18.102 Assignment 3

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Problem 1

(a)

Proof. We want to show that $u \in M'$. First, we show that u is linear.

Let $a, b \in M$ and let $\lambda \in \mathbb{C}$. Then

$$u(\lambda a) = \lim_{k\to\infty} (\lambda a_k) = \lambda \cdot \lim_{k\to\infty} a_k = \lambda u(a) \text{ , and} \tag{1}$$

$$u(a+b) = \lim_{k\to\infty} (a_k + b_k) = \lim_{k\to\infty} a_k + \lim_{k\to\infty} b_k = u(a) + u(b). \tag{2}$$

So u is linear on M. Next, we show that u is bounded.

Let $a \in M$, i.e. $\lim_{k\to\infty} a_k$ exists. Then a is bounded, so $\exists B \geq 0$ such that $\forall k \in \mathbb{N}$, $|a_k| \leq B$. Then by continuity of the norm,

$$\begin{aligned}
||u|| &\leq |u(a)| && (3)\\
&= \left|\lim_{k\to\infty} a_k\right| && (4)\\
&= \lim_{k\to\infty} |a_k| && (5)\\
&\leq B, && (6)
\end{aligned}$$

so u is bounded.

Then we conclude that u is a bounded linear functional on M. $\square$

(b)

Proof. (By contradiction). Suppose instead that $\exists b \in \ell^1$ such that $\forall a \in \ell^\infty$,

$$v(a) = \sum_{k=1}^{\infty} a_k b_k. \tag{7}$$

Define $e_n := \{\delta_{kn}\}_k \in \ell^\infty$, for fixed $n \in \mathbb{N}$. Then $\lim_{k\to\infty} \delta_{nk} = 0$, so $e_n \in M$ as well. By equation (7), we have

$$v(e_n) = \sum_{k=1}^{\infty} \delta_{kn} b_k = b_n. \tag{8}$$

By the Hahn-Banach theorem, $v|_M = u$. But $u(e_n) = \lim_{k\to\infty} \delta_{kn} = 0$, and since $e_n \in M$, we have

$$b_n = v(e_n) = u(e_n) = 0. \tag{9}$$

This must hold for any $n \in \mathbb{N}$, so $b_n = 0 \ \forall n \in \mathbb{N}$. Then $b = \{b_k\}_k = (0, 0, ...)$, so $v = 0$ by definition. But

$$0 = v(1, 1, ...) = u(1, 1, ...) = 1, \quad (\Rightarrow\Leftarrow) \tag{10}$$

so we arrive at a contradiction to the initial assumption.

Therefore $\nexists b \in \ell^1$ such that $\forall a \in \ell^\infty$, $v(a) = \sum_k a_k b_k$. $\square$

Problem 2

(a)

Proof. First we show that $\|T^\dagger\| \leq \|T\|$. We have

$$\begin{aligned}
\|T^\dagger\| &= \sup_{||f||=1} \|T^\dagger f\| && (11)\\
&= \sup_{\|f\|=1} \|f \circ T\| && (12)\\
&\leq \sup_{\|f\|=1} \|f\|\|T\| && (13)\\
&\leq \|T\|. && (14)
\end{aligned}$$

So, $T^\dagger : W' \to V'$ is bounded, i.e. $T^\dagger \in \mathcal{B}(W', V')$.

Next we show that $\|T^\dagger\| \geq \|T\|$.

Let $x \in V$ with $\|x\| = 1$. Since W is a normed space, then by the theorem from lecture 6 (corollary to Hahn-Banach Thm.), $\exists f \in W'$ such that $\|f\| = 1$ and $f(w) = \|w\| \ \forall w \in W\backslash\{0\}$.

Since $T : V \to W$, then $w = Tx$ for $x \in V$, so $f(Tx) = \|Tx\|$. Then we have

$$\begin{aligned}
\|T^\dagger\| &= \sup_{\|f\|=1} \|T^\dagger f\| && (15)\\
&= \sup_{\|f\|=1} \sup_{\|x\|=1} |(f \circ T)(x)| && (16)\\
&= \sup_{\|f\|=1} \sup_{\|x\|=1} |f(Tx)| && (17)\\
&\geq \sup_{\|x\|=1} \|Tx\| && (18)\\
&= \|T\|, && (19)
\end{aligned}$$

as desired.

