18.102 Assignment 5

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In the problems to follow, we denote by $\mathcal{M}$ the set of all Lebesgue-measurable subsets of $\mathbb{R}$.

Problem 1

(a)

Proof. Since $E, M \in \mathcal{M}$, then $E \cup M \in \mathcal{M}$ and $E \cap M \in \mathcal{M}$. We can express the union as

$$E \cup F = (E \cap F) \cup (E \backslash F) \cup (F \backslash E). \tag{1}$$

Then since E and $F \backslash E$ are disjoint, we have

$$\begin{aligned} m(E \cup F) + m(E \cap F) &= m\left[(E \cap F) \cup (E \backslash F)\right] + m(F \backslash E) + m(E \cap F) & (2) \\ &= m\left[(E \cap F) \cup (E \backslash F)\right] + m\left[(F \backslash E) \cup (E \cap F)\right] & (3) \\ &= m(E) + m(F), & (4) \end{aligned}$$

as desired. $\square$

(b)

Proof. Since $m(E_1) < \infty$ and $E_{k+1} \subset E_k\ \forall k \in \mathbb{N}$, then $m(E_k) < m(E_1) < \infty$. Then

$$m\left(\bigcap_{k=1}^{\infty} E_k\right) < \infty, \tag{5}$$

since $\bigcap_{k=1}^{\infty} E_k \subset E_k$.

Additionally, we have

$$m\left[E_1 \backslash \left(\bigcap_{k=1}^{\infty} E_k\right)\right] = m(E_1) - m\left(\cap_{k=1}^{\infty} E_k\right). \tag{6}$$

Also, we note that

$$E_1 \backslash \left(\bigcap_{k=1}^{\infty} E_k \right) = E_1 \cap \left(\bigcap_{k=1}^{\infty} E_k \right)^c \tag{7}$$

$$= E_1 \cap \left(\bigcup_{k=1}^{\infty} E_k^c \right) \tag{8}$$

$$= \bigcup_{k=1}^{\infty} (E_1 \cap E_k^c) \tag{9}$$

$$= \bigcup_{k=1}^{\infty} (E_1 \backslash E_k) \,. \tag{10}$$

Define $U_k = E_1 \backslash E_k$ for each $k \in \mathbb{N}$. Then $\forall k \in \mathbb{N}$, $U_k \subset U_{k+1}$ since $E_{k+1} \subset E_k$. By (10), we have

$$m \left(E_1 \backslash \bigcap_{k=1}^{\infty} E_k \right) = m \left[\bigcup_{k=1}^{\infty} (E_1 \backslash E_k) \right] \tag{11}$$

$$= \lim_{k \to \infty} m(E_1 \backslash E_k) \tag{12}$$

$$= \lim_{k \to \infty} \left[m(E_1) - m(E_k) \right] \tag{13}$$

$$= m(E_1) - \lim_{k \to \infty} m(E_k), \tag{14}$$

where we get to line (12) using the theorem proven in appendix A. Then using equation (6), this gives

$$m(E_1) - \lim_{k \to \infty} m(E_k) = m(E_1) - m \left(\bigcap_{k=1}^{\infty} E_k \right) . \tag{15}$$

Therefore, we conclude that $m \left(\bigcap_{k=1}^{\infty} E_k \right) = \lim_{k \to \infty} m(E_k)$.[1] $\square$

Problem 2

(a)

Proof. Let $a \in \mathbb{R}$.

We can write

$$fg = \frac{1}{4} \left[(f+g)^2 - (f-g)^2 \right] . \tag{16}$$

We showed in lecture 9 that linear combinations of measurable functions are measurable, so we need only show that f^2 and g^2 are measurable.

[1]The assignment states this result as *continuity from below* for the Lebesgue measure, but I believe this is actually *continuity from above*.

Case 1: $\alpha < 0$. Then,

$$(f^2)^{-1}((\alpha, \infty]) = (f^2)^{-1}([0, \infty]) = E \in \mathcal{M}. \tag{17}$$

Case 2: $\alpha \geq 0$. Then $\forall x \in E$,

$$f^2(x) > \alpha \iff f(x) > \sqrt{\alpha} \text{ or } f(x) < -\sqrt{a} \tag{18}$$
$$\implies (f^2)^{-1}((\alpha, \infty]) = [-\infty, -\sqrt{\alpha}) \cup (\sqrt{\alpha}, \infty] \in \mathcal{M}. \tag{19}$$

So f^2 is measurable, and by the same reasoning g^2 is measurable.

