18.102 Assignment 4

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Problem 1

Proof. Let $A \subset \mathbb{R}$ and let $E \in \mathcal{A}$. Then $f^{-1}(E)$ is measurable, so

$$m^*\left(A \cap f^{-1}(E)\right) + m^*\left(A \cap f^{-1}(E)^c\right) \leq m^*(A). \tag{1}$$

We will first show that $\mathcal{A}$ is closed under taking complements so we must show that $E^c \in \mathcal{A}$ for every $E \in \mathcal{A}$; i.e. $f^{-1}(E^c)$ is measurable.

We will use the fact that $f^{-1}(E^c) = f^{-1}(E)^c$, which is proven in appendix A.1.

Since $f^{-1}(E)$ is measurable, we have

$$\begin{aligned}
m^*(A) &\geq m^*\left(A \cap f^{-1}(E)\right) + m^*\left(A \cap f^{-1}(E)^c\right) && (2)\\
&= m^*\left[A \cap \left(f^{-1}(E)^c\right)^c\right] + m^*\left(A \cap f^{-1}(E^c)\right) && (3)\\
&= m^*\left(A \cap f^{-1}(E^c)^c\right) = m^*\left(A \cap f^{-1}(E^c)\right). && (4)
\end{aligned}$$

Hence, $f^{-1}(E^c)$ is measurable, so $E^c \in \mathcal{A}$. Thus, $\mathcal{A}$ is closed under taking complements.

Now we show that $\mathcal{A}$ is closed under taking countable unions. We use the fact that $f^{-1}\left(\bigcup_n E_n\right) = \bigcup_n f^{-1}(E_n)$, which is proven in appendix A.2.

Let $\{E_n\}_n \subset \mathcal{A}$ be a sequence of sets in $\mathcal{A}$, and let $A \subset \mathbb{R}$. Then

$$\begin{aligned}
m^*\left[A \cap f^{-1}\left(\bigcup_n E_n\right)\right] &= m^*\left[A \cap \left(\bigcup_n f^{-1}(E_n)\right)\right] && (5)\\
&= m^*\left[\bigcup_n \left(A \cap f^{-1}(E_n)\right)\right] && (6)\\
&\leq \sum_n m^*\left(A \cap f^{-1}(E_n)\right) && (7)\\
&\leq m^*\left(A \cap f^{-1}(E_n)\right), && (8)
\end{aligned}$$

by countable subadditivity and positive-definiteness of the outer measure m^*. Similarly, we have

$$m^*\left[A \cap f^{-1}\left(\bigcup_n E_n\right)^c\right] = m^*\left[A \cap \left(\bigcup_n f^{-1}(E_n)\right)^c\right] \tag{9}$$

$$= m^*\left[A \cap \left(\bigcap_n f^{-1}(E_n)^c\right)\right] \tag{10}$$

$$\leq m^*\left(A \cap f^{-1}(E_n)^c\right), \tag{11}$$

by monotonicity of m^*, because $\bigcap_n f^{-1}(E_n)^c \subseteq f^{-1}(E_n)^c$.

Finally, since $E_n \in \mathcal{A}$, then $f^{-1}(E_n)$ is Lebesgue measurable, so we have

$$m^*\left[A \cap f^{-1}\left(\bigcup_n E_n\right)\right] +$$

$$m^*\left[A \cap f^{-1}\left(\bigcup_n E_n\right)^c\right] \leq m^*\left(A \cap f^{-1}(E_n)\right) + m^*(A \cap f^{-1}(E_n)^c) \tag{12}$$

$$\leq m^*(A). \tag{13}$$

So, $f^{-1}\left(\bigcup_n E_n\right)$ is measurable, hence $\bigcup_n E_n \in \mathcal{A}$. Thus, $\mathcal{A}$ is closed under taking countable unions.

Therefore, $\mathcal{A}$ is a σ-algebra. $\square$

Problem 2

Proof. ($\Rightarrow$) Suppose E is measurable.

