

## 18.102 Assignment 2

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### Problem 1

(a)

*Proof.* Let  $B$  be a Banach space. Suppose  $T \in \mathcal{B}(B, B)$  and  $\|I - T\| < 1$ . Then by Geometric series,

$$\sum_{n=0}^{\infty} \|(I - T)^n\| \leq \sum_{n=0}^{\infty} \|I - T\|^n = \frac{1}{1 - \|I - T\|} < \infty. \quad (1)$$

So the series  $\sum_{n=0}^{\infty} (I - T)^n$  converges absolutely, which implies that it converges. Fix  $m \in \mathbb{N}$ . Then

$$T \sum_{n=0}^m (I - T)^n = [I - (I - T)] \sum_{n=0}^m (I - T)^n \quad (2)$$

$$= \sum_{n=0}^m (I - T)^n - \sum_{n=0}^m (I - T)^{n+1} \quad (3)$$

$$= I - (I - T)^{m+1}, \text{ by telescoping sum.} \quad (4)$$

By continuity of  $T$ ,

$$T \sum_{n=0}^{\infty} (I - T)^n = T \left( \lim_{m \rightarrow \infty} \sum_{n=0}^m (I - T)^n \right) \quad (5)$$

$$= \lim_{m \rightarrow \infty} T \sum_{n=0}^m (I - T)^n \quad (6)$$

$$= \lim_{m \rightarrow \infty} [I - (I - T)^{m+1}] \quad (7)$$

$$= I, \quad (8)$$

since  $\|I - T\| < 1$ . We can similarly show that  $\sum_{n=0}^{\infty} (I - T)^n = I$ .

Thus,  $T$  is indeed invertible, and  $\sum_{n=0}^{\infty} (I - T)^n \rightarrow T^{-1}$  in  $\mathcal{B}(B, B)$ .  $\square$

(b)

*Proof.* Let  $\mathcal{I} := \{T \in \mathcal{B}(B, B) | T^{-1} \text{ exists}\}$ . We want to show that  $\forall T \in \mathcal{I}$ ,  $\exists \delta > 0$  such that if  $\|S - T\| < \delta \implies S \in \mathcal{I}$ .

Choose  $\delta = \frac{1}{\|T^{-1}\|}$ , and write

$$S = T - (T - S) = T [I - T^{-1}(T - S)] . \quad (9)$$

If  $\|S - T\| < \delta = \frac{1}{\|T^{-1}\|}$ , then

$$\frac{1}{\|T^{-1}\|} > \|S - T\| \quad (10)$$

$$= \|T - T [I - T^{-1}(T - S)]\| \quad (11)$$

$$= \|T\| \cdot \|I - [I - T^{-1}(T - S)]\| \quad (12)$$

$$\implies \|I - [I - T^{-1}(T - S)]\| < \frac{1}{\|T^{-1}\| \cdot \|T\|} = 1 \quad (13)$$

$$\implies \|T^{-1}(T - S)\| = \|I - T^{-1}S\| < 1. \quad (14)$$

So by (a),  $T^{-1}S$  is invertible, which implies that  $S$  is invertible. Thus,  $\exists \delta > 0$  such that if  $S \in B_\delta(T)$ , then  $S \in \mathcal{I}$ .

Therefore,  $\mathcal{I}$  is open.  $\square$

## Problem 2

(a)

*Proof.* To show that  $\|v + W\|$  is a norm, we will show that it obeys positive definiteness, homogeneity, and the triangle inequality.

First, suppose that  $0 = \|v + W\| = \inf_{w \in W} \|v + w\|$ . Then since  $\|\cdot\|_V$  is a norm on  $V$ ,

$$\|w + w\| = 0 \iff v + w = 0 \implies v = -w. \quad (15)$$

So  $\exists$  a sequence  $\{w_k\}_k \subset W$  such that  $w_k \rightarrow -v$ . Since  $W$  is closed,  $-v \in W \implies v \in V$ . But then  $v + W = 0 + W$  because  $v \in W$ .

Thus,  $\|v + W\| = 0 \iff v = 0$  (definiteness).

Also,  $\|v + W\| = \inf_{w \in W} \|v + w\| \geq 0$  because  $\|\cdot\|_V$  is a norm, and  $\|v + w\| \geq 0 \forall w \in W$ .