Therefore, fg is measurable. □

(b)

Proof. Let $\alpha \in \mathbb{R}$.

Case 1: $\alpha = +\infty$. Then,

$$h^{-1}(\{\infty\}) = \{a\} = \bigcap_{n=1}^{\infty} \left[a, a + \frac{1}{n}\right] \in \mathcal{M}. \tag{20}$$

Case 2: $\alpha \neq \infty$. We have

$$h^{-1}((\alpha, \infty]) = (f + g)^{-1}((\alpha, \infty]). \tag{21}$$

Then $x \in (f+g)^{-1}((\alpha, \infty]) \iff f(x) + g(x) > \alpha$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists r \in \mathbb{Q}$ such that $f(x) > r > \alpha - g(x)$. Then since f and g are measurable, we have

$$(f + g)^{-1}((\alpha, \infty]) = \bigcup_{r \in \mathbb{Q}} \left[f^{-1}((r, \infty]) \cap g^{-1}((\alpha - r, \infty])\right] \in \mathcal{M}. \tag{22}$$

Therefore, h is measurable. □

Problem 3

(a)

Proof. ($\Rightarrow$) Suppose f is measurable.

Let $\alpha \in \mathbb{R}$. We may express the preimage of the set $(\alpha, \infty]$ under the inverse of the restriction of f to E as follows:

$$f^{-1}\big|_E((\alpha, \infty])) = f^{-1}((\alpha, \infty])) \cap E, \tag{23}$$

and similarly for F:

$$f^{-1}\big|_F((\alpha, \infty])) = f^{-1}((\alpha, \infty])) \cap F. \tag{24}$$

Since f is measurable, then $f^{-1}((\alpha, \infty])) \in \mathcal{M}$. By assumption, E and F are also measurable. Hence, the intersections in (23) and (24) are also measurable.

Therefore, $f\big|_E$ and $f\big|_F$ are measurable.

($\Leftarrow$) Suppose $f\big|_E$ and $f\big|_F$ are measurable.

Then for ever $\alpha \in \mathbb{R}$, $f^{-1}\big|_E((\alpha, \infty])) \in \mathcal{M}$ and $f^{-1}\big|_F((\alpha, \infty])) \in \mathcal{M}$. Since $\mathcal{M}$ is closed under taking finite unions, then the union of each of these sets is also measurable, i.e.

$$f^{-1}\big|_E((\alpha, \infty])) \cup f^{-1}\big|_F((\alpha, \infty])) \in \mathcal{M}. \tag{25}$$

We also have that $E, F \in \mathcal{M}$, so $E \cup F \in \mathcal{M}$. Then we have

$$\begin{aligned} & f^{-1}\big|_E((\alpha, \infty])) \cup f^{-1}\big|_F((\alpha, \infty])) && (26) \\ &= \left(f^{-1}((\alpha, \infty])) \cap E\right) \cup \left(f^{-1}((\alpha, \infty])) \cap F\right) \\ &= f^{-1}((\alpha, \infty]) \cap (E \cup F) && (27) \\ &= f^{-1}((\alpha, \infty]) \in \mathcal{M}, && (28) \end{aligned}$$

where in line (28) we used the fact that $f^{-1}((\alpha, \infty]) \subset (E \cup F)$.

Therefore, as desired, f must be measurable. $\square$

(b)

Proof. ($\Rightarrow$) Suppose f is measurable.

We define the indicator function χ_E on E via

$$\chi_E(x) := \begin{cases} 1, \ x \in E \\ 0, \ x \in E^c. \end{cases} \tag{29}$$

Then we can express g as the product

$$g(x) = f(x) \cdot \chi_E(x). \tag{30}$$

In problem **2a**, we showed that the product of measurable functions is measurable. By assumption, f is measurable. so we need only check that χ_E is measurable.

Let $\alpha \in \mathbb{R}$.

