

# 18.102 Assignment 1

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## Problem 1

(a)

[Hölder's Inequality]

*Proof.* Let  $A, B > 0$  and  $t \in (0, 1)$ . We claim that

$$A^t B^{1-t} \leq tA + (1-t)B. \quad (1)$$

For  $x > 0$ , define

$$f(x) := tx + (1-t)B - x^t B^{1-t}.$$

Computing the first and second derivatives of  $f$ , we find

$$f'(x) = t - tx^{t-1} B^{1-t}, \quad \text{and}$$

$$f''(x) = -t(t-1)x^{t-2} B^{1-t}.$$

Then at  $x = B$ , we have  $f'(B) = 0$  and  $f''(B) = -t(t-1)\frac{1}{B} > 0$  for  $0 < t < 1$ . Hence,  $f(x)$  has a minimum at  $x = B$  by the second derivative test. Since  $f(B) = 0$ , we conclude that  $f$  attains a minimum value of 0 at  $x = B$ . If  $A \neq B$ , then it follows that

$$f(A) \geq f(B) = 0 \quad (2)$$

$$\implies tA + (1-t)B - A^t B^{1-t} \geq 0 \quad (3)$$

$$\implies A^t B^{1-t} \leq tA + (1-t)B, \quad (4)$$

and the claim is proven.

Now let  $A = \frac{\sum_{k=1}^n |a_k|^p}{\sum_{k=1}^n |a_k|^p}$  and  $B = \frac{\sum_{k=1}^n |b_k|^p}{\sum_{k=1}^n |b_k|^p}$  for  $n \in \mathbb{N}$ . Note that these choices satisfy the positivity conditions required in the previous claim. Then by (1), letting  $t = \frac{1}{p}$ , we have

$$A^{\frac{1}{p}} B^{\frac{1}{q}} \leq \frac{A}{p} + \frac{B}{q} \quad (5)$$

Substituting the expressions for  $A$  and  $B$  gives

$$\frac{|a_k||b_k|}{(\sum_{k=1}^n |a_k|^p)^{\frac{1}{p}} (\sum_{k=1}^n |b_k|^q)^{\frac{1}{q}}} \leq \frac{|a_k|^p}{p \sum_{k=1}^n |a_k|^p} + \frac{|b_k|^q}{q \sum_{k=1}^n |b_k|^q}. \quad (6)$$

Summing from  $k = 1$  to  $n$  on both sides of the inequality, we find

$$\sum_{k=1}^n \frac{|a_k||b_k|}{(\sum_{k=1}^n |a_k|^p)^{\frac{1}{p}} (\sum_{k=1}^n |b_k|^q)^{\frac{1}{q}}} \leq \frac{1}{p} \sum_{k=1}^n \frac{|a_k|^p}{\sum_{k=1}^n |a_k|^p} + \frac{1}{q} \sum_{k=1}^n \frac{|b_k|^q}{\sum_{k=1}^n |b_k|^q} \quad (7)$$

$$= \frac{1}{p} + \frac{1}{q} = 1 \quad (8)$$

$$\Rightarrow \sum_{k=1}^n \frac{|a_k||b_k|}{(\sum_{k=1}^n |a_k|^p)^{\frac{1}{p}} (\sum_{k=1}^n |b_k|^q)^{\frac{1}{q}}} \leq 1. \quad (9)$$

Multiplying both sides by the product  $(\sum_{k=1}^n |a_k|^p)^{\frac{1}{p}} (\sum_{k=1}^n |b_k|^q)^{\frac{1}{q}}$ , we obtain the desired result,

$$\sum_{k=1}^n |a_k b_k| \leq \left[ \sum_{k=1}^n |a_k|^p \right]^{\frac{1}{p}} \left[ \sum_{k=1}^n |b_k|^q \right]^{\frac{1}{q}}. \quad (10)$$

□

(b)

[Minkowski's Inequality]

*Proof.* By the triangle inequality, we have

$$\sum_{k=1}^n |a_k + b_k|^p = \sum_{k=1}^n |a_k + b_k| |a_k + b_k|^{p-1} \quad (11)$$

$$\leq \sum_{k=1}^n |a_k| |a_k + b_k|^{p-1} + \sum_{k=1}^n |b_k| |a_k + b_k|^{p-1}. \quad (12)$$