Thus $T^\dagger \in \mathcal{B}(W', V')$ and $\|T^\dagger\| = \|T\|$. $\square$

(b)

Proof. Let $a \in \ell^p$, and $b_k = Ra_k$. We want to show that $b = \{b_k\}_k \in \ell^p$. Since $a \in \ell^p$, then a is bounded, which implies that $\exists B \geq 0$ such that $\forall k \in \mathbb{N}$, $|a_k| \leq B$. Then $b_k = Ra_k := a_{k-1}$, with $b_1 = a_0 := 0$.

Thus $\forall k \in \mathbb{N}$, $|b_k| = |a_{k-1}| \leq B$, so b is also bounded. We have

$$\|b\|_\infty = \sup_k |b_k| = \sup_k |a_{k-1}|. \tag{20}$$

So, $R : \ell^p \to \ell^p$. Next we compute the operator norm of R:

$$\begin{aligned}
\|R\| &= \sup_{\|a\|=1} \|Ra\| && (21)\\
&= \sup_{\|a\|=1} \sup_k |Ra_k| && (22)\\
&= \sup_{\|a\|=1} \sup_k |a_{k-1}| && (23)\\
&= \sup_{\|a\|=1} \sup_k \{0, |a_1|, |a_2|, ...\} && (24)\\
&= \sup_{\|a\|=1} \sup_k |a_k| && (25)\\
&= \sup_{\|a\|=1} \|a\| && (26)\\
&= 1. && (27)
\end{aligned}$$

Therefore, $R \in \mathcal{B}(\ell^p, \ell^p)$ with $\|R\| = 1$. $\square$

(c)

Suppose $1 \leq p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. From assignment 1, we identify $(\ell^p)'$ with ℓ^q via the pairing: $f \in (\ell^p)' \iff \exists b \in \ell^q$ such that $\forall a \in \ell^p$,

$$f(a) = \sum_{k=1}^{\infty} a_k b_k, \tag{28}$$

and $\|f\| = \|b\|_q$.

For example, let $b = \{b_k\}_k = e_1$ defined by $e_1 := \{\delta_{1k}\}_k = \{1, 0, 0, ...\} \in \ell^q$. Then $\forall a \in ell^p$,

$$\begin{aligned}
\left(R^\dagger e_1\right)(a) &:= \sum_{k=1}^{\infty} R a_k \delta_{1k} && (29)\\
&= R a_1 && (30)\\
&= a_0 && (31)\\
&= 0 && (32)\\
&= \sum_{k=1}^{\infty} a_k \cdot 0. && (33)
\end{aligned}$$

Thus $R^\dagger e_1 = 0 \in \ell^q$.

Now let $a \in \ell^p$. Then

$$\begin{aligned}
\left(R^\dagger b\right)(a) &= \sum_{k=1}^{\infty} (Ra)_k\, b_k && (34)\\
&= \sum_{k=1}^{\infty} a_{k-1} b_k && (35)\\
&= 0 \cdot b_k + \sum_{k=2}^{\infty} a_{k-1} b_k && (36)\\
&= \sum_{k=1}^{\infty} a_k b_{k+1}. && (37)
\end{aligned}$$

Hence, $\{\left(R^\dagger b\right)_k\}_k = \{b_{k+1}\}_k$. Therefore, where R was the right-shift operator, we can identify $R^\dagger$ as a left-shift operator.

Problem 3

Proof. To show that $m^*(E + x) = m^*(E)$, we will show both that $m^*(E + x) \geq m^*(E)$ and $m^*(E + x) \leq m^*(E)$.

Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of open intervals that covers E. Then $\{I_n + x\}_n$ covers $E + x$. Since interval length is invariant under translation, we have

$$m^*(E + x) \leq \sum_n \ell(I_n + x) = \sum_n \ell(I_n). \tag{38}$$

So for every such sequence of intervals $\{I_n\}_n$, we have $m^*(E + x) \leq \sum_n \ell(I_n)$. Thus,

$$m^*(E + x) \leq m^*(E). \tag{39}$$

Now let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of open intervals that covers $E + x$. Then $\{I_n - x\}_n$ covers E. Once again, by the translation-invariance of interval length, we have

$$m^*(E) \leq \sum_n \ell(I_n - x) = \sum_n \ell(I_n). \tag{40}$$

Since this is true for every such sequence of intervals $\{I_n\}$, then

$$m^*(E) \leq m^*(E + x). \tag{41}$$

From equations (39) and (41), we conclude that $m^*(E + x) = m^*(E)$.

