

18.102 Assignment 6

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Problem 1

(a)

Proof. Let $\epsilon > 0$, and define $c_1 := a$ and $c_{n+1} := b$.

Since ψ is a step function on $[a, b]$, $\exists c_1 \leq c_2 \leq \dots \leq c_n \leq c_{n+1} \in [a, b]$ such that $\forall i = 1, \dots, n$,

$$\psi^{-1}(\{a_i\}) = (c_i, c_{i+1}], \quad (1)$$

where each a_i is one of the finitely many values that ψ takes on.

Choose $\delta > 0$ such that $\delta < \frac{\epsilon}{2n}$. Define $g : [a, b] \rightarrow \mathbb{R}$ via

$$g(x) := \begin{cases} \frac{a_i + a_{i+1}}{2} + \left(\frac{a_i - a_{i+1}}{2\delta} \right) (x - c_i), & x \in (c_i - \delta, c_i + \delta) \\ a_i, & x \in [c_i + \delta, c_{i+1} - \delta] \\ -\frac{a_n}{2\delta} (x - b), & x \in (c_{n-1} - \delta, b], \end{cases} \quad (2)$$

where $a_0 := -a_1$. Then

$$g(a) = \frac{a_1 + a_0}{2} + \left(\frac{a_1 - a_0}{2\delta} \right) (c_1 - c_1) = \frac{a_1 - a_1}{2} = 0, \quad (3)$$

and

$$g(b) = -\frac{a_n}{2\delta} (b - b) = 0, \quad (4)$$

as desired. We also see that since g is piecewise linear, it is continuous.

Now consider the difference $|\psi(x) - g(x)|$.

Case 1: $x \in [c_i + \delta, c_{i+1} - \delta]$. Then by (1), we have

$$|\psi(x) - g(x)| = |\psi((c_i, c_{i+1} - \delta)) - a_i| = |a_i - a_i| = 0. \quad (5)$$

Case 2: $x \in (c_i - \delta, c_i + \delta)$. Then

$$|\psi(x) - g(x)| = \left| \frac{a_i + a_{i-1}}{2} + \left(\frac{a_i - a_{i-1}}{2\delta} \right) (x - c_i) - \psi(x) \right| \quad (6)$$

$$< \left| \frac{a_i + a_{i-1}}{2} + \left(\frac{a_i - a_{i-1}}{2\delta} \right) \delta - \psi(x) \right| \quad (7)$$

$$= \left| \frac{a_i + a_{i-1}}{2} + \frac{a_i + a_{i-1}}{2} - \psi(x) \right| \quad (8)$$

$$= |a_i - \psi(x)| \quad (9)$$

$$= 0 \text{ or } |a_{i+1} - a_i|. \quad (10)$$

Case 3: $x \in (c_n - \delta, b)$. Then

$$|\psi(x) - g(x)| = \left| -\frac{a_n}{2\delta}(x - b) - \psi(x) \right| \quad (11)$$

$$< \left| -\frac{a_n}{2\delta}\delta - \psi(x) \right| \quad (12)$$

$$= \left| -\frac{a_n}{2} - \psi(x) \right| \quad (13)$$

$$= \frac{3a_n}{2}. \quad (14)$$

So in all three cases, we have that either $|g(x) - \psi(x)| = 0$, or $|g(x) - \psi(x)| < \frac{3a}{2}$, or $|g(x) - \psi(x)| < |a_{i+1} - a_i|$.

Define the set E to be the collection of points in $[a, b]$ for which $|\psi(x) - g(x)| \neq 0$. Then

$$E := \bigcup_{k=1}^n (c_k - \delta, c_k + \delta). \quad (15)$$

By definition, $\forall x \in E^c$,

$$|\psi(x) - g(x)| = 0 < \epsilon. \quad (16)$$

Since E is a countable union of intervals, we have

$$m(E) = m \left[\bigcup_{k=1}^n (c_k - \delta, c_k + \delta) \right] \quad (17)$$

$$\leq \sum_{k=1}^n m[(c_k - \delta, c_k + \delta)] \quad (18)$$

$$= \sum_{k=1}^n \ell(c_k - \delta, c_k + \delta) \quad (19)$$

$$= \sum_{k=1}^n 2\delta \quad (20)$$

$$= 2n\delta \quad (21)$$

$$< 2n \frac{\epsilon}{2n} \quad (22)$$

$$= \epsilon, \quad (23)$$

as desired. \square

(b)

