18.102 Assignment 5

Octavio Vega

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We denote by $\mathcal{M}$ the set of all Lebesgue-measurable subsets of $\mathbb{R}$.

Problem 1

TODO TODO TODO

Problem 2

TODO TODO TODO

Problem 3

(a)

Proof. ($\Rightarrow$) Suppose f is measurable.

Let $\alpha \in \mathbb{R}$. We may express the preimage of the set $(\alpha, \infty]$ under the inverse of the restriction of f to E as follows:

$$f^{-1}\big|_E\left((\alpha,\infty]\right) = f^{-1}\left((\alpha,\infty]\right) \cap E, \tag{1}$$

and similarly for F:

$$f^{-1}\big|_F\left((\alpha,\infty]\right) = f^{-1}\left((\alpha,\infty]\right) \cap F. \tag{2}$$

Since f is measurable, then $f^{-1}\left((\alpha,\infty]\right) \in \mathcal{M}$. By assumption, E and F are also measurable. Hence, the intersections in (1) and (2) are also measurable.

Therefore, $f\big|_E$ and $f\big|_F$ are measurable.

($\Leftarrow$) Suppose $f\big|_E$ and $f\big|_F$ are measurable.

Then for ever $\alpha \in \mathbb{R}$, $f^{-1}\big|_E\left((\alpha,\infty]\right) \in \mathcal{M}$ and $f^{-1}\big|_F\left((\alpha,\infty]\right) \in \mathcal{M}$. Since $\mathcal{M}$ is closed under taking finite unions, then the union of each of these sets is also measurable, i.e.

$$f^{-1}\big|_E\left((\alpha,\infty]\right) \cup f^{-1}\big|_F\left((\alpha,\infty]\right) \in \mathcal{M}. \tag{3}$$

We also have that $E, F \in \mathcal{M}$, so $E \cup F \in \mathcal{M}$. Then we have

$$\begin{aligned} & f^{-1}\big|_E\left((\alpha,\infty]\right)\right) \cup f^{-1}\big|_F\left((\alpha,\infty]\right)\right) && (4) \\ &= \left(f^{-1}\left((\alpha,\infty]\right)\right) \cap E\right) \cup \left(f^{-1}\left((\alpha,\infty]\right)\right) \cap F\right) \\ &= f^{-1}\left((\alpha,\infty]\right) \cap (E \cup F) && (5) \\ &= f^{-1}\left((\alpha,\infty]\right) \in \mathcal{M}, && (6) \end{aligned}$$

where in line (6) we used the fact that $f^{-1}\left((\alpha,\infty]\right) \subset (E \cup F)$.

Therefore, as desired, f must be measurable. $\square$