18.102 Assignment 7

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Problem 1

(a)

Proof. Let $E = [a, b]$ and let $f : E \to \mathbb{C}$. Suppose $f \in L^p([a, b])$. Then

$$\Rightarrow \int_E |f|^p < \infty \tag{1}$$

$$\Rightarrow \left(\int_E |f|^p\right)^{\frac{1}{p}} < \infty \tag{2}$$

$$\iff ||f||_{L^p(E)} < \infty. \tag{3}$$

Now let $1 \leq q \leq p$. Then by Hölder's inequality, since $1 : E \to \mathbb{C}$ is measurable, then

$$\begin{aligned}
||f||^q_{L^q(E)} &= \int_E |f|^q && (4)\\
&= |||f|^q||_{L^1(E)} && (5)\\
&= ||1 \cdot |f|^q||_{L^1(E)} && (6)\\
&\leq ||1||_{L^{\frac{p}{p-q}}(E)} |||f|^q||_{L^{\frac{p}{q}}(E)} && (7)\\
&= \left(\int_E 1\right)^{\frac{p-q}{p}} \left(\int_E \left||f|^q\right|^{\frac{p}{q}}\right)^{\frac{q}{p}} && (8)\\
&= (b-a)^{\frac{p-q}{p}} \left(\int_E |f|^p\right)^{\frac{q}{p}} && (9)\\
&= (b-a)^{\frac{p-q}{p}} ||f||^q_{L^p(E)} && (10)\\
&< \infty. && (11)
\end{aligned}$$

Hence, $f \in L^p([a, b]) \Rightarrow f \in L^q([a, b])$.

Therefore $L^p([a, b]) \subset L^q([a, b])$. □

(b)

Proof. Let $f \in L^p([a,b])$ and $\epsilon > 0$. Choose N such that

$$||f - f\chi_{[-f^{-1}(N), f^{-1}(N)]}||_p < \frac{\epsilon}{2}. \tag{12}$$

Let $f_n = f\chi_{[-f^{-1}(n), f^{-1}(n)]}$. From PS6.2, Littlewood's third principle tells us that "every measurable function is nearly continuous." This gives us a closed set F such that

$$m([a,b]\backslash F) < \left(\frac{\epsilon}{4N}\right)^p, \tag{13}$$

and the restriction $f_n\big|_F$ is continuous with $f_N(a) = f_N(b) = 0$. So, we have

$$\begin{aligned} ||f_n - f||^p &= \int_{[a,b]\backslash F} |f_n - f|_p^p & (14)\\ &\leq (2N)^p m([a,b]\backslash F) & (15)\\ &< (2N)^p \frac{\epsilon^p}{(4N)^p} & (16)\\ &= \left(\frac{\epsilon}{2}\right)^p, & (17) \end{aligned}$$

i.e. $||f_n - f||_p < \epsilon$.

Since χ is a step function, the theorem given in the assignment tells us that we can find a $g \in C([a,b])$ with $g(a) = g(b) = 0$. Let $g = \lim_{n\to\infty} f_n$. Then $|f - g| < \epsilon$. Thus, $C([a,b])$ is dense in $L^p([a,b])$.

Therefore $L^p([a,b])$ is separable. □

(c)

Proof. For each $n \in \mathbb{N}$, define $f_n := f\chi_{[-n,n]}$. Since $f \in L^p(\mathbb{R})$, then $f \in L^p([-n,n])$, so by the given theorem, f_n is a step function and $\exists g_n \in C([-n,n])$ with $g(-n) = g(n) = 0$ such that

$$||f - f_n||_p + ||f - g_n||_p < \epsilon. \tag{18}$$

We compute

$$\begin{aligned} ||f - f_n||_p^p &= \int_{\mathbb{R}} |f - f_n|^p & (19)\\ &= \int_{\mathbb{R}} |f|^p |1 - \chi_{[-n,n]}|^p & (20)\\ &= \int_{\mathbb{R}\backslash[-n,n]} |f|^p |1 - \chi_{[-n,n]}|^p + \int_{[-n,n]} |f|^p |\chi_{[-n,n]}|^p & (21)\\ &= \int_{\mathbb{R}\backslash[-n,n]} |f|^p. & (22) \end{aligned}$$

Sending $n \to \infty$, we get

$$||f - f_n||_p^p = \int_{\varnothing} |f|^p = 0. \tag{23}$$

Thus, $||f_n - f||_p \to 0$ as $n \to \infty$. Choosing R sufficiently large, we are left with $||f - g_n||_p < \epsilon$ as $n \to \infty$. Take $g = \lim_{n\to\infty} g_n$, and we are done.

Therefore $||f - g||_p < \epsilon$. □

(d)

Proof. From part **(c)**, we know that for every $f \in L^p(\mathbb{R})$ and $\epsilon > 0$, we can find a $g \in C(\mathbb{R})$ such that $\forall |x| > R$, $g(x) = 0$ and $||f - g||_p < \epsilon$. So, $C(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

Let $\{g_n\}_n \subset C(\mathbb{R})$. Then we are done, since this sequence is countable.

Therefore $L^p(\mathbb{R})$ is separable. □