18.102 Assignment 8

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Problem 1

(a)

Proof. ($\Rightarrow$) Suppose $w \in \overline{W}$. Then since $w \in H$ and since $\{e_n\}_n \subset H$ is a countably infinite orthonormal subset, we have

$$w = \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n. \tag{1}$$

Computing the norm gives

$$\begin{aligned}
||w||^2 &= \left\langle \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n, \sum_{k=1}^{\infty} \langle u, e_k \rangle e_k \right\rangle && (2)\\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \langle u, e_n \rangle \overline{\langle u, e_k \rangle} \langle e_n, e_k \rangle && (3)\\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \langle u, e_n \rangle \overline{\langle u, e_k \rangle} \delta_{nk} && (4)\\
&= \sum_{n=1}^{\infty} \langle u, e_n \rangle \overline{\langle u, e_n \rangle} && (5)\\
&= \sum_{n=1}^{\infty} |\langle u, e_n \rangle|^2. && (6)
\end{aligned}$$

Thus,

$$||w|| = \left(\sum_{n=1}^{\infty} |\langle u, e_n \rangle|^2 \right)^{\frac{1}{2}} < \infty, \tag{7}$$

so defining $c_n := \langle u, e_n \rangle$ yields a sequence $\{c_n\}_n \in \ell^2(\mathbb{N})$, as desired.

($\Leftarrow$) Let $\{c_n\}_{n=1}^{\infty} \in \ell^2$ such that $w = \sum_{n=1}^{\infty} c_n e_n$.

Define $w_N := \sum_{n=1}^{N} c_n e_n$. Then $w = \lim_{N\to\infty} w_N$, and for each $N \in N$, $w_N \in W$ since it is a finite linear combination of elements in $\{e_n\}_n$.

Thus, since $\overline{W}$ contains all the limit points of W, then $w \in \overline{W}$, as desired. $\square$

(b)

Proof. Let $w \in \overline{W}$ and $u \in H$. Then by **(a)**, we may write $w = \sum_{n=1}^{\infty} c_n e_n$ for $\{c_n\}_n \in \ell^2(\mathbb{N})$. Suppose $c_n = \langle u, e_n \rangle$. Then

$$||u - \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n|| = ||u - \sum_{n=1}^{\infty} c_n e_n|| \tag{8}$$

$$= ||u - v||. \tag{9}$$

Now suppose $c_n \neq \langle u, e_n \rangle$. Then we compute

$$||u - w||^2 = \left\| u - \sum_{n=1}^{\infty} c_n e_n \right\|^2 \tag{10}$$

$$= \left\| u - \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n + \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n - \sum_{n=1}^{\infty} c_n e_n \right\|^2 \tag{11}$$

$$= \left\| u - \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n + \sum_{n=1}^{\infty} (\langle u, e_n \rangle - c_n) e_n \right\|^2 \tag{12}$$

$$\geq \left\| u - \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n \right\|^2 . \tag{13}$$

Therefore, $\|u - \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n\| \leq ||u - w||$, with equality only if $w = \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n$, and we are done. $\square$

Problem 2

(a)

Proof. Let $\{u_k\}_k \subset W^{\perp}$ be a sequence such that $u_k \to u$ as $k \to \infty$. Then $\forall k \in \mathbb{N}$ and $\forall w \in W$, we have $\langle u_k, w \rangle = 0$.

By continuity of the inner product,

$$0 = \lim_{k\to\infty} \langle u_k, w \rangle \tag{14}$$

$$= \left\langle \lim_{k\to\infty} u_k, w \right\rangle \tag{15}$$

$$= \langle u, w \rangle. \tag{16}$$

Thus, $u \in W^{\perp}$, so $W^{\perp} \subset H$ is closed. $\square$

(b)

Proof. We note that

$$\left(W^{\perp}\right)^{\perp} := \{v \in H \mid \langle v, u\rangle = 0\,\forall u \in W^{\perp}\}. \tag{17}$$

Let $w \in W$. Then $\forall u \in W^{\perp}$,

$$\begin{aligned} \langle w, u\rangle &= 0 && (18)\\ \implies w &\in \left(W^{\perp}\right)^{\perp} && (19)\\ \implies W &\subseteq \left(W^{\perp}\right)^{\perp}. && (20) \end{aligned}$$

Since $\left(W^{\perp}\right)^{\perp}$ is an orthogonal complement, then by **(a)** it is a closed linear subspace of H. So, $\left(W^{\perp}\right)^{\perp}$ must also be complete.

