18.102 Assignment 6

Octavio Vega

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Problem 1

(a)

Proof. Let $\epsilon > 0$, and define $c_1 := a$ and $c_{n+1} := b$.

Since ψ is a step function on $[a, b]$, $\exists c_1 \leq c_2 \leq \cdots \leq c_n \leq c_{n+1} \in [a, b]$ such that $\forall i = 1, \cdots, n$,

$$\psi^{-1}(\{a_i\}) = (c_i, c_{i+1}], \tag{1}$$

where each a_i is one of the finitely many values that ψ takes on.

Choose $\delta > 0$ such that $\delta < \frac{\epsilon}{2n}$. Define $g : [a, b] \to \mathbb{R}$ via

$$g(x) := \begin{cases} \frac{a_i + a_{i-1}}{2} + \left(\frac{a_i - a_{i-1}}{2\delta}\right)(x - c_i), & x \in (c_i - \delta, c_i + \delta) \\ a_i, & x \in [c_i + \delta, c_{i+1} - \delta] \\ -\frac{a_n}{2\delta}(x - b), & x \in (c_{n-1} - \delta, b], \end{cases} \tag{2}$$

where $a_0 := -a_1$. Then

$$g(a) = \frac{a_1 + a_0}{2} + \left(\frac{a_1 - a_0}{2\delta}\right)(c_1 - c_1) = \frac{a_1 - a_1}{2} = 0, \tag{3}$$

and

$$g(b) = -\frac{a_n}{2\delta}(b - b) = 0, \tag{4}$$

as desired. We also see that since g is piecewise linear, it is continuous.

Now consider the difference $|\psi(x) - g(x)|$.

<u>Case 1</u>: $x \in [c_i + \delta, c_{i+1} - \delta]$. Then by (1), we have

$$|\psi(x) - g(x)| = |\psi((c_i, c_{i+1} - \delta)) - a_i| = |a_i - a_i| = 0. \tag{5}$$

Case 2: $x \in (c_i - \delta, c_i + \delta)$. Then

$$
\begin{aligned}
|\psi(x) - g(x)| &= \left| \frac{a_i + a_{i-1}}{2} + \left(\frac{a_i - a_{i-1}}{2\delta} \right) (x - c_i) - \psi(x) \right| && (6) \\
&< \left| \frac{a_i + a_{i-1}}{2} + \left(\frac{a_i - a_{i-1}}{2\delta} \right) \delta - \psi(x) \right| && (7) \\
&= \left| \frac{a_i + a_{i-1}}{2} + \frac{a_i + a_{i-1}}{2} - \psi(x) \right| && (8) \\
&= |a_i - \psi(x)| && (9) \\
&= 0 \text{ or } |a_{i+1} - a_i|. && (10)
\end{aligned}
$$

Case 3: $x \in (c_n - \delta, b)$. Then

$$
\begin{aligned}
|\psi(x) - g(x)| &= \left| -\frac{a_n}{2\delta}(x - b) - \psi(x) \right| && (11) \\
&< \left| -\frac{a_n}{2\delta}\delta - \psi(x) \right| && (12) \\
&= \left| -\frac{a_n}{2} - \psi(x) \right| && (13) \\
&= \frac{3a_n}{2}. && (14)
\end{aligned}
$$

So in all three cases, we have that either $|g(x) - \psi(x)| = 0$, or $|g(x) - \psi(x)| < \frac{3a}{2}$, or $|g(x) - \psi(x)| < |a_{i+1} - a_i|$.

