18.102 Assignment 2

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February 14, 2023

Problem 1

(a)

Proof. Let B be a Banach space. Suppose $T \in \mathcal{B}(B, B)$ and $||I - T|| < 1$. Then by Geometric series,

$$\sum_{n=0}^{\infty} ||(I-T)^n|| \leq \sum_{n=0}^{\infty} ||I-T||^n = \frac{1}{1 - ||I-T||} < \infty. \tag{1}$$

So the series $\sum_{n=0}^{\infty}(I-T)^n$ converges absolutely, which implies that it converges. Fix $m \in \mathbb{N}$. Then

$$\begin{aligned} T\sum_{n=0}^{m}(I-T)^n &= [I - (I-T)]\sum_{n=0}^{m}(I-T)^n & (2) \\ &= \sum_{n=0}^{m}(I-T)^n - \sum_{n=0}^{m}(I-T)^{n+1} & (3) \\ &= I - (I-T)^{m+1}, \text{ by telescoping sum.} & (4) \end{aligned}$$

By continuity of T,

$$\begin{aligned} T\sum_{n=0}^{\infty}(I-T)^n &= T\left(\lim_{m\to\infty}\sum_{n=0}^{m}(I-T)^n\right) & (5) \\ &= \lim_{m\to\infty} T\sum_{n=0}^{m}(I-T)^n & (6) \\ &= \lim_{m\to\infty}\left[I - (I-T)^{m+1}\right] & (7) \\ &= I, & (8) \end{aligned}$$

since $||I - T|| < 1$. We can similarly show that $\sum_{n=0}^{\infty}(I-T)^n = I$.

Thus, T is indeed invertible, and $\sum_{n=0}^{\infty}(I-T)^n \to T^{-1}$ in $\mathcal{B}(B, B)$. $\square$

(b)

Proof. Let $\mathcal{I} := \{T \in \mathcal{B}(B, B) | T^{-1} \text{ exists}\}$. We want to show that $\forall T \in \mathcal{I}$, $\exists \delta > 0$ such that if $||S - T|| < \delta \implies S \in \mathcal{I}$.

Choose $\delta = \frac{1}{||T^{-1}||}$, and write

$$S = T - (T - S) = T\left[I - T^{-1}(T - S)\right]. \tag{9}$$

If $||S - T|| < \delta = \frac{1}{||T^{-1}||}$, then

$$\begin{aligned} \frac{1}{||T^{-1}||} &> ||S - T|| && (10)\\ &= ||T - T\left[I - T^{-1}(T - S)\right]|| && (11)\\ &= ||T|| \cdot ||I - \left[I - T^{-1}(T - S)\right]|| && (12) \end{aligned}$$

$$\begin{aligned} &\implies ||I - \left[I - T^{-1}(T - S)\right]|| < \frac{1}{||T^{-1}|| \cdot ||T||} = 1 && (13)\\ &\implies ||T^{-1}(T - S)|| = ||I - T^{-1}S|| < 1. && (14) \end{aligned}$$

So by (a), $T^{-1}S$ is invertible, which implies that S is invertible. Thus, $\exists \delta > 0$ such that if $S \in B_\delta(T)$, then $S \in \mathcal{I}$.

Therefore, $\mathcal{I}$ is open. $\square$

Problem 2

(a)

Proof. To show that $||v + W||$ is a norm, we will show that it obeys positive definiteness, homogeneity, and the triangle inequality.

First, suppose that $0 = ||v + W|| = \inf_{w \in W} ||v + w||$. Then since $|| \cdot ||_V$ is a norm on V,

$$||w + w|| = 0 \iff v + w = 0 \implies v = -w. \tag{15}$$

So $\exists$ a sequence $\{w_k\}_k \subset W$ such that $w_k \to -v$. Since W is closed, $-v \in W \implies v \in V$. But then $v + W = 0 + W$ because $v \in W$.

Thus, $||v + W|| = 0 \iff v = 0$ (definiteness).

Also, $||v + W|| = \inf_{w \in W} ||v + w|| \geq 0$ because $|| \cdot ||_V$ is a norm, and $||v + w|| \geq 0$ $\forall w \in W$.

