

18.102 Assignment 4

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Problem 1

Proof. Let $A \subset \mathbb{R}$ and let $E \in \mathcal{A}$. Then $f^{-1}(E)$ is measurable, so

$$m^*(A \cap f^{-1}(E)) + m^*(A \cap f^{-1}(E)^c) \leq m^*(A). \quad (1)$$

We will first show that \mathcal{A} is closed under taking complements so we must show that $E^c \in \mathcal{A}$ for every $E \in \mathcal{A}$; i.e. $f^{-1}(E^c)$ is measurable.

We will use the fact that $f^{-1}(E^c) = f^{-1}(E)^c$, which is proven in appendix [A](#).

Since $f^{-1}(E)$ is measurable, we have

$$m^*(A) \geq m^*(A \cap f^{-1}(E)) + m^*(A \cap f^{-1}(E)^c) \quad (2)$$

$$= m^*\left[A \cap (f^{-1}(E)^c)^c\right] + m^*(A \cap f^{-1}(E^c)) \quad (3)$$

$$= m^*(A \cap f^{-1}(E^c)^c) = m^*(A \cap f^{-1}(E^c)). \quad (4)$$

Hence, $f^{-1}(E^c)$ is measurable, so $E^c \in \mathcal{A}$. Thus, \mathcal{A} is closed under taking complements.

Now we show that \mathcal{A} is closed under taking countable unions.

Let $\{E_n\}_n \subset \mathcal{A}$ be a sequence of sets in \mathcal{A} , and let $A \subset \mathbb{R}$. Then

$$m^*\left[A \cap f^{-1}\left(\bigcup_n E_n\right)\right] = m^*\left[A \cap \left(\bigcup_n f^{-1}(E_n)\right)\right] \quad (5)$$

$$= m^*\left[\bigcup_n (A \cap f^{-1}(E_n))\right] \quad (6)$$

$$\leq \sum_n m^*(A \cap f^{-1}(E_n)) \quad (7)$$

$$\leq m^*(A \cap f^{-1}(E_n)), \quad (8)$$

by countable subadditivity and positive-definiteness of the outer measure m^* . Similarly, we also have

$$m^* \left[A \cap f^{-1} \left(\bigcup_n E_n \right)^c \right] = m^* \left[A \cap \left(\bigcup_n f^{-1}(E_n) \right)^c \right] \quad (9)$$

$$= m^* \left[A \cap \left(\bigcap_n f^{-1}(E_n)^c \right) \right] \quad (10)$$

$$\leq m^* (A \cap f^{-1}(E_n)^c), \quad (11)$$

by monotonicity of m^* , because $\bigcap_n f^{-1}(E_n)^c \subseteq f^{-1}(E_n)^c$.

Finally, since $E_n \in \mathcal{A}$, then $f^{-1}(E_n)$ is Lebesgue measurable, so we have

$$m^* \left[A \cap f^{-1} \left(\bigcup_n E_n \right) \right] + \quad (12)$$

$$m^* \left[A \cap f^{-1} \left(\bigcup_n E_n \right)^c \right] \leq m^* (A \cap f^{-1}(E_n)) + m^* (A \cap f^{-1}(E_n)^c) \quad (13)$$

$$\leq m^*(A). \quad (14)$$

So, $f^{-1}(\bigcup_n E_n)$ is measurable, hence $\bigcup_n E_n \in \mathcal{A}$. Thus, \mathcal{A} is closed under taking countable unions.

Therefore, \mathcal{A} is a σ -algebra. \square

Problem 2

TODO TODO TODO

Appendices

A Appendix A

Theorem: Let $f : \mathbb{R} \rightarrow \mathbb{R}$. If $E \subset \mathbb{R}$, then $f^{-1}(E^c) = f^{-1}(E)^c$.

Proof. Let $x \in f^{-1}(E^c)$. Then

$$\implies f(x) \notin E^c \iff f(x) \notin E \quad (15)$$

$$\implies x \notin f^{-1}(E) \iff x \in f^{-1}(E)^c. \quad (16)$$

Thus, we have

$$f^{-1}(E^c) \subseteq f^{-1}(E)^c. \quad (17)$$

Now let $x \in f^{-1}(E)^c$. Then

$$\implies x \notin f^{-1}(E) \iff f(x) \notin E \tag{18}$$

$$\implies f(x) \in E^c \iff x \in f^{-1}(E^c). \tag{19}$$

So, we also have

$$f^{-1}(E)^c \subseteq f^{-1}(E^c). \tag{20}$$

Taking equations (17) and (20) together allows us to conclude that $f^{-1}(E^c) = f^{-1}(E)^c$, as desired. \square