

18.102 Assignment 2

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Problem 1

(a)

Proof. Let B be a Banach space. Suppose $T \in \mathcal{B}(B, B)$ and $\|I - T\| < 1$. Then by Geometric series,

$$\sum_{n=0}^{\infty} \|(I - T)^n\| \leq \sum_{n=0}^{\infty} \|I - T\|^n = \frac{1}{1 - \|I - T\|} < \infty. \quad (1)$$

So the series $\sum_{n=0}^{\infty} (I - T)^n$ converges absolutely, which implies that it converges. Fix $m \in \mathbb{N}$. Then

$$T \sum_{n=0}^m (I - T)^n = [I - (I - T)] \sum_{n=0}^m (I - T)^n \quad (2)$$

$$= \sum_{n=0}^m (I - T)^n - \sum_{n=0}^m (I - T)^{n+1} \quad (3)$$

$$= I - (I - T)^{m+1}, \text{ by telescoping sum.} \quad (4)$$

By continuity of T ,

$$T \sum_{n=0}^{\infty} (I - T)^n = T \left(\lim_{m \rightarrow \infty} \sum_{n=0}^m (I - T)^n \right) \quad (5)$$

$$= \lim_{m \rightarrow \infty} T \sum_{n=0}^m (I - T)^n \quad (6)$$

$$= \lim_{m \rightarrow \infty} [I - (I - T)^{m+1}] \quad (7)$$

$$= I, \quad (8)$$

since $\|I - T\| < 1$. We can similarly show that $\sum_{n=0}^{\infty} (I - T)^n = I$.

Thus, T is indeed invertible, and $\sum_{n=0}^{\infty} (I - T)^n \rightarrow T^{-1}$ in $\mathcal{B}(B, B)$. \square

(b)

Proof. Let $\mathcal{I} := \{T \in \mathcal{B}(B, B) | T^{-1} \text{ exists}\}$. We want to show that $\forall T \in \mathcal{I}$, $\exists \delta > 0$ such that if $\|S - T\| < \delta \implies S \in \mathcal{I}$.

Choose $\delta = \frac{1}{\|T^{-1}\|}$, and write

$$S = T - (T - S) = T [I - T^{-1}(T - S)] . \quad (9)$$

If $\|S - T\| < \delta = \frac{1}{\|T^{-1}\|}$, then

$$\frac{1}{\|T^{-1}\|} > \|S - T\| \quad (10)$$

$$= \|T - T [I - T^{-1}(T - S)]\| \quad (11)$$

$$= \|T\| \cdot \|I - [I - T^{-1}(T - S)]\| \quad (12)$$

$$\implies \|I - [I - T^{-1}(T - S)]\| < \frac{1}{\|T^{-1}\| \cdot \|T\|} = 1 \quad (13)$$

$$\implies \|T^{-1}(T - S)\| = \|I - T^{-1}S\| < 1. \quad (14)$$

So by (a), $T^{-1}S$ is invertible, which implies that S is invertible. Thus, $\exists \delta > 0$ such that if $S \in B_\delta(T)$, then $S \in \mathcal{I}$.

Therefore, \mathcal{I} is open. \square

Problem 2

(a)

Proof. To show that $\|v + W\|$ is a norm, we will show that it obeys positive definiteness, homogeneity, and the triangle inequality.

First, suppose that $0 = \|v + W\| = \inf_{w \in W} \|v + w\|$. Then since $\|\cdot\|_V$ is a norm on V ,

$$\|w + w\| = 0 \iff v + w = 0 \implies v = -w. \quad (15)$$

So \exists a sequence $\{w_k\}_k \subset W$ such that $w_k \rightarrow -v$. Since W is closed, $-v \in W \implies v \in V$. But then $v + W = 0 + W$ because $v \in W$.

Thus, $\|v + W\| = 0 \iff v = 0$ (definiteness).

Also, $\|v + W\| = \inf_{w \in W} \|v + w\| \geq 0$ because $\|\cdot\|_V$ is a norm, and $\|v + w\| \geq 0 \forall w \in W$.