Therefore, the outer measure m^* is translation-invariant. □

Problem 4

(a)

Proof. Let $x \in U$. Then $a_x \leq b_x$, and both $(a_x, x] \subset U$ and $[x, b_x) \subset U$. Thus,

$$(a_x, b_x) = (a_x, x] \cup [x, b_x) \subset U, \tag{42}$$

as desired. □

(b)

Proof. Let $x, y \in U$ and suppose $y \in (a_x, b_x)$.

<u>Case 1</u>: $y \in (a_x, x]$. Then since $y < x$,

$$\begin{aligned}
a_y &= \inf\{a \in \mathbb{R} \mid (a, y] \subset U\} &(43)\\
&= \inf\{a \in \mathbb{R} \mid (a, y] \cup [y, x] \subset U\} &(44)\\
&= \inf\{a \in \mathbb{R} \mid (a, x] \subset U\} &(45)\\
&= a_x. &(46)
\end{aligned}$$

Similarly,

$$\begin{aligned}
b_y &= \sup\{b \in \mathbb{R} \mid [y, b) \subset U\} &(47)\\
&= \sup\{b \in \mathbb{R} \mid [x, y] \cup [y, b) \subset U\} &(48)\\
&= \sup\{b \in \mathbb{R} \mid [x, b) \subset U\} &(49)\\
&= b_x. &(50)
\end{aligned}$$

Case 2: $y \in [x, b_x)$. Then since $y > x$,

$$\begin{aligned} a_y &= \inf\{a \in \mathbb{R} \mid (a, y] \subset U\} && (51)\\ &= \inf\{a \in \mathbb{R} \mid (a, x] \cup [x, y] \subset U\} && (52)\\ &= \inf\{a \in \mathbb{R} \mid (a, x] \subset U\} && (53)\\ &= a_x. && (54) \end{aligned}$$

Similarly,

$$\begin{aligned} b_y &= \sup\{b \in \mathbb{R} \mid [y, b) \subset U\} && (55)\\ &= \sup\{b \in \mathbb{R} \mid [x, y] \cup [y, b) \subset U\} && (56)\\ &= \sup\{b \in \mathbb{R} \mid [x, b) \subset U\} && (57)\\ &= b_x. && (58) \end{aligned}$$

So we conclude that $(a_x, b_x) = (a_y, b_y)$ □

(c)

Proof. Let $x \in \bigcup_{q \in U \cap \mathbb{Q}} (a_q, b_q)$.

Then for at least one $q \in U \cap \mathbb{Q}$, $x \in (a_q, b_q)$. By part **(a)**, since $q \in U$, then $(a_q, b_q) \subset U \implies x \in U$. Thus,

$$\bigcup_{q \in U \cap \mathbb{Q}} (a_q, b_q) \subseteq U. \tag{59}$$

Now let $x \in U$.

Since U is open, then $\exists \epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then $\exists q, r \in U \cap \mathbb{Q}$ such that $x - \epsilon < q < x$ and $x < r < x + \epsilon$, i.e.

$$q < x < r. \tag{60}$$

By **(a)**, $(a_q, b_q) \subset U$. Then since $x > q$, we have $x \in (a_q, b_q)$
$\implies x \in \bigcup_{q \in U \cap \mathbb{Q}} (a_q, b_q)$. Thus,

$$U \subseteq \bigcup_{q \in U \cap \mathbb{Q}} (a_q, b_q). \tag{61}$$

Then equations (59) and (61) imply the final result,

$$U = \bigcup_{q \in U \cap \mathbb{Q}} (a_q, b_q), \tag{62}$$

as desired. □

In other words, having completed these proofs, we've shown that every open set $U \subset \mathbb{R}$ can be written as a countable union of open intervals.