Let $\epsilon > 0$. Then by definition of the outer measure m^*, $\exists$ a collection of open intervals $\{I_n\}_n$ with $E \subset \bigcup_n I_n$ such that

$$\sum_n \ell(I_n) < m^*(E) + \frac{\epsilon}{2} \tag{14}$$

$$\Longrightarrow \sum_n \ell(I_n) - m^*(E) < \frac{\epsilon}{2}, \tag{15}$$

where $\ell(I_n)$ denotes the length of the interval I_n. Also, since $m^*(E) < \infty$, then

$$\sum_n m^*(I_n) = \sum_n \ell(I_n) < \infty. \tag{16}$$

By the countable subbaditivity of m^*, we have

$$m^*\left(\bigcup_n I_n\right) \leq \sum_n m^*(I_n) = \sum_n \ell(I_n) < \infty. \tag{17}$$

Then by removing E from the union $\cup_n I_n$, we get

$$
\begin{aligned}
m^*\left[\left(\bigcup_n I_n\right)\setminus E\right] &= m^*\left(\bigcup_n I_n\right) - m^*(E) && (18)\\
&\leq \sum_n \ell(I_n) - m^*(E) && (19)\\
&< \frac{\epsilon}{2}. && (20)
\end{aligned}
$$

For any $N \in \mathbb{N}$, $\bigcup_{n=1}^N I_n \subseteq \bigcup_{n=1}^\infty I_n$. Thus, $\left(\bigcup_{n=1}^N I_n\right)\setminus E \subseteq \left(\bigcup_{n=1}^\infty I_n\right)\setminus E$. Then by the monotonicity of m^*,

$$
m^*\left[\left(\bigcup_{n=1}^N I_n\right)\setminus E\right] \leq m^*\left[\left(\bigcup_n I_n\right)\setminus E\right] < \frac{\epsilon}{2}. \tag{21}
$$

Since $\sum_n m^*(I_n) < \infty$, then the series converges, so $\exists N \in \mathbb{N}$ such that $\sum_{n=N+1}^\infty m^*(I_n) < \frac{\epsilon}{2}$. Because $E \subset \bigcup_n I_n$, we have

$$
E\setminus\left(\bigcup_{n=1}^N I_n\right) \subset \left(\bigcup_{n=1}^\infty I_n\right)\setminus\left(\bigcup_{n=1}^N I_n\right) \subseteq \bigcup_{n=N+1}^\infty I_n. \tag{22}
$$

Then by monotonicity of m^*,

$$
\begin{aligned}
m^*\left[E\setminus\left(\bigcup_{n=1}^\infty I_n\right)\right] &\leq m^*\left(\bigcup_{n=N+1}^\infty I_n\right)\\
&\leq \sum_{n=N+1}^\infty m^*(I_n)\\
&< \frac{\epsilon}{2}. && (23)
\end{aligned}
$$

Combining equations (21) and (23), we have

$$
\begin{aligned}
m^*\left(E\Delta\bigcup_{n=1}^N I_n\right) &= m^*\left[E\setminus\left(\bigcup_{n=1}^N I_n\right)\right] + m^*\left[\left(\bigcup_{n=1}^N I_n\right)\setminus E\right] && (24)\\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} && (25)\\
&= \epsilon. && (26)
\end{aligned}
$$

Letting $U = \bigcup_{n=1}^N I_n$, we see that

$$
m^*(U\Delta E) < \epsilon, \tag{27}
$$

as desired.

($\Leftarrow$) Suppose that for every $\epsilon > 0$, $\exists$ a finite union of open intervals U such that $m^*(U \Delta E) < \epsilon$.

