18.100A Assignment 12

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Problem 1

(a)

Proof. Suppose $\exists c \in [a, b]$ such that $f(c) > 0$. Since f is continuous, $\exists \delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(x)| < \frac{f(c)}{2}$, i.e. $\frac{f(c)}{2} < f(x)$. We compute

$$0 > \int_a^b f \tag{1}$$

$$> \int_a^b \frac{f(c)}{2} \tag{2}$$

$$= \frac{f(c)}{2}(b - a) \tag{3}$$

$$> 0, \tag{4}$$

i.e. $0 > 0$ ($\Rightarrow\Leftarrow$), which is cleary a contradiction.

Therefore $f(x) = 0 \ \forall x \in [a, b]$. $\square$

(b)

Proof. Let $E = \int_a^b (u')^2 \mathrm{d}x$. Then $E \geq 0$ since $(u')^2 \geq 0$. Using integration by parts, we have

$$E = \int_a^b u'u' \mathrm{d}x \tag{5}$$

$$= uu'\Big|_a^b - \int_a^b uu'' \mathrm{d}x \tag{6}$$

$$= u'(b)u(b) - u'(a)u(a) - \int_a^b u(Vu) \mathrm{d}x \tag{7}$$

$$= -\int_a^b Vu^2 \mathrm{d}x. \tag{8}$$

But $V(x) \geq 0$ and $u^2 \geq 0$, so $-(Vu^2) \leq 0$, hence $E \leq 0$. Thus $E = 0$, which must mean that

$$\int_a^b (u')^2 \mathrm{d}x = 0, \tag{9}$$

and by part **(a)**, this implies that $(u')^2 = 0 \ \forall x \in [a, b]$. Hence $u'(x) = 0$ for all x, and since $u(a) = 0$, then u remains constant at 0; i.e. $u = 0$ everywhere. □

Problem 2

We compute:

$$\int_{-x}^{x} e^{s^2} \mathrm{d}s = \int_{-x}^{0} e^{s^2} \mathrm{d}s + \int_{0}^{x} e^{s^2} \mathrm{d}s \tag{10}$$

$$= \int_{0}^{x} e^{s^2} \mathrm{d}s - \int_{0}^{-x} e^{s^2} \mathrm{d}s. \tag{11}$$

Differentiating, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-x}^{x} e^{s^2} \mathrm{d}s\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{0}^{x} e^{s^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{0}^{-x} e^{s^2}\right) \tag{12}$$

$$= e^{x^2} + e^{x^2}. \tag{13}$$

Thus $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-x}^{x} e^{s^2} \mathrm{d}s\right) = 2e^{x^2}$.

Problem 3

Proof. Define $G(x) = \int_a^x f(t)\mathrm{d}t$. By the fundamental theorem of calculus, G is continuous on $[a, b]$. Note that $G(x) = 0 \ \forall x \in \mathbb{Q} \cap [a, b]$. We now claim that $G(x) = 0$ on $[a, b]$.

Suppose $\exists c \in [a, b]$ such that $G(c) \neq 0$. For some $x \in [a, b]$, let $\epsilon = \frac{|G(x)|}{2}$ and let $\delta > 0$. Then $\forall x$ such that $|x - c| < \delta$, we have $|G(x) - G(c)| < \epsilon$ since G is continuous. $\exists c \in [a, b] \cap \mathbb{Q}$ such that $|x - c| < \delta$. But then $|G(x) - G(c)| = |G(x)| > \frac{|G(x)|}{2} = \epsilon$, which is a contradiction, since we assumed G to be continuous. Thus $G(x) = 0$ on $[a, b]$, which proves the claim.

Thus, $G(x) = \int_a^x f(t)\mathrm{d}t = 0$ on $[a, b]$. Since G is constant, then $G' = 0$ on $[a, b]$. By the fundamental theorem of calculus, $G'(x) = f(x) = 0 \ \forall x \in [a, b]$, and we are done. □

Problem 4

0.1 (a)

Let $f_n(x) = \frac{e^{\frac{x}{n}}}{n}$ for each $n \in \mathbb{N}$. Then by continuity, we have

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} e^{\frac{x}{n}} \cdot \lim_{n\to\infty} \frac{1}{n} && (14)\\
&= e^{x \lim_{n\to\infty} \frac{1}{n}} \cdot \lim_{n\to\infty} \frac{1}{n} && (15)\\
&= e^0 \cdot 0 && (16)\\
&= 0. && (17)
\end{aligned}$$

Therefore $f_n \to 0$ pointwise.

(b)

Let $M \in \mathbb{N}$. Choose $\epsilon_0 = 1$, $x = n\ln(n)$, and $n = M$. Then

$$\begin{aligned}
|f_n(x) - f(x)| &= \left|\frac{e^{\frac{x}{n}}}{n}\right| && (18)\\
&= \frac{e^{\ln(M)}}{M} && (19)\\
&= \frac{M}{M} && (20)\\
&= 1 && (21)\\
&= \epsilon_0. && (22)
\end{aligned}$$

Therefore the limit is NOT uniform on $\mathbb{R}$.

(c)

Let $\epsilon > 0$. Choose $M = \frac{1}{\log(\epsilon)}$. Then $\forall x \in [0,1]$ and $\forall n \geq M$, we have

$$\begin{aligned}
|f_n(x) - 0| &= \left|\frac{e^{\frac{x}{n}}}{n}\right| && (23)\\
&< \frac{e^{\frac{1}{n}}}{n} && (24)\\
&< e^{\frac{1}{n}} && (25)\\
&< e^{\log(\epsilon)} && (26)\\
&= \epsilon. && (27)
\end{aligned}$$

Therefore the limit is uniform on $[0,1]$.

Problem 5