \left[\sum_{j=1}^n |b_j|^p \right]^{\frac{1}{p}} \quad (24)$$

Hence, $\|a + b\|_p \leq \|a\|_p + \|b\|_p$ [triangle inequality]. Thus we have proven that $\|\cdot\|_p$ is a norm on ℓ^p , so we conclude that ℓ^p is a normed space.

Next we show that ℓ^p is complete.

Let $\{a^{(n)}\}_n$ be a Cauchy sequence in ℓ^p [i.e. $\{a_j^{(n)}\}_{j=1}^\infty \in \ell^p$ and $\{a^{(n)}\}_n = \{\{a_j^{(n)}\}_j\}_n$]. Let $\epsilon > 0$. Then $\exists N_0 \in \mathbb{N}$ such that $\forall n, m \geq N_0$,

$$\|a^{(n)} - a^{(m)}\|_p < \epsilon. \quad (25)$$

Then this implies

$$\left[\sum_{j=1}^\infty |a_j^{(n)} - a_j^{(m)}|^p \right]^{\frac{1}{p}} = \|a^{(n)} - a^{(m)}\|_p < \epsilon \quad (26)$$

$$\implies \sum_{j=1}^\infty |a_j^{(n)} - a_j^{(m)}|^p = \|a^{(n)} - a^{(m)}\|_p^p < \epsilon^p. \quad (27)$$

Then for any $j \in \mathbb{N}$,

$$|a_j^{(n)} - a_j^{(m)}|^p < \sum_{j=1}^\infty |a_j^{(n)} - a_j^{(m)}|^p < \epsilon^p. \quad (28)$$

Hence, the sequence $\{a_j^{(n)}\}_n \subset \ell^p$ is Cauchy. By completeness of \mathbb{R} , $\forall j \in \mathbb{N} \exists a_j$ such that $\lim_{n \rightarrow \infty} a_j^{(n)} = a_j \in \mathbb{R}$.

Fix $k \in \mathbb{N}$. Then for $m, n > N_0$,

$$\sum_{j=1}^k |a_j^{(n)} - a_j^{(m)}|^p \leq \sum_{j=1}^\infty |a_j^{(n)} - a_j^{(m)}|^p \quad (29)$$

$$= \|a^{(n)} - a^{(m)}\|_p^p < \epsilon^p \quad (30)$$

$$\xrightarrow{n \rightarrow \infty} \sum_{j=1}^k |a_j^{(m)} - a_j|^p < \epsilon^p. \quad (31)$$

By Minkowski's inequality for $\|\cdot\|_p$ in \mathbb{R}^k , for $m > N_0$, we have

$$\left[\sum_{j=1}^k |a_j|^p \right]^{\frac{1}{p}} \leq \left[\sum_{j=1}^k |a_j^{(m)} - a_j|^p \right]^{\frac{1}{p}} + \left[\sum_{j=1}^k |a_j^{(m)}|^p \right]^{\frac{1}{p}} \quad (32)$$

$$< \epsilon + \left[\sum_{j=1}^k |a_j^{(m)}|^p \right]^{\frac{1}{p}} \quad (33)$$

$$\xrightarrow{n \rightarrow \infty} \|a\|_p \leq \epsilon + \|a^{(m)}\|_p. \quad (34)$$

Hence, $a \in \ell^p$. Then by letting $k \rightarrow \infty$ in (31),

$$\sum_{j=1}^{\infty} |a_j^{(m)} - a_j|^p = \|a^{(m)} - a\|_p^p < \epsilon^p \quad (35)$$

$$\implies \|a^{(m)} - a\|_p < \epsilon. \quad (36)$$

Then $\lim_{m \rightarrow \infty} \|a^{(m)} - a\|_p = 0$, which implies that the sequence $\{a^{(m)}\}_m \subset \ell^p$ converges to $a \in \ell^p$. Thus, ℓ^p is complete.

Since ℓ^p is a complete normed space, we conclude that ℓ^p is a Banach space. \square

Problem 3

Proof. We will show that $\ell^p \setminus c_0 := \{b \in \ell^p \mid \lim_{k \rightarrow \infty} b_k \neq 0\}$ is open. \square