18.100A Final

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June 21, 2023

Problem 1

We complete the following **negations**:

(i)

Let $S \subset \mathbb{R}$. A function $f : S \to \mathbb{R}$ is **not continuous** at $c \in S$ if $\exists \epsilon_0 > 0$ such that $\forall \delta > 0$ if $|x - c| < \delta$, $|f(x) - f(c)| \geq \epsilon_0$.

(ii)

Let $S \subset \mathbb{R}$. A function $f : S \to \mathbb{R}$ is **not uniformly continuous** on S if $\exists x_0 \in S$ such that $\forall \delta > 0\ \exists \epsilon_0 > 0$ such that if $|x_0 - x| < \delta$, then $|f(x) - f(x_0)| \geq \epsilon_0$.

(iii)

Let $S \subset \mathbb{R}$. A sequence of functions $f_n : S \to \mathbb{R}$ **does not converge uniformly** to $f : S \to \mathbb{R}$ if $\exists \epsilon_0 > 0$ such that $\forall M \in \mathbb{N}\ \exists n \geq M$ and $x \in S$ such that $|f_n(x) - f(x)| \geq \epsilon_0$.

Problem 2

(a)

(i)

A continunous function on $(0, 1)$ with neither a global minimum or maximum:

Let $f(x) = 1\ \forall x \in (0, 1)$. Then $\forall x, y \in (0, 1)$, $f(x) = f(y)$ so f has no absolute maximum or minimum, and f is constant and therefore continuous.

(ii)

A function on $[0, 1]$ with absolute minimum at 0, absolute maximum at 1, and such that $\exists y \in (f(0), f(1))$ not in the range of f:

Define f via

$$f(x) := \begin{cases} x, & x \in \left(0, \frac{1}{2}\right) \cap \left(\frac{1}{2}, 1\right] \\ -1, & x = 0 \\ -\frac{1}{2}, & x = \frac{1}{2}. \end{cases} \tag{1}$$

Then, for example, $-\frac{3}{4} \in (-1, 1) = (f(0), f(1))$, but $-\frac{3}{4}$ is not in the range of f. Also, f has an absolute minimum and maximum at 0 and 1, respectively.

(b)

Proof. Let $\epsilon > 0$. Since f is continuous, then $\exists \delta_0 > 0$ such that if $|x - c| < \delta_0$ then $|f(x) - f(c)| < \frac{\epsilon}{2}$. Similarly, since g is continuous, then $\exists \delta_1 > 0$ such that if $|x - c| < \delta_1$ then $|g(x) - g(c)| < \frac{\epsilon}{2}$.

Choose δ_0, δ_1 such that $|f(x)| + |g(c)| < 2$, and let $\delta = \min\{\delta_0, \delta_1\}$. Then if $|x - c| < \delta$, we have

$$\begin{aligned} |f(x)g(x) - f(c)g(c)| &= |f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)| && (2) \\ &\leq |f(x)||g(x) - g(c)| + |g(c)||f(x) - f(c) && (3) \\ &< |f(x)|\frac{\epsilon}{2} + |g(c)|\frac{\epsilon}{2} && (4) \\ &< \epsilon. && (5) \end{aligned}$$

Therefore, the product fg is continuous at c. $\square$

Problem 3

(a)

Proof. Let $\{x_n\}_n$ be a sequence of elements in $[a, b]$. Choose $B = \max\{|a|, |b|\}$. Then $\forall n \in \mathbb{N}$, $|x_n| \leq B$, so the sequence $\{x_n\}_n$ is bounded.

By the Bolzano-Weierstrass theorem, $\exists$ a subsequence $\{x_{n_k}\}_k \subset [a, b]$ that converges, i.e. such that $x_{n_k} \to x$ as $k \to \infty$ for some $x \in \mathbb{R}$. Since $[a, b]$ is closed, then $x \in [a, b]$.

Therefore $[a, b]$ is compact. $\square$

(b)

Proof. Let $\{x_n\}_n$ be a sequence of elements in $[a, b]$. Since $[a, b]$ is compact, $\exists \{x_{n_k}\}_k \subset [a, b]$ such that $x_{n_k} \to x$ for some $x \in [a, b]$.

Then $\{f(x_{n_k})\}_k$ is a subsequence in $[a, b]$, and since f is continuous we have

$$\begin{aligned} \lim_{k \to \infty} f(x_{n_k}) &= f\left(\lim_{k \to \infty} x_{n_k}\right) && (6) \\ &= f(x) \in f([a, b]), && (7) \end{aligned}$$

so the subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ converges in $[a, b]$.

Therefore $f([a, b])$ is compact. $\square$