18.102 Assignment 1

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Problem 1

(a)

[Hölder's Inequality]

Proof. Let $A, B > 0$ and $t \in (0,1)$. We claim that

$$A^t B^{1-t} \leq tA + (1-t)B. \tag{1}$$

For $x > 0$, define

$$f(x) := tx + (1-t)B - x^t B^{1-t}.$$

Computing the first and second derivatives of f, we find

$$f'(x) = t - tx^{t-1}B^{1-t}, \quad \text{and}$$

$$f''(x) = -t(t-1)x^{t-2}B^{1-t}.$$

Then at $x = B$, we have $f'(B) = 0$ and $f''(B) = -t(t-1)\frac{1}{B} > 0$ for $0 < t < 1$. Hence, $f(x)$ has a minimum at $x = B$ by the second derivative test. Since $f(B) = 0$, we conclude that f attains a minimum value of 0 at $x = B$. If $A \neq B$, then it follows that

$$f(A) \geq f(B) = 0 \tag{2}$$

$$\implies tA + (1-t)B - A^t B^{1-t} \geq 0 \tag{3}$$

$$\implies A^t B^{1-t} \leq tA + (1-t)B, \tag{4}$$

and the claim is proven.

Now let $A = \frac{|a_k|^p}{\sum_{k=1}^n |a_k|^p}$ and $B = \frac{|b_k|^p}{\sum_{k=1}^n |b_k|^p}$ for $n \in \mathbb{N}$. Note that these choices satisfy the positivity conditions required in the previous claim. Then by (1), letting $t = \frac{1}{p}$, we have

$$A^{\frac{1}{p}} B^{\frac{1}{q}} \leq \frac{A}{p} + \frac{B}{q} \tag{5}$$

Substituting the expressions for A and B gives

$$\frac{|a_k||b_k|}{\left(\sum_{k=1}^n |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n |b_k|^q\right)^{\frac{1}{q}}} \leq \frac{|a_k|^p}{p\sum_{k=1}^n |a_k|^p} + \frac{|b_k|^q}{q\sum_{k=1}^n |b_k|^q}. \tag{6}$$

Summing from $k = 1$ to n on both sides of the inequality, we find

$$\sum_{k=1}^n \frac{|a_k||b_k|}{\left(\sum_{k=1}^n |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n |b_k|^q\right)^{\frac{1}{q}}} \leq \frac{1}{p}\sum_{k=1}^n \frac{|a_k|^p}{\sum_{k=1}^n |a_k|^p} + \frac{1}{q}\sum_{k=1}^n \frac{|b_k|^q}{\sum_{k=1}^n |b_k|^q} \tag{7}$$

$$= \frac{1}{p} + \frac{1}{q} = 1 \tag{8}$$

$$\implies \sum_{k=1}^n \frac{|a_k||b_k|}{\left(\sum_{k=1}^n |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n |b_k|^q\right)^{\frac{1}{q}}} \leq 1. \tag{9}$$

Multiplying both sides by the product $\left(\sum_{k=1}^n |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n |b_k|^q\right)^{\frac{1}{q}}$, we obtain the desired result,

$$\sum_{k=1}^n |a_k b_k| \leq \left[\sum_{k=1}^n |a_k|^p\right]^{\frac{1}{p}} \left[\sum_{k=1}^n |b_k|^q\right]^{\frac{1}{q}}. \tag{10}$$

□

(b)

[Minkowski's Inequality]

Proof. By the triangle inequality, we have

$$\sum_{k=1}^n |a_k + b_k|^p = \sum_{k=1}^n |a_k + b_k||a_k + b_k|^{p-1} \tag{11}$$

$$\leq \sum_{k=1}^n |a_k||a_k + b_k|^{p-1} + \sum_{k=1}^n |b_k||a_k + b_k|^{p-1}. \tag{12}$$

