

18.102 Assignment 4

Octavio Vega

March 2, 2023

Problem 1

Proof. Let $A \subset \mathbb{R}$ and let $E \in \mathcal{A}$. Then $f^{-1}(E)$ is measurable, so

$$m^*(A \cap f^{-1}(E)) + m^*(A \cap f^{-1}(E)^c) \leq m^*(A). \quad (1)$$

We will first show that \mathcal{A} is closed under taking complements so we must show that $E^c \in \mathcal{A}$ for every $E \in \mathcal{A}$; i.e. $f^{-1}(E^c)$ is measurable.

We will use the fact that $f^{-1}(E^c) = f^{-1}(E)^c$, which is proven in appendix A.

Since $f^{-1}(E)$ is measurable, we have

$$m^*(A) \geq m^*(A \cap f^{-1}(E)) + m^*(A \cap f^{-1}(E)^c) \quad (2)$$

$$= m^*\left[A \cap (f^{-1}(E)^c)^c\right] + m^*(A \cap f^{-1}(E^c)) \quad (3)$$

$$= m^*(A \cap f^{-1}(E^c)^c) = m^*(A \cap f^{-1}(E^c)). \quad (4)$$

Hence, $f^{-1}(E^c)$ is measurable, so $E^c \in \mathcal{A}$. Thus, \mathcal{A} is closed under taking complements.

Now we show that \mathcal{A} is closed under taking countable unions. We use the fact that $f^{-1}(\bigcup_n E_n) = \bigcup_n f^{-1}(E_n)$, which is proven in appendix B.

Let $\{E_n\}_n \subset \mathcal{A}$ be a sequence of sets in \mathcal{A} , and let $A \subset \mathbb{R}$. Then

$$m^*\left[A \cap f^{-1}\left(\bigcup_n E_n\right)\right] = m^*\left[A \cap \left(\bigcup_n f^{-1}(E_n)\right)\right] \quad (5)$$

$$= m^*\left[\bigcup_n (A \cap f^{-1}(E_n))\right] \quad (6)$$

$$\leq \sum_n m^*(A \cap f^{-1}(E_n)) \quad (7)$$

$$\leq m^*(A \cap f^{-1}(E_n)), \quad (8)$$

by countable subadditivity and positive-definiteness of the outer measure m^* . Similarly, we have

$$m^* \left[A \cap f^{-1} \left(\bigcup_n E_n \right)^c \right] = m^* \left[A \cap \left(\bigcup_n f^{-1}(E_n) \right)^c \right] \quad (9)$$

$$= m^* \left[A \cap \left(\bigcap_n f^{-1}(E_n)^c \right) \right] \quad (10)$$

$$\leq m^* (A \cap f^{-1}(E_n)^c), \quad (11)$$

by monotonicity of m^* , because $\bigcap_n f^{-1}(E_n)^c \subseteq f^{-1}(E_n)^c$.

Finally, since $E_n \in \mathcal{A}$, then $f^{-1}(E_n)$ is Lebesgue measurable, so we have

$$m^* \left[A \cap f^{-1} \left(\bigcup_n E_n \right) \right] + m^* \left[A \cap f^{-1} \left(\bigcup_n E_n \right)^c \right] \leq m^* (A \cap f^{-1}(E_n)) + m^* (A \cap f^{-1}(E_n)^c) \quad (12)$$

$$\leq m^*(A). \quad (13)$$

So, $f^{-1}(\bigcup_n E_n)$ is measurable, hence $\bigcup_n E_n \in \mathcal{A}$. Thus, \mathcal{A} is closed under taking countable unions.

Therefore, \mathcal{A} is a σ -algebra. \square

Problem 2

Proof. (\Rightarrow) Suppose E is measurable.

