18.102 Midterm

Octavio Vega

April 24, 2023

Problem 1

Proof. We will show that $\Lambda([a,b])$ is a proper closed subspace of $C([a,b])$, which we know is a Banach space. Let $\{f_n\}_n$ be a cauchy sequence in $\Lambda([a,b])$ such that $f_n \to f$ pointwise. Then for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $||f - f_n|| < \epsilon$. This is equivalent to

$$\sup_{x\in[a,b]} |f(x) - f_n(x)| + \sup_{x\neq y\in[a,b]} \frac{|f(x) - f_n(x) - f(y) + f_n(y)|}{|x-y|} < \epsilon. \tag{1}$$

Since both terms on the left hand side are non-negative, this implies

$$\sup_{x\neq y\in[a,b]} \frac{|f(x) - f_n(x) - f(y) + f_n(y)|}{|x-y|} < \epsilon. \tag{2}$$

Then for any $x \neq y \in [a,b]$, we have

$$|f(x) - f_n(x) - f(y) + f_n(y)| < \epsilon|x-y|, \tag{3}$$

which confirms that for each $n \geq N$, the function $f - f_n$ is Lipschitz continuous. By assumtion, f_n is Lipschitz continuous $\forall n \in \mathbb{N}$, and the sum of Lipschitz continuous functions is also Lipschitz, thus $f = f_n + (f - f_n)$ is Lipschitz continuous.

So, $\lim_{n\to\infty} f_n = f \in \Lambda([a,b])$, which proves that $\Lambda([a,b])$ is a proper closed subspace of $C([a,b])$.

Therefore, $\Lambda([a,b])$ is a Banach space. $\square$

Problem 2

Proof. First we show that $||a + c_0||_{\ell^\infty/c_0} \leq \limsup_{n\to\infty} |a_n|$.

Let $a = \{a_n\}_n \in \ell^\infty$. For each $n \in \mathbb{N}$, let $b_n = (a_1, a_2, ..., a_n, 0, 0, ...) \in c_0$. Then

$$\begin{aligned} \inf_{b \in c_0} ||a + b||_\infty &\leq \inf_n ||a - b_n||_\infty &(4)\\ &= \inf_n \sup_{m \in \mathbb{N}} |a_m - b_m| &(5)\\ &= \inf_n \sup_{m \geq n} |a_m| &(6)\\ &= \limsup_{n \to \infty} |a_n|. &(7) \end{aligned}$$

Thus,

$$||a + c_0||_{\ell^\infty / c_0} \leq \limsup_{n \to \infty} |a_n|. \tag{8}$$

Let $b = (b_1, b_2, b_3, ...) \in c_0$. Then for every $\epsilon > 0$, $\exists n \in N$ such that $\forall m \geq n$, $|b_m| < \epsilon$, so

$$\begin{aligned} ||a + b||_\infty &\geq \sup_{m \geq n} |a_m| - \epsilon &(9)\\ &\geq \limsup_{n \to \infty} |a_n| - \epsilon, &(10) \end{aligned}$$

hence $\limsup_{n \to \infty} |a_n| < ||a + c_0||_{\ell^\infty / c_0} + \epsilon$.

Therefore, $||a + c_0||_{\ell^\infty / c_0} = \limsup_{n \to \infty} |a_n|$. □

Problem 3

(a)

Proof. Since $\lim_{n \to \infty} T_n x = Tx$, then for every $\epsilon > 0$ $\exists N \in N$ such that $\forall n \geq N$,

$$||T_n x - Tx|| < \epsilon. \tag{11}$$

By linearity of T, this is equivalent to

$$||(T_n - T)x|| < \epsilon. \tag{12}$$

Choose $\epsilon = ||x||$. With a sufficiently large choice of N, we have $\forall n \geq N$ and $\forall x \in V$,

$$||(T_n - T)x|| < ||x||. \tag{13}$$

The above equation implies that the operator $T_n - T$ is continuous. Since $\{T_n\}_n$ is assumed to be a sequence in $\mathcal{B}(V, W)$, then $T_n - (T_n - T) = T$ is continuous.

Therefore, T is a bounded linear operator. □

(b)

Proof. Since V is a Banach space with respect to both norms $||\cdot||_1$ and $||\cdot||_2$, we may regard the spaces $V_1 := (V, ||\cdot||_1)$ and $V_2 := (V, ||\cdot||_2)$ as separate Banach spaces.

Consider the identity mapping $\mathbb{1} \in \mathcal{B}(V_1, V_2)$. Since $\mathbb{1}$ is a bounded linear operator, then $\exists C > 0$ such that $\forall v \in V_1$,

$$||v||_2 = ||\mathbb{1}v||_2 \leq C||v||_1, \tag{14}$$

and we are done. □

Problem 4

(a)

Proof. For each $n \in \mathbb{N}$, define the set $F_n \subset E$ via

$$F_n := \left\{ x \in E \mid |f(x)| > ||f||_\infty + \frac{1}{n} \right\}. \tag{15}$$

Then by definition of the essential supremum of f, $\forall n \in \mathbb{N}$, $m(F_n) = 0$. So for almost every $x \in E$ (i.e. $\forall x \in E \backslash F_n$), we have

$$|f(x)| \leq ||f||_\infty + \frac{1}{n}. \tag{16}$$

Now consider $\bigcup_{n\in\mathbb{N}} F_n$. Since $\forall n \in \mathbb{N}$ we have $F_{n+1} \subset F_n$, then
$(E\backslash F_n) \subset (E\backslash F_{n+1})$.