Let $A \subset \mathbb{R}$. Since U is a finite union of open intervals, then U is measurable. Thus,

$$m^*(A \cap U) + m^*(A \cap U^c) \leq m^*(A). \tag{28}$$

We can express the set $A \cap E$ as follows:

$$\begin{align} A \cap E &= A \cap [(E \cap U) \cup (E \cap U^c)] \tag{29} \\ &= (A \cap E \cap U) \cup (A \cap E \cap U^c). \tag{30} \end{align}$$

Then by exclusivity,

$$m^*(A \cap E) = m^*(A \cap E \cap U) + m^*(A \cap E \cap U^c) \tag{31}$$

Similarly, we can write

$$\begin{align} A \cap E^c &= A \cap [(E^c \cap U) \cup (E^c \cap U^c)] \tag{32} \\ &= (A \cap E^c \cap U) \cup (A \cap E^c \cap U^c). \tag{33} \end{align}$$

This gives

$$m^*(A \cap E^c) = m^*(A \cap E^c \cap U) + m^*(A \cap E^c \cap U^c). \tag{34}$$

Adding equations (31) and (34) yields

$$\begin{align} m^*(A \cap E) + m^*(A \cap E^c) &= m^*(A \cap E \cap U) + m^*(A \cap E \cap U^c) \nonumber \\ &\quad + m^*(A \cap E^c \cap U) + m^*(A \cap E^c \cap U^c) \tag{35} \\ &= m^*(A \cap E \cap U) + m^*\left[A \cap (E \backslash U)\right] \nonumber \\ &\quad + m^*\left[A \cap (U \backslash E)\right] + m^*(A \cap E^c \cap U^c). \tag{36} \end{align}$$

Since $A \cap E \cap U \subseteq A \cap U$ and $A \cap E^c \cap U^c \subseteq A \cap U^c$, and since $(U \backslash E)$ and $(E \backslash U)$ are disjoint, we have

$$\begin{align} m^*(A \cap E) + m^*(A \cap E^c) &\leq m^*(A \cap U) + m^*(A \cap U^c) \nonumber \\ &\quad + m^*\left(A \cap [(E \backslash U) \cup (U \cap E)]\right) \tag{37} \\ &= m^*(A \cap U) + m^*(A \cap U^c) + m^*(U \Delta E) \tag{38} \\ &\leq m^*(A) + \epsilon. \tag{39} \end{align}$$

Since this holds for every $\epsilon > 0$, we can take ϵ to be arbitrarily small:

$$\xrightarrow{\epsilon \to 0} m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A). \tag{40}$$

Therefore, E is measurable. $\square$

Problem 3

(a)

Proof. Let $A \subset \mathbb{R}$. We can express the intersection

$$(A + x) \cap (E + x) = (A \cap E) + x. \tag{41}$$

Similarly,

$$(A + x) \cap (E + x)^c = (A \cap E^c) + x. \tag{42}$$

In PS3.3, we showed that the outer measure m^* is translation invariant. Thus, computing the outer measures of sets (41) and (42) gives

$$m^* [(A \cap E) + x] = m^*(A \cap E), \tag{43}$$

and by the same token

$$m^* [(A \cap E^c) + x] = m^*(A \cap E^c). \tag{44}$$

Adding equations (43) and (44) results in

$$m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A), \tag{45}$$

which holds since E is assumed to be measurable.

Since $A \subset \mathbb{R}$, then $A + x \subset \mathbb{R}$ also. Then from the equations above, it follows that

$$m^* [(A + x) \cap (E + x)] + m^* [(A + x) \cap (E + x)^c] \leq m^*(A) = m^*(A + x). \tag{46}$$

Therefore, we conclude that $E + x$ is measurable. □

(b)

Proof. We proceed using the fact that for any $U \subset \mathbb{R}$ and $r > 0$, the outer measure respects scalar multiplication; i.e. that

$$m^*(rU) = r \cdot m^*(U). \tag{47}$$

We prove this statement in appendix B.1.

Let $r > 0$, and let $A \subset \mathbb{R}$. We may express the intersection of the scaled sets

$$(rA) \cap (rE) = r(A \cap E), \tag{48}$$

and similarly for the complement E^c,

$$(rA) \cap (rE^c) = r(A \cap E^c). \tag{49}$$

By equation (47), computing the measures of sets (48) and (49) gives

$$m^* [r(A \cap E)] = r \cdot m^*(A \cap E), \tag{50}$$

and similarly

$$m^*\left[r(A \cap E^c)\right] = r \cdot m^*(A \cap E^c). \tag{51}$$

Since E is assumed to be measurable, then adding equations (50) and (51) yields

$$r \cdot m^*(A \cap E) + r \cdot m^*(A \cap E^c) \leq r \cdot m^*(A). \tag{52}$$