Proof. Suppose E is Lebesgue measurable and $m(E) < \infty$. Then by Littlewood's first principle, $\forall \delta > 0 \exists$ a finite collection of open intervals $\{U_i\}_{i=1}^n$ such that

$$m \left(E \Delta \bigcup_{i=1}^n U_i \right) < \delta. \quad (24)$$

Let $\epsilon > 0$. Since φ is a simple function, then we can express it as

$$\varphi = \sum_{i=1}^n a_i \chi_{E_i}, \quad (25)$$

where we take U_i such that the symmetric difference

$$m(U_i \setminus E_i) + m(E_i \setminus U_i) < \frac{\epsilon}{n} \quad (26)$$

for each $i \in \{1, \dots, n\}$.

Let $\psi = \sum_{i=1}^n a_i \chi_{U_i}$. Then $\forall x \in U \cap E = (U \Delta E)^c$,

$$|\varphi(x) - \psi(x)| = 0 < \epsilon. \quad (27)$$

We have

$$m(U \Delta E) = \sum_{i=1}^n m(U_i \setminus E_i) + \sum_{i=1}^n m(E_i \setminus U_i) \quad (28)$$

$$< n \frac{\epsilon}{n} = \epsilon, \quad (29)$$

as desired. \square

Problem 2

Proof. Let $c_0 = a$, $c_n = b$, and partition $[a, b]$ into disjoint intervals as follows:

$$[a, b] = [c_0, c_1) \cup [c_1, c_2) \cup \dots \cup [c_{n-1}, c_n) \cup \{b\}. \quad (30)$$

Letting $U_k = [c_{k-1}, c_k)$ for each $k \in \{1, \dots, n\}$, we have

$$\bigcup_{k=1}^n U_k = [a, b]. \quad (31)$$

On each interval U_k , extract a closed set F_k such that f is continuous on F_k and such that

$$m\left(\bigcup_{k=1}^n F_k^c\right) < \epsilon. \quad (32)$$

Then for each $k \in \{1, \dots, n\}$, $\exists F_k \subseteq U_k$ closed such that $m(E \setminus F_k) < \frac{\epsilon}{2^n}$ and such that $f|_{F_k}$ is continuous.

Since F_k is closed, then F_k^c is open, so the union $\bigcup_{k=1}^n F_k^c$ is open as well. Then $E = \bigcup_{k=1}^n F_k^c$ can be expressed as a union of disjoint intervals:

$$E = \bigcup_{k=1}^n (a_k, b_k). \quad (33)$$

Define $g : [a, b] \rightarrow \mathbb{R}$ via

$$g(x) := \begin{cases} \frac{x-a_k}{b_k-a_k} f(b_k) + f(a_k), & x \in E \\ f(x), & x \in E^c. \end{cases} \quad (34)$$

Then

$$\sup_{x \in [a, b]} |g(x)| \leq \sup_{x \in E} |g(x)| \leq |f(a_k)| + |f(b_k)|, \quad (35)$$

and

$$\sup_{x \in [a, b]} |g(x)| \leq \sup_{x \in E^c} |g(x)| \leq \sup_{x \in E^c} |f(x)| \leq B. \quad (36)$$

Also,

$$m(E) = \sum_{k=1}^n m(F_k^c) < \sum_{k=1}^n \frac{\epsilon}{2^n} = \epsilon. \quad (37)$$

Finally, $\forall x \in E^c$, $g(x) = f(x)$, so $|f(x) - g(x)| = 0 < \epsilon$.

Therefore, the function g meets the desired requirements. \square

Problem 3

(a)

Proof. Since f is Lebesgue integrable, then f is measurable. For each $n \in \mathbb{N}$, let $E_n = f^{-1}([-n, n])$. Then $[-n, n] = [-\infty, n] \cap [-n, \infty]$ and since f is measurable, then $f^{-1}([-\infty, n])$ and $f^{-1}([-n, \infty])$ are measurable. Thus $f^{-1}([-\infty, n] \cap [-n, \infty])$ is measurable. Hence $f^{-1}([-n, n])$ is measurable $\iff E_n$ is measurable.

