18.102 Assignment 2

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Problem 1

(a)

Proof. Let B be a Banach space. Suppose $T \in \mathcal{B}(B, B)$ and $||I - T|| < 1$. Then by Geometric series,

$$\sum_{n=0}^{\infty} ||(I-T)^n|| \leq \sum_{n=0}^{\infty} ||I-T||^n = \frac{1}{1 - ||I-T||} < \infty. \tag{1}$$

So the series $\sum_{n=0}^{\infty}(I-T)^n$ converges absolutely, which implies that it converges. Fix $m \in \mathbb{N}$. Then

$$\begin{aligned} T\sum_{n=0}^{m}(I-T)^n &= [I-(I-T)]\sum_{n=0}^{m}(I-T)^n \quad (2)\\ &= \sum_{n=0}^{m}(I-T)^n - \sum_{n=0}^{m}(I-T)^{n+1} \quad (3)\\ &= I-(I-T)^{m+1}, \text{ by telescoping sum.} \quad (4)\end{aligned}$$

By continuity of T,

$$\begin{aligned} T\sum_{n=0}^{\infty}(I-T)^n &= T\left(\lim_{m\to\infty}\sum_{n=0}^{m}(I-T)^n\right) \quad (5)\\ &= \lim_{m\to\infty} T\sum_{n=0}^{m}(I-T)^n \quad (6)\\ &= \lim_{m\to\infty}\left[I-(I-T)^{m+1}\right] \quad (7)\\ &= I, \quad (8)\end{aligned}$$

since $||I-T|| < 1$. We can similarly show that $\sum_{n=0}^{\infty}(I-T)^n = I$.

Thus, T is indeed invertible, and $\sum_{n=0}^{\infty}(I-T)^n \to T^{-1}$ in $\mathcal{B}(B, B)$. $\square$

(b)

Proof. Let $\mathcal{I} := \{T \in \mathcal{B}(B,B) | T^{-1} \text{ exists}\}$. We want to show that $\forall T \in \mathcal{I}$, $\exists \delta > 0$ such that if $||S - T|| < \delta \implies S \in \mathcal{I}$.

Choose $\delta = \frac{1}{||T^{-1}||}$, and write

$$S = T - (T - S) = T\left[I - T^{-1}(T - S)\right]. \tag{9}$$

If $||S - T|| < \delta = \frac{1}{||T^{-1}||}$, then

$$\begin{aligned} \frac{1}{||T^{-1}||} &> ||S - T|| && (10)\\ &= ||T - T\left[I - T^{-1}(T - S)\right]|| && (11)\\ &= ||T|| \cdot ||I - \left[I - T^{-1}(T - S)\right]|| && (12) \end{aligned}$$

$$\begin{aligned} &\implies ||I - \left[I - T^{-1}(T - S)\right]|| < \frac{1}{||T^{-1}|| \cdot ||T||} = 1 && (13)\\ &\implies ||T^{-1}(T - S)|| = ||I - T^{-1}S|| < 1. && (14) \end{aligned}$$

So by (a), $T^{-1}S$ is invertible, which implies that S is invertible. Thus, $\exists \delta > 0$ such that if $S \in B_\delta(T)$, then $S \in \mathcal{I}$.

Therefore, $\mathcal{I}$ is open. $\square$

Problem 2

(a)

Proof. To show that $||v + W||$ is a norm, we will show that it obeys positive definiteness, homogeneity, and the triangle inequality.

First, suppose that $0 = ||v + W|| = \inf_{w \in W} ||v + w||$. Then since $|| \cdot ||_V$ is a norm on V,

$$||w + w|| = 0 \iff v + w = 0 \implies v = -w. \tag{15}$$

So $\exists$ a sequence $\{w_k\}_k \subset W$ such that $w_k \to -v$. Since W is closed, $-v \in W \implies v \in V$. But then $v + W = 0 + W$ because $v \in W$.

Thus, $||v + W|| = 0 \iff v = 0$ (definiteness).

Also, $||v + W|| = \inf_{w \in W} ||v + w|| \geq 0$ because $|| \cdot ||_V$ is a norm, and $||v + w|| \geq 0$ $\forall w \in W$.

Let $\lambda \in \mathbb{K}$. Then since $\lambda W = W$,

$$\begin{align}
||\lambda(v+W)|| &= ||\lambda v + W|| \tag{16}\\
&= \inf_{w\in W} ||\lambda v + w|| \tag{17}\\
&= \inf_{w\in W} |\lambda| \cdot ||v + \frac{w}{\lambda}|| \tag{18}\\
&= |\lambda| \inf_{w\in W} ||v + w|| \tag{19}\\
&= |\lambda| \cdot ||v + W|| \quad \text{(homogeneity).} \tag{20}
\end{align}$$

Now let $u + W$, $v + W \in V/W$. Then

$$\begin{align}
||(u+W) + (v+W)|| &= ||u + v + W|| \tag{21}\\
&= \inf_{w\in W} ||u + v + w|| \tag{22}\\
&= \inf_{w\in W} ||u + v + 2w|| \tag{23}\\
&= \inf_{w\in W} ||u + w + v + w|| \tag{24}\\
&\leq \inf_{w\in W} (||u + w|| + ||v + w||) \tag{25}\\
&\leq \inf_{w\in W} ||u + w|| + \inf_{w\in W} ||v + w|| \tag{26}\\
&= ||u + W|| + ||v + W|| \quad \text{(triangle inequality).} \tag{27}
\end{align}$$

Thus, $||v + W||$ is a norm on V/W. ☐

(b)

TODO TODO TODO TODO

Problem 3

Proof. Let $\{v_n\}_n$ be a sequence of elements in V. Suppose that the series $\sum_n (v_n + W)$ is absolutely summable, i.e. that $\sum_n ||v_n + W||$ converges. Since $||v_n + W|| = \inf_{w\in W} ||v_n + w||$, then for each $n \in \mathbb{N}$, $\exists w_n \in W$ such that

$$\begin{align}
||v_n + w_n|| &\leq ||v_n + W|| + 2^{-n} \tag{28}\\
\implies \sum_n ||v_n + w_n|| &\leq \sum_n ||v_n + W|| + \sum_n 2^{-n} \tag{29}\\
&= \sum_n ||v_n + W|| + 1. \tag{30}
\end{align}$$

Then by comparison, $\sum_n ||v_n + w_n||$ converges, so $\sum_n (v_n + w_n)$ converges.

Since V is a Banach space, then, by closure, $\exists v \in V$ such that $v = \sum_n (v_n + w_n)$. Then

$$\lim_{N\to\infty} v + W - \sum_{n=1}^{N} (v_n + W) = \sum_{n=1}^{\infty} (v_n + w_n) + W - \lim_{N\to\infty} \sum_{n=1}^{N} (v_n + W) \tag{31}$$

$$= \sum_{n=1}^{\infty} v_n + \sum_{n=1}^{\infty} w_n + W - \lim_{N\to\infty} \sum_{n=1}^{N} (v_n + W) \tag{32}$$

$$= \sum_{n=1}^{\infty} v_n + W - \lim_{N\to\infty} \sum_{n=1}^{N} (v_n + W) \tag{33}$$

$$= \sum_{n=1}^{\infty} v_n + W - \lim_{N\to\infty} \sum_{n=1}^{N} v_n - W \tag{34}$$

$$= \sum_{n=1}^{\infty} v_n - \lim_{N\to\infty} \sum_{n=1}^{N} = 0. \tag{35}$$

So $\sum_n (v_n + W) = v + W$, thus $\sum_n (v_n + W)$ converges in V/W.

Therefore V/W is a Banach space. $\square$