Let  $\lambda \in \mathbb{K}$ . Then since  $\lambda W = W$ ,

$$\|\lambda(v + W)\| = \|\lambda v + W\| \quad (16)$$

$$= \inf_{w \in W} \|\lambda v + w\| \quad (17)$$

$$= \inf_{w \in W} |\lambda| \cdot \left\| v + \frac{w}{\lambda} \right\| \quad (18)$$

$$= |\lambda| \inf_{w \in W} \|v + w\| \quad (19)$$

$$= |\lambda| \cdot \|v + W\| \quad (\text{homogeneity}). \quad (20)$$

Now let  $u + W, v + W \in V/W$ . Then

$$\|(u + W) + (v + W)\| = \|u + v + W\| \quad (21)$$

$$= \inf_{w \in W} \|u + v + w\| \quad (22)$$

$$= \inf_{w \in W} \|u + v + 2w\| \quad (23)$$

$$= \inf_{w \in W} \|u + w + v + w\| \quad (24)$$

$$\leq \inf_{w \in W} (\|u + w\| + \|v + w\|) \quad (25)$$

$$\leq \inf_{w \in W} \|u + w\| + \inf_{w \in W} \|v + w\| \quad (26)$$

$$= \|u + W\| + \|v + W\| \quad (\text{triangle inequality}). \quad (27)$$

Thus,  $\|v + W\|$  is a norm on  $V/W$ .  $\square$

(b)

TODO TODO TODO TODO

### Problem 3

*Proof.* Let  $\{v_n\}_n$  be a sequence of elements in  $V$ . Suppose that the series  $\sum_n (v_n + W)$  is absolutely summable, i.e. that  $\sum_n \|v_n + W\|$  converges. Since  $\|v_n + W\| = \inf_{w \in W} \|v_n + w\|$ , then for each  $n \in \mathbb{N}$ ,  $\exists w_n \in W$  such that

$$\|v_n + w_n\| \leq \|v_n + W\| + 2^{-n} \quad (28)$$

$$\implies \sum_n \|v_n + w_n\| \leq \sum_n \|v_n + W\| + \sum_n 2^{-n} \quad (29)$$

$$= \sum_n \|v_n + W\| + 1. \quad (30)$$

Then by comparison,  $\sum_n \|v_n + w_n\|$  converges, so  $\sum_n (v_n + w_n)$  converges.

Since  $V$  is a Banach space, then, by closure,  $\exists v \in V$  such that  $v = \sum_n (v_n + w_n)$ . Then

$$\lim_{N \rightarrow \infty} v + W - \sum_{n=1}^N (v_n + W) = \sum_{n=1}^{\infty} (v_n + w_n) + W - \lim_{N \rightarrow \infty} \sum_{n=1}^N (v_n + W) \quad (31)$$

$$= \sum_{n=1}^{\infty} v_n + \sum_{n=1}^{\infty} w_n + W - \lim_{N \rightarrow \infty} \sum_{n=1}^N (v_n + W) \quad (32)$$

$$= \sum_{n=1}^{\infty} v_n + W - \lim_{N \rightarrow \infty} \sum_{n=1}^N (v_n + W) \quad (33)$$

$$= \sum_{n=1}^{\infty} v_n + W - \lim_{N \rightarrow \infty} \sum_{n=1}^N v_n - W \quad (34)$$

$$= \sum_{n=1}^{\infty} v_n - \lim_{N \rightarrow \infty} \sum_{n=1}^N v_n = 0. \quad (35)$$

So  $\sum_n (v_n + W) = v + W$ , thus  $\sum_n (v_n + W)$  converges in  $V/W$ .

Therefore  $V/W$  is a Banach space.  $\square$

## Problem 4

(a)

*Proof.* Let  $\{v_n\}_n$  be a sequence of elements in  $\ker(T)$  such that  $v_n \rightarrow v \in V$  and  $Tv_n \rightarrow w \in W$ . Then  $\forall n \in \mathbb{N}$ ,

$$\implies Tv_n = 0 \quad (36)$$

$$\implies \{Tv_n\}_n \rightarrow w = 0. \quad (37)$$

By continuity of  $T$ ,

$$0 = \lim_{n \rightarrow \infty} Tv_n = T \left( \lim_{n \rightarrow \infty} v_n \right) = Tv, \quad (38)$$

so  $v \in \ker(T)$ . Hence  $\ker(T)$  is closed.  $\square$

(b)

*Proof.* ( $\Rightarrow$ ) Suppose  $V/\ker(T)$  is isomorphic to  $\text{range}(T)$ . Then  $\exists$  isomorphism  $S : V/\ker(T) \rightarrow \text{range}(T)$ . We claim that the operator defined via  $S(v + \ker(T)) = Tv$  satisfies this.

