

18.102 Assignment 3

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Problem 1

(a)

Proof. We want to show that $u \in M'$. First, we show that u is linear.

Let $a, b \in M$ and let $\lambda \in \mathbb{C}$. Then

$$u(\lambda a) = \lim_{k \rightarrow \infty} (\lambda a_k) = \lambda \cdot \lim_{k \rightarrow \infty} a_k = \lambda u(a), \text{ and} \quad (1)$$

$$u(a + b) = \lim_{k \rightarrow \infty} (a_k + b_k) = \lim_{k \rightarrow \infty} a_k + \lim_{k \rightarrow \infty} b_k = u(a) + u(b). \quad (2)$$

So u is linear on M . Next, we show that u is bounded.

Let $a \in M$, i.e. $\lim_{k \rightarrow \infty} a_k$ exists. Then a is bounded, so $\exists B \geq 0$ such that $\forall k \in \mathbb{N}$, $|a_k| \leq B$. Then by continuity of the norm,

$$\|u\| \leq |u(a)| \quad (3)$$

$$= \left| \lim_{k \rightarrow \infty} a_k \right| \quad (4)$$

$$= \lim_{k \rightarrow \infty} |a_k| \quad (5)$$

$$\leq B, \quad (6)$$

so u is bounded.

Then we conclude that u is a bounded linear functional on M . \square

(b)

Proof. (By contradiction). Suppose instead that $\exists b \in \ell^1$ such that $\forall a \in \ell^\infty$,

$$v(a) = \sum_{k=1}^{\infty} a_k b_k. \quad (7)$$

Define $e_n := \{\delta_{kn}\}_k \in \ell^\infty$, for fixed $n \in \mathbb{N}$. Then $\lim_{k \rightarrow \infty} \delta_{kn} = 0$, so $e_n \in M$ as well. By equation (7), we have

$$v(e_n) = \sum_{k=1}^{\infty} \delta_{kn} b_k = b_n. \quad (8)$$

By the Hahn-Banach theorem, $v|_M = u$. But $u(e_n) = \lim_{k \rightarrow \infty} \delta_{kn} = 0$, and since $e_n \in M$, we have

$$b_n = v(e_n) = u(e_n) = 0. \quad (9)$$

This must hold for any $n \in \mathbb{N}$, so $b_n = 0 \forall n \in \mathbb{N}$. Then $b = \{b_k\}_k = (0, 0, \dots)$, so $v = 0$ by definition. But

$$0 = v(1, 1, \dots) = u(1, 1, \dots) = 1, \quad (\Rightarrow \Leftarrow) \quad (10)$$

so we arrive at a contradiction to the initial assumption.

Therefore $\nexists b \in \ell^1$ such that $\forall a \in \ell^\infty, v(a) = \sum_k a_k b_k$. \square

Problem 2

(a)

Proof. First we show that $\|T^\dagger\| \leq \|T\|$. We have

$$\|T^\dagger\| = \sup_{\|f\|=1} \|T^\dagger f\| \quad (11)$$

$$= \sup_{\|f\|=1} \|f \circ T\| \quad (12)$$

$$\leq \sup_{\|f\|=1} \|f\| \|T\| \quad (13)$$

$$\leq \|T\|. \quad (14)$$

So, $T^\dagger : W' \rightarrow V'$ is bounded, i.e. $T^\dagger \in \mathcal{B}(W', V')$.

Next we show that $\|T^\dagger\| \geq \|T\|$.

Let $x \in V$ with $\|x\| = 1$. Since W is a normed space, then by the theorem from lecture 6 (corollary to Hahn-Banach Thm.), $\exists f \in W'$ such that $\|f\| = 1$ and $f(w) = \|w\| \forall w \in W \setminus \{0\}$.

Since $T : V \rightarrow W$, then $w = Tx$ for $x \in V$, so $f(Tx) = \|Tx\|$. Then we have

$$\|T^\dagger\| = \sup_{\|f\|=1} \|T^\dagger f\| \quad (15)$$

$$= \sup_{\|f\|=1} \sup_{\|x\|=1} |(f \circ T)(x)| \quad (16)$$

$$= \sup_{\|f\|=1} \sup_{\|x\|=1} |f(Tx)| \quad (17)$$

$$\geq \sup_{\|x\|=1} \|Tx\| \quad (18)$$

$$= \|T\|, \quad (19)$$

as desired.

Thus $T^\dagger \in \mathcal{B}(W', V')$ and $\|T^\dagger\| = \|T\|$. \square

(b)

Proof. Let $a \in \ell^p$, and $b_k = Ra_k$. We want to show that $b = \{b_k\}_k \in \ell^p$. Since $a \in \ell^p$, then a is bounded, which implies that $\exists B \geq 0$ such that $\forall k \in \mathbb{N}$, $|a_k| \leq B$. Then $b_k = Ra_k := a_{k-1}$, with $b_1 = a_0 := 0$.

