18.102 Midterm

Octavio Vega

April 23, 2023

Problem 1

Proof. We will show that $\Lambda([a,b])$ is a proper closed subspace of $C([a,b])$, which we know is a Banach space. Let $\{f_n\}_n$ be a cauchy sequence in $\Lambda([a,b])$ such that $f_n \to f$ pointwise. Then for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $||f - f_n|| < \epsilon$. This is equivalent to

$$\sup_{x\in[a,b]} |f(x) - f_n(x)| + \sup_{x\neq y\in[a,b]} \frac{|f(x) - f_n(x) - f(y) + f_n(y)|}{|x-y|} < \epsilon. \tag{1}$$

Since both terms on the left hand side are non-negative, this implies

$$\sup_{x\neq y\in[a,b]} \frac{|f(x) - f_n(x) - f(y) + f_n(y)|}{|x-y|} < \epsilon. \tag{2}$$

Then for any $x \neq y \in [a,b]$, we have

$$|f(x) - f_n(x) - f(y) + f_n(y)| < \epsilon|x-y|, \tag{3}$$

which confirms that for each $n \geq N$, the function $f - f_n$ is Lipschitz continuous. By assumtion, f_n is Lipschitz continuous $\forall n \in \mathbb{N}$, and the sum of Lipschitz continuous functions is also Lipschitz, thus $f = f_n + (f - f_n)$ is Lipschitz continuous.

So, $\lim_{n\to\infty} f_n = f \in \Lambda([a,b])$, which proves that $\Lambda([a,b])$ is a proper closed subspace of $C([a,b])$.

Therefore, $\Lambda([a,b])$ is a Banach space. $\square$

Problem 2

Proof. First we show that $||a + c_0||_{\ell^\infty/c_0} \leq \limsup_{n\to\infty} |a_n|$.

Let $a = \{a_n\}_n \in \ell^\infty$. For each $n \in \mathbb{N}$, let $b_n = (a_1, a_2, ..., a_n, 0, 0, ...) \in c_0$. Then

$$\begin{aligned} \inf_{b \in c_0} ||a + b||_\infty &\leq \inf_n ||a - b_n||_\infty && (4) \\ &= \inf_n \sup_{m \in \mathbb{N}} |a_m - b_m| && (5) \\ &= \inf_n \sup_{m \geq n} |a_m| && (6) \\ &= \limsup_{n \to \infty} |a_n|. && (7) \end{aligned}$$

Thus,

$$||a + c_0||_{\ell^\infty / c_0} \leq \limsup_{n \to \infty} |a_n|. \tag{8}$$

Let $b = (b_1, b_2, b_3, ...) \in c_0$. Then for every $\epsilon > 0$, $\exists n \in N$ such that $\forall m \geq n$, $|b_m| < \epsilon$, so

$$\begin{aligned} ||a + b||_\infty &\geq \sup_{m \geq n} |a_m| - \epsilon && (9) \\ &\geq \limsup_{n \to \infty} |a_n| - \epsilon, && (10) \end{aligned}$$

hence $\limsup_{n \to \infty} |a_n| < ||a + c_0||_{\ell^\infty / c_0} + \epsilon$.

Therefore, $||a + c_0||_{\ell^\infty / c_0} = \limsup_{n \to \infty} |a_n|$. □

Problem 3

(a)

Proof. Since $\lim_{n \to \infty} T_n x = Tx$, then for every $\epsilon > 0$ $\exists N \in N$ such that $\forall n \geq N$,

$$||T_n x - Tx|| < \epsilon. \tag{11}$$

By linearity of T, this is equivalent to

$$||(T_n - T)x|| < \epsilon. \tag{12}$$

Choose $\epsilon = ||x||$. With a sufficiently large choice of N, we have $\forall n \geq N$ and $\forall x \in V$,

$$||(T_n - T)x|| < ||x||. \tag{13}$$

The above equation implies that the operator $T_n - T$ is continuous. Since $\{T_n\}_n$ is assumed to be a sequence in $\mathcal{B}(V, W)$, then $T_n - (T_n - T) = T$ is continuous.

Therefore, T is a bounded linear operator. □

(b)

Proof. Since V is a Banach space with respect to both norms $||\cdot||_1$ and $||\cdot||_2$, we may regard the spaces $V_1 := (V, ||\cdot||_1)$ and $V_2 := (V, ||\cdot||_2)$ as separate Banach spaces.

Consider the identity mapping $\mathbb{1} \in \mathcal{B}(V_1, V_2)$. Since $\mathbb{1}$ is a bounded linear operator, then $\exists C > 0$ such that $\forall v \in V_1$,

$$||v||_2 = ||\mathbb{1}v||_2 \leq C||v||_1, \tag{14}$$

and we are done. $\square$