18.102 Assignment 5

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We denote by $\mathcal{M}$ the set of all Lebesgue-measurable subsets of $\mathbb{R}$.

Problem 1

(a)

Proof. Since $E, M \in \mathcal{M}$, then $E \cup M \in \mathcal{M}$ and $E \cap M \in \mathcal{M}$. We can express the union as

$$E \cup F = (E \cap F) \cup (E \backslash F) \cup (F \backslash E). \tag{1}$$

Then since E and $F \backslash E$ are disjoint, we have

$$\begin{aligned} m(E \cup F) + m(E \cap F) &= m\left[(E \cap F) \cup (E \backslash F)\right] + m(F \backslash E) + m(E \cap F) && (2) \\ &= m\left[(E \cap F) \cup (E \backslash F)\right] + m\left[(F \backslash E) \cup (E \cap F)\right] && (3) \\ &= m(E) + m(F), && (4) \end{aligned}$$

as desired. □

Problem 2

(a)

Proof. Let $a \in \mathbb{R}$.

We can write

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]. \tag{5}$$

We showed in lecture 9 that linear combinations of measurable functions are measurable, so we need only show that f^2 and g^2 are measurable.

<u>Case 1</u>: $\alpha < 0$. Then,

$$(f^2)^{-1}\left((\alpha, \infty]\right) = (f^2)^{-1}\left([0, \infty]\right) = E \in \mathcal{M}. \tag{6}$$

Case 2: $\alpha \geq 0$. Then $\forall x \in E$,

$$f^2(x) > \alpha \iff f(x) > \sqrt{\alpha} \text{ or } f(x) < -\sqrt{a} \tag{7}$$
$$\implies (f^2)^{-1}((\alpha,\infty]) = [-\infty, -\sqrt{\alpha}) \cup (\sqrt{\alpha}, \infty] \in \mathcal{M}. \tag{8}$$

So f^2 is measurable, and by the same reasoning g^2 is measurable.

Therefore, fg is measurable. □

(b)

Proof. Let $\alpha \in \mathbb{R}$.

Case 1: $\alpha = +\infty$. Then,

$$h^{-1}(\{\infty\}) = \{a\} = \bigcap_{n=1}^{\infty} \left[a, a + \frac{1}{n}\right] \in \mathcal{M}. \tag{9}$$

Case 2: $\alpha \neq \infty$. We have

$$h^{-1}((\alpha,\infty]) = (f+g)^{-1}((\alpha,\infty]). \tag{10}$$

Then $x \in (f+g)^{-1}((\alpha,\infty]) \iff f(x) + g(x) > \alpha$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists r \in \mathbb{Q}$ such that $f(x) > r > \alpha - g(x)$. Then since f and g are measurable, we have

$$(f+g)^{-1}((\alpha,\infty]) = \bigcup_{r \in \mathbb{Q}} \left[f^{-1}((r,\infty]) \cap g^{-1}((\alpha - r, \infty])\right] \in \mathcal{M}. \tag{11}$$

Therefore, h is measurable. □

Problem 3

(a)

Proof. ($\Rightarrow$) Suppose f is measurable.

Let $\alpha \in \mathbb{R}$. We may express the preimage of the set $(\alpha,\infty]$ under the inverse of the restriction of f to E as follows:

$$f^{-1}\big|_E((\alpha,\infty])) = f^{-1}((\alpha,\infty])) \cap E, \tag{12}$$

and similarly for F:

$$f^{-1}\big|_F((\alpha,\infty])) = f^{-1}((\alpha,\infty])) \cap F. \tag{13}$$

Since f is measurable, then $f^{-1}((\alpha,\infty])) \in \mathcal{M}$. By assumption, E and F are also measurable. Hence, the intersections in (12) and (13) are also measurable.

Therefore, $f\big|_E$ and $f\big|_F$ are measurable.

($\Leftarrow$) Suppose $f\big|_E$ and $f\big|_F$ are measurable.

