18.100A Assignment 5

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Problem 1

Proof. We have that

$$L = \lim_{n\to\infty} \frac{|x_{n+1} - x|}{|x_n - x|} < 1. \tag{1}$$

Then for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\left| \frac{x_{n+1} - x}{x_n - x} \right| - 1 < \epsilon. \tag{2}$$

Rearranging gives

$$|x_{n+1} - x| < (1 + \epsilon)|x_n - x|. \tag{3}$$

By taking ϵ to be arbitrarily small, we have, $\forall n \geq N$,

$$\xRightarrow{\epsilon\to 0} |x_{n+1} - x| < |x_n - x|. \tag{4}$$

Define $y_n := |x_n - x|$. Then $\forall n \geq N$,

$$0 \leq y_{n+1} < y_n, \tag{5}$$

so $\{y_n\}_n$ is a decreasing sequence bounded below by 0. Hence,

$$\implies y_n \to 0 \tag{6}$$
$$\implies |x_n - x| \to 0 \tag{7}$$
$$\implies x_n \to x. \tag{8}$$

Therefore, $\{x_n\}_n$ converges to x. $\square$

Problem 2

(a)

Let $x_n = \frac{(-1)^n}{n}$. Then $\forall n \in \mathbb{N}$,

$$-\frac{1}{n} \leq x_n \leq \frac{1}{n}. \tag{9}$$

Allowing $n \to \infty$ on all sides of the inequality gives

$$0 = \lim_{n\to\infty} \left(-\frac{1}{n}\right) \leq \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0. \tag{10}$$

So by the Squeeze Theorem, $\lim_{n\to\infty} x_n = 0$. Finally, by the theorem from lecture 9, we conclude that $\liminf x_n = \limsup x_n = 0$.

(b)

Let $x_n = (-1)^n \frac{(n-1)}{n}$. Define

$$a_n := \sup\{x_k \mid k \geq n\}, \text{ and } b_n := \sup\{x_k \mid k \geq n\}. \tag{11}$$

Then $\forall n \in \mathbb{N}$, we have

$$\begin{aligned} |x_n| &= \left|\frac{n-1}{n}\right| \cdot |(-1)^n| && (12)\\ &= \frac{n-1}{n} && (13)\\ &= 1 - \frac{1}{n} && (14)\\ &\leq 1. && (15) \end{aligned}$$

Thus, x_n is bounded and $-1 \leq x_n \leq 1$.

Let $n_k = 2k$ and $m_k = 2k-1$ for $k \in \mathbb{N}$. Then we construct two subsequences $\{x_{n_k}\}_k$ and $\{x_{m_k}\}_k$ of $\{x_n\}_n$ via

$$x_{n_k} := \frac{2k-1}{2k}(-1)^{2k} = \frac{2k-1}{2k}, \tag{16}$$

and

$$x_{m_k} := \frac{2k-2}{2k-1}(-1)^{2k-1} = \frac{2-2k}{2k-1}. \tag{17}$$

Taking the limit of the first subsequence gives

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} \left(1 - \frac{1}{2k}\right) = 1, \tag{18}$$

and for the second subsequence we get

$$\lim_{k\to\infty} x_{m_k} = \lim_{k\to\infty} \left(\frac{1}{2k-1} - 1\right) = -1. \tag{19}$$

Thus, $\{x_{n_k}\}_k$ converges to 1, and $\{x_{m_k}\}_k$ converges to -1. Then $\sup\{x_k \mid k \geq n\} \geq 1$. But $-1 \leq x_n \leq 1$, so it must also be true that

$$\sup\{x_k \mid k \geq n\} \leq \sup x_n = 1. \tag{20}$$

Thus, $\sup\{x_k \mid k \geq n\} = 1$, and we can conclude that $\limsup x_n = 1$.

Similarly, $\inf\{x_k \mid k \geq n\} \leq -1$, but $-1 \leq x_n \leq 1$, so we must have

$$\inf\{x_k \mid k \geq n\} \geq \inf x_n = -1, \tag{21}$$

so $\inf\{x_k \mid k \geq n\} = 1$. Therefore, we conclude also that $\liminf x_n = -1$.

Problem 3

Proof. (i) Since $x_n \leq y_n \forall n \in \mathbb{N}$, then for each n, we have

$$\sup\{x_k \mid k \geq n\} \leq \sup\{y_k \mid k \geq n\}. \tag{22}$$

Taking the limit on both sides of (22) gives

$$\xRightarrow{n\to\infty} \limsup x_n \leq \limsup x_n, \tag{23}$$

as desired.

(ii) Since $x_n \leq y_n \forall n \in \mathbb{N}$, then for each n, we have

$$\inf\{x_k \mid k \geq n\} \leq \inf\{y_k \mid k \geq n\}. \tag{24}$$

Taking the limit in (24) yields

$$\xRightarrow{n\to\infty} \liminf x_n \leq \liminf y_n, \tag{25}$$

and we are done. □

Problem 4

(a)

Proof. Since $\{x_n\}_n$ and $\{y_n\}_n$ are bounded sequences, then $\exists B_0 \geq 0$ and $B_1 \geq 0$ such that $\forall n \in \mathbb{N}$, $|x_n| \leq B_0$ and $|y_n| \leq B_1$.