Let $\epsilon > 0$. Then by definition of the outer measure m^* , \exists a collection of open intervals $\{I_n\}_n$ with $E \subset \bigcup_n I_n$ such that

$$\sum_n \ell(I_n) < m^*(E) + \frac{\epsilon}{2} \quad (14)$$

$$\Rightarrow \sum_n \ell(I_n) - m^*(E) < \frac{\epsilon}{2}, \quad (15)$$

where $\ell(I_n)$ denotes the length of the interval I_n . Also, since $m^*(E) < \infty$, then

$$\sum_n m^*(I_n) = \sum_n \ell(I_n) < \infty. \quad (16)$$

By the countable subbadditivity of m^* , we have

$$m^* \left(\bigcup_n I_n \right) \leq \sum_n m^*(I_n) = \sum_n \ell(I_n) < \infty. \quad (17)$$

Then by removing E from the union $\cup_n I_n$, we get

$$m^* \left[\left(\bigcup_n I_n \right) \setminus E \right] = m^* \left(\bigcup_n I_n \right) - m^*(E) \quad (18)$$

$$\leq \sum_n \ell(I_n) - m^*(E) \quad (19)$$

$$< \frac{\epsilon}{2}. \quad (20)$$

For any $N \in \mathbb{N}$, $\bigcup_{n=1}^N I_n \subseteq \bigcup_{n=1}^\infty I_n$. Thus, $\left(\bigcup_{n=1}^N I_n \right) \setminus E \subseteq \left(\bigcup_{n=1}^\infty I_n \right) \setminus E$. Then by the monotonicity of m^* ,

$$m^* \left[\left(\bigcup_{n=1}^N I_n \right) \setminus E \right] \leq m^* \left[\left(\bigcup_n I_n \right) \setminus E \right] < \frac{\epsilon}{2}. \quad (21)$$

Since $\sum_n m^*(I_n) < \infty$, then the series converges, so $\exists N \in \mathbb{N}$ such that $\sum_{n=N+1}^\infty m^*(I_n) < \frac{\epsilon}{2}$. Because $E \subset \bigcup_n I_n$, we have

$$E \setminus \left(\bigcup_{n=1}^N I_n \right) \subset \left(\bigcup_{n=1}^\infty I_n \right) \setminus \left(\bigcup_{n=1}^N I_n \right) \subseteq \bigcup_{n=N+1}^\infty I_n. \quad (22)$$

Then by monotonicity of m^* ,

$$\begin{aligned} m^* \left[E \setminus \left(\bigcup_{n=1}^\infty I_n \right) \right] &\leq m^* \left(\bigcup_{n=N+1}^\infty I_n \right) \\ &\leq \sum_{n=N+1}^\infty m^*(I_n) \\ &< \frac{\epsilon}{2}. \end{aligned} \quad (23)$$

Combining equations (21) and (23), we have

$$m^* \left(E \Delta \bigcup_{n=1}^N I_n \right) = m^* \left[E \setminus \left(\bigcup_{n=1}^N I_n \right) \right] + m^* \left[\left(\bigcup_{n=1}^N I_n \right) \setminus E \right] \quad (24)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad (25)$$

$$= \epsilon. \quad (26)$$

Letting $U = \bigcup_{n=1}^N I_n$, we see that

$$m^*(U \Delta E) < \epsilon, \quad (27)$$

as desired.

(\Leftarrow) Suppose that for every $\epsilon > 0$, \exists a finite union of open intervals U such that $m^*(U \Delta E) < \epsilon$.

Let $A \subset \mathbb{R}$. Since U is a finite union of open intervals, then U is measurable. Thus,

$$m^*(A \cap U) + m^*(A \cap U^c) \leq m^*(A). \quad (28)$$

We can express the set $A \cap E$ as follows:

$$A \cap E = A \cap [(E \cap U) \cup (E \cap U^c)] \quad (29)$$

$$= (A \cap E \cap U) \cup (A \cap E \cap U^c). \quad (30)$$

Then by exclusivity,

$$m^*(A \cap E) = m^*(A \cap E \cap U) + m^*(A \cap E \cap U^c) \quad (31)$$

Similarly, we can write

$$A \cap E^c = A \cap [(E^c \cap U) \cup (E^c \cap U^c)] \quad (32)$$

$$= (A \cap E^c \cap U) \cup (A \cap E^c \cap U^c). \quad (33)$$

This gives

$$m^*(A \cap E^c) = m^*(A \cap E^c \cap U) + m^*(A \cap E^c \cap U^c). \quad (34)$$

