18.102 Assignment 2

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Problem 1

(a)

Proof. Let B be a Banach space. Suppose $T \in \mathcal{B}(B, B)$ and $||I - T|| < 1$. Then by Geometric series,

$$\sum_{n=0}^{\infty} ||(I-T)^n|| \leq \sum_{n=0}^{\infty} ||I-T||^n = \frac{1}{1 - ||I-T||} < \infty. \tag{1}$$

So the series $\sum_{n=0}^{\infty}(I-T)^n$ converges absolutely, which implies that it converges. Fix $m \in \mathbb{N}$. Then

$$\begin{aligned} T\sum_{n=0}^{m}(I-T)^n &= [I - (I-T)]\sum_{n=0}^{m}(I-T)^n && (2)\\ &= \sum_{n=0}^{m}(I-T)^n - \sum_{n=0}^{m}(I-T)^{n+1} && (3)\\ &= I - (I-T)^{m+1}, \text{ by telescoping sum.} && (4) \end{aligned}$$

By continuity of T,

$$\begin{aligned} T\sum_{n=0}^{\infty}(I-T)^n &= T\left(\lim_{m\to\infty}\sum_{n=0}^{m}(I-T)^n\right) && (5)\\ &= \lim_{m\to\infty} T\sum_{n=0}^{m}(I-T)^n && (6)\\ &= \lim_{m\to\infty}\left[I - (I-T)^{m+1}\right] && (7)\\ &= I, && (8) \end{aligned}$$

since $||I - T|| < 1$. We can similarly show that $\sum_{n=0}^{\infty}(I-T)^n = I$.

Thus, T is indeed invertible, and $\sum_{n=0}^{\infty}(I-T)^n \to T^{-1}$ in $\mathcal{B}(B, B)$. □