

# 18.100A Assignment 5

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## Problem 1

*Proof.* We have that

$$L = \lim_{n \rightarrow \infty} \frac{|x_{n+1} - x|}{|x_n - x|} < 1. \quad (1)$$

Then for every  $\epsilon > 0$ ,  $\exists N \in \mathbb{N}$  such that  $\forall n \geq N$ ,

$$\left| \frac{x_{n+1} - x}{x_n - x} \right| - 1 < \epsilon. \quad (2)$$

Rearranging gives

$$|x_{n+1} - x| < (1 + \epsilon)|x_n - x|. \quad (3)$$

By taking  $\epsilon$  to be arbitrarily small, we have,  $\forall n \geq N$ ,

$$\xrightarrow{\epsilon \rightarrow 0} |x_{n+1} - x| < |x_n - x|. \quad (4)$$

Define  $y_n := |x_n - x|$ . Then  $\forall n \geq N$ ,

$$0 \leq y_{n+1} < y_n, \quad (5)$$

so  $\{y_n\}_n$  is a decreasing sequence bounded below by 0. Hence,

$$\implies y_n \rightarrow 0 \quad (6)$$

$$\implies |x_n - x| \rightarrow 0 \quad (7)$$

$$\implies x_n \rightarrow x. \quad (8)$$

Therefore,  $\{x_n\}_n$  converges to  $x$ .  $\square$

## Problem 2

(a)

Let  $x_n = \frac{(-1)^n}{n}$ . Then  $\forall n \in \mathbb{N}$ ,

$$-\frac{1}{n} \leq x_n \leq \frac{1}{n}. \quad (9)$$

Allowing  $n \rightarrow \infty$  on all sides of the inequality gives

$$0 = \lim_{n \rightarrow \infty} \left( -\frac{1}{n} \right) \leq \lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} \left( \frac{1}{n} \right) = 0. \quad (10)$$

So by the Squeeze Theorem,  $\lim_{n \rightarrow \infty} x_n = 0$ . Finally, by the theorem from lecture 9, we conclude that  $\liminf x_n = \limsup x_n = 0$ .

(b)

Let  $x_n = (-1)^n \frac{(n-1)}{n}$ . Define

$$a_n := \sup\{x_k \mid k \geq n\}, \text{ and } b_n := \sup\{x_k \mid k \geq n\}. \quad (11)$$

Then  $\forall n \in \mathbb{N}$ , we have

$$|x_n| = \left| \frac{n-1}{n} \right| \cdot |(-1)^n| \quad (12)$$

$$= \frac{n-1}{n} \quad (13)$$

$$= 1 - \frac{1}{n} \quad (14)$$

$$\leq 1. \quad (15)$$

Thus,  $x_n$  is bounded and  $-1 \leq x_n \leq 1$ .

Let  $n_k = 2k$  and  $m_k = 2k-1$  for  $k \in \mathbb{N}$ . Then we construct two subsequences  $\{x_{n_k}\}_k$  and  $\{x_{m_k}\}_k$  of  $\{x_n\}_n$  via

$$x_{n_k} := \frac{2k-1}{2k} (-1)^{2k} = \frac{2k-1}{2k}, \quad (16)$$

and

$$x_{m_k} := \frac{2k-2}{2k-1} (-1)^{2k-1} = \frac{2-2k}{2k-1}. \quad (17)$$

Taking the limit of the first subsequence gives

$$\lim_{k \rightarrow \infty} x_{n_k} = \lim_{k \rightarrow \infty} \left( 1 - \frac{1}{2k} \right) = 1, \quad (18)$$

and for the second subsequence we get

$$\lim_{k \rightarrow \infty} x_{m_k} = \lim_{k \rightarrow \infty} \left( \frac{1}{2k-1} - 1 \right) = -1. \quad (19)$$

Thus,  $\{x_{n_k}\}_k$  converges to 1, and  $\{x_{m_k}\}_k$  converges to -1. Then  $\sup\{x_k \mid k \geq n\} \geq 1$ . But  $-1 \leq x_n \leq 1$ , so it must also be true that

$$\sup\{x_k \mid k \geq n\} \leq \sup x_n = 1. \quad (20)$$

Thus,  $\sup\{x_k \mid k \geq n\} = 1$ , and we can conclude that  $\limsup x_n = 1$ .

Similarly,  $\inf\{x_k \mid k \geq n\} \leq -1$ , but  $-1 \leq x_n \leq 1$ , so we must have

$$\inf\{x_k \mid k \geq n\} \geq \inf x_n = -1, \quad (21)$$

so  $\inf\{x_k \mid k \geq n\} = -1$ . Therefore, we conclude also that  $\liminf x_n = -1$ .

### Problem 3

*Proof.* (i) Since  $x_n \leq y_n \forall n \in \mathbb{N}$ , then for each  $n$ , we have

$$\sup\{x_k \mid k \geq n\} \leq \sup\{y_k \mid k \geq n\}. \quad (22)$$

Taking the limit on both sides of (22) gives

$$\xrightarrow{n \rightarrow \infty} \limsup x_n \leq \limsup y_n, \quad (23)$$

as desired.

(ii) Since  $x_n \leq y_n \forall n \in \mathbb{N}$ , then for each  $n$ , we have

$$\inf\{x_k \mid k \geq n\} \leq \inf\{y_k \mid k \geq n\}. \quad (24)$$

Taking the limit in (24) yields

$$\xrightarrow{n \rightarrow \infty} \liminf x_n \leq \liminf y_n, \quad (25)$$

and we are done.  $\square$