

18.100A Assignment 10

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Problem 1

(a)

Proof. Suppose $\exists C \geq 0$ such that $\forall x, y \in I$,

$$|f(x) - f(y)| \leq C|x - y|^\alpha. \quad (1)$$

Let $\epsilon > 0$. Choose $\delta = \left(\frac{\epsilon}{C}\right)^{\frac{1}{\alpha}}$. Then if $|x - y| < \delta$, we get

$$|f(x) - f(y)| \leq C|x - y|^\alpha \quad (2)$$

$$< C\delta^\alpha \quad (3)$$

$$= C\frac{\epsilon}{C} \quad (4)$$

$$= \epsilon. \quad (5)$$

Therefore f is uniformly continuous on I . \square

(b)

Proof. Suppose $\exists C \geq 0$ such that $\forall x, y \in I$, $|f(x) - f(y)| \leq C|x - y|^\alpha$.

Since $\alpha > 1$, then $\alpha = 1 + r$ for some $0 < r$, we have

$$\implies 0 \leq |f(x) - f(y)| \leq C|x - y|^{1+r} \quad (6)$$

$$\implies 0 \leq \frac{|f(x) - f(y)|}{|x - y|} \leq C|x - y|^r \quad (7)$$

$$\implies \lim_{x \rightarrow y} 0 \leq \lim_{x \rightarrow y} \frac{|f(x) - f(y)|}{|x - y|} \leq C \lim_{x \rightarrow y} |x - y|^r \quad (8)$$

$$\implies 0 \leq \lim_{x \rightarrow y} \frac{|f(x) - f(y)|}{|x - y|} \leq 0. \quad (9)$$

Then by the squeeze theorem, $\lim_{x \rightarrow y} \frac{|f(x) - f(y)|}{|x - y|} = 0$. Thus $\forall y \in I$, $f'(y) = 0$.

Therefore f is constant. \square

Problem 2

Proof. We compute:

$$L = \lim_{x \rightarrow c} \frac{h(x) - h(c)}{x - c} \quad (10)$$

$$= \lim_{x \rightarrow c} \frac{f(x)g(x) - f(c)g(c)}{x - c} \quad (11)$$

$$= \lim_{x \rightarrow c} \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x - c} \quad (12)$$

$$= \lim_{x \rightarrow c} \left[f(x) \left(\frac{g(x) - g(c)}{x - c} \right) \right] + g(c) \lim_{x \rightarrow c} \left(\frac{f(x) - f(c)}{x - c} \right) \quad (13)$$

$$(14)$$

Since f is continuous at c , and both f and g are differentiable at c , this gives us

$$L = f(c)g'(c) + g(c)f'(c), \quad (15)$$

which exists.

Therefore $f(x)g(x)$ is differentiable at c . \square

Problem 3

Proof. (\Rightarrow) Suppose f is Lipschitz. Then $\exists L \geq 0$ such that $\forall x, y \in \mathbb{R}$,

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq L. \quad (16)$$

Since f is differentiable, we have

$$|f'(y)| = \lim_{x \rightarrow y} \left| \frac{f(x) - f(y)}{x - y} \right| \leq \lim_{x \rightarrow y} L, \quad (17)$$

so $\forall y \in \mathbb{R}$, $|f'(y)| \leq L$.

Therefore f' is bounded.

(\Leftarrow) Suppose f' is bounded.

Then $\exists B \geq 0$ such that $|f'(x)| \leq B \forall x \in \mathbb{R}$. Let $x, y \in \mathbb{R}$. Then by the mean value theorem, $\exists c \in \mathbb{R}$ such that $f(x) - f(y) = (x - y)f'(c)$, i.e.

$$|f(x) - f(y)| = |x - y||f'(c)| \quad (18)$$

$$\leq |x - y|B. \quad (19)$$

Choose $L = B$, and we see

$$|f(x) - f(y)| \leq L|x - y|. \quad (20)$$

Thus, we conclude f is Lipschitz $\iff f'$ is bounded. \square