Then for ever $\alpha \in \mathbb{R}$, $f^{-1}\big|_E((\alpha,\infty])) \in \mathcal{M}$ and $f^{-1}\big|_F((\alpha,\infty])) \in \mathcal{M}$. Since $\mathcal{M}$ is closed under taking finite unions, then the union of each of these sets is also measurable, i.e.

$$f^{-1}\big|_E((\alpha,\infty])) \cup f^{-1}\big|_F((\alpha,\infty])) \in \mathcal{M}. \tag{14}$$

We also have that $E, F \in \mathcal{M}$, so $E \cup F \in \mathcal{M}$. Then we have

$$\begin{aligned} f^{-1}\big|_E((\alpha,\infty])) \cup f^{-1}\big|_F((\alpha,\infty])) \quad &(15)\\ = \left(f^{-1}((\alpha,\infty])) \cap E\right) \cup \left(f^{-1}((\alpha,\infty])) \cap F\right) &\\ = f^{-1}((\alpha,\infty]) \cap (E \cup F) \quad &(16)\\ = f^{-1}((\alpha,\infty]) \in \mathcal{M}, \quad &(17) \end{aligned}$$

where in line (17) we used the fact that $f^{-1}((\alpha,\infty]) \subset (E \cup F)$.

Therefore, as desired, f must be measurable. □

(b)

Proof. ($\Rightarrow$) Suppose f is measurable.

We define the indicator function χ_E on E via

$$\chi_E(x) := \begin{cases} 1, & x \in E \\ 0, & x \in E^c. \end{cases} \tag{18}$$

Then we can express g as the product

$$g(x) = f(x) \cdot \chi_E(x). \tag{19}$$

In problem **2a**, we showed that the product of measurable functions is measurable. By assumption, f is measurable. so we need only check that χ_E is measurable.

Let $\alpha \in \mathbb{R}$.

<u>Case 1</u>: $1 \le \alpha \le \infty$. Then,

$$\chi_E^{-1}((\alpha,\infty]) = \emptyset \in \mathcal{M}. \tag{20}$$

<u>Case 2</u>: $0 \le \alpha < 1$. Then,

$$\chi_E^{-1}((\alpha,\infty]) = E \in \mathcal{M}. \tag{21}$$

<u>Case 3</u>: $-\infty \le \alpha < 0$. Then,

$$\chi_E^{-1}((\alpha,\infty]) = \mathbb{R} \in \mathcal{M}. \tag{22}$$

Hence, χ_E is measurable, so $f \cdot \chi_E$ is also measurable.

Therefore, g is measurable.

($\Leftarrow$) Suppose g is measurable. Since $g : E \cup E^c = \mathbb{R} \to [-\infty, \infty]$ is defined by

$$g(x) := \begin{cases} f(x), & x \in E \\ 0, & x \in E^c, \end{cases} \tag{23}$$

then by restricting g to E we get $g\big|_E(x) = f(x)$. By part **(a)**, $g\big|_E$ must be measurable.

Therefore, f is measurable. □

(c)

Proof. We have already shown in class that sums and products of measurable functions are measurable. So if u and v are measurable, then both u^2 and v^2 are measurable, which implies that $u^2 + v^2$ is measurable.

Define

$$f(x) := u^2(x) + v^2(x). \tag{24}$$

Then we need only check that $f^{\frac{1}{2}}$ is measurable.

Let $g(x) = x^{\frac{1}{2}}$. Then $f^{\frac{1}{2}}(x) = (g \circ f)(x)$, and $f : E \to [0, \infty]$
$\implies g : [0, \infty] \to [0, \infty]$. We use the fact that the composition of measurable functions is measurable, proven in appendix A.1, to show that $f^{\frac{1}{2}}$ is measurable.

Let $\alpha \in \mathbb{R}$.

Case 1: $0 \leq \alpha \leq \infty$. Then,

$$g^{-1}\left((\alpha, \infty]\right) = (\alpha^2, \infty) \in \mathcal{M}. \tag{25}$$

Case 2: $-\infty \leq \alpha < 0$. Then,

$$g^{-1}\left((\alpha, \infty]\right) = \emptyset \in \mathcal{M}. \tag{26}$$

Hence, g is measurable, so by A.1, $f^{\frac{1}{2}}$ is measurable.

Therefore, $(u^2 + v^2)^{\frac{1}{2}}$ is measurable. □

Problem 4

TODO TODO TODO

Appendices

A Appendix A

A.1

TODO TODO TODO