18.102 Assignment 6

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Problem 1

(a)

Proof. Let $\epsilon > 0$, and define $c_1 := a$ and $c_{n+1} := b$.

Since ψ is a step function on $[a, b]$, $\exists c_1 \le c_2 \le \cdots \le c_n \le c_{n+1} \in [a, b]$ such that $\forall i = 1, \cdots, n$,

$$\psi^{-1}(\{a_i\}) = (c_i, c_{i+1}], \tag{1}$$

where each a_i is one of the finitely many values that ψ takes on.

Choose $\delta > 0$ such that $\delta < \frac{\epsilon}{2n}$. Define $g : [a, b] \to \mathbb{R}$ via

$$g(x) := \begin{cases} \frac{a_i + a_{i-1}}{2} + \left(\frac{a_i - a_{i-1}}{2\delta}\right)(x - c_i), & x \in (c_i - \delta, c_i + \delta) \\ a_i, & x \in [c_i + \delta, c_{i+1} - \delta] \\ -\frac{a_n}{2\delta}(x - b), & x \in (c_{n-1} - \delta, b], \end{cases} \tag{2}$$

where $a_0 := -a_1$. Then

$$g(a) = \frac{a_1 + a_0}{2} + \left(\frac{a_1 - a_0}{2\delta}\right)(c_1 - c_1) = \frac{a_1 - a_1}{2} = 0, \tag{3}$$

and

$$g(b) = -\frac{a_n}{2\delta}(b - b) = 0, \tag{4}$$

as desired. We also see that since g is piecewise linear, it is continuous.

Now consider the difference $|\psi(x) - g(x)|$.

<u>Case 1</u>: $x \in [c_i + \delta, c_{i+1} - \delta]$. Then by (1), we have

$$|\psi(x) - g(x)| = |\psi((c_i, c_{i+1} - \delta)) - a_i| = |a_i - a_i| = 0. \tag{5}$$

Case 2: $x \in (c_i - \delta, c_i + \delta)$. Then

$$\begin{aligned} |\psi(x) - g(x)| &= \left| \frac{a_i + a_{i-1}}{2} + \left(\frac{a_i - a_{i-1}}{2\delta} \right) (x - c_i) - \psi(x) \right| && (6) \\ &< \left| \frac{a_i + a_{i-1}}{2} + \left(\frac{a_i - a_{i-1}}{2\delta} \right) \delta - \psi(x) \right| && (7) \\ &= \left| \frac{a_i + a_{i-1}}{2} + \frac{a_i + a_{i-1}}{2} - \psi(x) \right| && (8) \\ &= |a_i - \psi(x)| && (9) \\ &= 0 \text{ or } |a_{i+1} - a_i|. && (10) \end{aligned}$$

Case 3: $x \in (c_n - \delta, b)$. Then

$$\begin{aligned} |\psi(x) - g(x)| &= \left| -\frac{a_n}{2\delta}(x - b) - \psi(x) \right| && (11) \\ &< \left| -\frac{a_n}{2\delta}\delta - \psi(x) \right| && (12) \\ &= \left| -\frac{a_n}{2} - \psi(x) \right| && (13) \\ &= \frac{3a_n}{2}. && (14) \end{aligned}$$

So in all three cases, we have that either $|g(x) - \psi(x)| = 0$, or $|g(x) - \psi(x)| < \frac{3a}{2}$, or $|g(x) - \psi(x)| < |a_{i+1} - a_i|$.

Define the set E to be the collection of points in $[a, b]$ for which $|\psi(x) - g(x)| \neq 0$. Then

$$E := \bigcup_{k=1}^{n} (c_k - \delta, c_k + \delta). \tag{15}$$

By definition, $\forall x \in E^c$,

$$|\psi(x) - g(x)| = 0 < \epsilon. \tag{16}$$

Since E is a countable union of intervals, we have

$$m(E) = m\left[\bigcup_{k=1}^{n}(c_k-\delta, c_k+\delta)\right] \tag{17}$$

$$\leq \sum_{k=1}^{n} m\left[(c_k-\delta, c_k+\delta)\right] \tag{18}$$

$$= \sum_{k=1}^{n} \ell(c_k-\delta, c_k+\delta) \tag{19}$$

$$= \sum_{k=1}^{n} 2\delta \tag{20}$$

$$= 2n\delta \tag{21}$$

$$< 2n\frac{\epsilon}{2n} \tag{22}$$

$$= \epsilon, \tag{23}$$

as desired. □

(b)

