18.102 Assignment 7

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Problem 1

(a)

Proof. Let $E = [a, b]$ and let $f : E \to \mathbb{C}$. Suppose $f \in L^p([a, b])$. Then

$$\Rightarrow \int_E |f|^p < \infty \tag{1}$$

$$\Rightarrow \left(\int_E |f|^p\right)^{\frac{1}{p}} < \infty \tag{2}$$

$$\iff ||f||_{L^p(E)} < \infty. \tag{3}$$

Now let $1 \leq q \leq p$. Then by Hölder's inequality, since $1 : E \to \mathbb{C}$ is measurable, then

$$
\begin{aligned}
||f||^q_{L^q(E)} &= \int_E |f|^q && (4)\\
&= |||f|^q||_{L^1(E)} && (5)\\
&= ||1 \cdot |f|^q||_{L^1(E)} && (6)\\
&\leq ||1||_{L^{\frac{p}{p-q}}(E)} |||f|^q||_{L^{\frac{p}{q}}(E)} && (7)\\
&= \left(\int_E 1\right)^{\frac{p-q}{p}} \left(\int_E \left||f|^q\right|^{\frac{p}{q}}\right)^{\frac{q}{p}} && (8)\\
&= (b-a)^{\frac{p-q}{p}} \left(\int_E |f|^p\right)^{\frac{q}{p}} && (9)\\
&= (b-a)^{\frac{p-q}{p}} ||f||^q_{L^p(E)} && (10)\\
&< \infty. && (11)
\end{aligned}
$$

Hence, $f \in L^p([a, b]) \Rightarrow f \in L^q([a, b])$.

Therefore $L^p([a, b]) \subset L^q([a, b])$. □

(b)

Proof. Let $f \in L^p([a,b])$ and $\epsilon > 0$. Choose N such that

$$||f - f\chi_{[-f^{-1}(N), f^{-1}(N)]}||_p < \frac{\epsilon}{2}. \tag{12}$$

Let $f_n = f\chi_{[-f^{-1}(n), f^{-1}(n)]}$. From PS6.2, Littlewood's third principle tells us that "every measurable function is nearly continuous." This gives us a closed set F such that

$$m([a,b]\backslash F) < \left(\frac{\epsilon}{4N}\right)^p, \tag{13}$$

and the restriction $f_n\big|_F$ is continuous with $f_N(a) = f_N(b) = 0$. So, we have

$$\begin{aligned} ||f_n - f||^p &= \int_{[a,b]\backslash F} |f_n - f|_p^p && (14)\\ &\leq (2N)^p m([a,b]\backslash F) && (15)\\ &< (2N)^p \frac{\epsilon^p}{(4N)^p} && (16)\\ &= \left(\frac{\epsilon}{2}\right)^p, && (17) \end{aligned}$$

i.e. $||f_n - f||_p < \epsilon$.

Since χ is a step function, the theorem given in the assignment tells us that we can find a $g \in C([a,b])$ with $g(a) = g(b) = 0$. Let $g = \lim_{n\to\infty} f_n$. Then $|f - g| < \epsilon$. Thus, $C([a,b])$ is dense in $L^p([a,b])$.

Therefore $L^p([a,b])$ is separable. □

(c)

Proof. For each $n \in \mathbb{N}$, define $f_n := f\chi_{[-n,n]}$. Since $f \in L^p(\mathbb{R})$, then $f \in L^p([-n,n])$, so by the given theorem, f_n is a step function and $\exists g_n \in C([-n,n])$ with $g(-n) = g(n) = 0$ such that

$$||f - f_n||_p + ||f - g_n||_p < \epsilon. \tag{18}$$

We compute

$$\begin{aligned} ||f - f_n||_p^p &= \int_{\mathbb{R}} |f - f_n|^p && (19)\\ &= \int_{\mathbb{R}} |f|^p |1 - \chi_{[-n,n]}|^p && (20)\\ &= \int_{\mathbb{R}\backslash[-n,n]} |f|^p |1 - \chi_{[-n,n]}|^p + \int_{[-n,n]} |f|^p |\chi_{[-n,n]}|^p && (21)\\ &= \int_{\mathbb{R}\backslash[-n,n]} |f|^p. && (22) \end{aligned}$$

