18.102 Assignment 5

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We denote by $\mathcal{M}$ the set of all Lebesgue-measurable subsets of $\mathbb{R}$.

Problem 1

TODO TODO TODO

Problem 2

(a)

Proof. Let $a \in \mathbb{R}$.

We can write

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]. \tag{1}$$

We showed in lecture 9 that linear combinations of measurable functions are measurable, so we need only show that f^2 and g^2 are measurable.

<u>Case 1</u>: $\alpha < 0$. Then,

$$(f^2)^{-1}\left((\alpha, \infty]\right) = (f^2)^{-1}\left([0, \infty]\right) = E \in \mathcal{M}. \tag{2}$$

<u>Case 2</u>: $\alpha \geq 0$. Then $\forall x \in E$,

$$f^2(x) > \alpha \iff f(x) > \sqrt{\alpha} \text{ or } f(x) < -\sqrt{a} \tag{3}$$

$$\implies (f^2)^{-1}\left((\alpha, \infty]\right) = [-\infty, -\sqrt{\alpha}) \cup (\sqrt{\alpha}, \infty] \in \mathcal{M}. \tag{4}$$

So f^2 is measurable, and by the same reasoning g^2 is measurable.

Therefore, fg is measurable. $\square$

Problem 3

(a)

Proof. ($\Rightarrow$) Suppose f is measurable.

Let $\alpha \in \mathbb{R}$. We may express the preimage of the set $(\alpha, \infty]$ under the inverse of the restriction of f to E as follows:

$$f^{-1}\big|_E\left((\alpha,\infty]\right)\right) = f^{-1}\left((\alpha,\infty]\right)\right) \cap E, \tag{5}$$

and similarly for F:

$$f^{-1}\big|_F\left((\alpha,\infty]\right)\right) = f^{-1}\left((\alpha,\infty]\right)\right) \cap F. \tag{6}$$

Since f is measurable, then $f^{-1}\left((\alpha,\infty]\right)\right) \in \mathcal{M}$. By assumption, E and F are also measurable. Hence, the intersections in (5) and (6) are also measurable.

Therefore, $f\big|_E$ and $f\big|_F$ are measurable.

($\Leftarrow$) Suppose $f\big|_E$ and $f\big|_F$ are measurable.

Then for ever $\alpha \in \mathbb{R}$, $f^{-1}\big|_E\left((\alpha,\infty]\right)\right) \in \mathcal{M}$ and $f^{-1}\big|_F\left((\alpha,\infty]\right)\right) \in \mathcal{M}$. Since $\mathcal{M}$ is closed under taking finite unions, then the union of each of these sets is also measurable, i.e.

$$f^{-1}\big|_E\left((\alpha,\infty]\right)\right) \cup f^{-1}\big|_F\left((\alpha,\infty]\right)\right) \in \mathcal{M}. \tag{7}$$

We also have that $E, F \in \mathcal{M}$, so $E \cup F \in \mathcal{M}$. Then we have

$$\begin{aligned}
&f^{-1}\big|_E\left((\alpha,\infty]\right)\right) \cup f^{-1}\big|_F\left((\alpha,\infty]\right)\right) && (8)\\
&= \big(f^{-1}\left((\alpha,\infty]\right)\right) \cap E\big) \cup \big(f^{-1}\left((\alpha,\infty]\right)\right) \cap F\big)\\
&= f^{-1}\left((\alpha,\infty]\right) \cap (E \cup F) && (9)\\
&= f^{-1}\left((\alpha,\infty]\right) \in \mathcal{M}, && (10)
\end{aligned}$$

where in line (10) we used the fact that $f^{-1}\left((\alpha,\infty]\right) \subset (E \cup F)$.

Therefore, as desired, f must be measurable. $\square$

(b)

Proof. ($\Rightarrow$) Suppose f is measurable.

We define the indicator function χ_E on E via

$$\chi_E(x) := \begin{cases} 1, \ x \in E \\ 0, \ x \in E^c. \end{cases} \tag{11}$$

Then we can express g as the product

$$g(x) = f(x) \cdot \chi_E(x). \tag{12}$$

In problem **2a**, we showed that the product of measurable functions is measurable. By assumption, f is measurable. so we need only check that χ_E is measurable.

Let $\alpha \in \mathbb{R}$.

Case 1: $1 \leq \alpha \leq \infty$. Then,

$$\chi_E^{-1}\left((\alpha, \infty]\right) = \emptyset \in \mathcal{M}. \tag{13}$$

Case 2: $0 \leq \alpha < 1$. Then,

$$\chi_E^{-1}\left((\alpha, \infty]\right) = E \in \mathcal{M}. \tag{14}$$

Case 3: $-\infty \leq \alpha < 0$. Then,

$$\chi_E^{-1}\left((\alpha, \infty]\right) = \mathbb{R} \in \mathcal{M}. \tag{15}$$

Hence, χ_E is measurable, so $f \cdot \chi_E$ is also measurable.

Therefore, g is measurable.

($\Leftarrow$) Suppose g is measurable. Since $g : E \cup E^c = \mathbb{R} \to [-\infty, \infty]$ is defined by

$$g(x) := \begin{cases} f(x), & x \in E \\ 0, & x \in E^c, \end{cases} \tag{16}$$

then by restricting g to E we get $g\big|_E(x) = f(x)$. By part **(a)**, $g\big|_E$ must be measurable.

Therefore, f is measurable. $\square$

(c)

Proof. We have already shown in class that sums and products of measurable functions are measurable. So if u and v are measurable, then both u^2 and v^2 are measurable, which implies that $u^2 + v^2$ is measurable.

Define

$$f(x) := u^2(x) + v^2(x). \tag{17}$$

Then we need only check that $f^{\frac{1}{2}}$ is measurable.

Let $g(x) = x^{\frac{1}{2}}$. Then $f^{\frac{1}{2}}(x) = (g \circ f)(x)$, and $f : E \to [0, \infty]$
$\implies g : [0, \infty] \to [0, \infty]$. We use the fact that the composition of measurable functions is measurable, proven in appendix A.1, to show that $f^{\frac{1}{2}}$ is measurable.

Let $\alpha \in \mathbb{R}$.

Case 1: $0 \leq \alpha \leq \infty$. Then,

$$g^{-1}\left((\alpha, \infty]\right) = (\alpha^2, \infty) \in \mathcal{M}. \tag{18}$$

Case 2: $-\infty \leq \alpha < 0$. Then,

$$g^{-1}\left((\alpha, \infty]\right) = \emptyset \in \mathcal{M}. \tag{19}$$

Hence, g is measurable, so by A.1, $f^{\frac{1}{2}}$ is measurable.

Therefore, $(u^2 + v^2)^{\frac{1}{2}}$ is measurable. $\square$

Problem 4

TODO TODO TODO

Appendices

A Appendix A

A.1

TODO TODO TODO