18.102 Assignment 8

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Problem 1

(a)

Proof. ($\Rightarrow$) Suppose $w \in \overline{W}$. Then since $w \in H$ and since $\{e_n\}_n \subset H$ is a countably infinite orthonormal subset, we have

$$w = \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n. \tag{1}$$

Computing the norm gives

$$\begin{aligned}
||w||^2 &= \left\langle \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n, \sum_{k=1}^{\infty} \langle u, e_k \rangle e_k \right\rangle &(2)\\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \langle u, e_n \rangle \overline{\langle u, e_k \rangle} \langle e_n, e_k \rangle &(3)\\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \langle u, e_n \rangle \overline{\langle u, e_k \rangle} \delta_{nk} &(4)\\
&= \sum_{n=1}^{\infty} \langle u, e_n \rangle \overline{\langle u, e_n \rangle} &(5)\\
&= \sum_{n=1}^{\infty} |\langle u, e_n \rangle|^2. &(6)
\end{aligned}$$

Thus,

$$||w|| = \left(\sum_{n=1}^{\infty} |\langle u, e_n \rangle|^2 \right)^{\frac{1}{2}} < \infty, \tag{7}$$

so defining $c_n := \langle u, e_n \rangle$ yields a sequence $\{c_n\}_n \in \ell^2(\mathbb{N})$, as desired.

($\Leftarrow$) Let $\{c_n\}_{n=1}^{\infty} \in \ell^2$ such that $w = \sum_{n=1}^{\infty} c_n e_n$.

Define $w_N := \sum_{n=1}^{N} c_n e_n$. Then $w = \lim_{N\to\infty} w_N$, and for each $N \in N$, $w_N \in W$ since it is a finite linear combination of elements in $\{e_n\}_n$.

Thus, since $\overline{W}$ contains all the limit points of W, then $w \in \overline{W}$, as desired. □

(b)

Proof. Let $w \in \overline{W}$ and $u \in H$. Then by **(a)**, we may write $w = \sum_{n=1}^{\infty} c_n e_n$ for $\{c_n\}_n \in \ell^2(\mathbb{N})$. Suppose $c_n = \langle u, e_n\rangle$. Then

$$||u - \sum_{n=1}^{\infty}\langle u, e_n\rangle e_n|| = ||u - \sum_{n=1}^{\infty} c_n e_n|| \tag{8}$$

$$= ||u - v||. \tag{9}$$

Now suppose $c_n \neq \langle u, e_n\rangle$. Then we compute

$$||u - w||^2 = \left\| u - \sum_{n=1}^{\infty} c_n e_n \right\|^2 \tag{10}$$

$$= \left\| u - \sum_{n=1}^{\infty}\langle u, e_n\rangle e_n + \sum_{n=1}^{\infty}\langle u, e_n\rangle e_n - \sum_{n=1}^{\infty} c_n e_n \right\|^2 \tag{11}$$

$$= \left\| u - \sum_{n=1}^{\infty}\langle u, e_n\rangle e_n + \sum_{n=1}^{\infty}(\langle u, e_n\rangle - c_n) e_n \right\|^2 \tag{12}$$

$$\geq \left\| u - \sum_{n=1}^{\infty}\langle u, e_n\rangle e_n \right\|^2. \tag{13}$$

Therefore, $\|u - \sum_{n=1}^{\infty}\langle u, e_n\rangle e_n\| \leq ||u - w||$, with equality only if $w = \sum_{n=1}^{\infty}\langle u, e_n\rangle e_n$, and we are done. □

Problem 2

(a)

Proof. Let $\{u_k\}_k \subset W^{\perp}$ be a sequence such that $u_k \to u$ as $k \to \infty$. Then $\forall k \in \mathbb{N}$ and $\forall w \in W$, we have $\langle u_k, w\rangle = 0$.

By continuity of the inner product,

$$0 = \lim_{k\to\infty}\langle u_k, w\rangle \tag{14}$$

$$= \left\langle \lim_{k\to\infty} u_k, w \right\rangle \tag{15}$$

$$= \langle u, w\rangle. \tag{16}$$

Thus, $u \in W^{\perp}$, so $W^{\perp} \subset H$ is closed. □

(b)

Proof. We note that

$$\left(W^{\perp}\right)^{\perp} := \{v \in H \mid \langle v, u\rangle = 0 \, \forall u \in W^{\perp}\}. \tag{17}$$

Let $w \in W$. Then $\forall u \in W^{\perp}$,

$$\begin{aligned} \langle w, u\rangle &= 0 && (18) \\ \implies w &\in \left(W^{\perp}\right)^{\perp} && (19) \\ \implies W &\subseteq \left(W^{\perp}\right)^{\perp}. && (20) \end{aligned}$$

Since $\left(W^{\perp}\right)^{\perp}$ is an orthogonal complement, then by **(a)** it is a closed linear subspace of H. So, $\left(W^{\perp}\right)^{\perp}$ must also be complete.