Sending $n \to \infty$, we get

$$||f - f_n||_p^p = \int_{\varnothing} |f|^p = 0. \tag{23}$$

Thus, $||f_n - f||_p \to 0$ as $n \to \infty$. Choosing R sufficiently large, we are left with $||f - g_n||_p < \epsilon$ as $n \to \infty$. Take $g = \lim_{n\to\infty} g_n$, and we are done.

Therefore $||f - g||_p < \epsilon$. ☐

(d)

Proof. From part **(c)**, we know that for every $f \in L^p(\mathbb{R})$ and $\epsilon > 0$, we can find a $g \in C(\mathbb{R})$ such that $\forall |x| > R$, $g(x) = 0$ and $||f - g||_p < \epsilon$. So, $C(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

Let $\{g_n\}_n \subset C(\mathbb{R})$. Then we are done, since this sequence is countable.

Therefore $L^p(\mathbb{R})$ is separable. ☐

Problem 2

(a)

Proof. Let $f \in L^\infty(E)$. The norm is given by the essential supremum, i.e.

$$||f||_\infty = \inf\{C \geq 0 \mid |f(x)| \leq C \text{ a.e.}\} \tag{24}$$

In problem 4b of the midterm, we showed that the ess. sup satisfied homogeneity and the triangle inequality. In 4a, we showed that $||f||_\infty \geq |f(x)|$ almost everywhere on E, so if $||f||_\infty = 0$, then $f = 0$ a.e., so $||\cdot||_\infty$ is a well-defined norm. Thus, $L^\infty(E)$ is a normed vector space.

Let $\epsilon > 0$. Then $\exists N_0 \in \mathbb{N}$ such that $\forall n \geq N_0$ and $x \in F$, $|f_n(x) - f(x)| < \epsilon$. Then

$$m(\{x \in E \mid |f_n(x) - f(x)| > \epsilon\}) = 0. \tag{25}$$

So $\forall n \geq N_0$, $||f_n - f|| < \epsilon$. Thus if $\exists F \subset E$ such that $m(F^c) = 0$ and $f_n \to f$ uniformly on F, then $||f_n - f|| \to 0$.

Let $\{f_n\}_n \subset L^\infty(E)$ be a Cauchy sequence. Then for $\epsilon > 0$, $\exists N_0(\epsilon) \in \mathbb{N}$ such that $\forall n, m \geq N_0(\epsilon)$, we have $||f_n - f_m|| < \frac{\epsilon}{2}$. Then $\exists F^\epsilon \subset E$ such that $m(F^\epsilon) = 0$ and for $x \notin F^\epsilon$, $|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$.

Let $F = \cup_n F^{\frac{1}{n}}$. Then $m(F) = 0$ and $\forall x \in F^c$, $\{f_n(x)\}_n$ is Cauchy. Let $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in F^c$, and let $\epsilon > 0$. Then $\exists k_0 \in \mathbb{N}$ such that $\forall k \geq k_0$, $\frac{1}{k} < \frac{\epsilon}{2}$. Then $\forall n, m \geq N_0(k_0)$, we have

$$|f_n(x) - f_m(x)| < \frac{1}{k} < \frac{\epsilon}{2}. \tag{26}$$

Sending $m \to \infty$ gives us $|f_n(x) - f(x)| < \epsilon$, hence $||f_n - f|| \to 0$.