Proof. Suppose E is Lebesgue measurable and $m(E) < \infty$. Then by Littlewood's first principle, $\forall\delta > 0$ $\exists$ a finite collection of open intervals $\{U_i\}_{i=1}^{n}$ such that

$$m\left(E\Delta\bigcup_{i=1}^{n} U_i\right) < \delta. \tag{24}$$

Let $\epsilon > 0$. Since φ is a simple function, then we can express it as

$$\varphi = \sum_{i=1}^{n} a_i\chi_{E_i}, \tag{25}$$

where we take U_i such that the symmetric difference

$$m(U_i \backslash E_i) + m(E_i \backslash U_i) < \frac{\epsilon}{n} \tag{26}$$

for each $i \in \{1, ..., n\}$.

Let $\psi = \sum_{i=1}^{n} a_i\chi_{U_i}$. Then $\forall x \in U \cap E = (U\Delta E)^c$,

$$|\varphi(x) - \psi(x)| = 0 < \epsilon. \tag{27}$$

We have

$$m(U\Delta E) = \sum_{i=1}^{n} m(U_i \backslash E_i) + \sum_{i=1}^{n} m(E_i \backslash U_i) \tag{28}$$

$$< n\frac{\epsilon}{n} = \epsilon, \tag{29}$$

as desired. □

Problem 2

Proof. Let $c_0 = a$, $c_n = b$, and partition $[a, b]$ into disjoint intervals as follows:

$$[a, b] = [c_0, c_1) \cup [c_1, c_2) \cup ... \cup [c_{n-1}, c_n) \cup \{b\}. \tag{30}$$

Letting $U_k = [c_{k-1}, c_k)$ for each $k \in \{1, ...n\}$, we have

$$\bigcup_{k=1}^{n} U_k = [a, b). \tag{31}$$

On each interval U_k, extract a closed set F_k such that f is continuous on F_k and such that

$$m\left(\bigcup_{k=1}^{n} F_k^c\right) < \epsilon. \tag{32}$$

Then for each $k \in \{1, ..., n\}$, $\exists F_k \subseteq U_k$ closed such that $m(E \backslash F_k) < \frac{\epsilon}{2^n}$ and such that $f\big|_{F_k}$ is continuous.

Since F_k is closed, then F_k^c is open, so the union $\bigcup_{k=1}^{n} F_k^c$ is open as well. Then $E = \bigcup_{k=1}^{n} F_k^c$ can be expressed as a union of disjoint intervals:

$$E = \bigcup_{k=1}^{n} (a_k, b_k). \tag{33}$$

Define $g : [a, b] \to \mathbb{R}$ via

$$g(x) := \begin{cases} \frac{x-a_k}{b_k-a_k} f(b_k) + f(a_k), & x \in E \\ f(x), & x \in E^c. \end{cases} \tag{34}$$

Then

$$\sup_{x\in[a,b]} |g(x)| \leq \sup_{x\in E} |g(x)| \leq |f(a_k)| + |f(b_k)|, \tag{35}$$

and

$$\sup_{x\in[a,b]} |g(x)| \leq \sup_{x\in E^c} |g(x)| \leq \sup_{x\in E^c} |f(x)| \leq B. \tag{36}$$

Also,

$$m(E) = \sum_{k=1}^{n} m(F_k^c) < \sum_{k=1}^{n} \frac{\epsilon}{2^n} = \epsilon. \tag{37}$$

Finally, $\forall x \in E^c$, $g(x) = f(x)$, so $|f(x) - g(x)| = 0 < \epsilon$.

Therefore, the function g meets the desired requirements. $\square$

Problem 3

(a)

Proof. Since f is Lebesgue integrable, then f is measurable. For each $n \in \mathbb{N}$, let $E_n = f^{-1}([-n, n])$. Then $[-n, n] = [-\infty, n] \cap [-n, \infty]$ and since f is measurable, then $f^{-1}([-\infty, n])$ and $f^{-1}([-n, \infty])$ are measurable. Thus $f^{-1}([-\infty, n] \cap [-n, \infty])$ is measurable. Hence $f^{-1}([-n, n])$ is measurable $\iff$ E_n is measurable.

