18.102 Assignment 8

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Problem 1

(a)

Proof. ($\Rightarrow$) Suppose $w \in \overline{W}$. Then since $w \in H$ and since $\{e_n\}_n \subset H$ is a countably infinite orthonormal subset, we have

$$w = \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n. \tag{1}$$

Computing the norm gives

$$\begin{aligned}
||w||^2 &= \left\langle \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n, \sum_{k=1}^{\infty} \langle u, e_k \rangle e_k \right\rangle & (2)\\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \langle u, e_n \rangle \overline{\langle u, e_k \rangle} \langle e_n, e_k \rangle & (3)\\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \langle u, e_n \rangle \overline{\langle u, e_k \rangle} \delta_{nk} & (4)\\
&= \sum_{n=1}^{\infty} \langle u, e_n \rangle \overline{\langle u, e_n \rangle} & (5)\\
&= \sum_{n=1}^{\infty} |\langle u, e_n \rangle|^2. & (6)
\end{aligned}$$

Thus,

$$||w|| = \left(\sum_{n=1}^{\infty} |\langle u, e_n \rangle|^2 \right)^{\frac{1}{2}} < \infty, \tag{7}$$

so defining $c_n := \langle u, e_n \rangle$ yields a sequence $\{c_n\}_n \in \ell^2(\mathbb{N})$, as desired.

($\Leftarrow$) Let $\{c_n\}_{n=1}^{\infty} \in \ell^2$ such that $w = \sum_{n=1}^{\infty} c_n e_n$.

Define $w_N := \sum_{n=1}^N c_n e_n$. Then $w = \lim_{N\to\infty} w_N$, and for each $N \in N$, $w_N \in W$ since it is a finite linear combination of elements in $\{e_n\}_n$.

Thus, since $\overline{W}$ contains all the limit points of W, then $w \in \overline{W}$, as desired. $\square$

(b)

Proof. Let $w \in \overline{W}$ and $u \in H$. Then by **(a)**, we may write $w = \sum_{n=1}^\infty c_n e_n$ for $\{c_n\}_n \in \ell^2(\mathbb{N})$. Suppose $c_n = \langle u, e_n\rangle$. Then

$$
\begin{aligned}
||u - \sum_{n=1}^{\infty} \langle u, e_n\rangle e_n|| &= ||u - \sum_{n=1}^{\infty} c_n e_n|| && (8)\\
&= ||u - v||. && (9)
\end{aligned}
$$

Now suppose $c_n \neq \langle u, e_n\rangle$. Then we compute

$$
\begin{aligned}
||u - w||^2 &= \left\| u - \sum_{n=1}^{\infty} c_n e_n \right\|^2 && (10)\\
&= \left\| u - \sum_{n=1}^{\infty} \langle u, e_n\rangle e_n + \sum_{n=1}^{\infty} \langle u, e_n\rangle e_n - \sum_{n=1}^{\infty} c_n e_n \right\|^2 && (11)\\
&= \left\| u - \sum_{n=1}^{\infty} \langle u, e_n\rangle e_n + \sum_{n=1}^{\infty} (\langle u, e_n\rangle - c_n) e_n \right\|^2 && (12)\\
&\geq \left\| u - \sum_{n=1}^{\infty} \langle u, e_n\rangle e_n \right\|^2 . && (13)
\end{aligned}
$$

Therefore, $\|u - \sum_{n=1}^\infty \langle u, e_n\rangle e_n\| \leq ||u - w||$, with equality only if $w = \sum_{n=1}^\infty \langle u, e_n\rangle e_n$, and we are done. $\square$