Then by Hölder's inequality [proved in (a)],

$$\sum_{k=1}^n |a_k| |a_k + b_k|^{p-1} \leq \left[ \sum_{k=1}^n |a_k|^p \right]^{\frac{1}{p}} \left[ \sum_{k=1}^n |a_k + b_k|^p \right]^{\frac{p-1}{p}}, \quad \text{and} \quad (13)$$

$$\sum_{k=1}^n |b_k| |a_k + b_k|^{p-1} \leq \left[ \sum_{k=1}^n |b_k|^p \right]^{\frac{1}{p}} \left[ \sum_{k=1}^n |a_k + b_k|^p \right]^{\frac{p-1}{p}}, \quad (14)$$

where we have identified  $q = \frac{p}{p-1}$ . Then

$$\sum_{k=1}^n |a_k + b_k|^p \leq \left( \left[ \sum_{k=1}^n |a_k|^p \right]^{\frac{1}{p}} + \left[ \sum_{k=1}^n |b_k|^p \right]^{\frac{1}{p}} \right) \left[ \sum_{k=1}^n |a_k + b_k|^p \right]^{\frac{p-1}{p}} \quad (15)$$

$$\Rightarrow \left[ \sum_{k=1}^n |a_k + b_k|^p \right] \left[ \sum_{k=1}^n |a_k + b_k|^p \right]^{\frac{1-p}{p}} \leq \left[ \sum_{k=1}^n |a_k|^p \right]^{\frac{1}{p}} + \left[ \sum_{k=1}^n |b_k|^p \right]^{\frac{1}{p}}. \quad (16)$$

Combining exponents on the left side, we arrive at

$$\left[ \sum_{k=1}^n |a_k + b_k|^p \right]^{\frac{1}{p}} \leq \left[ \sum_{k=1}^n |a_k|^p \right]^{\frac{1}{p}} + \left[ \sum_{k=1}^n |b_k|^p \right]^{\frac{1}{p}}. \quad (17)$$

□

## Problem 2

*Proof.* We first show that  $\ell^p$  is a normed space.

Let  $a = \{a_j\}_{j=1}^\infty$  and  $b = \{b_j\}_{j=1}^\infty$  be sequences in  $\ell^p$ . Suppose  $\|a\|_p = 0$ . Then by Hölder's inequality, letting  $b_j = n^{-\frac{1}{p}}$  for  $n \in \mathbb{N}$ ,  $\forall j \in \mathbb{N}$ , we have

$$0 = \left[ \sum_{j=1}^n |a_j|^p \right]^{\frac{1}{p}} = \left[ \sum_{j=1}^n |a_j|^p \right]^{\frac{1}{p}} \left[ \sum_{j=1}^n \frac{1}{n} \right]^{\frac{1}{p}} \quad (18)$$

$$\geq \sum_{j=1}^n |a_j n^{-\frac{1}{p}}| = n^{-\frac{1}{p}} \sum_{j=1}^n |a_j| \quad (19)$$

$$\geq 0. \quad (20)$$

Thus, we have that

$$0 \leq \sum_{j=1}^n |a_j| \leq 0, \quad (21)$$

but since  $|a_j|$  is always nonnegative, this must imply that  $a_j = 0 \forall j \in \mathbb{N}$ . Going in the opposite direction, suppose  $a = 0$  [i.e.  $a_j = 0 \forall j \in \mathbb{N}$ ]. Then

$$\|a\|_p = \left[ \sum_{j=1}^n |a_j|^p \right]^{\frac{1}{p}} = \left[ \sum_{j=1}^n 0 \right]^{\frac{1}{p}} = 0^{\frac{1}{p}} = 0. \quad (22)$$

Hence, we have shown  $\|a\|_p = 0 \iff a = 0$  [definiteness]. Now let  $\lambda \in \mathbb{K}$  [an element in a field of scalars,  $\mathbb{R}$  or  $\mathbb{C}$ ]. Then

$$\|\lambda a\|_p = \left[ \sum_{j=1}^n |\lambda a_j|^p \right]^{\frac{1}{p}} = \left[ |\lambda|^p \sum_{j=1}^n |a_j|^p \right]^{\frac{1}{p}} = |\lambda| \left[ \sum_{j=1}^n |a_j|^p \right]^{\frac{1}{p}}. \quad (23)$$

Hence,  $\|\lambda a\|_p = |\lambda| \cdot \|a\|_p$  [homogeneity]. Now consider the norm of the sum,  $\|a + b\|_p$ . By Minkowski's inequality, we have

$$\|a + b\|_p = \left[ \sum_{j=1}^n |a_j + b_j|^p \right]^{\frac{1}{p}} \leq \left[ \sum_{j=1}^n |a_j|^p \right]^{\frac{1}{p}} + \left[ \sum_{j=1}^n |b_j|^p \right]^{\frac{1}{p}} \quad (24)$$

Hence,  $\|a + b\|_p \leq \|a\|_p + \|b\|_p$  [triangle inequality]. Thus we have proven that  $\|\cdot\|_p$  is a norm on  $\ell^p$ , so we conclude that  $\ell^p$  is a normed space.