Then by Hölder's inequality [proved in **(a)**],

$$\sum_{k=1}^n |a_k||a_k + b_k|^{p-1} \leq \left[\sum_{k=1}^n |a_k|^p\right]^{\frac{1}{p}} \left[\sum_{k=1}^n |a_k + b_k|^p\right]^{\frac{p-1}{p}}, \quad \text{and} \tag{13}$$

$$\sum_{k=1}^n |b_k||a_k + b_k|^{p-1} \leq \left[\sum_{k=1}^n |b_k|^p\right]^{\frac{1}{p}} \left[\sum_{k=1}^n |a_k + b_k|^p\right]^{\frac{p-1}{p}}, \tag{14}$$

where we have identified $q = \frac{p}{p-1}$. Then

$$\sum_{k=1}^{n} |a_k + b_k|^p \leq \left(\left[\sum_{k=1}^{n} |a_k|^p \right]^{\frac{1}{p}} + \left[\sum_{k=1}^{n} |b_k|^p \right]^{\frac{1}{p}} \right) \left[\sum_{k=1}^{n} |a_k + b_k|^p \right]^{\frac{p-1}{p}} \tag{15}$$

$$\implies \left[\sum_{k=1}^{n} |a_k + b_k|^p \right] \left[\sum_{k=1}^{n} |a_k + b_k|^p \right]^{\frac{1-p}{p}} \leq \left[\sum_{k=1}^{n} |a_k|^p \right]^{\frac{1}{p}} + \left[\sum_{k=1}^{n} |b_k|^p \right]^{\frac{1}{p}}. \tag{16}$$

Combining exponents on the left side, we arrive at

$$\left[\sum_{k=1}^{n} |a_k + b_k|^p \right]^{\frac{1}{p}} \leq \left[\sum_{k=1}^{n} |a_k|^p \right]^{\frac{1}{p}} + \left[\sum_{k=1}^{n} |b_k|^p \right]^{\frac{1}{p}}. \tag{17}$$

□

Problem 2

Proof. We first show that ℓ^p is a normed space.

Let $a = \{a_j\}_{j=1}^{\infty}$ and $b = \{b_j\}_{j=1}^{\infty}$ be sequences in ℓ^p. Suppose $||a||_p = 0$. Then by Hölder's inequality, letting $b_j = n^{-\frac{1}{p}}$ for $n \in \mathbb{N}$, $\forall j \in \mathbb{N}$, we have

$$0 = \left[\sum_{j=1}^{n} |a_j|^p \right]^{\frac{1}{p}} = \left[\sum_{j=1}^{n} |a_j|^p \right]^{\frac{1}{p}} \left[\sum_{j=1}^{n} \frac{1}{n} \right]^{\frac{1}{p}} \tag{18}$$

$$\geq \sum_{j=1}^{n} |a_j n^{-\frac{1}{p}}| = n^{-\frac{1}{p}} \sum_{j=1}^{n} |a_j| \tag{19}$$

$$\geq 0. \tag{20}$$

Thus, we have that

$$0 \leq \sum_{j=1}^{n} |a_j| \leq 0, \tag{21}$$

but since $|a_j|$ is always nonnegative, this must imply that $a_j = 0 \ \forall j \in \mathbb{N}$. Going in the opposite direction, suppose $a = 0$ [i.e. $a_j = 0 \ \forall j \in \mathbb{N}$]. Then

$$||a||_p = \left[\sum_{j=1}^{n} |a_j|^p \right]^{\frac{1}{p}} = \left[\sum_{j=1}^{n} 0 \right]^{\frac{1}{p}} = 0^{\frac{1}{p}} = 0. \tag{22}$$

Hence, we have shown $||a||_p = 0 \iff a = 0$ [definiteness]. Now let $\lambda \in \mathbb{K}$ [an element in a field of scalars, $\mathbb{R}$ or $\mathbb{C}$]. Then

$$||\lambda a||_p = \left[\sum_{j=1}^{n} |\lambda a_j|^p\right]^{\frac{1}{p}} = \left[|\lambda|^p \sum_{j=1}^{n} |a_j|^p\right]^{\frac{1}{p}} = |\lambda| \left[\sum_{j=1}^{n} |a_j|^p\right]^{\frac{1}{p}}. \tag{23}$$

Hence, $||\lambda a||_p = |\lambda| \cdot ||a||_p$ [homogeneity]. Now consider the norm of the sum, $||a+b||_p$. By Minkowski's inequality, we have

$$||a+b||_p = \left[\sum_{j=1}^{n} |a_j + b_j|^p\right]^{\frac{1}{p}} \leq \left[\sum_{j=1}^{n} |a_j|^p\right]^{\frac{1}{p}} + \left[\sum_{j=1}^{n} |b_j|^p\right]^{\frac{1}{p}} \tag{24}$$

Hence, $||a+b||_p \leq ||a||_p + ||b||_p$ [triangle inequality]. Thus we have proven that $||\cdot||_p$ is a norm on ℓ^p, so we conclude that ℓ^p is a normed space.