Let $\{v_n\}_n \subset W$ be a Cauchy sequence. Since $W \subseteq \left(W^{\perp}\right)^{\perp}$, then $\{v_n\}_n \subset \left(W^{\perp}\right)^{\perp}$, so $\{v_n\}_n$ converges in $\left(W^{\perp}\right)^{\perp}$, i.e.

$$\lim_{n\to\infty} v_n = v \in \left(W^{\perp}\right)^{\perp}. \tag{21}$$

Thus the Cauchy sequence $\{v_n\}_n$ converges to v in $\left(W^{\perp}\right)^{\perp}$.

We conclude that the closure $\overline{W} = \left(W^{\perp}\right)^{\perp}$. □

Problem 3

(a)

Proof. Since $f \in C^k([-\pi, \pi])$, then f must be bounded. Let $B \geq 0$ be such a bound for f. Then we have

$$\begin{aligned} \|f\|_{L^2}^2 &= \int_{-\pi}^{\pi} |f(t)|^2 \mathrm{d}t && (22)\\ &\leq \int_{-\pi}^{\pi} B^2 \mathrm{d}t && (23)\\ &= 2\pi B^2 && (24)\\ &< \infty. && (25) \end{aligned}$$

Thus $f \in L^2([-\pi, \pi])$.

Next, we prove the following claim:

<u>Claim</u>: Let $k \in \mathbb{N}$. For each $j \in \{0, 1, ..., k\}$,

$$\hat{f}(n) = \left(-\frac{i}{n}\right)^j \widehat{f^{(j)}}(n). \tag{26}$$

Proof of claim: (By induction on j).

<u>Base case</u>: ($j = 0$)

$$\hat{f}(n) = \widehat{f^{(0)}}(n).$$

<u>Inductive step:</u> Assume $\hat{f}(n) = \left(-\frac{i}{n}\right)^j \hat{f^{(j)}}(n)$. Then integrating by parts, we get

$$\begin{aligned}
\hat{f}(n) &= \left(-\frac{i}{n}\right)^j \frac{1}{2\pi} \int_{-\pi}^{\pi} f^{(j)}(t) e^{-int} \mathrm{d}t && (27) \\
&= \left(-\frac{i}{n}\right)^j \frac{1}{2\pi} \left[f^{(j)}(t) \frac{e^{-int}}{(-in)} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f^{(j+1)}(t) \frac{e^{-int}}{(-in)} \mathrm{d}t \right] && (28) \\
&= 0 - \frac{i}{n} \left(-\frac{i}{n}\right)^j \frac{1}{2\pi} \int_{-\pi}^{\pi} f^{(j+1)} e^{-int} \mathrm{d}t && (29) \\
&= \left(-\frac{i}{n}\right)^{j+1} \widehat{f^{(j+1)}}(n), && (30)
\end{aligned}$$

and the claim is proven. □

So for each $j \in \{0, 1, ..., k\}$,

$$\begin{aligned}
\left|\hat{f}(n)\right| &= \left|(-i)^j \frac{1}{n^j} \widehat{f^{(j)}}(n)\right| && (31) \\
&= \frac{1}{n^j} \left|\widehat{f^{(j)}}(n)\right|. && (32)
\end{aligned}$$

Let $0 \leq s \leq k$ and $N \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{|n| \leq N} \left|\hat{f}(n)\right|^2 \left(1 + |n|^2\right)^s &= \sum_{|n| \leq N} \frac{1}{n^{2j}} \left|\widehat{f^{(j)}}(n)\right|^2 \left(1 + |n|^2\right)^s && (33) \\
&= \sum_{|n| \leq N} \frac{\left(1 + |n|^2\right)^s}{n^{2j}} \left|\widehat{f^{(j)}}(n)\right|^2. && (34)
\end{aligned}$$

Since this holds $\forall 0 \leq j \leq k$, we can choose $j = k$, giving

$$\begin{aligned}
\sum_{|n| \leq N} \left|\hat{f}(n)\right|^2 \left(1 + |n|^2\right)^s &= \sum_{|n| \leq N} \frac{\left(1 + |n|^2\right)^s}{n^{2k+2}} \left|\widehat{f^{(k)}}(n)\right|^2 && (35) \\
&\leq \sum_{|n| \leq N} \left|\widehat{f^{(k)}}(n)\right|^2 && (36) \\
&< \infty. && (37)
\end{aligned}$$

Sending $N \to \infty$, we get the desired result.

