

18.102 Midterm

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Problem 1

Proof. We will show that $\Lambda([a, b])$ is a proper closed subspace of $C([a, b])$, which we know is a Banach space. Let $\{f_n\}_n$ be a Cauchy sequence in $\Lambda([a, b])$ such that $f_n \rightarrow f$ pointwise. Then for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $\|f - f_n\| < \epsilon$. This is equivalent to

$$\sup_{x \in [a, b]} |f(x) - f_n(x)| + \sup_{x \neq y \in [a, b]} \frac{|f(x) - f_n(x) - f(y) + f_n(y)|}{|x - y|} < \epsilon. \quad (1)$$

Since both terms on the left hand side are non-negative, this implies

$$\sup_{x \neq y \in [a, b]} \frac{|f(x) - f_n(x) - f(y) + f_n(y)|}{|x - y|} < \epsilon. \quad (2)$$

Then for any $x \neq y \in [a, b]$, we have

$$|f(x) - f_n(x) - f(y) + f_n(y)| < \epsilon|x - y|, \quad (3)$$

which confirms that for each $n \geq N$, the function $f - f_n$ is Lipschitz continuous. By assumption, f_n is Lipschitz continuous $\forall n \in \mathbb{N}$, and the sum of Lipschitz continuous functions is also Lipschitz, thus $f = f_n + (f - f_n)$ is Lipschitz continuous.

So, $\lim_{n \rightarrow \infty} f_n = f \in \Lambda([a, b])$, which proves that $\Lambda([a, b])$ is a proper closed subspace of $C([a, b])$.

Therefore, $\Lambda([a, b])$ is a Banach space. □

Problem 2

Proof. First we show that $\|a + c_0\|_{\ell^\infty / c_0} \leq \limsup_{n \rightarrow \infty} |a_n|$.

Let $a = \{a_n\}_n \in \ell^\infty$. For each $n \in \mathbb{N}$, let $b_n = (a_1, a_2, \dots, a_n, 0, 0, \dots) \in c_0$. Then

$$\inf_{b \in c_0} \|a + b\|_\infty \leq \inf_n \|a - b_n\|_\infty \quad (4)$$

$$= \inf_n \sup_{m \in \mathbb{N}} |a_m - b_m| \quad (5)$$

$$= \inf_n \sup_{m \geq n} |a_m| \quad (6)$$

$$= \limsup_{n \rightarrow \infty} |a_n|. \quad (7)$$

Thus,

$$\|a + c_0\|_{\ell^\infty/c_0} \leq \limsup_{n \rightarrow \infty} |a_n|. \quad (8)$$

Let $b = (b_1, b_2, b_3, \dots) \in c_0$. Then for every $\epsilon > 0$, $\exists n \in \mathbb{N}$ such that $\forall m \geq n$, $|b_m| < \epsilon$, so

$$\|a + b\|_\infty \geq \sup_{m \geq n} |a_m| - \epsilon \quad (9)$$

$$\geq \limsup_{n \rightarrow \infty} |a_n| - \epsilon, \quad (10)$$

hence $\limsup_{n \rightarrow \infty} |a_n| < \|a + c_0\|_{\ell^\infty/c_0} + \epsilon$.

Therefore, $\|a + c_0\|_{\ell^\infty/c_0} = \limsup_{n \rightarrow \infty} |a_n|$. \square

Problem 3

(a)

Proof. Since $\lim_{n \rightarrow \infty} T_n x = Tx$, then for every $\epsilon > 0 \exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\|T_n x - Tx\| < \epsilon. \quad (11)$$

By linearity of T , this is equivalent to

$$\|(T_n - T)x\| < \epsilon. \quad (12)$$

Choose $\epsilon = \|x\|$. With a sufficiently large choice of N , we have $\forall n \geq N$ and $\forall x \in V$,

$$\|(T_n - T)x\| < \|x\|. \quad (13)$$

The above equation implies that the operator $T_n - T$ is continuous. Since $\{T_n\}_n$ is assumed to be a sequence in $\mathcal{B}(V, W)$, then $T_n - (T_n - T) = T$ is continuous.

Therefore, T is a bounded linear operator. \square

(b)

Proof. Since V is a Banach space with respect to both norms $\|\cdot\|_1$ and $\|\cdot\|_2$, we may regard the spaces $V_1 := (V, \|\cdot\|_1)$ and $V_2 := (V, \|\cdot\|_2)$ as separate Banach spaces.

