18.102 Assignment 2

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Problem 1

(a)

Proof. Let B be a Banach space. Suppose $T \in \mathcal{B}(B, B)$ and $||I-T|| < 1$. Then by Geometric series,

$$\sum_{n=0}^{\infty} ||(I-T)^n|| \leq \sum_{n=0}^{\infty} ||I-T||^n = \frac{1}{1-||I-T||} < \infty. \tag{1}$$

So the series $\sum_{n=0}^{\infty}(I-T)^n$ converges absolutely, which implies that it converges. Fix $m \in \mathbb{N}$. Then

$$\begin{aligned} T\sum_{n=0}^{m}(I-T)^n &= [I-(I-T)]\sum_{n=0}^{m}(I-T)^n && (2)\\ &= \sum_{n=0}^{m}(I-T)^n - \sum_{n=0}^{m}(I-T)^{n+1} && (3)\\ &= I-(I-T)^{m+1}, \text{ by telescoping sum.} && (4)\end{aligned}$$

By continuity of T,

$$\begin{aligned} T\sum_{n=0}^{\infty}(I-T)^n &= T\left(\lim_{m\to\infty}\sum_{n=0}^{m}(I-T)^n\right) && (5)\\ &= \lim_{m\to\infty} T\sum_{n=0}^{m}(I-T)^n && (6)\\ &= \lim_{m\to\infty}\left[I-(I-T)^{m+1}\right] && (7)\\ &= I, && (8)\end{aligned}$$

since $||I-T|| < 1$. We can similarly show that $\sum_{n=0}^{\infty}(I-T)^n = I$.

Thus, T is indeed invertible, and $\sum_{n=0}^{\infty}(I-T)^n \to T^{-1}$ in $\mathcal{B}(B, B)$. $\square$

(b)

Proof. Let $\mathcal{I} := \{T \in \mathcal{B}(B,B) | T^{-1} \text{ exists}\}$. We want to show that $\forall T \in \mathcal{I}$, $\exists \delta > 0$ such that if $||S - T|| < \delta \implies S \in \mathcal{I}$.

Choose $\delta = \frac{1}{||T^{-1}||}$, and write

$$S = T - (T - S) = T \left[I - T^{-1} (T - S) \right]. \tag{9}$$

If $||S - T|| < \delta = \frac{1}{||T^{-1}||}$, then

$$\begin{aligned} \frac{1}{||T^{-1}||} &> ||S - T|| && (10) \\ &= ||T - T \left[I - T^{-1}(T - S) \right] || && (11) \\ &= ||T|| \cdot ||I - \left[I - T^{-1}(T - S) \right] || && (12) \end{aligned}$$

$$\begin{aligned} &\implies ||I - \left[I - T^{-1}(T - S) \right] || < \frac{1}{||T^{-1}|| \cdot ||T||} = 1 && (13) \\ &\implies ||T^{-1}(T - S)|| = ||I - T^{-1}S|| < 1. && (14) \end{aligned}$$

So by (a), $T^{-1}S$ is invertible, which implies that S is invertible. Thus, $\exists \delta > 0$ such that if $S \in B_\delta(T)$, then $S \in \mathcal{I}$.

Therefore, $\mathcal{I}$ is open. □

Problem 2

(a)

Proof. To show that $||v + W||$ is a norm, we will show that it obeys positive definiteness, homogeneity, and the triangle inequality.

First, suppose that $0 = ||v + W|| = \inf_{w \in W} ||v + w||$. Then since $|| \cdot ||_V$ is a norm on V,

$$||w + w|| = 0 \iff v + w = 0 \implies v = -w. \tag{15}$$

So $\exists$ a sequence $\{w_k\}_k \subset W$ such that $w_k \to -v$. Since W is closed, $-v \in W \implies v \in V$. But then $v + W = 0 + W$ because $v \in W$.

Thus, $||v + W|| = 0 \iff v = 0$ (definiteness).

Also, $||v + W|| = \inf_{w \in W} ||v + w|| \geq 0$ because $|| \cdot ||_V$ is a norm, and $||v + w|| \geq 0$ $\forall w \in W$.

Let $\lambda \in \mathbb{K}$. Then since $\lambda W = W$,

$$\begin{aligned}
||\lambda(v+W)|| &= ||\lambda v + W|| && (16)\\
&= \inf_{w\in W} ||\lambda v + w|| && (17)\\
&= \inf_{w\in W} |\lambda| \cdot ||v + \frac{w}{\lambda}|| && (18)\\
&= |\lambda| \inf_{w\in W} ||v + w|| && (19)\\
&= |\lambda| \cdot ||v + W|| \quad \text{(homogeneity).} && (20)
\end{aligned}$$

Now let $u + W$, $v + W \in V/W$. Then

$$\begin{aligned}
||(u+W)+(v+W)|| &= ||u + v + W|| && (21)\\
&= \inf_{w\in W} ||u + v + w|| && (22)\\
&= \inf_{w\in W} ||u + v + 2w|| && (23)\\
&= \inf_{w\in W} ||u + w + v + w|| && (24)\\
&\le \inf_{w\in W} (||u + w|| + ||v + w||) && (25)\\
&\le \inf_{w\in W} ||u + w|| + \inf_{w\in W} ||v + w|| && (26)\\
&= ||u + W|| + ||v + W|| \quad \text{(triangle inequality).} && (27)
\end{aligned}$$

Thus, $||v + W||$ is a norm on V/W. $\square$