Next we show that ℓ^p is complete.

Let $\{a^{(n)}\}_n$ be a Cauchy sequence in ℓ^p [i.e. $\{a_j^{(n)}\}_{j=1}^{\infty} \in \ell^p$ and $\{a^{(n)}\}_n = \{\{a_j^{(n)}\}_j\}_n$]. Let $\epsilon > 0$. Then $\exists N_0 \in \mathbb{N}$ such that $\forall n, m \geq N_0$,

$$||a^{(n)} - a^{(m)}|| < \epsilon. \tag{25}$$

Then this implies

$$\left[\sum_{j=1}^{\infty} |a_j^{(n)} - a_j^{(m)}|^p\right]^{\frac{1}{p}} = ||a^{(n)} - a^{(m)}||_p < \epsilon \tag{26}$$

$$\implies \sum_{j=1}^{\infty} |a_j^{(n)} - a_j^{(m)}|^p = ||a^{(n)} - a^{(m)}||_p^p < \epsilon^p. \tag{27}$$

Then for any $j \in \mathbb{N}$,

$$|a_j^{(n)} - a_j^{(m)}|^p < \sum_{j=1}^{\infty} |a_j^{(n)} - a_j^{(m)}|^p < \epsilon^p. \tag{28}$$

Hence, the sequence $\{a_j^{(n)}\}_n \subset \ell^p$ is Cauchy. By completeness of $\mathbb{R}$, $\forall j \in \mathbb{N}\ \exists a_j$ such that $\lim_{n\to\infty} a_j^{(n)} = a_j \in \mathbb{R}$.

Fix $k \in \mathbb{N}$. Then for $m, n > N_0$,

$$\sum_{j=1}^{k} |a_j^{(n)} - a_j^{(m)}|^p \leq \sum_{j=1}^{\infty} |a_j^{(n)} - a_j^{(m)}|^p \tag{29}$$

$$= ||a^{(n)} - a^{(m)}||_p < \epsilon^p \tag{30}$$

$$\overset{n\to\infty}{\implies} \sum_{j=1}^{k} |a_j^{(m)} - a_j|^p < \epsilon^p. \tag{31}$$

By Minkowski's inequality for $||\cdot||_p$ in $\mathbb{R}^k$, for $m > N_0$, we have

$$\left[\sum_{j=1}^{k} |a_j|^p\right]^{\frac{1}{p}} \leq \left[\sum_{j=1}^{k} |a_j^{(m)} - a_j|^p\right]^{\frac{1}{p}} + \left[\sum_{j=1}^{k} |a_j^{(m)}|^p\right]^{\frac{1}{p}} \tag{32}$$

$$< \epsilon + \left[\sum_{j=1}^{k} |a_j^{(m)}|^p\right]^{\frac{1}{p}} \tag{33}$$

$$\overset{n\to\infty}{\Longrightarrow} ||a||_p \leq \epsilon + ||a^{(m)}||_p. \tag{34}$$

Hence, $a \in \ell^p$. Then by letting $k \to \infty$ in (31),

$$\sum_{j=1}^{\infty} |a_j^{(m)} - a_j|^p = ||a^{(m)} - a||_p^p < \epsilon^p \tag{35}$$

$$\Longrightarrow ||a^{(m)} - a||_p < \epsilon. \tag{36}$$

Then $\lim_{m\to\infty} ||a^{(m)} - a||_p = 0$, which implies that the sequence $\{a^{(m)}\}_m \subset \ell^p$ converges to $a \in \ell^p$. Thus, ℓ^p is complete.

Since ℓ^p is a complete normed space, we conclude that ℓ^p is a Banach space. □

Problem 3

Proof. We will show that $\ell^p \setminus c_0 := \{b \in \ell^p | \lim_{k\to\infty} b_k \neq 0\}$ is open. □

Problem 5

(a)

Proof. We consider two cases.

Case 1: $p = 1$ [i.e. $q = \infty$]. If $\{a_k\}_k \in \ell^1$, then $||a||_1 < \infty$, so $\sum_{k=1}^{\infty} |a_k|$ converges. Similarly, if $\{b_k\}_k \in \ell^\infty$, then $||b||_\infty = \sup_{1\leq k<\infty} |b_k| < \infty$. Then

$$\sum_{k=1}^{\infty} |a_k b_k| \leq \sum_{k=1}^{\infty} |a_k| \sup_{k\in\mathbb{N}} |b_k| \tag{37}$$

$$= ||b||_\infty \left[\sum_{k=1}^{\infty} |a_k|\right]^1 \tag{38}$$

$$= ||a||_1 ||b||_\infty, \tag{39}$$

as desired.

