18.102 Midterm

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Problem 1

Proof. We will show that $\Lambda([a,b])$ is a proper closed subspace of $C([a,b])$, which we know is a Banach space. Let $\{f_n\}_n$ be a cauchy sequence in $\Lambda([a,b])$ such that $f_n \to f$ pointwise. Then for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $||f - f_n|| < \epsilon$. This is equivalent to

$$\sup_{x\in[a,b]} |f(x) - f_n(x)| + \sup_{x\neq y\in[a,b]} \frac{|f(x) - f_n(x) - f(y) + f_n(y)|}{|x-y|} < \epsilon. \tag{1}$$

Since both terms on the left hand side are non-negative, this implies

$$\sup_{x\neq y\in[a,b]} \frac{|f(x) - f_n(x) - f(y) + f_n(y)|}{|x-y|} < \epsilon. \tag{2}$$

Then for any $x \neq y \in [a,b]$, we have

$$|f(x) - f_n(x) - f(y) + f_n(y)| < \epsilon|x-y|, \tag{3}$$

which confirms that for each $n \geq N$, the function $f - f_n$ is Lipschitz continuous. By assumtion, f_n is Lipschitz continuous $\forall n \in \mathbb{N}$, and the sum of Lipschitz continuous functions is also Lipschitz, thus $f = f_n + (f - f_n)$ is Lipschitz continuous.

So, $\lim_{n\to\infty} f_n = f \in \Lambda([a,b])$, which proves that $\Lambda([a,b])$ is a proper closed subspace of $C([a,b])$.

Therefore, $\Lambda([a,b])$ is a Banach space. $\square$

Problem 2

Proof. First we show that $||a + c_0||_{\ell^\infty/c_0} \leq \limsup_{n\to\infty} |a_n|$.

Let $a = \{a_n\}_n \in \ell^\infty$. For each $n \in \mathbb{N}$, let $b_n = (a_1, a_2, ..., a_n, 0, 0, ...) \in c_0$. Then

$$\begin{aligned}
\inf_{b \in c_0} ||a + b||_\infty &\leq \inf_n ||a - b_n||_\infty && (4)\\
&= \inf_n \sup_{m \in \mathbb{N}} |a_m - b_m| && (5)\\
&= \inf_n \sup_{m \geq n} |a_m| && (6)\\
&= \limsup_{n \to \infty} |a_n|. && (7)
\end{aligned}$$

Thus,

$$||a + c_0||_{\ell^\infty / c_0} \leq \limsup_{n \to \infty} |a_n|. \tag{8}$$

Let $b = (b_1, b_2, b_3, ...) \in c_0$. Then for every $\epsilon > 0$, $\exists n \in N$ such that $\forall m \geq n$, $|b_m| < \epsilon$, so

$$\begin{aligned}
||a + b||_\infty &\geq \sup_{m \geq n} |a_m| - \epsilon && (9)\\
&\geq \limsup_{n \to \infty} |a_n| - \epsilon, && (10)
\end{aligned}$$

hence $\limsup_{n \to \infty} |a_n| < ||a + c_0||_{\ell^\infty / c_0} + \epsilon$.

Therefore, $||a + c_0||_{\ell^\infty / c_0} = \limsup_{n \to \infty} |a_n|$. $\square$