Let $\lambda \in \mathbb{K}$. Then since $\lambda W = W$,

$$\|\lambda(v + W)\| = \|\lambda v + W\| \quad (16)$$

$$= \inf_{w \in W} \|\lambda v + w\| \quad (17)$$

$$= \inf_{w \in W} |\lambda| \cdot \left\| v + \frac{w}{\lambda} \right\| \quad (18)$$

$$= |\lambda| \inf_{w \in W} \|v + w\| \quad (19)$$

$$= |\lambda| \cdot \|v + W\| \quad (\text{homogeneity}). \quad (20)$$

Now let $u + W, v + W \in V/W$. Then

$$\|(u + W) + (v + W)\| = \|u + v + W\| \quad (21)$$

$$= \inf_{w \in W} \|u + v + w\| \quad (22)$$

$$= \inf_{w \in W} \|u + v + 2w\| \quad (23)$$

$$= \inf_{w \in W} \|u + w + v + w\| \quad (24)$$

$$\leq \inf_{w \in W} (\|u + w\| + \|v + w\|) \quad (25)$$

$$\leq \inf_{w \in W} \|u + w\| + \inf_{w \in W} \|v + w\| \quad (26)$$

$$= \|u + W\| + \|v + W\| \quad (\text{triangle inequality}). \quad (27)$$

Thus, $\|v + W\|$ is a norm on V/W . \square

(b)

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Problem 3

Proof. Let $\{v_n\}_n$ be a sequence of elements in V . Suppose that the series $\sum_n (v_n + W)$ is absolutely summable, i.e. that $\sum_n \|v_n + W\|$ converges. Since $\|v_n + W\| = \inf_{w \in W} \|v_n + w\|$, then for each $n \in \mathbb{N}$, $\exists w_n \in W$ such that

$$\|v_n + w_n\| \leq \|v_n + W\| + 2^{-n} \quad (28)$$

$$\implies \sum_n \|v_n + w_n\| \leq \sum_n \|v_n + W\| + \sum_n 2^{-n} \quad (29)$$

$$= \sum_n \|v_n + W\| + 1. \quad (30)$$

Then by comparison, $\sum_n \|v_n + w_n\|$ converges, so $\sum_n (v_n + w_n)$ converges.

Since V is a Banach space, then, by closure, $\exists v \in V$ such that $v = \sum_n (v_n + w_n)$. Then

$$\lim_{N \rightarrow \infty} v + W - \sum_{n=1}^N (v_n + W) = \sum_{n=1}^{\infty} (v_n + w_n) + W - \lim_{N \rightarrow \infty} \sum_{n=1}^N (v_n + W) \quad (31)$$

$$= \sum_{n=1}^{\infty} v_n + \sum_{n=1}^{\infty} w_n + W - \lim_{N \rightarrow \infty} \sum_{n=1}^N (v_n + W) \quad (32)$$

$$= \sum_{n=1}^{\infty} v_n + W - \lim_{N \rightarrow \infty} \sum_{n=1}^N (v_n + W) \quad (33)$$

$$= \sum_{n=1}^{\infty} v_n + W - \lim_{N \rightarrow \infty} \sum_{n=1}^N v_n - W \quad (34)$$

$$= \sum_{n=1}^{\infty} v_n - \lim_{N \rightarrow \infty} \sum_{n=1}^N v_n = 0. \quad (35)$$

So $\sum_n (v_n + W) = v + W$, thus $\sum_n (v_n + W)$ converges in V/W .

Therefore V/W is a Banach space. \square

Problem 4

(a)

Proof. Let $\{v_n\}_n$ be a sequence of elements in $\ker(T)$ such that $v_n \rightarrow v \in V$ and $Tv_n \rightarrow w \in W$. Then $\forall n \in \mathbb{N}$,

$$\implies Tv_n = 0 \quad (36)$$

$$\implies \{Tv_n\}_n \rightarrow w = 0. \quad (37)$$

By continuity of T ,

$$0 = \lim_{n \rightarrow \infty} Tv_n = T \left(\lim_{n \rightarrow \infty} v_n \right) = Tv, \quad (38)$$

so $v \in \ker(T)$. Hence $\ker(T)$ is closed. \square