18.102 Assignment 5

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We denote by $\mathcal{M}$ the set of all Lebesgue-measurable subsets of $\mathbb{R}$.

Problem 1

TODO TODO TODO

Problem 2

TODO TODO TODO

Problem 3

(a)

Proof. ($\Rightarrow$) Suppose f is measurable.

Let $\alpha \in \mathbb{R}$. We may express the preimage of the set $(\alpha, \infty]$ under the inverse of the restriction of f to E as follows:

$$f^{-1}\big|_E ((\alpha, \infty])) = f^{-1} ((\alpha, \infty])) \cap E, \tag{1}$$

and similarly for F:

$$f^{-1}\big|_F ((\alpha, \infty])) = f^{-1} ((\alpha, \infty])) \cap F. \tag{2}$$

Since f is measurable, then $f^{-1} ((\alpha, \infty])) \in \mathcal{M}$. By assumption, E and F are also measurable. Hence, the intersections in (1) and (2) are also measurable.

Therefore, $f\big|_E$ and $f\big|_F$ are measurable.

($\Leftarrow$) Suppose $f\big|_E$ and $f\big|_F$ are measurable.

Then for ever $\alpha \in \mathbb{R}$, $f^{-1}\big|_E ((\alpha, \infty])) \in \mathcal{M}$ and $f^{-1}\big|_F ((\alpha, \infty])) \in \mathcal{M}$. Since $\mathcal{M}$ is closed under taking finite unions, then the union of each of these sets is also measurable, i.e.

$$f^{-1}\big|_E ((\alpha, \infty])) \cup f^{-1}\big|_F ((\alpha, \infty])) \in \mathcal{M}. \tag{3}$$

We also have that $E, F \in \mathcal{M}$, so $E \cup F \in \mathcal{M}$. Then we have

$$\begin{aligned} & f^{-1}\big|_E ((\alpha,\infty])) \cup f^{-1}\big|_F ((\alpha,\infty])) && (4)\\ &= \left(f^{-1}((\alpha,\infty])) \cap E\right) \cup \left(f^{-1}((\alpha,\infty])) \cap F\right) \\ &= f^{-1}((\alpha,\infty]) \cap (E \cup F) && (5)\\ &= f^{-1}((\alpha,\infty]) \in \mathcal{M}, && (6) \end{aligned}$$

where in line (6) we used the fact that $f^{-1}((\alpha,\infty]) \subset (E \cup F)$.

Therefore, as desired, f must be measurable. $\square$

(b)

Proof. ($\Rightarrow$) Suppose f is measurable.

We define the indicator function χ_E on E via

$$\chi_E(x) := \begin{cases} 1, & x \in E \\ 0, & x \in E^c. \end{cases} \tag{7}$$

Then we can express g as the product

$$g(x) = f(x) \cdot \chi_E(x). \tag{8}$$

In problem **2a**, we showed that the product of measurable functions is measurable. By assumption, f is measurable. so we need only check that χ_E is measurable.

Let $\alpha \in \mathbb{R}$.

<u>Case 1</u>: $1 \leq \alpha \leq \infty$. Then,

$$\chi_E^{-1}((\alpha,\infty]) = \emptyset \in \mathcal{M}. \tag{9}$$

<u>Case 2</u>: $0 \leq \alpha < 1$. Then,

$$\chi_E^{-1}((\alpha,\infty]) = E \in \mathcal{M}. \tag{10}$$

<u>Case 3</u>: $-\infty \leq \alpha < 0$. Then,

$$\chi_E^{-1}((\alpha,\infty]) = \mathbb{R} \in \mathcal{M}. \tag{11}$$

Hence, χ_E is measurable, so $f \cdot \chi_E$ is also measurable.

Therefore, g is measurable.

($\Leftarrow$) Suppose g is measurable. Since $g : E \cup E^c = \mathbb{R} \to [-\infty, \infty]$ is defined by

$$g(x) := \begin{cases} f(x), & x \in E \\ 0, & x \in E^c, \end{cases} \tag{12}$$

then by restricting g to E we get $g\big|_E(x) = f(x)$. By part **(a)**, $g\big|_E$ must be measurable.

Therefore, f is measurable. $\square$