

18.102 Assignment 2

Octavio Vega

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Problem 1

(a)

Proof. Let B be a Banach space. Suppose $T \in \mathcal{B}(B, B)$ and $\|I - T\| < 1$. Then by Geometric series,

$$\sum_{n=0}^{\infty} \|(I - T)^n\| \leq \sum_{n=0}^{\infty} \|I - T\|^n = \frac{1}{1 - \|I - T\|} < \infty. \quad (1)$$

So the series $\sum_{n=0}^{\infty} (I - T)^n$ converges absolutely, which implies that it converges. Fix $m \in \mathbb{N}$. Then

$$T \sum_{n=0}^m (I - T)^n = [I - (I - T)] \sum_{n=0}^m (I - T)^n \quad (2)$$

$$= \sum_{n=0}^m (I - T)^n - \sum_{n=0}^m (I - T)^{n+1} \quad (3)$$

$$= I - (I - T)^{m+1}, \text{ by telescoping sum.} \quad (4)$$

By continuity of T ,

$$T \sum_{n=0}^{\infty} (I - T)^n = T \left(\lim_{m \rightarrow \infty} \sum_{n=0}^m (I - T)^n \right) \quad (5)$$

$$= \lim_{m \rightarrow \infty} T \sum_{n=0}^m (I - T)^n \quad (6)$$

$$= \lim_{m \rightarrow \infty} [I - (I - T)^{m+1}] \quad (7)$$

$$= I, \quad (8)$$

since $\|I - T\| < 1$. We can similarly show that $\sum_{n=0}^{\infty} (I - T)^n = I$.

Thus, T is indeed invertible, and $\sum_{n=0}^{\infty} (I - T)^n \rightarrow T^{-1}$ in $\mathcal{B}(B, B)$. \square

(b)

Proof. Let $\mathcal{I} := \{T \in \mathcal{B}(B, B) | T^{-1} \text{ exists}\}$. We want to show that $\forall T \in \mathcal{I}$, $\exists \delta > 0$ such that if $\|S - T\| < \delta \implies S \in \mathcal{I}$.

Choose $\delta = \frac{1}{\|T^{-1}\|}$, and write

$$S = T - (T - S) = T [I - T^{-1}(T - S)]. \quad (9)$$

If $\|S - T\| < \delta = \frac{1}{\|T^{-1}\|}$, then

$$\frac{1}{\|T^{-1}\|} > \|S - T\| \quad (10)$$

$$= \|T - T [I - T^{-1}(T - S)]\| \quad (11)$$

$$= \|T\| \cdot \|I - [I - T^{-1}(T - S)]\| \quad (12)$$

$$\implies \|I - [I - T^{-1}(T - S)]\| < \frac{1}{\|T^{-1}\| \cdot \|T\|} = 1 \quad (13)$$

$$\implies \|T^{-1}(T - S)\| = \|I - T^{-1}S\| < 1. \quad (14)$$

So by (a), $T^{-1}S$ is invertible, which implies that S is invertible. Thus, $\exists \delta > 0$ such that if $S \in B_\delta(T)$, then $S \in \mathcal{I}$.

Therefore, \mathcal{I} is open. \square