18.102 Assignment 6

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Problem 1

(a)

Proof. Let $\epsilon > 0$, and define $c_1 := a$ and $c_{n+1} := b$.

Since ψ is a step function on $[a, b]$, $\exists c_1 \leq c_2 \leq \cdots \leq c_n \leq c_{n+1} \in [a, b]$ such that $\forall i = 1, \cdots, n$,

$$\psi^{-1}(\{a_i\}) = (c_i, c_{i+1}], \tag{1}$$

where each a_i is one of the finitely many values that ψ takes on.

Choose $\delta > 0$ such that $\delta < \frac{\epsilon}{2n}$. Define $g : [a, b] \to \mathbb{R}$ via

$$g(x) := \begin{cases} \frac{a_i + a_{i-1}}{2} + \left(\frac{a_i - a_{i-1}}{2\delta}\right)(x - c_i), & x \in (c_i - \delta, c_i + \delta) \\ a_i, & x \in [c_i + \delta, c_{i+1} - \delta] \\ -\frac{a_n}{2\delta}(x - b), & x \in (c_{n-1} - \delta, b], \end{cases} \tag{2}$$

where $a_0 := -a_1$. Then

$$g(a) = \frac{a_1 + a_0}{2} + \left(\frac{a_1 - a_0}{2\delta}\right)(c_1 - c_1) = \frac{a_1 - a_1}{2} = 0, \tag{3}$$

and

$$g(b) = -\frac{a_n}{2\delta}(b - b) = 0, \tag{4}$$

as desired. We also see that since g is piecewise linear, it is continuous.

Now consider the difference $|\psi(x) - g(x)|$.

<u>Case 1</u>: $x \in [c_i + \delta, c_{i+1} - \delta]$. Then by (1), we have

$$|\psi(x) - g(x)| = |\psi((c_i, c_{i+1} - \delta)) - a_i| = |a_i - a_i| = 0. \tag{5}$$

Case 2: $x \in (c_i - \delta, c_i + \delta)$. Then

$$
\begin{aligned}
|\psi(x) - g(x)| &= \left| \frac{a_i + a_{i-1}}{2} + \left(\frac{a_i - a_{i-1}}{2\delta} \right) (x - c_i) - \psi(x) \right| && (6) \\
&< \left| \frac{a_i + a_{i-1}}{2} + \left(\frac{a_i - a_{i-1}}{2\delta} \right) \delta - \psi(x) \right| && (7) \\
&= \left| \frac{a_i + a_{i-1}}{2} + \frac{a_i + a_{i-1}}{2} - \psi(x) \right| && (8) \\
&= |a_i - \psi(x)| && (9) \\
&= 0 \text{ or } |a_{i+1} - a_i|. && (10)
\end{aligned}
$$

Case 3: $x \in (c_n - \delta, b)$. Then

$$
\begin{aligned}
|\psi(x) - g(x)| &= \left| -\frac{a_n}{2\delta}(x - b) - \psi(x) \right| && (11) \\
&< \left| -\frac{a_n}{2\delta}\delta - \psi(x) \right| && (12) \\
&= \left| -\frac{a_n}{2} - \psi(x) \right| && (13) \\
&= \frac{3a_n}{2}. && (14)
\end{aligned}
$$

So in all three cases, we have that either $|g(x) - \psi(x)| = 0$, or $|g(x) - \psi(x)| < \frac{3a}{2}$, or $|g(x) - \psi(x)| < |a_{i+1} - a_i|$.

Define the set E to be the collection of points in $[a, b]$ for which $|\psi(x) - g(x)| \neq 0$. Then

$$E := \bigcup_{k=1}^{n} (c_k - \delta, c_k + \delta). \tag{15}$$

By definition, $\forall x \in E^c$,

$$|\psi(x) - g(x)| = 0 < \epsilon. \tag{16}$$

Since E is a countable union of intervals, we have

$$
\begin{aligned}
m(E) &= m\left[\bigcup_{k=1}^{n}(c_k-\delta, c_k+\delta)\right] && (17)\\
&\leq \sum_{k=1}^{n} m\left[(c_k-\delta, c_k+\delta)\right] && (18)\\
&= \sum_{k=1}^{n} \ell(c_k-\delta, c_k+\delta) && (19)\\
&= \sum_{k=1}^{n} 2\delta && (20)\\
&= 2n\delta && (21)\\
&< 2n\frac{\epsilon}{2n} && (22)\\
&= \epsilon, && (23)
\end{aligned}
$$

as desired. $\square$