Therefore $L^\infty(E)$ is a Banach space. $\square$

(b)

Proof. Let $f \in L^\infty([a,b])$. We have

$$
\begin{aligned}
||f||_{L^\infty([a,b])} &= \inf\{C \geq 0 \mid m(x \in [a,b] \mid |f(x)| > C) = 0\} &(27)\\
&= \inf\{C \geq 0 \mid |f(x)| \leq C \text{ a.e. on } [a,b]\}. &(28)
\end{aligned}
$$

Note that $\forall x \in [a,b]$, $|f(x)| \leq \sup_{x\in[a,b]} |f(x)|$, so

$$
\begin{aligned}
||f||_{L^\infty([a,b])} &= \inf\{C \geq 0 \mid C \geq \sup_{x\in[a,b]} |f(x)|\} &(29)\\
&= \sup_{x\in[a,b]} |f(x)| &(30)\\
&= ||f||_\infty. &(31)
\end{aligned}
$$

Consider the function $f = \chi_{[a,b]}$. Then

$$
||f||_\infty = \sup_{x\in[a,b]} |\chi_{[a,b]}(x)| = 1, \tag{32}
$$

so $f \in L^\infty([a,b])$. Now suppose $\exists\{f_n\}_n \subset C([a,b])$ such that $||f_n - f||_\infty \to 0$. Then for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $||f_n - f||_\infty < \epsilon$. Choose $\epsilon = \frac{1}{2}$. Then $\exists N$ such that $\forall n \geq N$, $||f_n - f||_\infty < \frac{1}{2}$. In particular, $\exists N$ such that $||f_n - f||_\infty < \frac{1}{2}$ and such that $|f_n(x) - f(x)| > \frac{1}{2}$ for some $x \in [a,b]$. But this contradicts our conclusion above that $||f||_\infty = ||f||_{L^\infty([a,b])}$, so this f cannot be approximated arbitrarily closely by continuous functions.

Therefore $C([a,b])$ is not dense in $L^\infty([a,b])$. $\square$

Problem 3

Proof. First, we show that $T : L^p([a,b]) \to L^p([a,b])$, defined via

$$
Tf(x) := f(x)g(x) \tag{33}
$$

for $g \in L^\infty([a,b])$, is linear.

Let $f_1, f_2 \in L^p([a,b])$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. Then

$$
\begin{aligned}
T(\lambda_1 f_1 + \lambda_2 f_2) &= (\lambda_1 f_1 + \lambda_2 f_2)g &(34)\\
&= \lambda_1 f_1 g + \lambda_2 f_2 g &(35)\\
&= \lambda_1 Tf_1 + \lambda_2 Tf_2. &(36)
\end{aligned}
$$

Thus, T is linear. We can also check that T does indeed map $L^p([a,b])$ into itself; let $f \in L^p([a,b])$. Then

$$\begin{aligned}
\int_a^b |Tf|^p &= \int_a^b |fg|^p && (37)\\
&= \int_a^b |f|^p |g|^p && (38)\\
&\leq \sup_{x\in[a,b]} |g(x)|^p \int_a^b |f|^p && (39)\\
&= ||g||_\infty^p ||f||_\infty^p && (40)\\
&< \infty, && (41)
\end{aligned}$$

hence $Tf \in L^p([a,b])$.

Next we show that T is bounded:

$$\begin{aligned}
||T|| &= \sup_{||f||_p=1} ||fg||_p && (42)\\
&= \sup_{||f||_p=1} \left(\int_a^b |fg|^p\right)^{\frac{1}{p}} && (43)\\
&= \sup_{||f||_p=1} \left(||g||_\infty^p \int_a^b |f|^p\right)^{\frac{1}{p}} && (44)\\
&= \sup_{||f||_p=1} ||g||_\infty ||f||_p && (45)\\
&= ||g||_{L^\infty}. && (46)
\end{aligned}$$

Thus we conclude that $T \in \mathcal{B}(L^p([a,b]), L^p([a,b]))$ with $||T|| = ||g||_{L^\infty}$. $\square$