<u>Case 1</u>: $1 \leq \alpha \leq \infty$. Then,

$$\chi_E^{-1}((\alpha, \infty]) = \emptyset \in \mathcal{M}. \tag{31}$$

<u>Case 2</u>: $0 \leq \alpha < 1$. Then,

$$\chi_E^{-1}((\alpha, \infty]) = E \in \mathcal{M}. \tag{32}$$

Case 3: $-\infty \le \alpha < 0$. Then,

$$\chi_E^{-1}\left((\alpha, \infty]\right) = \mathbb{R} \in \mathcal{M}. \tag{33}$$

Hence, χ_E is measurable, so $f \cdot \chi_E$ is also measurable.

Therefore, g is measurable.

($\Leftarrow$) Suppose g is measurable. Since $g : E \cup E^c = \mathbb{R} \to [-\infty, \infty]$ is defined by

$$g(x) := \begin{cases} f(x), & x \in E \\ 0, & x \in E^c, \end{cases} \tag{34}$$

then by restricting g to E we get $g\big|_E(x) = f(x)$. By part **(a)**, $g\big|_E$ must be measurable.

Therefore, f is measurable. $\square$

(c)

Proof. We have already shown in class that sums and products of measurable functions are measurable. So if u and v are measurable, then both u^2 and v^2 are measurable, which implies that $u^2 + v^2$ is measurable.

Define

$$f(x) := u^2(x) + v^2(x). \tag{35}$$

Then we need only check that $f^{\frac{1}{2}}$ is measurable.

Let $g(x) = x^{\frac{1}{2}}$. Then $f^{\frac{1}{2}}(x) = (g \circ f)(x)$, and $f : E \to [0, \infty]$
$\implies g : [0, \infty] \to [0, \infty]$. We use the fact that the composition of a continuous function (g) with a measurable function (f) is measurable to show that $f^{\frac{1}{2}}$ is measurable.

Let $\alpha \in \mathbb{R}$.

Case 1: $0 \le \alpha \le \infty$. Then,

$$g^{-1}\left((\alpha, \infty]\right) = (\alpha^2, \infty) \in \mathcal{M}. \tag{36}$$

Case 2: $-\infty \le \alpha < 0$. Then,

$$g^{-1}\left((\alpha, \infty]\right) = \emptyset \in \mathcal{M}. \tag{37}$$

Hence, g is measurable, so $f^{\frac{1}{2}}$ is measurable.

Therefore, $(u^2 + v^2)^{\frac{1}{2}}$ is measurable. $\square$

Problem 4

(a)

Proof. We define S as the set of elements in E for which f_n does not converge to f, via

$$S := \{x \in E \mid \lim_{n\to\infty} f_n(x) \neq f(x)\}. \tag{38}$$

If $\lim_{n\to\infty} f_n(x) \neq f(x)$, then for every $n \in \mathbb{N}$, $\exists \epsilon_0$ such that $\forall m > n$, $|f_m(x) - f(x)| \geq \epsilon_0$. Since this statement holds for every n, then letting $\epsilon_0 = \frac{1}{k}$ allows us to express S as a countable union:

$$S = \bigcup_{m=1}^{\infty} \{x \in E \mid |f_m(x) - f(x)| \geq k^{-1}\} = F_1(k). \tag{39}$$

But since $f_n \to f$ almost everywhere (a.e.) on E, then S must have measure 0, i.e.

$$m(F_1(k)) = 0 < \infty. \tag{40}$$

We can also check that

$$\begin{aligned} F_{n+1}(k) &= \bigcup_{m=n+1}^{\infty} \{x \in E \mid |f_m(x) - f(x)| \geq k^{-1}\} && (41)\\ &\subset \bigcup_{m=n}^{\infty} \{x \in E \mid |f_m(x) - f(x)| \geq k^{-1}\} && (42)\\ &= F_n(k). && (43) \end{aligned}$$

So $\forall n \in \mathbb{N}$,

$$F_{n+1}(k) \subset F_n(k). \tag{44}$$

By equations (40) and (44), we appeal to problem **1b** to deduce that

$$\lim_{n\to\infty} m(F_n(k)) = m\left(\bigcap_{n=1}^{\infty} F_n(k)\right). \tag{45}$$