Case 2: $1 < p < \infty$. Since $\frac{1}{p} + \frac{1}{q} = 1$, then the result for this case follows immediately from Hölder's inequality. For all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} |a_k b_k| \leq \left[\sum_{k=1}^{n} |a_k|^p\right]^{\frac{1}{p}} + \left[\sum_{k=1}^{n} |b_k|^q\right]^{\frac{1}{q}}. \tag{40}$$

Taking $n \to \infty$ in the above inequality, we achieve the desired result:

$$\sum_{k=1}^{\infty} |a_k b_k| \leq ||a||_p ||b||_q. \tag{41}$$

□

(b)

Proof. By the result proven in part **(a)**,

$$\begin{align} \sum_{k=1}^{\infty} a_k b_k &\leq \sum_{k=1}^{\infty} |a_k b_k| \tag{42} \\ &\leq ||a||_p ||b||_q \in \mathbb{C}. \tag{43} \end{align}$$

Hence, F_b maps ℓ^p into the scalar field $\mathbb{K} = \mathbb{C}$. So we conclude that $F_b \in (\ell^p)'$.

Next, we compute the operator norm of F_b. Let $a \in \ell^p$. Note that

$$\begin{align} |F_b(a)| &= \left|\sum_{k=1}^{\infty} a_k b_k\right| \tag{44} \\ &\leq \sum_{k=1}^{\infty} |a_k b_k| \tag{45} \\ &\leq ||a||_p ||b||_q. \tag{46} \end{align}$$

Then taking the supremum over the set of $||a||_p = 1$,

$$\begin{align} ||F_b|| &= \sup_{||a||_p = 1} |F_b(a)| \tag{47} \\ &= \sup_{||a||_p = 1} ||a||_p ||b||_q \tag{48} \\ &= ||b||_q. \tag{49} \end{align}$$

Thus we have shown that $F_b \in (\ell^p)'$ and $||F_b|| = ||b||_{\ell^q}$, as desired. □

(c)

Proof. We first show that F is linear.

[Note: $F(b) = F_b = \sum_{k=1}^{\infty} b_k \in (\ell^p)'$.]

Let b and c be sequences in ℓ^q, and let $\lambda \in \mathbb{K}$. Then by the linearity of sums,

$$F(b+c) = \sum_{k=1}^{\infty}(b_k + c_k) = \sum_{k=1}^{\infty} b_k + \sum_{k=1}^{\infty} c_k = F(b) + F(c), \quad \text{and} \tag{50}$$

$$F(\lambda b) = \sum_{k=1}^{\infty} \lambda b_k = \lambda \sum_{k=1}^{\infty} b_k = \lambda F(b). \tag{51}$$

Thus, F is linear.

Next, we show that F is bijective. Suppose $F(b) = F(c)$. Then by part **(b)**,

$$0 = ||F_b - F_c|| = ||b - c||_{\ell^q}. \tag{52}$$

But since $|| \cdot ||_{\ell^q}$ is a norm, this must imply that

$$b - c = 0 \tag{53}$$

$$\implies b = c. \tag{54}$$

Hence, F is injective. Now suppose we are given $F_b = \sum_{k=1}^{\infty} b_k \in (\ell^p)'$. Then by definition, we choose $b = \{b_k\}_k \in \ell^q$ so that $F(b) = F_b$. We can make this choice for any such series in $(\ell^p)'$, since by what we proved in **(b)**, it always holds that the sequence b exists in ℓ^q. Hence, F is surjective, which means that F is bijective.

We now show that F is bounded. The proof of this claim follows immediately from the result in **(b)**. Let $b \in \ell^q$. Then

$$||F(b)|| = ||F_b|| \tag{55}$$

$$= ||b||_{\ell^q} \tag{56}$$

$$\leq c||b||_{\ell^q}, \quad \text{for } c = 1. \tag{57}$$

So, F is continuous (bounded).

Therefore, $F : \ell^q \to (\ell^p)'$ is a bijective, bounded, linear operator. $\square$