Consider the identity mapping $\mathbb{1} \in \mathcal{B}(V_1, V_2)$. Since $\mathbb{1}$ is a bounded linear operator, then $\exists C > 0$ such that $\forall v \in V_1$,

$$\|v\|_2 = \|\mathbb{1}v\|_2 \leq C\|v\|_1, \quad (14)$$

and we are done. \square

Problem 4

(a)

Proof. For each $n \in \mathbb{N}$, define the set $F_n \subset E$ via

$$F_n := \left\{ x \in E \mid |f(x)| > \|f\|_\infty + \frac{1}{n} \right\}. \quad (15)$$

Then by definition of the essential supremum of f , $\forall n \in \mathbb{N}$, $m(F_n) = 0$. So for almost every $x \in E$ (i.e. $\forall x \in E \setminus F_n$), we have

$$|f(x)| \leq \|f\|_\infty + \frac{1}{n}. \quad (16)$$

Now consider $\bigcup_{n \in \mathbb{N}} F_n$. Since $\forall n \in \mathbb{N}$ we have $F_{n+1} \subset F_n$, then $(E \setminus F_n) \subset (E \setminus F_{n+1})$.

By continuity from below (proved in [PS5.1b](#)), we have that

$$m\left(E \setminus \bigcup_n F_n\right) = m\left(\bigcap_n F_n^c\right) \quad (17)$$

$$= \lim_{n \rightarrow \infty} m(F_n^c) \quad (18)$$

$$= \lim_{n \rightarrow \infty} m(E \setminus F_n) \quad (19)$$

$$= m(E) - \lim_{n \rightarrow \infty} m(F_n) \quad (20)$$

$$= m(E). \quad (21)$$

This is equivalent to the statement

$$m\left(\bigcup_n F_n\right) = 0. \quad (22)$$

Therefore, $|f(x)| \leq \|f\|_\infty$ almost everywhere on E . \square

(b)

Proof. (i) Let $c \in \mathbb{R}$. Then the function cf is measurable, and by definition,

$$\|cf\|_\infty = \inf \{B \geq 0 \mid m(\{x \in E \mid |cf(x)| > B\}) = 0\} \quad (23)$$

$$= \inf \left\{ |c| \frac{B}{|c|} \mid m \left(\left\{ x \in E \mid |f(x)| \geq \frac{B}{|c|} \right\} \right) = 0 \right\} \quad (24)$$

$$= |c| \inf \left\{ \frac{B}{|c|} \mid m \left(\left\{ x \in E \mid |f(x)| > \frac{B}{|c|} \right\} \right) = 0 \right\} \quad (25)$$

$$= |c| \cdot \|f\|_\infty, \quad (26)$$

as desired.

(ii) By definition, we have

$$\|f + g\|_\infty = \inf \{C > 0 \mid m(\{x \in E \mid |f(x) + g(x)| > C\}) = 0\}. \quad (27)$$

Note that by the triangle inequality, $|f(x) + g(x)| \leq |f(x)| + |g(x)|$, so if $|f(x) + g(x)| > C \Rightarrow |f(x)| + |g(x)| > C$. This means

$$\{x \in E \mid |f(x) + g(x)| > C\} \subseteq \{x \in E \mid |f(x)| + |g(x)| > C\}. \quad (28)$$

By monotonicity of the Lebesgue measure, this gives

$$m(\{x \in E \mid |f(x) + g(x)| > C\}) \leq m(\{x \in E \mid |f(x)| + |g(x)| > C\}), \quad (29)$$

which implies that

$$\|f + g\|_\infty \leq \inf \{C > 0 \mid m(\{x \in E \mid |f(x)| + |g(x)| > C\}) = 0\}. \quad (30)$$

Note that for $C > 0$, if for some $x \in E$ $|f(x)| + |g(x)| > C$, then it follows that either $|f(x)| > C$ or $|g(x)| > C$. Thus we can express the set on the right side of (28) as follows:

$$\{x \in E \mid |f(x)| + |g(x)| > C\} \subseteq \{x \in E \mid |f(x)| > C\} \cup \{x \in E \mid |g(x)| > C\}. \quad (31)$$

Once again using monotonicity of $m(\cdot)$, we can conclude that

$$\begin{aligned} \|f + g\|_\infty &\leq \inf \{C > 0 \mid m(\{x \in E \mid |f(x)| > C\}) = 0\} + \\ &\quad \inf \{C > 0 \mid m(\{x \in E \mid |g(x)| > C\}) = 0\} \end{aligned} \quad (32)$$

$$= \|f\|_\infty + \|g\|_\infty. \quad (33)$$

Therefore the essential supremum satisfies both homogeneity and the triangle inequality. \square