18.102 Assignment 4

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March 1, 2023

Problem 1

Proof. Let $A \subset \mathbb{R}$ and let $E \in \mathcal{A}$. Then $f^{-1}(E)$ is measurable, so

$$m^*\left(A \cap f^{-1}(E)\right) + m^*\left(A \cap f^{-1}(E)^c\right) \le m^*(A). \tag{1}$$

We will first show that $\mathcal{A}$ is closed under taking complements so we must show that $E^c \in \mathcal{A}$ for every $E \in \mathcal{A}$; i.e. $f^{-1}(E^c)$ is measurable.

We will use the fact that $f^{-1}(E^c) = f^{-1}(E)^c$, which is proven in appendix A.

Since $f^{-1}(E)$ is measurable, we have

$$\begin{aligned} m^*(A) &\ge m^*\left(A \cap f^{-1}(E)\right) + m^*\left(A \cap f^{-1}(E)^c\right) &(2)\\ &= m^*\left[A \cap \left(f^{-1}(E)^c\right)^c\right] + m^*\left(A \cap f^{-1}(E^c)\right) &(3)\\ &= m^*\left(A \cap f^{-1}(E^c)^c\right) = m^*\left(A \cap f^{-1}(E^c)\right). &(4) \end{aligned}$$

Hence, $f^{-1}(E^c)$ is measurable, so $E^c \in \mathcal{A}$. Thus, $\mathcal{A}$ is closed under taking complements.

Now we show that $\mathcal{A}$ is closed under taking countable unions. We use the fact that $f^{-1}\left(\cup_n E_n\right) = \cup_n f^{-1}(E_n)$, which is proven in appendix B.

Let $\{E_n\}_n \subset \mathcal{A}$ be a sequence of sets in $\mathcal{A}$, and let $A \subset \mathbb{R}$. Then

$$\begin{aligned} m^*\left[A \cap f^{-1}\left(\bigcup_n E_n\right)\right] &= m^*\left[A \cap \left(\bigcup_n f^{-1}(E_n)\right)\right] &(5)\\ &= m^*\left[\bigcup_n \left(A \cap f^{-1}(E_n)\right)\right] &(6)\\ &\le \sum_n m^*\left(A \cap f^{-1}(E_n)\right) &(7)\\ &\le m^*\left(A \cap f^{-1}(E_n)\right), &(8) \end{aligned}$$

by countable subadditivity and positive-definiteness of the outer measure m^*. Similarly, we have

$$
\begin{aligned}
m^*\left[A \cap f^{-1}\left(\bigcup_n E_n\right)^c\right] &= m^*\left[A \cap \left(\bigcup_n f^{-1}(E_n)\right)^c\right] && (9)\\
&= m^*\left[A \cap \left(\bigcap_n f^{-1}(E_n)^c\right)\right] && (10)\\
&\leq m^*\left(A \cap f^{-1}(E_n)^c\right), && (11)
\end{aligned}
$$

by monotonicity of m^*, because $\bigcap_n f^{-1}(E_n)^c \subseteq f^{-1}(E_n)^c$.

Finally, since $E_n \in \mathcal{A}$, then $f^{-1}(E_n)$ is Lebesgue measurable, so we have

$$
\begin{aligned}
& m^*\left[A \cap f^{-1}\left(\bigcup_n E_n\right)\right] + && (12)\\
& m^*\left[A \cap f^{-1}\left(\bigcup_n E_n\right)^c\right] \leq m^*\left(A \cap f^{-1}(E_n)\right) + m^*(A \cap f^{-1}(E_n)^c) && (13)\\
& \leq m^*(A). && (14)
\end{aligned}
$$

So, $f^{-1}\left(\bigcup_n E_n\right)$ is measurable, hence $\bigcup_n E_n \in \mathcal{A}$. Thus, $\mathcal{A}$ is closed under taking countable unions.

Therefore, $\mathcal{A}$ is a σ-algebra. $\square$

Problem 2

TODO TODO TODO

Appendices

A Appendix A

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$. If $E \subset \mathbb{R}$, then $f^{-1}(E^c) = f^{-1}(E)^c$.

Proof. Let $x \in f^{-1}(E^c)$. Then

$$
\begin{aligned}
&\implies f(x) \in E^c \iff f(x) \notin E && (15)\\
&\implies x \notin f^{-1}(E) \iff x \in f^{-1}(E)^c. && (16)
\end{aligned}
$$

Thus, we have

$$
f^{-1}(E^c) \subseteq f^{-1}(E)^c. \qquad (17)
$$

Now let $x \in f^{-1}(E)^c$. Then

$$\implies x \notin f^{-1}(E) \iff f(x) \notin E \tag{18}$$
$$\implies f(x) \in E^c \iff x \in f^{-1}(E^c). \tag{19}$$

So, we also have

$$f^{-1}(E)^c \subseteq f^{-1}(E^c). \tag{20}$$

Taking equations (17) and (20) together allows us to conclude that
$f^{-1}(E^c) = f^{-1}(E)^c$, as desired. $\square$

B Appendix B

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ and let $\{E_n\}_n$ be a collection of subsets $E_n \subset \mathbb{R}$. Then

$$f^{-1}\left(\bigcup_n E_n\right) = \bigcup_n f^{-1}(E_n). \tag{21}$$

Proof. Let $x \in f^{-1}(\cup_n E_n)$. Then $f(x) \in \cup_n E_n$, so $f(x)$ is in any of E_n for $n \in \mathbb{N}$. This is equivalent to saying that $x \in f^{-1}(E_n)$ for some $n \in \mathbb{N}$, which implies that $x \in \cup_n f^{-1}(E_n)$. Hence,

$$f^{-1}\left(\bigcup_n E_n\right) \subseteq \bigcup_n f^{-1}(E_n). \tag{22}$$

Now let $x \in \cup_n f^{-1}(E_n)$. Then $x \in f^{-1}(E_n)$ for some $n \in \mathbb{N}$, which is equivalent to saying that $f(x) \in E_n$ for some n. Then $f(x) \in \cup_n E_n$, so $x \in f^{-1}(\cup_n E_n)$. Thus,

$$\bigcup_n f^{-1}(E_n) \subseteq f^{-1}\left(\bigcup_n E_n\right). \tag{23}$$

Therefore, both sets are subsets of one another, so they are equal. $\square$