Let $\lambda \in \mathbb{K}$. Then since $\lambda W = W$,

$$
\begin{aligned}
||\lambda(v+W)|| &= ||\lambda v + W|| && (16)\\
&= \inf_{w\in W} ||\lambda v + w|| && (17)\\
&= \inf_{w\in W} |\lambda| \cdot ||v + \frac{w}{\lambda}|| && (18)\\
&= |\lambda| \inf_{w\in W} ||v + w|| && (19)\\
&= |\lambda| \cdot ||v + W|| \quad \text{(homogeneity)}. && (20)
\end{aligned}
$$

Now let $u + W$, $v + W \in V/W$. Then

$$
\begin{aligned}
||(u+W) + (v+W)|| &= ||u + v + W|| && (21)\\
&= \inf_{w\in W} ||u + v + w|| && (22)\\
&= \inf_{w\in W} ||u + v + 2w|| && (23)\\
&= \inf_{w\in W} ||u + w + v + w|| && (24)\\
&\leq \inf_{w\in W} (||u + w|| + ||v + w||) && (25)\\
&\leq \inf_{w\in W} ||u + w|| + \inf_{w\in W} ||v + w|| && (26)\\
&= ||u + W|| + ||v + W|| \quad \text{(triangle inequality)}. && (27)
\end{aligned}
$$

Thus, $||v + W||$ is a norm on V/W. □

(b)

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Problem 3

Proof. Let $\{v_n\}_n$ be a sequence of elements in V. Suppose that the series $\sum_n (v_n + W)$ is absolutely summable, i.e. that $\sum_n ||v_n + W||$ converges. Since $||v_n + W|| = \inf_{w\in W} ||v_n + w||$, then for each $n \in \mathbb{N}$, $\exists w_n \in W$ such that

$$
\begin{aligned}
||v_n + w_n|| &\leq ||v_n + W|| + 2^{-n} && (28)\\
\implies \sum_n ||v_n + w_n|| &\leq \sum_n ||v_n + W|| + \sum_n 2^{-n} && (29)\\
&= \sum_n ||v_n + W|| + 1. && (30)
\end{aligned}
$$

Then by comparison, $\sum_n ||v_n + w_n||$ converges, so $\sum_n (v_n + w_n)$ converges.

Since V is a Banach space, then, by closure, $\exists v \in V$ such that $v = \sum_n (v_n + w_n)$. Then

$$\lim_{N\to\infty} v + W - \sum_{n=1}^{N}(v_n + W) = \sum_{n=1}^{\infty}(v_n + w_n) + W - \lim_{N\to\infty}\sum_{n=1}^{N}(v_n + W) \tag{31}$$

$$= \sum_{n=1}^{\infty} v_n + \sum_{n=1}^{\infty} w_n + W - \lim_{N\to\infty}\sum_{n=1}^{N}(v_n + W) \tag{32}$$

$$= \sum_{n=1}^{\infty} v_n + W - \lim_{N\to\infty}\sum_{n=1}^{N}(v_n + W) \tag{33}$$

$$= \sum_{n=1}^{\infty} v_n + W - \lim_{N\to\infty}\sum_{n=1}^{N} v_n - W \tag{34}$$

$$= \sum_{n=1}^{\infty} v_n - \lim_{N\to\infty}\sum_{n=1}^{N} = 0. \tag{35}$$

So $\sum_n (v_n + W) = v + W$, thus $\sum_n (v_n + W)$ converges in V/W.

Therefore V/W is a Banach space. $\square$

Problem 4

(a)

Proof. Let $\{v_n\}_n$ be a sequence of elements in $\ker(T)$ such that $v_n \to v \in V$ and $Tv_n \to w \in W$. Then $\forall n \in \mathbb{N}$,

$$\implies Tv_n = 0 \tag{36}$$

$$\implies \{Tv_n\}_n \to w = 0. \tag{37}$$

By continuity of T,

$$0 = \lim_{n\to\infty} Tv_n = T\left(\lim_{n\to\infty} v_n\right) = Tv, \tag{38}$$

so $v \in \ker(T)$. Hence $\ker(T)$ is closed. $\square$

(b)

Proof. ($\Rightarrow$) Suppose $V/\ker(T)$ is isomorphic to range(T). Then $\exists$ isomorphism $S : V/\ker(T) \longrightarrow \text{range}(T)$. We claim that the operator defined via $S(v + \ker(T)) = Tv$ satisfies this.