Therefore $f \in H^s(\mathbb{T})$ $\forall 0 \leq s \leq k$. □

(b)

Proof. We first show that $H^s(\mathbb{T})$ is a vector space.

Let $f, g \in H^s(\mathbb{T})$. Consider the function $f+g$. For $N \in \mathbb{N}$, we have

$$\sum_{|n|\leq N} \left|\widehat{f+g}(n)\right|^2 \left(1+|n|^2\right)^s = \sum_{|n|\leq N} \left[\left|\hat{f}(n)\right|^2 + |\hat{g}(n)|^2 + \hat{f}(n)\overline{\hat{g}(n)} + \overline{\hat{f}(n)}\hat{g}(n)\right] \left(1+|n|^2\right)^s \tag{38}$$

$$= \sum_{|n|\leq N} \left[\left|\hat{f}(n)\right|^2 + |\hat{g}(n)|^2 + 2\Re\left(\hat{f}(n)\overline{\hat{g}(n)}\right)\right] \left(1+|n|^2\right)^s \tag{39}$$

$$\leq \sum_{|n|\leq N} \left[\left|\hat{f}(n)\right|^2 + |\hat{g}(n)|^2 + 2\max\left\{\left|\hat{f}(n)\right|^2, |\hat{g}(n)|^2\right\}\right] \left(1+|n|^2\right)^s \tag{40}$$

$$\tag{41}$$

Sending $N \to \infty$, we have

$$\sum_{|n|\in\mathbb{Z}} \left|\widehat{f+g}(n)\right|^2 \left(1+|n|^2\right)^s < \infty, \tag{42}$$

thus $f+g\ inH^s(\mathbb{T})$. Thus, $f+g \in H^s(\mathbb{T})$.

Scalar multiplication follows trivially, as does existence of additive inverses. The remaining vector space axioms hold automatically, since for any $f, g, h \in H^s(\mathbb{T})$, it follows $f, g, h \in L^2([-\pi, \pi])$, which is a vector space.

Next we show that $\langle f, g\rangle_{H^s(\mathbb{T})}$ is a Hermitian inner product on $H^s(\mathbb{T})$. We have

$$\langle f, f\rangle_{H^s(\mathbb{T})} = \sum_{|n|\leq N} \left|\hat{f}(n)\right|^2 \left(1+|n|^2\right)^s \geq 0, \tag{43}$$

since it is a sum of non-negative terms. Now suppose $\langle f, f\rangle_{H^s(\mathbb{T})} = 0$. Then

$$\sum_{|n|\leq N} \left|\hat{f}(n)\right|^2 \left(1+|n|^2\right)^s = 0 \tag{44}$$

Then $\forall n \in \mathbb{N}$,

$$\implies \left|\hat{f}(n)\right|^2 \left(1+|n|^2\right)^s = 0 \tag{45}$$

$$\implies \left|\hat{f}(n)\right|^2 \tag{46}$$

$$\implies \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\mathrm{d}t = 0 \tag{47}$$

$$\iff \int_{-\pi}^{\pi} f(t)\cos(t)\mathrm{d}t \quad \text{and} \quad \int_{-\pi}^{\pi} f(t)\sin(t)\mathrm{d}t. \tag{48}$$

The above two statements can only both be true if $f(t) = 0 \forall t \in [-\pi, \pi]$. Thus, $\langle f, f \rangle = 0 \implies f = 0$.