Next we show that  $\ell^p$  is complete.

Let  $\{a^{(n)}\}_n$  be a Cauchy sequence in  $\ell^p$  [i.e.  $\{a_j^{(n)}\}_{j=1}^\infty \in \ell^p$  and  $\{a^{(n)}\}_n = \{\{a_j^{(n)}\}_j\}_n$ ]. Let  $\epsilon > 0$ . Then  $\exists N_0 \in \mathbb{N}$  such that  $\forall n, m \geq N_0$ ,

$$\|a^{(n)} - a^{(m)}\|_p < \epsilon. \quad (25)$$

Then this implies

$$\left[ \sum_{j=1}^\infty |a_j^{(n)} - a_j^{(m)}|^p \right]^{\frac{1}{p}} = \|a^{(n)} - a^{(m)}\|_p < \epsilon \quad (26)$$

$$\implies \sum_{j=1}^\infty |a_j^{(n)} - a_j^{(m)}|^p = \|a^{(n)} - a^{(m)}\|_p^p < \epsilon^p. \quad (27)$$

Then for any  $j \in \mathbb{N}$ ,

$$|a_j^{(n)} - a_j^{(m)}|^p < \sum_{j=1}^\infty |a_j^{(n)} - a_j^{(m)}|^p < \epsilon^p. \quad (28)$$

Hence, the sequence  $\{a_j^{(n)}\}_n \subset \ell^p$  is Cauchy. By completeness of  $\mathbb{R}$ ,  $\forall j \in \mathbb{N} \exists a_j$  such that  $\lim_{n \rightarrow \infty} a_j^{(n)} = a_j \in \mathbb{R}$ .

Fix  $k \in \mathbb{N}$ . Then for  $m, n > N_0$ ,

$$\sum_{j=1}^k |a_j^{(n)} - a_j^{(m)}|^p \leq \sum_{j=1}^\infty |a_j^{(n)} - a_j^{(m)}|^p \quad (29)$$

$$= \|a^{(n)} - a^{(m)}\|_p^p < \epsilon^p \quad (30)$$

$$\xrightarrow{n \rightarrow \infty} \sum_{j=1}^k |a_j^{(m)} - a_j|^p < \epsilon^p. \quad (31)$$

By Minkowski's inequality for  $\|\cdot\|_p$  in  $\mathbb{R}^k$ , for  $m > N_0$ , we have

$$\left[ \sum_{j=1}^k |a_j|^p \right]^{\frac{1}{p}} \leq \left[ \sum_{j=1}^k |a_j^{(m)} - a_j|^p \right]^{\frac{1}{p}} + \left[ \sum_{j=1}^k |a_j^{(m)}|^p \right]^{\frac{1}{p}} \quad (32)$$

$$< \epsilon + \left[ \sum_{j=1}^k |a_j^{(m)}|^p \right]^{\frac{1}{p}} \quad (33)$$

$$\xrightarrow{n \rightarrow \infty} \|a\|_p \leq \epsilon + \|a^{(m)}\|_p. \quad (34)$$

Hence,  $a \in \ell^p$ . Then by letting  $k \rightarrow \infty$  in (31),

$$\sum_{j=1}^{\infty} |a_j^{(m)} - a_j|^p = \|a^{(m)} - a\|_p^p < \epsilon^p \quad (35)$$

$$\implies \|a^{(m)} - a\|_p < \epsilon. \quad (36)$$

Then  $\lim_{m \rightarrow \infty} \|a^{(m)} - a\|_p = 0$ , which implies that the sequence  $\{a^{(m)}\}_m \subset \ell^p$  converges to  $a \in \ell^p$ . Thus,  $\ell^p$  is complete.

Since  $\ell^p$  is a complete normed space, we conclude that  $\ell^p$  is a Banach space.  $\square$

### Problem 3

*Proof.* We will show that  $\ell^p \setminus c_0 := \{b \in \ell^p \mid \lim_{k \rightarrow \infty} b_k \neq 0\}$  is open.  $\square$

### Problem 4

### Problem 5

(a)

*Proof.* We consider two cases.

Case 1:  $p = 1$  [i.e.  $q = \infty$ ]. If  $\{a_k\}_k \in \ell^1$ , then  $\|a\|_1 < \infty$ , so  $\sum_{k=1}^{\infty} |a_k|$  converges. Similarly, if  $\{b_k\}_k \in \ell^{\infty}$ , then  $\|b\|_{\infty} = \sup_{1 \leq k < \infty} |b_k| < \infty$ . Then

$$\sum_{k=1}^{\infty} |a_k b_k| \leq \sum_{k=1}^{\infty} |a_k| \sup_{k \in \mathbb{N}} |b_k| \quad (37)$$

$$= \left[ \sum_{k=1}^{\infty} |a_k| \right] \|b\|_{\infty} \quad (38)$$

$$= \|a\|_1 \|b\|_{\infty}, \quad (39)$$

as desired.

