

# 18.102 Assignment 4

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## Problem 1

*Proof.* Let  $A \subset \mathbb{R}$  and let  $E \in \mathcal{A}$ . Then  $f^{-1}(E)$  is measurable, so

$$m^*(A \cap f^{-1}(E)) + m^*(A \cap f^{-1}(E)^c) \leq m^*(A). \quad (1)$$

We will first show that  $\mathcal{A}$  is closed under taking complements so we must show that  $E^c \in \mathcal{A}$  for every  $E \in \mathcal{A}$ ; i.e.  $f^{-1}(E^c)$  is measurable.

We will use the fact that  $f^{-1}(E^c) = f^{-1}(E)^c$ , which is proven in appendix A.

Since  $f^{-1}(E)$  is measurable, we have

$$m^*(A) \geq m^*(A \cap f^{-1}(E)) + m^*(A \cap f^{-1}(E)^c) \quad (2)$$

$$= m^*\left[A \cap (f^{-1}(E)^c)^c\right] + m^*(A \cap f^{-1}(E^c)) \quad (3)$$

$$= m^*(A \cap f^{-1}(E^c)^c) = m^*(A \cap f^{-1}(E^c)). \quad (4)$$

Hence,  $f^{-1}(E^c)$  is measurable, so  $E^c \in \mathcal{A}$ . Thus,  $\mathcal{A}$  is closed under taking complements.

Now we show that  $\mathcal{A}$  is closed under taking countable unions. We use the fact that  $f^{-1}(\cup_n E_n) = \cup_n f^{-1}(E_n)$ , which is proven in appendix B.

Let  $\{E_n\}_n \subset \mathcal{A}$  be a sequence of sets in  $\mathcal{A}$ , and let  $A \subset \mathbb{R}$ . Then

$$m^*\left[A \cap f^{-1}\left(\bigcup_n E_n\right)\right] = m^*\left[A \cap \left(\bigcup_n f^{-1}(E_n)\right)\right] \quad (5)$$

$$= m^*\left[\bigcup_n (A \cap f^{-1}(E_n))\right] \quad (6)$$

$$\leq \sum_n m^*(A \cap f^{-1}(E_n)) \quad (7)$$

$$\leq m^*(A \cap f^{-1}(E_n)), \quad (8)$$

by countable subadditivity and positive-definiteness of the outer measure  $m^*$ . Similarly, we have

$$m^* \left[ A \cap f^{-1} \left( \bigcup_n E_n \right)^c \right] = m^* \left[ A \cap \left( \bigcup_n f^{-1}(E_n) \right)^c \right] \quad (9)$$

$$= m^* \left[ A \cap \left( \bigcap_n f^{-1}(E_n)^c \right) \right] \quad (10)$$

$$\leq m^* (A \cap f^{-1}(E_n)^c), \quad (11)$$

by monotonicity of  $m^*$ , because  $\bigcap_n f^{-1}(E_n)^c \subseteq f^{-1}(E_n)^c$ .

Finally, since  $E_n \in \mathcal{A}$ , then  $f^{-1}(E_n)$  is Lebesgue measurable, so we have

$$m^* \left[ A \cap f^{-1} \left( \bigcup_n E_n \right) \right] + \quad (12)$$

$$m^* \left[ A \cap f^{-1} \left( \bigcup_n E_n \right)^c \right] \leq m^* (A \cap f^{-1}(E_n)) + m^* (A \cap f^{-1}(E_n)^c) \quad (13)$$

$$\leq m^* (A). \quad (14)$$

So,  $f^{-1}(\bigcup_n E_n)$  is measurable, hence  $\bigcup_n E_n \in \mathcal{A}$ . Thus,  $\mathcal{A}$  is closed under taking countable unions.

Therefore,  $\mathcal{A}$  is a  $\sigma$ -algebra.  $\square$

## Problem 2

*Proof.* ( $\Rightarrow$ ) Suppose  $E$  is measurable.