Since $A \subset \mathbb{R}$, then $rA \subset \mathbb{R}$ also. The equations above imply that

$$m^*\left[(rA) \cap (rE)\right] + m^*\left[(rA) \cap (rE^c)\right] \leq r \cdot m^*(A) = m^*(rA). \tag{53}$$

Therefore, we conclude that rE is measurable. □

Appendices

A Appendix A

A.1

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$. If $E \subset \mathbb{R}$, then $f^{-1}(E^c) = f^{-1}(E)^c$.

Proof. Let $x \in f^{-1}(E^c)$. Then

$$\implies f(x) \in E^c \iff f(x) \notin E \tag{54}$$

$$\implies x \notin f^{-1}(E) \iff x \in f^{-1}(E)^c. \tag{55}$$

Thus, we have

$$f^{-1}(E^c) \subseteq f^{-1}(E)^c. \tag{56}$$

Now let $x \in f^{-1}(E)^c$. Then

$$\implies x \notin f^{-1}(E) \iff f(x) \notin E \tag{57}$$

$$\implies f(x) \in E^c \iff x \in f^{-1}(E^c). \tag{58}$$

So, we also have

$$f^{-1}(E)^c \subseteq f^{-1}(E^c). \tag{59}$$

Taking equations (56) and (59) together allows us to conclude that $f^{-1}(E^c) = f^{-1}(E)^c$, as desired. □

A.2

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ and let $\{E_n\}_n$ be a collection of subsets $E_n \subset \mathbb{R}$. Then

$$f^{-1}\left(\bigcup_n E_n\right) = \bigcup_n f^{-1}(E_n). \tag{60}$$

Proof. Let $x \in f^{-1}(\cup_n E_n)$. Then $f(x) \in \cup_n E_n$, so $f(x)$ is in any of E_n for $n \in \mathbb{N}$. This is equivalent to saying that $x \in f^{-1}(E_n)$ for some $n \in \mathbb{N}$, which implies that $x \in \cup_n f^{-1}(E_n)$. Hence,

$$f^{-1}\left(\bigcup_n E_n\right) \subseteq \bigcup_n f^{-1}(E_n). \tag{61}$$

Now let $x \in \cup_n f^{-1}(E_n)$. Then $x \in f^{-1}(E_n)$ for some $n \in \mathbb{N}$, which is equivalent to saying that $f(x) \in E_n$ for some n. Then $f(x) \in \cup_n E_n$, so $x \in f^{-1}(\cup_n E_n)$. Thus,

$$\bigcup_n f^{-1}(E_n) \subseteq f^{-1}\left(\bigcup_n E_n\right). \tag{62}$$

Therefore, both sets are subsets of one another, so they are equal. □

B Appendix B

B.1

Theorem: Let $U \subset \mathbb{R}$ and $r > 0$. Then $m^*(rU) = rm^*(U)$.

Proof. Let $\{I_n\}_{n\in\mathbb{N}}$ be a countable collection of open intervals such that $U \subset \bigcup_n I_n$. Then $\{rI_n\}_n$ is also a collection of open intervals, and $rU \subset \bigcup_n rI_n$. By definition, we have

$$m^*(rU) = \inf\Big\{\sum_n \ell(rI_n) \mid rU \subset \bigcup_n rI_n\Big\} \tag{63}$$

$$= \inf\Big\{r\sum_n \ell(I_n) \mid U \subset \bigcup_n I_n\Big\}, \tag{64}$$

where we used the fact that intervall length respects scaling, i.e. $\ell(rIn) = r\ell(I_n)$. Then since $r > 0$,

$$m^*(rU) = r \cdot \inf\Big\{\sum_n \ell(I_n) \mid U \subset \bigcup_n I_n\Big\} \tag{65}$$

$$= r \cdot m^*(U). \tag{66}$$

Thus, the outer measure respects scalar multiplication. □