18.100A Assignment 11

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Problem 1

Proof. Let $f(x) = \frac{1}{1121}x^{1121} + \frac{1}{2021}x^{2021} + x + 1$. We compute the derivative:

$$f'(x) = x^{1120} + x^{2020} + 1. \tag{1}$$

Hence $f'(x) > 0 \ \forall x \in \mathbb{R}$, so $f(x)$ is increasing.

Suppose $f(x)$ has n real roots, where $n > 1$. Then $\exists x_1, x_2, ..., x_n$ such that $f(x_1) = \cdots = f(x_n) = 0$. Since $f(x)$ is polynomial, then f is continuous $\forall x$ and differentiable on $\mathbb{R}$. By Rolle's theorem, $\exists c \in (x_1, x_2)$ such that $f'(c) = 0$, which is a contradiction since $f'(x) > 0 \ \forall x$. Thus f cannot have more than one real root.

Now suppose $f(x)$ has no real roots. Then either $f(x) > 0 \ \forall x$ or $f(x) < 0 \ \forall x$. Choose $x_0 = 1$. Then $f(x_0) = \frac{1}{1121} + \frac{1}{2021} + 2 > 0$. Choose $x^* = -10$. Then $f(x^*) = -\frac{10^{1121}}{1121} - \frac{10^{2021}}{2021} + 2 < 0$. Then by continuity, $f(x)$ has at least one real root, which is a contradiction. Thus f must have at least one real root.

So we have the number of real roots $1 \leq n \leq 1$, so $n = 1$.

Therefore f has exactly one real root. $\square$

Problem 2

(a)

Let $f(x) = \sin(x)$ and $x_0 = 0$. We compute

$$P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \tag{2}$$

The derivatives are

$$\begin{aligned} f'(x) = \cos(x) &\implies f'(x_0) = \cos(0) = 1 && (3)\\ f''(x) = -\sin(x) &\implies f''(x_0) = -\sin(0) = 0 && (4)\\ f'''(x) = -\cos(x) &\implies f'''(x_0) = -\cos(0) = -1 && (5)\\ f^{(4)}(x) = \sin(x) &\implies f^{(4)}(x_0) = \sin(0) = 0. && (6) \end{aligned}$$

Therefore, the fourth Taylor polynomial is

$$P_4(x) = x - \frac{1}{3!}x^3. \tag{7}$$

(b)

Let $f(x) = \frac{1}{1-x}$ and $x_0 = -1$. The derivatives are

$$\begin{aligned} f'(x) = \frac{1}{(1-x)^2} &\implies f'(x_0) = \frac{1}{4} && (8)\\ f''(x) = \frac{2}{(1-x)^3} &\implies f''(x_0) = \frac{1}{4} && (9)\\ f'''(x) = \frac{6}{(1-x)^4} &\implies f'''(x_0) = \frac{3}{8} && (10)\\ f^{(4)}(x) = \frac{24}{(1-x)^5} &\implies f^{(4)}(x_0) = \frac{3}{4}. && (11) \end{aligned}$$

Therefore, the fourth Taylor polynomial is

$$P_4(x) = \frac{1}{2} + \frac{1}{4}(x+1) + \frac{1}{8}(x+1)^2 + \frac{1}{16}(x+1)^3 + \frac{1}{32}(x+1)^4. \tag{12}$$