Case 2:  $1 < p < \infty$ . Since  $\frac{1}{p} + \frac{1}{q} = 1$ , then the result for this case follows immediately from Hölder's inequality. For all  $n \in \mathbb{N}$ ,

$$\sum_{k=1}^n |a_k b_k| \leq \left[ \sum_{k=1}^n |a_k|^p \right]^{\frac{1}{p}} + \left[ \sum_{k=1}^n |b_k|^q \right]^{\frac{1}{q}}. \quad (40)$$

Taking  $n \rightarrow \infty$  in the above inequality, we achieve the desired result:

$$\sum_{k=1}^{\infty} |a_k b_k| \leq \|a\|_p \|b\|_q. \quad (41)$$

□

**(b)**

*Proof.* By the result proven in part **(a)**,

$$\sum_{k=1}^{\infty} a_k b_k \leq \sum_{k=1}^{\infty} |a_k b_k| \quad (42)$$

$$\leq \|a\|_p \|b\|_q \in \mathbb{C}. \quad (43)$$

Hence,  $F_b$  maps  $\ell^p$  into the scalar field  $\mathbb{K} = \mathbb{C}$ . So we conclude that  $F_b \in (\ell^p)'$ .

Next, we compute the operator norm of  $F_b$ . Let  $a \in \ell^p$ . Note that

$$|F_b(a)| = \left| \sum_{k=1}^{\infty} a_k b_k \right| \quad (44)$$

$$\leq \sum_{k=1}^{\infty} |a_k b_k| \quad (45)$$

$$\leq \|a\|_p \|b\|_q. \quad (46)$$

Then taking the supremum over the set of  $\|a\|_p = 1$ ,

$$\|F_b\| = \sup_{\|a\|_p=1} |F_b(a)| \quad (47)$$

$$= \sup_{\|a\|_p=1} \|a\|_p \|b\|_q \quad (48)$$

$$= \|b\|_q. \quad (49)$$

Thus we have shown that  $F_b \in (\ell^p)'$  and  $\|F_b\| = \|b\|_{\ell^q}$ , as desired. □

**(c)**

*Proof.* We first show that  $F$  is linear.

[Note:  $F(b) = F_b = \sum_{k=1}^{\infty} b_k \in (\ell^p)'$ .]

Let  $b$  and  $c$  be sequences in  $\ell^q$ , and let  $\lambda \in \mathbb{K}$ . Then by the linearity of sums,

$$F(b + c) = \sum_{k=1}^{\infty} (b_k + c_k) = \sum_{k=1}^{\infty} b_k + \sum_{k=1}^{\infty} c_k = F(b) + F(c), \quad \text{and} \quad (50)$$

$$F(\lambda b) = \sum_{k=1}^{\infty} \lambda b_k = \lambda \sum_{k=1}^{\infty} b_k = \lambda F(b). \quad (51)$$

Thus,  $F$  is linear.

Next, we show that  $F$  is bijective. Suppose  $F(b) = F(c)$ . Then by part **(b)**,

$$0 = \|F_b - F_c\| = \|b - c\|_{\ell^q}. \quad (52)$$

But since  $\|\cdot\|_{\ell^q}$  is a norm, this must imply that

$$b - c = 0 \quad (53)$$

$$\implies b = c. \quad (54)$$

Hence,  $F$  is injective. Now suppose we are given  $F_b = \sum_{k=1}^{\infty} b_k \in (\ell^p)'$ . Then by definition, we choose  $b = \{b_k\}_k \in \ell^q$  so that  $F(b) = F_b$ . We can make this choice for any such series in  $(\ell^p)'$ , since by what we proved in **(b)**, it always holds that the sequence  $b$  exists in  $\ell^q$ . Hence,  $F$  is surjective, which means that  $F$  is bijective.

We now show that  $F$  is bounded. The proof of this claim follows immediately from the result in **(b)**. Let  $b \in \ell^q$ . Then

$$\|F(b)\| = \|F_b\| \quad (55)$$

$$= \|b\|_{\ell^q} \quad (56)$$

$$\leq c\|b\|_{\ell^q}, \quad \text{for } c = 1. \quad (57)$$

So,  $F$  is continuous (bounded).

Therefore,  $F : \ell^q \rightarrow (\ell^p)'$  is a bijective, bounded, linear operator.  $\square$