Define $h_n = f\chi_{E_n}$. Then

$$\begin{align} 0 &\leq \int_a^b |f(x) - h_n(x)| \mathrm{d}x \tag{38} \\ &= \int_a^b |f(x) - f(x)\chi_{E_n}(x)| \mathrm{d}x \tag{39} \\ &= \int_a^b |f(x)||1 - \chi_{E_n}(x)| \mathrm{d}x \tag{40} \\ &= \int_a^b |f(x)| \chi_{E_n^c}(x) \mathrm{d}x \tag{41} \\ &= \int_{[a,b] \cap E_n^c} |f(x)| \mathrm{d}x, \tag{42} \end{align}$$

where we have used the simple fact that for a measurable set E, the characteristic function over the set's complement takes the form $\chi_{E^c} = 1 - \chi_E$.

We have $[a, b] \cap E_n^c = [a, b] \cap f^{-1}([-n, n])^c$. But as $n \to \infty$, $f^{-1}([-n, n]) \to [a, b]$, so $f^{-1}([-n, n])^c \to [a, b]^c$. Hence, $[a, b] \cap E_n^c \to \varnothing$.

Sending $n \to \infty$, we get

$$\begin{align} 0 &\leq \lim_{n\to\infty} \int_a^b |f(x) - h_n(x)| \mathrm{d}x \tag{43} \\ &= \lim_{n\to\infty} \int_{[a,b] \cap E_n^c} |f(x)| \mathrm{d}x \tag{44} \\ &= \int_{\varnothing} |f(x)| \mathrm{d}x \tag{45} \\ &= 0. \tag{46} \end{align}$$

Thus, $\lim_{n\to\infty} \int_a^b |f(x) - h_n(x)| \mathrm{d}x = 0$.

Let $h = \lim_{n\to\infty} h_n = f\chi_{[a,b]}$. Then h is a product of two measurable functions, so h is measurable. Also, h is the limit of a sequence of simple functions $\{h_n\}_n$ with $|h| \leq |h_n| \; \forall n \in \mathbb{N}$. So h is bounded.

Therefore $\int_a^b |f(x) - h(x)| \mathrm{d}x < \epsilon$, as desired. □

(b)

Proof. By **(a)**, we know that $\exists$ a bounded measurable function $h : [a, b] \to \mathbb{R}$ such that

$$\int_a^b |f(x) - h(x)| \mathrm{d}x < \frac{\epsilon}{3}. \tag{47}$$

Next, we approximate h by a simple function $\varphi : [a, b] \to \mathbb{R}$ such that

$$\int_a^b |h(x) - \phi(x)| \mathrm{d}x < \frac{\epsilon}{3}. \tag{48}$$

Finally, by problem **(1b)**, we can approximate φ by a step function ψ such that

$$|\varphi(x) - \psi(x)| < \frac{\epsilon}{3(b-a)}. \tag{49}$$

Integrating (49) over $[a, b]$, this is equivalent to

$$\int_a^b |\varphi(x) - \psi(x)| \mathrm{d}x < \frac{\epsilon}{3(b-a)} m([a, b]) = \frac{\epsilon}{3}. \tag{50}$$

So, by the triangle inequality we have

$$\begin{aligned}
\int_a^b |f(x) - \psi(x)| \mathrm{d}x &= \int_a^b |f(x) - h(x) + h(x) - \varphi(x) + \varphi(x) - \psi(x)| \mathrm{d}x & (51)\\
&\leq \int_a^b |f(x) - h(x)| \mathrm{d}x + \int_a^b |h(x) - \varphi(x)| \mathrm{d}x + \int_a^b |\varphi(x) - \psi(x)| \mathrm{d}x & (52)\\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} & (53)\\
&= \epsilon. & (54)
\end{aligned}$$

Thus, ψ suffices. $\square$

(c)

Proof. Let $E = \{x \in [a, b] \mid |f(x) - g(x)| \geq \frac{\epsilon}{b-a}\}$.