Define the set E to be the collection of points in $[a, b]$ for which $|\psi(x) - g(x)| \neq 0$. Then

$$E := \bigcup_{k=1}^{n} (c_k - \delta, c_k + \delta). \tag{15}$$

By definition, $\forall x \in E^c$,

$$|\psi(x) - g(x)| = 0 < \epsilon. \tag{16}$$

Since E is a countable union of intervals, we have

$$m(E) = m\left[\bigcup_{k=1}^{n}(c_k-\delta, c_k+\delta)\right] \tag{17}$$
$$\leq \sum_{k=1}^{n} m\left[(c_k-\delta, c_k+\delta)\right] \tag{18}$$
$$= \sum_{k=1}^{n} \ell(c_k-\delta, c_k+\delta) \tag{19}$$
$$= \sum_{k=1}^{n} 2\delta \tag{20}$$
$$= 2n\delta \tag{21}$$
$$< 2n\frac{\epsilon}{2n} \tag{22}$$
$$= \epsilon, \tag{23}$$

as desired. □

(b)

Proof. Suppose E is Lebesgue measurable and $m(E) < \infty$. Then by Littlewood's first principle, $\forall \delta > 0$ $\exists$ a finite collection of open intervals $\{U_i\}_{i=1}^{n}$ such that

$$m\left(E\Delta\bigcup_{i=1}^{n} U_i\right) < \delta. \tag{24}$$

Let $\epsilon > 0$. Since φ is a simple function, then we can express it as

$$\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}, \tag{25}$$

where we take U_i such that the symmetric difference

$$m(U_i \backslash E_i) + m(E_i \backslash U_i) < \frac{\epsilon}{n} \tag{26}$$

for each $i \in \{1, ..., n\}$.

Let $\psi = \sum_{i=1}^{n} a_i \chi_{U_i}$. Then $\forall x \in U \cap E = (U\Delta E)^c$,

$$|\varphi(x) - \psi(x)| = 0 < \epsilon. \tag{27}$$

We have

$$m(U\Delta E) = \sum_{i=1}^{n} m(U_i \backslash E_i) + \sum_{i=1}^{n} m(E_i \backslash U_i) \tag{28}$$
$$< n\frac{\epsilon}{n} = \epsilon, \tag{29}$$

as desired. □

Problem 2

Proof. Let $c_0 = a$, $c_n = b$, and partition $[a, b]$ into disjoint intervals as follows:

$$[a, b] = [c_0, c_1) \cup [c_1, c_2) \cup ... \cup [c_{n-1}, c_n) \cup \{b\}. \tag{30}$$

Letting $U_k = [c_{k-1}, c_k)$ for each $k \in \{1, ...n\}$, we have

$$\bigcup_{k=1}^{n} U_k = [a, b). \tag{31}$$

On each interval U_k, extract a closed set F_k such that f is continuous on F_k and such that

$$m\left(\bigcup_{k=1}^{n} F_k^c\right) < \epsilon. \tag{32}$$

Then for each $k \in \{1, ..., n\}$, $\exists F_k \subseteq U_k$ closed such that $m(E \backslash F_k) < \frac{\epsilon}{2^n}$ and such that $f\big|_{F_k}$ is continuous.

Since F_k is closed, then F_k^c is open, so the union $\bigcup_{k=1}^{n} F_k^c$ is open as well. Then $E = \bigcup_{k=1}^{n} F_k^c$ can be expressed as a union of disjoint intervals:

$$E = \bigcup_{k=1}^{n} (a_k, b_k). \tag{33}$$

Define $g : [a, b] \to \mathbb{R}$ via

$$g(x) := \begin{cases} \frac{x-a_k}{b_k-a_k} f(b_k) + f(a_k), & x \in E \\ f(x), & x \in E^c. \end{cases} \tag{34}$$

Then

$$\sup_{x\in[a,b]} |g(x)| \le \sup_{x\in E} |g(x)| \le |f(a_k)| + |f(b_k)|, \tag{35}$$

and

$$\sup_{x\in[a,b]} |g(x)| \le \sup_{x\in E^c} |g(x)| \le \sup_{x\in E^c} |f(x)| \le B. \tag{36}$$

Also,

$$m(E) = \sum_{k=1}^{n} m(F_k^c) < \sum_{k=1}^{n} \frac{\epsilon}{2^n} = \epsilon. \tag{37}$$

Finally, $\forall x \in E^c$, $g(x) = f(x)$, so $|f(x) - g(x)| = 0 < \epsilon$.

Therefore, the function g meets the desired requirements. □