First, we show that  $S$  is linear. Let  $v_1, v_2 \in V/\ker(T)$ . Then by linearity of  $T$ ,

$$S(v_1 + v_2 + \ker(T)) = T(v_1 + v_2) \quad (39)$$

$$= Tv_1 + Tv_2 \quad (40)$$

$$= S(v_1 + \ker(T)) + S(v_2 + \ker(T)). \quad (41)$$

Let  $\lambda \in \mathbb{K}$ . Then by linearity of  $T$  and since  $\lambda \cdot \ker(T) = \ker(T)$ ,

$$S(\lambda(v + \ker(T))) = S(\lambda v + \ker(T)) \quad (42)$$

$$= T(\lambda v) \quad (43)$$

$$= \lambda Tv \quad (44)$$

$$= \lambda S(v + \ker(T)). \quad (45)$$

Thus,  $S$  is linear.

Next, we show that  $S$  is bounded. We have

$$\|S\| = \sup_{\|v\|=1} \|S(v + \ker(T))\| \quad (46)$$

$$= \sup_{\|v\|=1} \|Tv\| \quad (47)$$

$$= \|T\|. \quad (48)$$

Thus  $S$  is bounded, since  $T \in \mathcal{B}(V, W)$ . So,  $S$  is indeed an isomorphism, which confirms that  $V/\ker(T)$  is isomorphic to  $\text{range}(T)$ .

Now we proceed to the main part of the proof, where we will show that the above implies that  $\text{range}(T)$  is closed.

Note that by problems 2 and 3, the space  $V/\ker(T)$  is a Banach space because we showed in **(a)** that  $\ker(T)$  is a proper closed subspace of  $V$ , and  $V$  is a Banach space.

Let  $\{w_j\}_j$  be a sequence in  $\text{range}(T)$  such that  $w_j \rightarrow w \in W$ . Then  $\{w_j\}_{j \in \mathbb{N}}$  is Cauchy. Since  $S^{-1}$  is a continuous linear operator, then  $\{S^{-1}(w_j)\}_j$  is also a Cauchy sequence in  $V/\ker(T)$ .

Since  $V/\ker(T)$  is a Banach space, then it is complete. So  $\exists v \in V/\ker(T)$  such that

$$S^{-1}(w_j) \rightarrow v. \quad (49)$$

By continuity,  $S(S^{-1}(w_j)) \rightarrow S(v)$ , then

$$\implies \lim_{j \rightarrow \infty} w_j = w = S(v) \quad (50)$$

$$\implies w \in \text{range}(T). \quad (51)$$

Thus,  $\text{range}(T)$  is closed in  $W$ .

( $\Leftarrow$ ) Suppose  $\text{range}(T)$  is closed. Then  $\text{range}(T) \subset W$  is a Banach space. The operator  $S : V/\ker(T) \rightarrow \text{range}(T)$  as defined before is a well-defined, bijective, bounded linear operator, i.e.  $S \in \mathcal{B}(V/\ker(T), \text{range}(T))$ . Then by the Open Mapping theorem,  $S^{-1} \in \mathcal{B}(\text{range}(T), V/\ker(T))$ .