Next, we compute

$$\begin{aligned} \langle g, f \rangle &= \sum_{|n| \in \mathbb{Z}} \hat{g}(n) \overline{\hat{f}(n)} \left(1 + |n|^2\right)^s && (49) \\ &= \sum_{|n| \in \mathbb{Z}} \overline{\hat{f}(n) \overline{\hat{g}(n)}} \left(1 + |n|^2\right)^s && (50) \\ &= \overline{\sum_{|n| \in \mathbb{Z}} \hat{f}(n) \overline{\hat{g} n} \left(1 + |n|^2\right)^s} && (51) \\ &= \overline{\langle f, g \rangle}. && (52) \end{aligned}$$

Similarly, for scalar multiples:

$$\begin{aligned} \langle \alpha f, g \rangle &= \sum_{|n| \in \mathbb{Z}} \widehat{\alpha f}(n) \overline{\hat{g}(n)} \left(1 + |n|^2\right)^s && (53) \\ &= \alpha \sum_{|n| \in \mathbb{Z}} \widehat{f}(n) \overline{\hat{g}(n)} \left(1 + |n|^2\right)^s && (54) \\ &= \alpha \langle f, g \rangle. && (55) \end{aligned}$$

Finally, consider

$$\begin{aligned} \langle f + h, g \rangle &= \sum_{|n| \in \mathbb{Z}} \widehat{(f + g)}(n) \overline{\hat{g}(n)} \left(1 + |n|^2\right)^s && (56) \\ &= \sum_{|n| \in \mathbb{Z}} \left(\hat{f} + \hat{h}\right) \overline{\hat{g}(n)} \left(1 + |n|^2\right)^s && (57) \\ &= \sum_{|n| \in \mathbb{Z}} \hat{f}(n) \overline{\hat{g}(n)} \left(1 + |n|^2\right)^s + \sum_{|n| \in \mathbb{Z}} \hat{h}(n) \overline{\hat{g}(n)} \left(1 + |n|^2\right)^s && (58) \\ &= \langle < f, g \rangle + \langle h, g \rangle, && (59) \end{aligned}$$

as desired.

Therefore $H^s(\mathbb{T})$ is a vector space with Hermitian inner product $\langle \cdot, \cdot \rangle_{H^s(\mathbb{T})}$. $\square$

(c)

Proof. We take $\| \cdot \|^2_{H^s(\mathbb{T})} := \langle \cdot, \cdot \rangle_{H^s(\mathbb{T})}$.

Let $\{f_k\}_k \subset H^s(\mathbb{T})$ be a Cauchy sequence. Then $\{f_k\}_k \subset L^2([-\pi, \pi])$ is Cauchy. Since $L^2([-\pi, \pi])$ is complete, then $f_k \to f$ for some $f \in L^2([-\pi, \pi])$. Then for every $\epsilon > 0$, $\exists N \in N$ such that $\forall k \geq N$, $\|f - f_k\|_{L^2} < \epsilon$. So

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2 \left(1+|n|^2\right)^s &= \sum_{n\in\mathbb{Z}} \left|\hat{f}(n) - \hat{f}_k(n) + \hat{f}_k(n)\right| \left(1+|n|^2\right)^s && (60)\\
&\leq \sum_{n\in\mathbb{Z}} \left|\hat{f}(n) - \hat{f}_k(n)\right|^2 \left(1+|n|^2\right)^s && (61)\\
&+ 2\sum_{n\in\mathbb{Z}} \left|\hat{f}(n) - \hat{f}_k(n)\right| \left|\hat{f}_k(n)\right| \left(1+|n|^2\right)^s && (62)\\
&+ 2\sum_{n\in\mathbb{Z}} \left|\hat{f}_k(n)\right|^2 \left(1+|n|^2\right)^s && (63)\\
&\leq \sum_{n\in\mathbb{Z}} \left|\hat{f}(n) - \hat{f}_k(n)\right|^2 \left(1+|n|^2\right)^s && (64)\\
&+ 2\sum_{n\in\mathbb{Z}} \max\{\left|\hat{f}_k(n)\right|^2, \left|\hat{f}(n) - \hat{f}_k(n)\right|^2\} \left(1+|n|^2\right)^s && (65)\\
&+ \sum_{n\in\mathbb{Z}} \left|\hat{f}_k(n)\right|^2 \left(1+|n|^2\right)^s && (66)\\
&< \infty, && (67)
\end{aligned}$$

since $f_k \in H^s(\mathbb{T})$ and since $f_k \to f$ in L^2, then $\hat{f}_k \to \hat{f}$. Thus $f \in H^s(\mathbb{T})$ so $H^s(\mathbb{T})$ is complete.

Therefore $H^s(\mathbb{T})$ is a Hilbert space. $\square$