18.100A Assignment 4

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Problem 1

(a)

Proof. Define the complement of $[a, b]$ via

$$[a, b]^c := \{x \in \mathbb{R} \mid x < a,\ x > b\}. \tag{1}$$

We can write this complement as the union of two sets:

$$\begin{aligned} [a, b]^c &= \{x \in \mathbb{R} \mid x < a\} \cup \{x \in \mathbb{R} \mid x > b\} &(2)\\ &= (-\infty, a) \cup (b, \infty). &(3) \end{aligned}$$

Both the sets $(-\infty, a)$ and (b, ∞) are open, as proved in PS3.5a. We also proved that the union of open sets is open. Thus, $[a, b]^c$ is open.

Therefore, we conclude that $[a, b]$ is closed. □

(b)

Proof. Consider the complement of the integers in the real numbers, $\mathbb{Z}^c = \mathbb{R}\backslash\mathbb{Z}$. We may write this complement as a union of open sets, where each of the open sets represents the set of numbers between (but not including) consecutive integers:

$$\mathbb{Z}^c = \bigcup_{n \in \mathbb{Z}} (n, n + 1). \tag{4}$$

Since the sets being unioned are all open, then so is the union, i.e. $\mathbb{Z}^c$ is open.

Thus, $\mathbb{Z}$ is closed. □

(c)

Proof. The complement of the rational numbers $\mathbb{Q}$ in the reals is the set of irrationals:

$$\mathbb{Q}^c = \mathbb{R}\backslash\mathbb{Q}. \tag{5}$$

Let $i \in \mathbb{Q}^c$. In class, we proved the density of $\mathbb{Q}$ in $\mathbb{R}$. Additionally, in PS3.1, we proved the density of the irrationals in $\mathbb{R}$. It follows that $\exists q, r \in \mathbb{Q}$ such that $q < i < r$.

Let $\epsilon > 0$. Since $i - \epsilon, i + \epsilon \in \mathbb{R}$, then $\exists p \in \mathbb{Q}$ such that $i - \epsilon < p < i + \epsilon$. This implies that $p \in (q, r)$, but $p \in \mathbb{Q} \implies p \notin \mathbb{Q}^c$, so $\mathbb{Q}^c$ is not open.

Therefore, $\mathbb{Q}$ is not closed. $\square$