By what we showed in problem **2**, we know $\exists$ such a continuous function $g : [a, b] \to \mathbb{R}$ such that $g(a) = g(b) = 0$ and

$$m(E) = 0. \tag{55}$$

Then we have

$$\begin{aligned}
\int_a^b |f(x) - g(x)| \mathrm{d}x &= \int_{E \cup E^c} |f(x) - g(x)| \mathrm{d}x & (56)\\
&= \int_E |f(x) - g(x)| \mathrm{d}x + \int_{E^c} |f(x) - g(x)| \mathrm{d}x. & (57)
\end{aligned}$$

But $m(E) = 0$, so $m(E^c) = b - a$. Then

$$\begin{aligned}
\int_a^b |f(x) - g(x)| \mathrm{d}x &= \int_{E^c} |f(x) - g(x)| \mathrm{d}x && (58) \\
&< \int_{E^c} \frac{\epsilon}{b - a} \mathrm{d}x && (59) \\
&= (b - a) \frac{\epsilon}{b - a} && (60) \\
&= \epsilon. && (61)
\end{aligned}$$

Hence, g satisfies the desired constraints. □

Problem 4

Proof. First, we prove the Riemann-Lebesgue Lemma for step functions.

Let $\psi : [-\pi, \pi] \to \mathbb{C}$ be a step function. Then given a partition $-\pi = x_0 < x_1 < ... < x_m = \pi$, we may write

$$\psi = \sum_{k=1}^{m} c_k \chi_{[x_{k-1}, x_k)}. \tag{62}$$

Consider each of the integrals

$$I_1 = \int_{-\pi}^{\pi} \psi \cos(nx) \mathrm{d}x \tag{63}$$

and

$$I_2 = \int_{-\pi}^{\pi} \psi \sin(nx) \mathrm{d}x. \tag{64}$$

Computing, we get

$$\begin{aligned}
I_1 &= \sum_{k=1}^{m} c_k \int_{-\pi}^{\pi} \chi_{[x_{k-1}, x_k)} \cos(nx) \mathrm{d}x && (65) \\
&= \sum_{k=1}^{m} c_k \int_{x_{k-1}}^{x_k} \cos(nx) \mathrm{d}x && (66) \\
&= \sum_{k=1}^{m} c_k \frac{1}{n} \left[\sin(x_k) - \sin(x_{k-1})\right]. && (67)
\end{aligned}$$

Similarly, we find

$$I_2 = -\sum_{k=1}^{m} c_k \frac{1}{n} \left[\cos(x_k) - \cos(x_{k-1})\right]. \tag{68}$$

Combining these results, we have

$$\hat{\psi}(n) = \frac{1}{2\pi}(I_1 - iI_2) \tag{69}$$
$$= \frac{1}{2\pi n}\sum_{k=1}^{m} c_k \left[\sin(x_k) - \sin(x_{k-1}) + i\left(\cos(x_k) - \cos(x_{k-1})\right)\right]. \tag{70}$$

Taking the magnitude, we are left with

$$|\hat{\psi}(n)| \leq \sum_{k=1}^{m} \frac{|c_k|}{n} \left|\sin(x_k) - \sin(x_{k-1}) + i\left(\cos(x_k) - \cos(x_{k-1})\right)\right| \tag{71}$$
$$\leq 4\sum_{k=1}^{m} \frac{|c_k|}{n} \xrightarrow{n\to\infty} 0. \tag{72}$$

Thus, $\lim_{|n|\to\infty} |\hat{\psi}(n)| = 0$.

By problem **3a**, this step function ψ exists for a Lebesgue integrable function $f : [-\pi, \pi] \to \mathbb{C}$, i.e. for $\epsilon > 0$, $\exists \psi$ such that

$$\int_{-\pi}^{\pi} \left(f(x) - \psi(x)\right) e^{-inx} \mathrm{d}x < \epsilon. \tag{73}$$

So we have

$$0 \leq \left| \int_{-\pi}^{\pi} f(x)e^{-inx}\mathrm{d}x - \int_{-\pi}^{\pi} \psi(x)e^{-inx}\mathrm{d}x \right| \tag{74}$$
$$\leq \int_{-\pi}^{\pi} \left|(f(x) - \psi(x))e^{-inx}\right| \mathrm{d}x \tag{75}$$
$$< \epsilon. \tag{76}$$

Therefore, $\lim_{|n|\to\infty} |\hat{f}(n)| = 0$. $\square$