

18.102 Assignment 5

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We denote by \mathcal{M} the set of all Lebesgue-measurable subsets of \mathbb{R} .

Problem 1

(a)

Proof. Since $E, M \in \mathcal{M}$, then $E \cup M \in \mathcal{M}$ and $E \cap M \in \mathcal{M}$. We can express the union as

$$E \cup F = (E \cap F) \cup (E \setminus F) \cup (F \setminus E). \quad (1)$$

Then since E and $F \setminus E$ are disjoint, we have

$$m(E \cup F) + m(E \cap F) = m[(E \cap F) \cup (E \setminus F)] + m(F \setminus E) + m(E \cap F) \quad (2)$$

$$= m[(E \cap F) \cup (E \setminus F)] + m[(F \setminus E) \cup (E \cap F)] \quad (3)$$

$$= m(E) + m(F), \quad (4)$$

as desired. \square

(b)

Proof. Since $m(E_1) < \infty$ and $E_{k+1} \subset E_k \forall k \in \mathbb{N}$, then $m(E_k) < m(E_1) < \infty$. Then

$$m\left(\bigcap_{k=1}^{\infty} E_k\right) < \infty, \quad (5)$$

since $\bigcap_{k=1}^{\infty} E_k \subset E_k$.

Additionally, we have

$$m\left[E_1 \setminus \left(\bigcap_{k=1}^{\infty} E_k\right)\right] = m(E_1) - m\left(\bigcap_{k=1}^{\infty} E_k\right). \quad (6)$$

Also, we note that

$$E_1 \setminus \left(\bigcap_{k=1}^{\infty} E_k \right) = E_1 \cap \left(\bigcap_{k=1}^{\infty} E_k \right)^c \quad (7)$$

$$= E_1 \cap \left(\bigcup_{k=1}^{\infty} E_k^c \right) \quad (8)$$

$$= \bigcup_{k=1}^{\infty} (E_1 \cap E_k^c) \quad (9)$$

$$= \bigcup_{k=1}^{\infty} (E_1 \setminus E_k). \quad (10)$$

Define $U_k = E_1 \setminus E_k$ for each $k \in \mathbb{N}$. Then $\forall k \in \mathbb{N}$, $U_k \subset U_{k+1}$ since $E_{k+1} \subset E_k$. By (10), we have

$$m \left(E_1 \setminus \bigcap_{k=1}^{\infty} E_k \right) = m \left[\bigcup_{k=1}^{\infty} (E_1 \setminus E_k) \right] \quad (11)$$

$$= \lim_{k \rightarrow \infty} m(E_1 \setminus E_k) \quad (12)$$

$$= \lim_{k \rightarrow \infty} [m(E_1) - m(E_k)] \quad (13)$$

$$= m(E_1) - \lim_{k \rightarrow \infty} m(E_k). \quad (14)$$

Then using equation (6), this gives

$$m(E_1) - \lim_{k \rightarrow \infty} m(E_k) = m(E_1) - m \left(\bigcap_{k=1}^{\infty} E_k \right). \quad (15)$$

Therefore, we conclude that $m \left(\bigcap_{k=1}^{\infty} E_k \right) = \lim_{k \rightarrow \infty} m(E_k)$.¹ □

Problem 2

(a)

Proof. Let $a \in \mathbb{R}$.

We can write

$$fg = \frac{1}{4} [(f+g)^2 - (f-g)^2]. \quad (16)$$

We showed in lecture 9 that linear combinations of measurable functions are measurable, so we need only show that f^2 and g^2 are measurable.

¹The assignment states this result as *continuity from below* for the Lebesgue measure, but I believe this is actually *continuity from above*.

Case 1: $\alpha < 0$. Then,

$$(f^2)^{-1}((\alpha, \infty]) = (f^2)^{-1}([0, \infty]) = E \in \mathcal{M}. \quad (17)$$

Case 2: $\alpha \geq 0$. Then $\forall x \in E$,

$$f^2(x) > \alpha \iff f(x) > \sqrt{\alpha} \text{ or } f(x) < -\sqrt{\alpha} \quad (18)$$

$$\implies (f^2)^{-1}((\alpha, \infty]) = [-\infty, -\sqrt{\alpha}) \cup (\sqrt{\alpha}, \infty] \in \mathcal{M}. \quad (19)$$

So f^2 is measurable, and by the same reasoning g^2 is measurable.

Therefore, fg is measurable. \square

(b)

Proof. Let $\alpha \in \mathbb{R}$.