Choose $B = B_0 + B_1$. Then by the triangle inequality,

$$|x_n + y_n| \leq |x_n| + |y_n| \leq B_0 + B_1 = B. \tag{26}$$

Therefore, the sequence $\{x_n + y_n\}_n$ is bounded. □

(b)

Proof. Let $\{x_{n_k}\}_k$ be a subsequence of $\{x_n\}_n$. Since the sequence $\{x_n\}_n$ is bounded, then $\{x_{n_k}\}_k$ must also be bounded. Then by the Bolzano-Weierstrass theorem, $\exists$ a (sub) subsequence $\{x_{n_{k_i}}\}_i$ of the subsequence $\{x_{n_k}\}_k$ such that $\{x_{n_{k_i}}\}_i$ converges; i.e. that $\lim_{i\to\infty} x_{n_{k_i}}$ exists. Then

$$\liminf x_n \leq \lim_{i\to\infty} x_{n_{k_i}}. \tag{27}$$

By the same logic as above, $\exists$ a (sub-sub) subsequence $\{y_{n_{k_{i_l}}}\}_l$ of $\{y_{n_{k_i}}\}_i$ such that $\lim_{l\to\infty} y_{n_{k_{i_l}}}$ exists. Then

$$\liminf y_n \leq \lim_{l\to\infty} y_{n_{k_{i_l}}}. \tag{28}$$

Thus the corresponding subsequence $\{x_{n_{k_{i_l}}} + y_{n_{k_{i_l}}}\}_l$ of $\{x_{n_k} + y_{n_k}\}_k$ is convergent, and

$$\lim_{l\to\infty} \left(x_{n_{k_{i_l}}} + y_{n_{k_{i_l}}}\right) = \lim_{k\to\infty} x_{n_k} + \lim_{k\to\infty} y_{n_k}. \tag{29}$$

But we also have

$$\begin{aligned} \lim_{l\to\infty} \left(x_{n_{k_{i_l}}} + y_{n_{k_{i_l}}}\right) &= \lim_{l\to\infty} x_{n_{k_{i_l}}} + \lim_{l\to\infty} y_{n_{k_{i_l}}} && (30) \\ &= \lim_{i\to\infty} x_{n_{k_i}} + \lim_{l\to\infty} y_{n_{k_{i_l}}} && (31) \\ &\geq \liminf x_n + \liminf y_n. && (32) \end{aligned}$$

Additionally,

$$\liminf(x_n + y_n) \geq \lim_{l\to\infty} \left(x_{n_{k_{i_l}}} + y_{n_{k_{i_l}}}\right). \tag{33}$$

Therefore, $\liminf x_n + \liminf y_n \leq \liminf(x_n + y_n)$. $\square$

(c)

Let $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$ for $n \in \mathbb{N}$. Then

$$x_n + y_n = (-1)^n + (-1)^{n+1} = (-1)^n(1-1) = 0. \tag{34}$$

Thus, $\liminf(x_n + y_n) = 0$. But $\liminf x_n = \liminf y_n = -1$, so

$$\liminf x_n + \liminf y_n = -1 - 1 = -2 < 0. \tag{35}$$

Therefore this example demonstrates a case where equality does not hold for the theorem in the previous sub-problem.

Problem 5

Proof. ($\Rightarrow$) Suppose $\lim_{n\to\infty} x_n = 0$.

Let $a_n = \sup\{|x_k| \mid k \geq n\}$. By PS3.4, since a_n is a supremum, then $\forall n$ $\exists x_n$ such that $a_n - \frac{1}{n} < |x_n| \leq a_n$; i.e.

$$-\frac{1}{n} < |x_n| - a_n \leq 0. \tag{36}$$

It becomes clear from the equation above that by the squeeze theorem, $\lim_{n\to\infty}(|x_n| - a_n) = 0$, hence

$$0 = \lim_{n\to\infty} |x_n| = \lim_{n\to\infty} a_n = \limsup |x_n|, \tag{37}$$

as desired.

($\Leftarrow$) Suppose $\limsup |x_n| = 0$. Then $\lim_{n\to\infty} \sup\{|x_k| \mid k \geq n\} = 0$.

Let $a_n = \sup\{|x_k| \mid k \geq n\}$. Then by definition of the supremum, we have

$$0 \leq |x_n| \leq a_n. \tag{38}$$

Taking the limit as $n \to \infty$ in the above inequality yields

$$0 \leq \lim_{n\to\infty} |x_n| \leq \limsup |x_n| = 0. \tag{39}$$

Thus, by the squeeze theorem, $|x_n| \to 0$. $\square$

Problem 6

Claim: There does not exist any sequence $\{x_n\}_n$ such that $\liminf x_n = -1$, $\lim_{n\to\infty} x_n = 0$, and $\limsup x_n = 1$.

Proof. Suppose $\{x_n\}_n$ is a sequence such that $x_n \to 0$. Then $\{x_n\}_n$ converges.

Now suppose $\liminf x_n = -1$ and $\limsup x_n = 1$. Thus,

$$\liminf x_n \neq \limsup x_n. \tag{40}$$

Then by the theorem from lecture, $\{x_n\}_n$ does not converge. ($\Rightarrow\Leftarrow$). This is a contradiction to the assumption that the sequence is convergent, so our initial assumption that such a sequence existed must be incorrect. $\square$

Problem 7

Proof. Let $\epsilon_0 > 0$. Since $\{x_n\}_n$ is a Cauchy sequence, then $\exists N_0 \in \mathbb{N}$ such that $\forall n \geq N_0 + 1$,

$$|x_n - x_{n-1}| < \epsilon_0. \tag{41}$$

Now let $\epsilon_1 = |x_n - x_{n-1}|$. Then since $\{x_n\}_n$ is Cauchy, $\exists N_1 \in \mathbb{N}$ such that

$$|x_{n+1} - x_n| < \epsilon_1. \tag{42}$$

Choose $M = N_1$. Then $\forall n \geq M$,

$$|x_{n+1} - x_n| < |x_n - x_{n-1}|, \tag{43}$$

as desired. $\square$