18.100A Assignment 7

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Problem 1

Proof. Since $\sum_n a_n$ and $\sum_n b_n$ converge absolutely, suppose that $\sum_n |a_n| < M$ and $\sum_n |b_n| < N$. Then

$$\begin{aligned}
\sum_{n=0}^{m} |c_n| &= \sum_{n=0}^{m} \left| \sum_{k=0}^{n} a_k b_{n-k} \right| && (1)\\
&\leq \sum_{n=0}^{m} \sum_{k=0}^{n} |a_k b_{n-k}| && (2)\\
&= |a_0 b_0| + (|a_0 b_1| + |a_1 b_0|) + \cdots + \\
&\quad (|a_0 b_m| + |a_1 b_{m-1}| + \cdots + |a_m b_0|) && (3)\\
&= \sum_{n=0}^{m} |a_n| \sum_{k=0}^{m-n} |b_k| && (4)\\
&< MN. && (5)
\end{aligned}$$

Thus $\sum_n |c_n|$ is bounded above and monotone, so it converges. $\square$

Problem 2

(a)

Let $a_n = 2^n x^n$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}x^{n+1}}{2^n x^n}\right| = 2|x|$.

By the ratio test, we must have

$$L = \lim_{n\to\infty} 2|x| < 1. \tag{6}$$

Thus, $\sum_{n=0}^{\infty} 2^n x^n$ converges for all $|x| < \frac{1}{2}$.

(b)

We have $a_n = nx^n$, so $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)x^{n+1}}{nx^n}\right| = \frac{n+1}{n}|x|$.

Thus, we require

$$\lim_{n\to\infty} \frac{n+1}{n}|x| < 1. \tag{7}$$

Therefore, $\sum_n nx^n$ converges for all $|x| < 1$.

(c)

Proceeding with the ratio test, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-10)^{n+1}(2n)!}{(2n+2)!(x-10)^n}\right| = \left|\frac{x-10}{(2n+2)(2n+1)}\right| \tag{8}$$

Then, we require

$$\lim_{n\to\infty} \left|\frac{x-10}{4n^2+6n+2}\right| = 0 < 1, \tag{9}$$

which is always satisfied. Thus, $\sum_n \frac{1}{(2n)!}(x-10)^n$ converges $\forall x \in \mathbb{R}$.

(d)

Letting $a_n = n!x^n$, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x|. \tag{10}$$

Thus we must have

$$\lim_{n\to\infty} (n+1)|x| < 1, \tag{11}$$

which is only satisfied for $x = 0$. Thus, $\sum_n n!x^n$ converges only for $x = 0$.

Problem 3

Proof. (i) Let $z_n = \max\{|x_n|, |y_n|\}$ for each $n \in \mathbb{N}$. Then

$$|x_n y_n| = |x_n||y_n| \le |x_n||z_n| \le |z_n|^2. \tag{12}$$

But we assumed that both $\sum_n |x_n|^2$ and $\sum_n |y_n|^2$ converge, so $\sum_n |z_n|^2$ converges. Thus, by (12) we see that $\sum_n |x_n y_n|$ converges by comparison.

Therefore, $\sum_n x_n y_n$ converges absolutely.

(ii) By the triangle inequality for infinite series, we have

$$\left|\sum_{n=1}^{\infty} x_n y_n\right| \le \sum_{n=1}^{\infty} |x_n y_n|. \tag{13}$$

Let $N \in \mathbb{N}$ and let $A = \sqrt{x_1^2 + \cdots + x_N^2}$ and $B = \sqrt{y_1^2 + \cdots + y_N^2}$. By the arithmetic mean - geometric mean inequality, for each $n \in \mathbb{N}$ we have

$$\sqrt{\frac{x_n^2 y_n^2}{A^2 B^2}} \leq \frac{1}{2}\left(\frac{x_n^2}{A^2} + \frac{y_n^2}{B^2}\right). \tag{14}$$

Summing over all $k = 1, ..., N$,

$$\sum_{n=1}^{N} \frac{x_n y_n}{AB} \leq \sum_{n=1}^{N} \frac{1}{2}\left(\frac{x_n^2}{A^2} + \frac{y_n^2}{B^2}\right) = 1 \tag{15}$$

Multiplying both sides of (15) by AB, we get

$$\sum_{n=1}^{N} x_n y_n \leq AB = \left(\sum_{n=1}^{N} x_n^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} y_n^2\right)^{\frac{1}{2}}. \tag{16}$$

Letting $N \to \infty$, since limits respect inequalities we arrive at

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} x_n^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} y_n^2\right)^{\frac{1}{2}}, \tag{17}$$

as desired. □

Problem 4

Proof. Let $x \in \mathbb{R}$ and let $\epsilon > 0$.

Consider the set $(x - \epsilon, x + \epsilon)$. We already showed that $\forall x,\ y \in \mathbb{R}$ with $x < y$, $\exists z \in \mathbb{R} \backslash \mathbb{Q}$ such that $x < z < y$.

Thus $\forall x \in \mathbb{R}$ and for every $\epsilon > 0$,

$$(x - \epsilon, x + \epsilon) \cap (\mathbb{R} \backslash \mathbb{Q}) \backslash \{x\} \neq \varnothing. \tag{18}$$

Therefore, every real number is a cluster point of the irrationals. □

Problem 5

Proof. Since c is a cluster point of S, then $\exists$ a sequence $\{y_k\}_k$ of elements in $S \backslash \{c\}$ such that $y_k \to c$.

Also, since f is bounded, then $\exists\ B \geq 0$ such that $\forall x \in S$, $|f(x)| \leq B$. Then the sequence $\{f(y_k)\}_k$ is bounded. By the Bolzano-Weierstrass theorem, $\exists$ a convergent subsequence $\{f(y_{k_n})\}_n$. Simply taking $x_n = y_{k_n}$ for each $n \in \mathbb{N}$, we see that $\{f(x_n)\}_n$ converges, as desired. □

Problem 6

(a)

Proof. Since $\lim_{x\to c} f(x) = L$, then for every $\epsilon > 0$ $\exists$ $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Let $\epsilon = |L|$, and suppose $0 < |x - c| < \delta$ with δ chosen appropriately. Then

$$\begin{aligned} |f(x)| &= |f(x) - L + L| && (19)\\ &\leq |f(x) - L| + |L| && (20)\\ &< \epsilon + |L| && (21)\\ &= 2|L|. && (22) \end{aligned}$$

Choosing $B = 2|L|$, we see that f is bounded, and we are done. □

(b)

Proof. Since $\lim_{x\to c} f(x) = L > 0$, then $\forall \epsilon > 0$, $\exists \delta_0 > 0$ such that if $|x - c| < \delta_0$, then $|f(x) - L| < \epsilon$. Equivalently, for $|x - c| < \delta_0$,

$$L - \epsilon < f(x) < L + \epsilon. \tag{23}$$

Let $0 < \epsilon < L$. Then $\exists \delta_1 > 0$ such that if $|x - c| < \delta_1$, then

$$0 < L - \epsilon < f(x) < L + \epsilon. \tag{24}$$

Choosing $\delta = \delta_1$, we see that $f(x)$ is positive for $|x - c| < \delta$, and we are done. □