By continuity from below (proved in PS5.1b), we have that

$$\begin{aligned} m\left(E\backslash \bigcup_n F_n\right) &= m\left(\bigcap_n F_n^c\right) && (17)\\ &= \lim_{n\to\infty} m(F_n^c) && (18)\\ &= \lim_{n\to\infty} m(E\backslash F_n) && (19)\\ &= m(E) - \lim_{n\to\infty} m(F_n) && (20)\\ &= m(E). && (21) \end{aligned}$$

This is equivalent to the statement

$$m\left(\bigcup_n F_n\right) = 0. \tag{22}$$

Therefore, $|f(x)| \leq ||f||_\infty$ almost everywhere on E. □

(b)

Proof. (i) Let $c \in \mathbb{R}$. Then the function cf is measurable, and by definition,

$$\begin{aligned}
||cf||_\infty &= \inf\left\{B \geq 0 \,\middle|\, m\left(\{x \in E \mid |cf(x)| > B\}\right) = 0\right\} && (23)\\
&= \inf\left\{|c|\frac{B}{|c|} \,\middle|\, m\left(\left\{x \in E \mid |f(x)| \geq \frac{B}{|c|}\right\}\right) = 0\right\} && (24)\\
&= |c| \inf\left\{\frac{B}{|c|} \,\middle|\, m\left(\left\{x \in E \mid |f(x)| > \frac{B}{|c|}\right\}\right) = 0\right\} && (25)\\
&= |c| \cdot ||f||_\infty, && (26)
\end{aligned}$$

as desired.

(ii) By definition, we have

$$||f+g||_\infty = \inf\{C > 0 \mid m(\{x \in E \mid |f(x)+g(x)| > C\} = 0)\}. \tag{27}$$

Note that by the triangle inequality, $|f(x)+g(x)| \leq |f(x)| + |g(x)|$, so if $|f(x)+g(x)| > C \Rightarrow |f(x)| + |g(x)| > C$. This means

$$\{x \in E \mid |f(x)+g(x)| > C\} \subseteq \{x \in E \mid |f(x)| + |g(x)| > C\}. \tag{28}$$

By monotonicity of the Lebesgue measure, this gives

$$m(\{x \in E \mid |f(x)+g(x)| > C\}) \leq m(\{x \in E \mid |f(x)| + |g(x)| > C\}), \tag{29}$$

which implies that

$$||f+g||_\infty \leq \inf\{C > 0 \mid m(\{x \in E \mid |f(x)| + |g(x)| > C\}) = 0\}. \tag{30}$$

Note that for $C > 0$, if for some $x \in E$ $|f(x)| + |g(x)| > C$, then it follows that either $|f(x)| > C$ or $|g(x)| > C$. Thus we can express the set on the right side of (28) as follows:

$$\{x \in E \mid |f(x)| + |g(x)| > C\} \subseteq \{x \in E \mid |f(x)| > C\} \cup \{x \in E \mid |g(x)| > C\}. \tag{31}$$

Once again using monotonicity of $m(\cdot)$, we can conclude that

$$\begin{aligned}
||f+g||_\infty \leq \inf\{C > 0 \mid m(\{x \in E \mid |f(x)| > C\}) = 0\} +& \\
\inf\{C > 0 \mid m(\{x \in E \mid |g(x)| > C\}) = 0\}& && (32)\\
= ||f||_\infty + ||g||_\infty.& && (33)
\end{aligned}$$

Therefore the essential supremum satisfies both homogeneity and the triangle inequality. $\square$

Problem 5

(a)

Proof. Using the standard representation of the simple function $\varphi \in L^+(E)$, we have

$$\int_F \varphi = \int_F \sum_{i=1}^n a_i \chi_{A_i} \tag{34}$$

$$= \sum_{i=1}^n a_i \int_F \chi_{A_i} \tag{35}$$

$$= \sum_{i=1}^n a_i \int_E \chi_{A_i \cup F}. \tag{36}$$

Note that the characteristic function $\chi_{A_i \cap F}$ takes value 1 when its argument is in both A_i and F. Equivalently, $\chi_{A_i \cap F}$ is zero when its argument is in either of the complements A_i^c or F^c, or in both. Following this logic, we may express $\chi_{A_i \cap F}$ as a product of two other characteristic functions:

$$\chi_{A_i \cap F} = \chi_{A_i} \chi_F. \tag{37}$$

Inserting this into (36) and using linearity of the Lebesgue integral, we have

$$\int_F \varphi = \sum_{i=1}^n a_i \int_E \chi_{A_i} \chi_F \tag{38}$$

$$= \int_E \left(\sum_{i=1}^n a_i \chi_{A_i} \right) \chi_F \tag{39}$$

$$= \int_E \varphi \chi_F, \tag{40}$$

as desired. $\square$

(b)

Proof. By the theorem proven in **(a)**, since the set $F \cup G \subset E$ is measurable, then

$$\int_{F \cup G} \varphi = \int_E \varphi \chi_{F \cup G}. \tag{41}$$

Since $F \cap G = \varnothing$, the characteristic function $\chi_{F \cup G}$ can only be 1 when its argument is in either of the sets F or G, where it cannot be in both simultaneously. By this logic, the characteristic function can be expressed as a sum:

$$\chi_{F \cup G} = \chi_F + \chi_G. \tag{42}$$

Inserting this into (41) and invoking the theorem in **(a)** once again, we find

$$\int_{F \cup G} \varphi = \int_E \varphi \left(\chi_F + \chi_G\right) \tag{43}$$

$$= \int_E \varphi \chi_E + \int_E \varphi \chi_G \tag{44}$$

$$= \int_F \varphi + \int_G \varphi, \tag{45}$$

which is the desired result. □