18.102 Midterm

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Problem 1

Proof. We will show that $\Lambda([a,b])$ is a proper closed subspace of $C([a,b])$, which we know is a Banach space. Let $\{f_n\}_n$ be a cauchy sequence in $\Lambda([a,b])$ such that $f_n \to f$ pointwise. Then for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $||f - f_n|| < \epsilon$. This is equivalent to

$$\sup_{x\in[a,b]} |f(x) - f_n(x)| + \sup_{x\neq y\in[a,b]} \frac{|f(x) - f_n(x) - f(y) + f_n(y)|}{|x-y|} < \epsilon. \tag{1}$$

Since both terms on the left hand side are non-negative, this implies

$$\sup_{x\neq y\in[a,b]} \frac{|f(x) - f_n(x) - f(y) + f_n(y)|}{|x-y|} < \epsilon. \tag{2}$$

Then for any $x \neq y \in [a,b]$, we have

$$|f(x) - f_n(x) - f(y) + f_n(y)| < \epsilon|x-y|, \tag{3}$$

which confirms that for each $n \geq N$, the function $f - f_n$ is Lipschitz continuous. By assumtion, f_n is Lipschitz continuous $\forall n \in \mathbb{N}$, and the sum of Lipschitz continuous functions is also Lipschitz, thus $f = f_n + (f - f_n)$ is Lipschitz continuous.

So, $\lim_{n\to\infty} f_n = f \in \Lambda([a,b])$, which proves that $\Lambda([a,b])$ is a proper closed subspace of $C([a,b])$.

Therefore, $\Lambda([a,b])$ is a Banach space. $\square$