Let $\{v_n\}_n \subset W$ be a Cauchy sequence. Since $W \subseteq \left(W^{\perp}\right)^{\perp}$, then $\{v_n\}_n \subset \left(W^{\perp}\right)^{\perp}$, so $\{v_n\}_n$ converges in $\left(W^{\perp}\right)^{\perp}$, i.e.

$$\lim_{n\to\infty} v_n = v \in \left(W^{\perp}\right)^{\perp}. \tag{21}$$

Thus the Cauchy sequence $\{v_n\}_n$ converges to v in $\left(W^{\perp}\right)^{\perp}$.

We conclude that the closure $\overline{W} = \left(W^{\perp}\right)^{\perp}$. □

Problem 3

(a)

Proof. Since $f \in C^k([-\pi, \pi])$, then f must be bounded. Let $B \geq 0$ be such a bound for f. Then we have

$$\begin{aligned} \|f\|_{L^2}^2 &= \int_{-\pi}^{\pi} |f(t)|^2 \mathrm{d}t && (22) \\ &\leq \int_{-\pi}^{\pi} B^2 \mathrm{d}t && (23) \\ &= 2\pi B^2 && (24) \\ &< \infty. && (25) \end{aligned}$$

Thus $f \in L^2([-\pi, \pi])$.

Next, we prove the following claim:

<u>Claim</u>: Let $k \in \mathbb{N}$. For each $j \in \{0, 1, ..., k\}$,

$$\hat{f}(n) = \left(-\frac{i}{n}\right)^j \widehat{f^{(j)}}(n). \tag{26}$$

Proof of claim: (By induction on j).

Base case: ($j = 0$)

$$\hat{f}(n) = \widehat{f^{(0)}}(n).$$

Inductive step: Assume $\hat{f}(n) = \left(-\frac{i}{n}\right)^j \hat{f^{(j)}}(n)$. Then integrating by parts, we get

$$\begin{aligned}
\hat{f}(n) &= \left(-\frac{i}{n}\right)^j \frac{1}{2\pi} \int_{-\pi}^{\pi} f^{(j)}(t) e^{-int} \mathrm{d}t && (27)\\
&= \left(-\frac{i}{n}\right)^j \frac{1}{2\pi} \left[f^{(j)}(t) \frac{e^{-int}}{(-in)}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f^{(j+1)}(t) \frac{e^{-int}}{(-in)} \mathrm{d}t \right] && (28)\\
&= 0 - \frac{i}{n} \left(-\frac{i}{n}\right)^j \frac{1}{2\pi} \int_{-\pi}^{\pi} f^{(j+1)} e^{-int} \mathrm{d}t && (29)\\
&= \left(-\frac{i}{n}\right)^{j+1} \widehat{f^{(j+1)}}(n), && (30)
\end{aligned}$$

and the claim is proven. $\square$

So for each $j \in \{0, 1, ..., k\}$,

$$\begin{aligned}
\left|\hat{f}(n)\right| &= \left|(-i)^j \frac{1}{n^j} \widehat{f^{(j)}}(n)\right| && (31)\\
&= \frac{1}{n^j} \left|\widehat{f^{(j)}}(n)\right|. && (32)
\end{aligned}$$

Let $0 \leq s \leq k$ and $N \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{|n| \leq N} \left|\hat{f}(n)\right|^2 \left(1 + |n|^2\right)^s &= \sum_{|n| \leq N} \frac{1}{n^{2j}} \left|\widehat{f^{(j)}}(n)\right|^2 \left(1 + |n|^2\right)^s && (33)\\
&= \sum_{|n| \leq N} \frac{\left(1 + |n|^2\right)^s}{n^{2j}} \left|\widehat{f^{(j)}}(n)\right|^2. && (34)
\end{aligned}$$

Since this holds $\forall 0 \leq j \leq k$, we can choose $j = k$, giving

$$\begin{aligned}
\sum_{|n| \leq N} \left|\hat{f}(n)\right|^2 \left(1 + |n|^2\right)^s &= \sum_{|n| \leq N} \frac{\left(1 + |n|^2\right)^s}{n^{2k+2}} \left|\widehat{f^{(k)}}(n)\right|^2 && (35)\\
&\leq \sum_{|n| \leq N} \left|\widehat{f^{(k)}}(n)\right|^2 && (36)\\
&< \infty. && (37)
\end{aligned}$$

Sending $N \to \infty$, we get the desired result.

Therefore $f \in H^s(\mathbb{T})\ \forall 0 \leq s \leq k$. $\square$