We also have that

$$\bigcap_{n=1}^{\infty} F_n(k) \subset F_n(k) \subset F_1(k), \tag{46}$$

so by the monotonicity of the Lebesgue measure,

$$m\left(\bigcap_{n=1}^{\infty} F_n(k)\right) \leq m(F_1(k)) = 0. \tag{47}$$

Therefore, $\lim_{n\to\infty} m(F_n(k)) = 0$. $\square$

(b)

Proof. (i) Since F is a countable union, then

$$m(F) = m\left(\bigcup_{k=1}^{\infty} F_{n_k}(k)\right) \tag{48}$$

$$\leq \sum_{k=1}^{\infty} m\left(F_{n_k}(k)\right) \tag{49}$$

$$< \sum_{k=1}^{\infty} 2^{-k}\epsilon \tag{50}$$

$$= \epsilon\left(\frac{1}{1-\frac{1}{2}} - 1\right) \tag{51}$$

$$= \epsilon. \tag{52}$$

Thus, $m(F) < \epsilon$.

(ii) The complement F^c is given by

$$F^c = \left(\bigcup_{k=1}^{\infty} F_{n_k}(k)\right)^c = \bigcap_{k=1}^{\infty} F_{n_k}(k)^c, \tag{53}$$

where

$$F_{n_k}(k) := \bigcup_{m=n_k}^{\infty} \{x \in E \mid |f_m(x) - f(x)| \geq k^{-1}\}. \tag{54}$$

Then,

$$F_{n_k}(k)^c = \bigcap_{m=n_k}^{\infty} \{x \in E \mid |f_n(x) - f(x)| < k^{-1}\} \tag{55}$$

$$\implies F^c = \bigcap_{k=1}^{\infty} \bigcap_{m=n_k}^{\infty} \{x \in E \mid |f_n(x) - f(x)| < k^{-1}\}. \tag{56}$$

Let $x \in F^c$. Then by the above equation, $\forall k \in \mathbb{N}$, $\exists n_k \in \mathbb{N}$ such that $\forall m \geq n_k$, $|f_m(x) - f(x)| < k^{-1}$. This implies that $f_n(x) \to f(x)$, and this is true for every $x \in F^c$.

Therefore, we conclude that $f_n \to f$ uniformly on F^c. $\square$

Remark: In this problem, we have effectively proven *Littlewood's second principle*, which is that every convergent sequence of measurable functions is uniformly convergent.

Appendices

A Appendix A

Theorem (*Continuity from Below*): If $\{E_n\}_{n=1}^{\infty}$ is a countable collection of measurable sets such that $\forall n \in \mathbb{N}$, $E_n \subset E_{n+1}$ and $m(E_n) < \infty$, then $m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$.

Proof. We can write the countable union of all the sets as

$$\bigcup_{n=1}^{\infty} E_n = E_1 \cup (E_2 \backslash E_1) \cup (E_3 \backslash E_2) \cup \cdots \tag{57}$$

$$= E_1 \cup \left(\bigcup_{n=2}^{\infty} (E_n \backslash E_{n-1}) \right). \tag{58}$$

Since $E_n \subset E_{n+1}$ $\forall n \in \mathbb{N}$, then E_1 and $\{E_n \backslash E_{n-1}\}_{n\geq 2}$ are all disjoint. Then taking the Lebesgue measure on both sides of (58) yields

$$m\left(\bigcup_{n=1}^{\infty} E_n \right) = m\left(E_1 \cup \left(\bigcup_{n=2}^{\infty} (E_n \backslash E_{n-1}) \right) \right) \tag{59}$$

$$= m(E_1) + \sum_{n=2}^{\infty} m(E_n \backslash E_{n-1}) \tag{60}$$

$$= m(E_1) + \sum_{n=2}^{\infty} [m(E_n) - m(E_{n-1})] \tag{61}$$

$$= m(E_1) + \lim_{n\to\infty} \sum_{k=2}^{n} [m(E_k) - m(E_{k-1})] \tag{62}$$

$$= m(E_1) + \lim_{n\to\infty} (m(E_n) - m(E_1)) \tag{63}$$

$$= \lim_{n\to\infty} m(E_n), \tag{64}$$

as desired. □