Thus  $S$  is an isomorphism, and we are done.  $\square$

## Problem 5

(a)

*Proof.* Let  $b \in \ell^1$ ,  $\epsilon > 0$ ,  $N \in \mathbb{N}$ . Define the truncated sequence

$$a := \{b_1, b_2, b_3, \dots, b_N, 0, 0, \dots\}. \quad (52)$$

Then  $\sum_{k=1}^{\infty} k|a_k| = \sum_{k=1}^N k|b_k| < \infty$ , so  $a \in W$ . We choose  $N$  such that  $\sum_{k=1}^N |b_k| > \sum_{k=1}^{\infty} |b_k| - \epsilon$ . [Note that this is always possible since the infinite series converges, so its sequence of partial sums also converges.] Then we have

$$\|a - b\|_1 = \sum_{k=1}^{\infty} |a_k - b_k| \quad (53)$$

$$= \sum_{k=1}^N |b_k - b_k| + \sum_{k=N+1}^{\infty} |0 - b_k| \quad (54)$$

$$= \sum_{k=N+1}^{\infty} |b_k| \quad (55)$$

$$= \sum_{k=1}^{\infty} |b_k| - \sum_{k=1}^N |b_k| \quad (56)$$

$$< \epsilon. \quad (57)$$

So we have shown that for every  $\epsilon > 0$  and  $b \in \ell^1$ ,  $\exists N \in \mathbb{N}$  such that  $\|a - b\|_1 < \epsilon$ , i.e.  $B(b, \epsilon) \cap W \neq \emptyset$ . Thus  $W$  is dense in  $\ell^1$ .

Now consider the sequence  $\{b_k\}_k$  given by  $b_k = \frac{1}{k^2}$ . Then  $\sum_k b_k$  converges absolutely ( $p > 1$ ), but

$$\sum_{k=1}^{\infty} k|b_k| = \sum_{k=1}^{\infty} k \cdot \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k} \quad (58)$$

diverges by Harmonic series. So  $b \in \ell^1$  but  $b \notin W$ . Hence  $\ell^1 \neq W$ .

Thus we conclude that  $W$  is a proper, dense subset of  $\ell^1$ .  $\square$

(b)

*Proof.* Define the graph of  $T$  by

$$\Gamma(T) := \{(a, Ta) \mid a \in W\} \subset W \times \ell^1. \quad (59)$$

Let  $\{x_n\}_n$  be a sequence in  $W$ . Then  $Tx_n \in \ell^1$ , i.e.  $(x_n, Tx_n) \in \Gamma(T)$ .

Suppose  $x_n \rightarrow x$  and  $Tx_n \rightarrow y$ . By definition of  $T$ , we have  $(Tx_n)_k = k(x_n)_k$ . Then

$$\lim_{n \rightarrow \infty} (Tx_n)_k = \lim_{n \rightarrow \infty} k(x_n)_k \quad (60)$$

$$= k \lim_{n \rightarrow \infty} (x_n)_k \quad (61)$$

$$= kx_k \quad (62)$$

$$= (Tx)_k \quad (63)$$

$$= y_k. \quad (64)$$

Thus  $Tx = y \in \ell^1$ . Also,  $\lim_{n \rightarrow \infty} x_n = x = \{x_k\}_k$ . Then

$$\implies \sum_k k|x_k| = \sum_k k \lim_{n \rightarrow \infty} (x_n)_k < \infty \quad (65)$$

$$\implies x \in W. \quad (66)$$

Hence, we have shown that  $\begin{pmatrix} x_n \\ Tx_n \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} \in \Gamma(T)$ .

Thus,  $\Gamma(T)$  is closed.

Now let  $e_n = \{\delta_{kn}\}_k$  for  $n \in \mathbb{N}$ . Then

$$\|e_n\| = \sum_k |\delta_{kn}| = 1, \quad (67)$$

so  $e_n \in \ell^1$ . Furthermore,

$$\sum_k k|\delta_{kn}| = n < \infty, \quad (68)$$

so  $e_n \in W$ . However,

$$\|Te_n\| = \|\{k\delta_{kn}\}_k\| = \sum_k k|\delta_{kn}| = n. \quad (69)$$

Since we can do this for any  $n \in \mathbb{N}$ , then  $\|T\| \geq n \implies \|T\| = \infty$ .

[*Note:* It is somewhat tempting to take the result of equation (69) to suggest that since  $n \in \mathbb{N}$  is finite, then  $\|T\| < \infty$  so  $T$  must be bounded. However, this is not necessarily true because even though each  $\sum_k k|a_k|$  is finite, the supremum over all  $\|a\|_1 = 1$  of the sum may not be. For instance,  $\sup_{n \in \mathbb{N}} n = \infty$ , even though each  $n$  is finite.]

Therefore we conclude that the graph of  $T$  is closed, but  $T$  is not bounded.

□