Adding equations (31) and (34) yields

$$\begin{aligned} m^*(A \cap E) + m^*(A \cap E^c) &= m^*(A \cap E \cap U) + m^*(A \cap E \cap U^c) \\ &\quad + m^*(A \cap E^c \cap U) + m^*(A \cap E^c \cap U^c) \end{aligned} \quad (35)$$

$$\begin{aligned} &= m^*(A \cap E \cap U) + m^*[A \cap (E \setminus U)] \\ &\quad + m^*[A \cap (U \setminus E)] + m^*(A \cap E^c \cap U^c). \end{aligned} \quad (36)$$

Since $A \cap E \cap U \subseteq A \cap U$ and $A \cap E^c \cap U^c \subseteq A \cap U^c$, and since $(U \setminus E)$ and $(E \setminus U)$ are disjoint, we have

$$\begin{aligned} m^*(A \cap E) + m^*(A \cap E^c) &\leq m^*(A \cap U) + m^*(A \cap U^c) \\ &\quad + m^*(A \cap [(E \setminus U) \cup (U \setminus E)]) \end{aligned} \quad (37)$$

$$= m^*(A \cap U) + m^*(A \cap U^c) + m^*(U \Delta E) \quad (38)$$

$$\leq m^*(A) + \epsilon. \quad (39)$$

Since this holds for every $\epsilon > 0$, we can take ϵ to be arbitrarily small:

$$\xrightarrow{\epsilon \rightarrow 0} m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A). \quad (40)$$

Therefore, E is measurable. \square

Problem 3

TODO TODO TODO

Appendices

A Appendix A

Theorem: Let $f : \mathbb{R} \rightarrow \mathbb{R}$. If $E \subset \mathbb{R}$, then $f^{-1}(E^c) = f^{-1}(E)^c$.

Proof. Let $x \in f^{-1}(E^c)$. Then

$$\implies f(x) \in E^c \iff f(x) \notin E \quad (41)$$

$$\implies x \notin f^{-1}(E) \iff x \in f^{-1}(E)^c. \quad (42)$$

Thus, we have

$$f^{-1}(E^c) \subseteq f^{-1}(E)^c. \quad (43)$$

Now let $x \in f^{-1}(E)^c$. Then

$$\implies x \notin f^{-1}(E) \iff f(x) \notin E \quad (44)$$

$$\implies f(x) \in E^c \iff x \in f^{-1}(E^c). \quad (45)$$

So, we also have

$$f^{-1}(E)^c \subseteq f^{-1}(E^c). \quad (46)$$

Taking equations (43) and (46) together allows us to conclude that $f^{-1}(E^c) = f^{-1}(E)^c$, as desired. \square

B Appendix B

Theorem: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and let $\{E_n\}_n$ be a collection of subsets $E_n \subset \mathbb{R}$. Then

$$f^{-1}\left(\bigcup_n E_n\right) = \bigcup_n f^{-1}(E_n). \quad (47)$$

Proof. Let $x \in f^{-1}(\cup_n E_n)$. Then $f(x) \in \cup_n E_n$, so $f(x)$ is in any of E_n for $n \in \mathbb{N}$. This is equivalent to saying that $x \in f^{-1}(E_n)$ for some $n \in \mathbb{N}$, which implies that $x \in \cup_n f^{-1}(E_n)$. Hence,

$$f^{-1}\left(\bigcup_n E_n\right) \subseteq \bigcup_n f^{-1}(E_n). \quad (48)$$

Now let $x \in \cup_n f^{-1}(E_n)$. Then $x \in f^{-1}(E_n)$ for some $n \in \mathbb{N}$, which is equivalent to saying that $f(x) \in E_n$ for some n . Then $f(x) \in \cup_n E_n$, so $x \in f^{-1}(\cup_n E_n)$. Thus,

$$\bigcup_n f^{-1}(E_n) \subseteq f^{-1}\left(\bigcup_n E_n\right). \quad (49)$$

Therefore, both sets are subsets of one another, so they are equal. \square