Thus $\forall k \in \mathbb{N}$, $|b_k| = |a_{k-1}| \leq B$, so b is also bounded. We have

$$\|b\|_\infty = \sup_k |b_k| = \sup_k |a_{k-1}|. \quad (20)$$

So, $R : \ell^p \rightarrow \ell^p$. Next we compute the operator norm of R :

$$\|R\| = \sup_{\|a\|=1} \|Ra\| \quad (21)$$

$$= \sup_{\|a\|=1} \sup_k |Ra_k| \quad (22)$$

$$= \sup_{\|a\|=1} \sup_k |a_{k-1}| \quad (23)$$

$$= \sup_{\|a\|=1} \sup_k \{0, |a_1|, |a_2|, \dots\} \quad (24)$$

$$= \sup_{\|a\|=1} \sup_k |a_k| \quad (25)$$

$$= \sup_{\|a\|=1} \|a\| \quad (26)$$

$$= 1. \quad (27)$$

Therefore, $R \in \mathcal{B}(\ell^p, \ell^p)$ with $\|R\| = 1$. \square

(c)

Suppose $1 \leq p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. From assignment 1, we identify $(\ell^p)'$ with ℓ^q via the pairing: $f \in (\ell^p)' \iff \exists b \in \ell^q$ such that $\forall a \in \ell^p$,

$$f(a) = \sum_{k=1}^{\infty} a_k b_k, \quad (28)$$

and $\|f\| = \|b\|_q$.

For example, let $b = \{b_k\}_k = e_1$ defined by $e_1 := \{\delta_{1k}\}_k = \{1, 0, 0, \dots\} \in \ell^q$. Then $\forall a \in \ell^p$,

$$(R^\dagger e_1)(a) := \sum_{k=1}^{\infty} R a_k \delta_{1k} \quad (29)$$

$$= R a_1 \quad (30)$$

$$= a_0 \quad (31)$$

$$= 0 \quad (32)$$

$$= \sum_{k=1}^{\infty} a_k \cdot 0. \quad (33)$$

Thus $R^\dagger e_1 = 0 \in \ell^q$.

Now let $a \in \ell^p$. Then

$$(R^\dagger b)(a) = \sum_{k=1}^{\infty} (R a)_k b_k \quad (34)$$

$$= \sum_{k=1}^{\infty} a_{k-1} b_k \quad (35)$$

$$= 0 \cdot b_1 + \sum_{k=2}^{\infty} a_{k-1} b_k \quad (36)$$

$$= \sum_{k=1}^{\infty} a_k b_{k+1}. \quad (37)$$

Hence, $\{(R^\dagger b)_k\}_k = \{b_{k+1}\}_k$. Therefore, where R was the right-shift operator, we can identify R^\dagger as a left-shift operator.

Problem 3

Proof. To show that $m^*(E + x) = m^*(E)$, we will show both that $m^*(E + x) \geq m^*(E)$ and $m^*(E + x) \leq m^*(E)$.

Let $\{I_n\}_{n \in \mathbb{N}}$ be a sequence of open intervals that covers E . Then $\{I_n + x\}_n$ covers $E + x$. Since interval length is invariant under translation, we have

$$m^*(E + x) \leq \sum_n \ell(I_n + x) = \sum_n \ell(I_n). \quad (38)$$

So for every such sequence of intervals $\{I_n\}_n$, we have $m^*(E + x) \leq \sum_n \ell(I_n)$. Thus,

$$m^*(E + x) \leq m^*(E). \quad (39)$$

Now let $\{I_n\}_{n \in \mathbb{N}}$ be a sequence of open intervals that covers $E + x$. Then $\{I_n - x\}_n$ covers E . Once again, by the translation-invariance of interval length, we have

$$m^*(E) \leq \sum_n \ell(I_n - x) = \sum_n \ell(I_n). \quad (40)$$

Since this is true for every such sequence of intervals $\{I_n\}$, then

$$m^*(E) \leq m^*(E + x). \quad (41)$$

From equations (39) and (41), we conclude that $m^*(E + x) = m^*(E)$.

Therefore, the outer measure m^* is translation-invariant. \square

Problem 4

(a)

Proof. Let $x \in U$. Then $a_x \leq b_x$, and both $(a_x, x] \subset U$ and $[x, b_x) \subset U$. Thus,

$$(a_x, b_x) = (a_x, x] \cup [x, b_x) \subset U, \quad (42)$$

as desired. \square