Let  $\epsilon > 0$ . Then by definition of the outer measure  $m^*$ ,  $\exists \{I_n\}_n$  a collection of open intervals with  $E \subset \bigcup_n I_n$  such that

$$\sum_n \ell(I_n) < m^*(E) + \frac{\epsilon}{2} \quad (15)$$

$$\Rightarrow \sum_n \ell(I_n) - m^*(E) < \frac{\epsilon}{2}, \quad (16)$$

where  $\ell(I_n)$  denotes the length of the interval  $I_n$ . Also, since  $m^*(E) < \infty$ , then

$$\sum_n m^*(I_n) = \sum_n \ell(I_n) < \infty. \quad (17)$$

By the countable subbadditivity of  $m^*$ , we have

$$m^* \left( \bigcup_n I_n \right) \leq \sum_n m^*(I_n) = \sum_n \ell(I_n) < \infty. \quad (18)$$

Then by removing  $E$  from the union  $\cup_n I_n$ , we get

$$m^* \left[ \left( \bigcup_n I_n \right) \setminus E \right] = m^* \left( \bigcup_n I_n \right) - m^*(E) \quad (19)$$

$$\leq \sum_n \ell(I_n) - m^*(E) \quad (20)$$

$$< \frac{\epsilon}{2}. \quad (21)$$

For any  $N \in \mathbb{N}$ ,  $\bigcup_{n=1}^N I_n \subseteq \bigcup_{n=1}^\infty I_n$ . Thus,  $\left( \bigcup_{n=1}^N I_n \right) \setminus E \subseteq \left( \bigcup_{n=1}^\infty I_n \right) \setminus E$ . Then by the monotonicity of  $m^*$ ,

$$m^* \left[ \left( \bigcup_{n=1}^N I_n \right) \setminus E \right] \leq m^* \left[ \left( \bigcup_n I_n \right) \setminus E \right] \quad (22)$$

$$< \frac{\epsilon}{2}. \quad (23)$$

Since  $\sum_n m^*(I_n) < \infty$ , then the series converges, so  $\exists N \in \mathbb{N}$  such that  $\sum_{n=N+1}^\infty m^*(I_n) < \frac{\epsilon}{2}$ . Because  $E \subseteq \bigcup_n I_n$ , we have

$$E \setminus \left( \bigcup_{n=1}^N I_n \right) \subset \left( \bigcup_{n=1}^\infty I_n \right) \setminus \left( \bigcup_{n=1}^N I_n \right) \quad (24)$$

$$\subseteq \bigcup_{n=N+1}^\infty I_n. \quad (25)$$

Then by monotonicity of  $m^*$ ,

$$m^* \left[ E \setminus \left( \bigcup_{n=1}^\infty I_n \right) \right] \leq m^* \left( \bigcup_{n=N+1}^\infty I_n \right) \quad (26)$$

$$\leq \sum_{n=N+1}^\infty m^*(I_n) \quad (27)$$

$$< \frac{\epsilon}{2}. \quad (28)$$

Combining equations (22) and (26), we have

$$m^* \left( E \Delta \bigcup_{n=1}^N I_n \right) = m^* \left[ E \setminus \left( \bigcup_{n=1}^N I_n \right) \right] + m^* \left[ \left( \bigcup_{n=1}^N I_n \right) \setminus E \right] \quad (29)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad (30)$$

$$= \epsilon. \quad (31)$$

Letting  $U = \bigcup_{n=1}^N I_n$ , we see that

$$m^*(U \Delta E) < \epsilon, \quad (32)$$

as desired.

( $\Leftarrow$ ) Suppose that for every  $\epsilon > 0$ ,  $\exists$  a finite union of open intervals  $U$  such that  $m^*(U \Delta E) < \epsilon$ .

Let  $A \subset \mathbb{R}$ . Since  $U$  is a finite union of open intervals, then  $U$  is measurable. Thus,

$$m^*(A \cap U) + m^*(A \cap U^c) \leq m^*(A). \quad (33)$$

We can express the set  $A \cap E$  as follows:

$$A \cap E = A \cap [(E \cap U) \cup (E \cap U^c)] \quad (34)$$

$$= (A \cap E \cap U) \cup (A \cap E \cap U^c). \quad (35)$$

Then by exclusivity,

$$m^*(A \cap E) = m^*(A \cap E \cap U) + m^*(A \cap E \cap U^c) \quad (36)$$

Similarly, we can write

$$A \cap E^c = A \cap [(E^c \cap U) \cup (E^c \cap U^c)] \quad (37)$$

$$= (A \cap E^c \cap U) \cup (A \cap E^c \cap U^c). \quad (38)$$

This gives

$$m^*(A \cap E^c) = m^*(A \cap E^c \cap U) + m^*(A \cap E^c \cap U^c). \quad (39)$$