First, we show that S is linear. Let $v_1, v_2 \in V/\ker(T)$. Then by linearity of T,

$$
\begin{aligned}
S(v_1 + v_2 + \ker(T)) &= T(v_1 + v_2) && (39)\\
&= Tv_1 + Tv_2 && (40)\\
&= S(v_1 + \ker(T)) + S(v_2 + \ker(T)). && (41)
\end{aligned}
$$

Let $\lambda \in \mathbb{K}$. Then by linearity of T and since $\lambda \cdot \ker(T) = \ker(T)$,

$$
\begin{aligned}
S(\lambda(v + \ker(T))) &= S(\lambda v + \ker(T)) && (42)\\
&= T(\lambda v) && (43)\\
&= \lambda T v && (44)\\
&= \lambda S(v + \ker(T)). && (45)
\end{aligned}
$$

Thus, S is linear.

Next, we show that S is bounded. We have

$$
\begin{aligned}
||S|| &= \sup_{||v||=1} ||S(v + \ker(T))|| && (46)\\
&= \sup_{||v||=1} ||Tv|| && (47)\\
&= ||T||. && (48)
\end{aligned}
$$

Thus S is bounded, since $T \in \mathcal{B}(V, W)$. So, S is indeed an isomorphism, which confirms that $V/\ker(T)$ is isomorphic to range(T).

Now we proceed to the main part of the proof, where we will show that the above implies that range(T) is closed.

Note that by problems 2 and 3, the space $V/\ker(T)$ is a Banach space because we showed in **(a)** that $\ker(T)$ is a proper closed supspace of V, and V is a Banach space.

Let $\{w_j\}_j$ be a sequence in range(T) such that $w_j \to w \in W$. Then $\{w_j\}_{j\in\mathbb{N}}$ is Cauchy. Since S^{-1} is a continuous linear operator, then $\{S^{-1}(w_j)\}_j$ is also a Cauchy sequence in $V/\ker(T)$.

Since $V/\ker(T)$ is a Banach space, then it is complete. So $\exists v \in V/\ker(T)$ such that

$$S^{-1}(w_j) \to v. \tag{49}$$

By continuity, $S\left(S^{-1}(w_j)\right) \to S(v)$, then

$$
\begin{aligned}
&\implies \lim_{j\to\infty} w_j = w = S(v) && (50)\\
&\implies w \in \text{range(T)}. && (51)
\end{aligned}
$$

Thus, range(T) is closed in W.

($\Leftarrow$) Suppose range(T) is closed. Then range(T) $\subset W$ is a Banach space. The operator $S : V/\ker(T) \longrightarrow$ range(T) as defined before is a well-defined, bijective, bounded linear operator, i.e. $S \in \mathcal{B}(V/\ker(T), \text{range}(T))$. Then by the Open Mapping theorem, $S^{-1} \in \mathcal{B}(\text{range}(T), V/\ker(T))$.

Thus S is an isomorphism, and we are done. $\square$

Problem 5

(a)

Proof. Let $b \in \ell^1$, $\epsilon > 0$, $N \in \mathbb{N}$. Define the truncated sequence

$$a := \{b_1, b_2, b_3, ..., b_N, 0, 0, ...\}. \tag{52}$$

Then $\sum_{k=1}^{\infty} k|a_k| = \sum_{k=1}^{N} k|b_k| < \infty$, so $a \in W$. We choose N such that $\sum_{k=1}^{N} |b_k| > \sum_{k=1}^{\infty} |b_k| - \epsilon$. [Note that this is always possible since the infinite series converges, so its sequence of partial sums also converges.] Then we have

$$\begin{aligned} ||a-b||_1 &= \sum_{k=1}^{\infty} |a_k - b_k| && (53)\\ &= \sum_{k=1}^{N} |b_k - b_k| + \sum_{k=N+1}^{\infty} |0 - b_k| && (54)\\ &= \sum_{k=N+1}^{\infty} |b_k| && (55)\\ &= \sum_{k=1}^{\infty} |b_k| - \sum_{k=1}^{N} |b_k| && (56)\\ &< \epsilon. && (57) \end{aligned}$$

So we have shown that for every $\epsilon > 0$ and $b \in \ell^1$, $\exists N \in \mathbb{N}$ such that $||a-b||_1 < \epsilon$, i.e. $B(b, \epsilon) \cap W \neq \emptyset$. Thus W is dense in ℓ^1.

Now consider the sequence $\{b_k\}_k$ given by $b_k = \frac{1}{k^2}$. Then $\sum_k b_k$ converges absolutely ($p > 1$), but

$$\sum_{k=1}^{\infty} k|b_k| = \sum_{k=1}^{\infty} k \cdot \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k} \tag{58}$$

diverges by Harmonic series. So $b \in \ell^1$ but $b \notin W$. Hence $\ell^1 \neq W$.

Thus we conclude that W is a proper, dense subset of ℓ^1. $\square$