Case 1: $\alpha = +\infty$. Then,

$$h^{-1}(\{\infty\}) = \{a\} = \bigcap_{n=1}^{\infty} \left[a, a + \frac{1}{n} \right] \in \mathcal{M}. \quad (20)$$

Case 2: $\alpha \neq \infty$. We have

$$h^{-1}((\alpha, \infty]) = (f + g)^{-1}((\alpha, \infty]). \quad (21)$$

Then $x \in (f + g)^{-1}((\alpha, \infty]) \iff f(x) + g(x) > \alpha$. By the density of \mathbb{Q} in \mathbb{R} , $\exists r \in \mathbb{Q}$ such that $f(x) > r > \alpha - g(x)$. Then since f and g are measurable, we have

$$(f + g)^{-1}((\alpha, \infty]) = \bigcup_{r \in \mathbb{Q}} [f^{-1}((r, \infty]) \cap g^{-1}((\alpha - r, \infty])] \in \mathcal{M}. \quad (22)$$

Therefore, h is measurable. \square

Problem 3

(a)

Proof. (\Rightarrow) Suppose f is measurable.

Let $\alpha \in \mathbb{R}$. We may express the preimage of the set $(\alpha, \infty]$ under the inverse of the restriction of f to E as follows:

$$f^{-1}|_E((\alpha, \infty]) = f^{-1}((\alpha, \infty]) \cap E, \quad (23)$$

and similarly for F :

$$f^{-1}|_F((\alpha, \infty]) = f^{-1}((\alpha, \infty]) \cap F. \quad (24)$$

Since f is measurable, then $f^{-1}((\alpha, \infty]) \in \mathcal{M}$. By assumption, E and F are also measurable. Hence, the intersections in (23) and (24) are also measurable.

Therefore, $f|_E$ and $f|_F$ are measurable.

(\Leftarrow) Suppose $f|_E$ and $f|_F$ are measurable.

Then for ever $\alpha \in \mathbb{R}$, $f^{-1}|_E((\alpha, \infty]) \in \mathcal{M}$ and $f^{-1}|_F((\alpha, \infty]) \in \mathcal{M}$. Since \mathcal{M} is closed under taking finite unions, then the union of each of these sets is also measurable, i.e.

$$f^{-1}|_E((\alpha, \infty]) \cup f^{-1}|_F((\alpha, \infty]) \in \mathcal{M}. \quad (25)$$

We also have that $E, F \in \mathcal{M}$, so $E \cup F \in \mathcal{M}$. Then we have

$$f^{-1}|_E((\alpha, \infty]) \cup f^{-1}|_F((\alpha, \infty]) \quad (26)$$

$$= (f^{-1}((\alpha, \infty]) \cap E) \cup (f^{-1}((\alpha, \infty]) \cap F) \quad (27)$$

$$= f^{-1}((\alpha, \infty]) \cap (E \cup F) \quad (28)$$

$$= f^{-1}((\alpha, \infty]) \in \mathcal{M},$$

where in line (28) we used the fact that $f^{-1}((\alpha, \infty]) \subset (E \cup F)$.

Therefore, as desired, f must be measurable. \square

(b)

Proof. (\Rightarrow) Suppose f is measurable.

We define the indicator function χ_E on E via

$$\chi_E(x) := \begin{cases} 1, & x \in E \\ 0, & x \in E^c. \end{cases} \quad (29)$$

Then we can express g as the product

$$g(x) = f(x) \cdot \chi_E(x). \quad (30)$$

In problem **2a**, we showed that the product of measurable functions is measurable. By assumption, f is measurable. so we need only check that χ_E is measurable.

Let $\alpha \in \mathbb{R}$.

Case 1: $1 \leq \alpha \leq \infty$. Then,

$$\chi_E^{-1}((\alpha, \infty]) = \emptyset \in \mathcal{M}. \quad (31)$$

Case 2: $0 \leq \alpha < 1$. Then,

$$\chi_E^{-1}((\alpha, \infty]) = E \in \mathcal{M}. \quad (32)$$

Case 3: $-\infty \leq \alpha < 0$. Then,

$$\chi_E^{-1}((\alpha, \infty]) = \mathbb{R} \in \mathcal{M}. \quad (33)$$

Hence, χ_E is measurable, so $f \cdot \chi_E$ is also measurable.

Therefore, g is measurable.

(\Leftarrow) Suppose g is measurable. Since $g : E \cup E^c = \mathbb{R} \rightarrow [-\infty, \infty]$ is defined by

$$g(x) := \begin{cases} f(x), & x \in E \\ 0, & x \in E^c, \end{cases} \quad (34)$$

then by restricting g to E we get $g|_E(x) = f(x)$. By part (a), $g|_E$ must be measurable.

Therefore, f is measurable. \square

(c)

Proof. We have already shown in class that sums and products of measurable functions are measurable. So if u and v are measurable, then both u^2 and v^2 are measurable, which implies that $u^2 + v^2$ is measurable.

Define

$$f(x) := u^2(x) + v^2(x). \quad (35)$$

Then we need only check that $f^{\frac{1}{2}}$ is measurable.

Let $g(x) = x^{\frac{1}{2}}$. Then $f^{\frac{1}{2}}(x) = (g \circ f)(x)$, and $f : E \rightarrow [0, \infty] \Rightarrow g : [0, \infty] \rightarrow [0, \infty]$. We use the fact that the composition of measurable functions is measurable, proven in appendix A.1, to show that $f^{\frac{1}{2}}$ is measurable.