Adding equations (36) and (39) yields

$$m^*(A \cap E) + m^*(A \cap E^c) = m^*(A \cap E \cap U) + m^*(A \cap E \cap U^c) \quad (40)$$

$$+ m^*(A \cap E^c \cap U) + m^*(A \cap E^c \cap U^c) \quad (41)$$

$$= m^*(A \cap E \cap U) + m^*[A \cap (E \setminus U)] \quad (42)$$

$$+ m^*[A \cap (U \setminus E)] + m^*(A \cap E^c \cap U^c). \quad (43)$$

Since  $A \cap E \cap U \subseteq A \cap U$  and  $A \cap E^c \cap U^c \subseteq A \cap U^c$ , and since  $(U \setminus E)$  and  $(E \setminus U)$  are disjoint, we have

$$m^*(A \cap E) = m^*(A \cap E \cap U) \leq m^*(A \cap U) + m^*(A \cap U^c) \quad (44)$$

$$+ m^*(A \cap [(E \setminus U) \cup (U \cap E)]) \quad (45)$$

$$= m^*(A \cap U) + m^*(A \cap U^c) + m^*(U \Delta E) \quad (46)$$

$$\leq m^*(A) + \epsilon. \quad (47)$$

Since this holds for every  $\epsilon > 0$ , we can take  $\epsilon$  to be arbitrarily small:

$$\xrightarrow{\epsilon \rightarrow 0} m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A). \quad (48)$$

Therefore,  $E$  is measurable.  $\square$

# Appendices

## A Appendix A

**Theorem:** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . If  $E \subset \mathbb{R}$ , then  $f^{-1}(E^c) = f^{-1}(E)^c$ .

*Proof.* Let  $x \in f^{-1}(E^c)$ . Then

$$\implies f(x) \in E^c \iff f(x) \notin E \quad (49)$$

$$\implies x \notin f^{-1}(E) \iff x \in f^{-1}(E)^c. \quad (50)$$

Thus, we have

$$f^{-1}(E^c) \subseteq f^{-1}(E)^c. \quad (51)$$

Now let  $x \in f^{-1}(E)^c$ . Then

$$\implies x \notin f^{-1}(E) \iff f(x) \notin E \quad (52)$$

$$\implies f(x) \in E^c \iff x \in f^{-1}(E^c). \quad (53)$$

So, we also have

$$f^{-1}(E)^c \subseteq f^{-1}(E^c). \quad (54)$$

Taking equations (51) and (54) together allows us to conclude that  $f^{-1}(E^c) = f^{-1}(E)^c$ , as desired.  $\square$

## B Appendix B

**Theorem:** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and let  $\{E_n\}_n$  be a collection of subsets  $E_n \subset \mathbb{R}$ . Then

$$f^{-1}\left(\bigcup_n E_n\right) = \bigcup_n f^{-1}(E_n). \quad (55)$$

*Proof.* Let  $x \in f^{-1}(\cup_n E_n)$ . Then  $f(x) \in \cup_n E_n$ , so  $f(x)$  is in any of  $E_n$  for  $n \in \mathbb{N}$ . This is equivalent to saying that  $x \in f^{-1}(E_n)$  for some  $n \in \mathbb{N}$ , which implies that  $x \in \cup_n f^{-1}(E_n)$ . Hence,

$$f^{-1}\left(\bigcup_n E_n\right) \subseteq \bigcup_n f^{-1}(E_n). \quad (56)$$

Now let  $x \in \cup_n f^{-1}(E_n)$ . Then  $x \in f^{-1}(E_n)$  for some  $n \in \mathbb{N}$ , which is equivalent to saying that  $f(x) \in E_n$  for some  $n$ . Then  $f(x) \in \cup_n E_n$ , so  $x \in f^{-1}(\cup_n E_n)$ . Thus,

$$\bigcup_n f^{-1}(E_n) \subseteq f^{-1}\left(\bigcup_n E_n\right). \quad (57)$$

Therefore, both sets are subsets of one another, so they are equal.  $\square$