Define $h_n = f\chi_{E_n}$. Then

$$0 \leq \int_a^b |f(x) - h_n(x)| dx \quad (38)$$

$$= \int_a^b |f(x) - f(x)\chi_{E_n}(x)| dx \quad (39)$$

$$= \int_a^b |f(x)| |1 - \chi_{E_n}(x)| dx \quad (40)$$

$$= \int_a^b |f(x)| \chi_{E_n^c}(x) dx \quad (41)$$

$$= \int_{[a,b] \cap E_n^c} |f(x)| dx, \quad (42)$$

where we have used the simple fact that for a measurable set E , the characteristic function over the set's complement takes the form $\chi_{E^c} = 1 - \chi_E$.

We have $[a, b] \cap E_n^c = [a, b] \cap f^{-1}([-n, n])^c$. But as $n \rightarrow \infty$, $f^{-1}([-n, n]) \rightarrow [a, b]$, so $f^{-1}([-n, n])^c \rightarrow [a, b]^c$. Hence, $[a, b] \cap E_n^c \rightarrow \emptyset$.

Sending $n \rightarrow \infty$, we get

$$0 \leq \lim_{n \rightarrow \infty} \int_a^b |f(x) - h_n(x)| dx \quad (43)$$

$$= \lim_{n \rightarrow \infty} \int_{[a,b] \cap E_n^c} |f(x)| dx \quad (44)$$

$$= \int_{\emptyset} |f(x)| dx \quad (45)$$

$$= 0. \quad (46)$$

Thus, $\lim_{n \rightarrow \infty} \int_a^b |f(x) - h_n(x)| dx = 0$.

Let $h = \lim_{n \rightarrow \infty} h_n = f\chi_{[a,b]}$. Then h is a product of two measurable functions, so h is measurable. Also, h is the limit of a sequence of simple functions $\{h_n\}_n$ with $|h| \leq |h_n| \forall n \in \mathbb{N}$. So h is bounded.

Therefore $\int_a^b |f(x) - h(x)| dx < \epsilon$, as desired. \square

(b)

Proof. By (a), we know that \exists a bounded measurable function $h : [a, b] \rightarrow \mathbb{R}$ such that

$$\int_a^b |f(x) - h(x)| dx < \frac{\epsilon}{3}. \quad (47)$$

Next, we approximate h by a simple function $\varphi : [a, b] \rightarrow \mathbb{R}$ such that

$$\int_a^b |h(x) - \varphi(x)| dx < \frac{\epsilon}{3}. \quad (48)$$

Finally, by problem (1b), we can approximate φ by a step function ψ such that

$$|\varphi(x) - \psi(x)| < \frac{\epsilon}{3(b-a)}. \quad (49)$$

Integrating (49) over $[a, b]$, this is equivalent to

$$\int_a^b |\varphi(x) - \psi(x)| dx < \frac{\epsilon}{3(b-a)} m([a, b]) = \frac{\epsilon}{3}. \quad (50)$$

So, by the triangle inequality we have

$$\int_a^b |f(x) - \psi(x)| dx = \int_a^b |f(x) - h(x) + h(x) - \varphi(x) + \varphi(x) - \psi(x)| dx \quad (51)$$

$$\leq \int_a^b |f(x) - h(x)| dx + \int_a^b |h(x) - \varphi(x)| dx + \int_a^b |\varphi(x) - \psi(x)| dx \quad (52)$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \quad (53)$$

$$= \epsilon. \quad (54)$$

Thus, ψ suffices. \square

(c)

Proof. Let $E = \{x \in [a, b] \mid |f(x) - g(x)| \geq \frac{\epsilon}{b-a}\}$.

By what we showed in problem 2, we know \exists such a continuous function $g : [a, b] \rightarrow \mathbb{R}$ such that $g(a) = g(b) = 0$ and

$$m(E) = 0. \quad (55)$$

Then we have

$$\int_a^b |f(x) - g(x)| dx = \int_{E \cup E^c} |f(x) - g(x)| dx \quad (56)$$

$$= \int_E |f(x) - g(x)| dx + \int_{E^c} |f(x) - g(x)| dx. \quad (57)$$

But $m(E) = 0$, so $m(E^c) = b - a$. Then

$$\int_a^b |f(x) - g(x)| dx = \int_{E^c} |f(x) - g(x)| dx \quad (58)$$

$$< \int_{E^c} \frac{\epsilon}{b-a} dx \quad (59)$$

$$= (b-a) \frac{\epsilon}{b-a} \quad (60)$$

$$= \epsilon. \quad (61)$$

Hence, g satisfies the desired constraints. \square