Let $\alpha \in \mathbb{R}$.

Case 1: $0 \leq \alpha \leq \infty$. Then,

$$g^{-1}((\alpha, \infty]) = (\alpha^2, \infty) \in \mathcal{M}. \quad (36)$$

Case 2: $-\infty \leq \alpha < 0$. Then,

$$g^{-1}((\alpha, \infty]) = \emptyset \in \mathcal{M}. \quad (37)$$

Hence, g is measurable, so by A.1, $f^{\frac{1}{2}}$ is measurable.

Therefore, $(u^2 + v^2)^{\frac{1}{2}}$ is measurable. \square

Problem 4

(a)

Proof. We define S as the set of elements in E for which f_n does not converge to f , via

$$S := \{x \in E \mid \lim_{n \rightarrow \infty} f_n(x) \neq f(x)\}. \quad (38)$$

If $\lim_{n \rightarrow \infty} f_n(x) \neq f(x)$, then for every $n \in \mathbb{N}$, $\exists \epsilon_0$ such that $\forall m > n$, $|f_m(x) - f(x)| \geq \epsilon_0$. Since this statement holds for every n , then letting $\epsilon_0 = \frac{1}{k}$ allows us to express S as a countable union:

$$S = \bigcup_{m=1}^{\infty} \{x \in E \mid |f_m(x) - f(x)| \geq k^{-1}\} = F_1(k). \quad (39)$$

But since $f_n \rightarrow f$ almost everywhere (a.e.) on E , then S must have measure 0, i.e.

$$m(F_1(k)) = 0 < \infty. \quad (40)$$

We can also check that

$$F_{n+1}(k) = \bigcup_{m=n+1}^{\infty} \{x \in E \mid |f_m(x) - f(x)| \geq k^{-1}\} \quad (41)$$

$$\subset \bigcup_{m=n}^{\infty} \{x \in E \mid |f_m(x) - f(x)| \geq k^{-1}\} \quad (42)$$

$$= F_n(k). \quad (43)$$

So $\forall n \in \mathbb{N}$,

$$F_{n+1}(k) \subset F_n(k). \quad (44)$$

By equations (40) and (44), we appeal to problem 1b to deduce that

$$\lim_{n \rightarrow \infty} m(F_n(k)) = m\left(\bigcap_{n=1}^{\infty} F_n(k)\right). \quad (45)$$

We also have that

$$\bigcap_{n=1}^{\infty} F_n(k) \subset F_n(k) \subset F_1(k), \quad (46)$$

so by the monotonicity of the Lebesgue measure,

$$m\left(\bigcap_{n=1}^{\infty} F_n(k)\right) \leq m(F_1(k)) = 0. \quad (47)$$

Therefore, $\lim_{n \rightarrow \infty} m(F_n(k)) = 0$. \square

(b)

Proof. (i) Since F is a countable union, then

$$m(F) = m\left(\bigcup_{k=1}^{\infty} F_{n_k}(k)\right) \quad (48)$$

$$\leq \sum_{k=1}^{\infty} m(F_{n_k}(k)) \quad (49)$$

$$< \sum_{k=1}^{\infty} 2^{-k} \epsilon \quad (50)$$

$$= \epsilon \left(\frac{1}{1 - \frac{1}{2}} - 1 \right) \quad (51)$$

$$= \epsilon. \quad (52)$$

Thus, $m(F) < \epsilon$.

(ii) The complement F^c is given by

$$F^c = \left(\bigcup_{k=1}^{\infty} F_{n_k}(k) \right)^c = \bigcap_{k=1}^{\infty} F_{n_k}(k)^c, \quad (53)$$

where

$$F_{n_k}(k) := \bigcup_{m=n_k}^{\infty} \{x \in E \mid |f_m(x) - f(x)| \geq k^{-1}\}. \quad (54)$$

Then,

$$F_{n_k}(k)^c = \bigcap_{m=n_k}^{\infty} \{x \in E \mid |f_m(x) - f(x)| < k^{-1}\} \quad (55)$$

$$\implies F^c = \bigcap_{k=1}^{\infty} \bigcap_{m=n_k}^{\infty} \{x \in E \mid |f_m(x) - f(x)| < k^{-1}\}. \quad (56)$$

Let $x \in F^c$. Then by the above equation, $\forall k \in \mathbb{N}$, $\exists n_k \in \mathbb{N}$ such that $\forall m \geq n_k$, $|f_m(x) - f(x)| < k^{-1}$. This implies that $f_n(x) \rightarrow f(x)$, and this is true for every $x \in F^c$.

Therefore, we conclude that $f_n \rightarrow f$ uniformly on F^c . \square

Remark: In this problem, we have effectively proven *Littlewood's second principle*, which is that every convergent sequence of measurable functions is uniformly convergent.

Appendices

A Appendix A

A.1

TODO TODO TODO