18.100A Assignment 11

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Problem 1

Proof. Let $f(x) = \frac{1}{1121}x^{1121} + \frac{1}{2021}x^{2021} + x + 1$. We compute the derivative:

$$f'(x) = x^{1120} + x^{2020} + 1. \tag{1}$$

Hence $f'(x) > 0 \ \forall x \in \mathbb{R}$, so $f(x)$ is increasing.

Suppose $f(x)$ has n real roots, where $n > 1$. Then $\exists x_1, x_2, ..., x_n$ such that $f(x_1) = \cdots = f(x_n) = 0$. Since $f(x)$ is polynomial, then f is continuous $\forall x$ and differentiable on $\mathbb{R}$. By Rolle's theorem, $\exists c \in (x_1, x_2)$ such that $f'(c) = 0$, which is a contradiction since $f'(x) > 0 \ \forall x$. Thus f cannot have more than one real root.

Now suppose $f(x)$ has no real roots. Then either $f(x) > 0 \ \forall x$ or $f(x) < 0 \ \forall x$. Choose $x_0 = 1$. Then $f(x_0) = \frac{1}{1121} + \frac{1}{2021} + 2 > 0$. Choose $x^* = -10$. Then $f(x^*) = -\frac{10^{1121}}{1121} - \frac{10^{2021}}{2021} + 2 < 0$. Then by continuity, $f(x)$ has at least one real root, which is a contradiction. Thus f must have at least one real root.

So we have the number of real roots $1 \leq n \leq 1$, so $n = 1$.

Therefore f has exactly one real root. $\square$

Problem 2

(a)

Let $f(x) = \sin(x)$ and $x_0 = 0$. We compute

$$P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \tag{2}$$

The derivatives are

$$\begin{align}
f'(x) = \cos(x) &\implies f'(x_0) = \cos(0) = 1 \tag{3}\\
f''(x) = -\sin(x) &\implies f''(x_0) = -\sin(0) = 0 \tag{4}\\
f'''(x) = -\cos(x) &\implies f'''(x_0) = -\cos(0) = -1 \tag{5}\\
f^{(4)}(x) = \sin(x) &\implies f^{(4)}(x_0) = \sin(0) = 0. \tag{6}
\end{align}$$

Therefore, the fourth Taylor polynomial is

$$P_4(x) = x - \frac{1}{3!}x^3. \tag{7}$$

(b)

Let $f(x) = \frac{1}{1-x}$ and $x_0 = -1$. The derivatives are

$$\begin{align}
f'(x) = \frac{1}{(1-x)^2} &\implies f'(x_0) = \frac{1}{4} \tag{8}\\
f''(x) = \frac{2}{(1-x)^3} &\implies f''(x_0) = \frac{1}{4} \tag{9}\\
f'''(x) = \frac{6}{(1-x)^4} &\implies f'''(x_0) = \frac{3}{8} \tag{10}\\
f^{(4)}(x) = \frac{24}{(1-x)^5} &\implies f^{(4)}(x_0) = \frac{3}{4}. \tag{11}
\end{align}$$

Therefore, the fourth Taylor polynomial is

$$P_4(x) = \frac{1}{2} + \frac{1}{4}(x+1) + \frac{1}{8}(x+1)^2 + \frac{1}{16}(x+1)^3 + \frac{1}{32}(x+1)^4. \tag{12}$$

Problem 3

(a)

We compute using L'Hopital:

$$\begin{align}
\lim_{x\to 0} \frac{x - \sin(x)}{x^3} &= \lim_{x\to 0} \frac{1-\cos(x)}{3x^2} \tag{13}\\
&= \lim_{x\to 0} \frac{\sin(x)}{6x} \tag{14}\\
&= \lim_{x\to 0} \frac{\cos(x)}{6} \tag{15}\\
&= \frac{1}{6}. \tag{16}
\end{align}$$

(b)

We proceed again using L'Hopital's rule:

$$\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin(x)}{\left(x - \frac{\pi}{2}\right)^2} = \lim_{x \to \frac{\pi}{2}} \frac{-\cos(x)}{2\left(x - \frac{\pi}{2}\right)} \tag{17}$$

$$= -\frac{1}{2} \lim_{x \to \frac{\pi}{2}} \sin(x) \tag{18}$$

$$= -\frac{1}{2}. \tag{19}$$

Problem 4

Proof. (By contradiction.) Assume without loss of generality that f has a local maximum at c. Then $\exists \delta > 0$ such that $\forall x \in (a, b)$, if $|x - c| < \delta$ then $f(c) \geq f(x)$.

<u>Case 1</u>: f is constant around c. Then $\forall |x - c| < \delta$, we have $f'(x) = 0$. By differentiating twice, then we must have $f'''(c) = 0$, which is a contradiction since we assumed f''' to be positive.

So f cannot have a local max at c.

<u>Case 2</u>: f is not constant around c. Then $\forall |x - c| < \delta$, $f(c) > f(x)$. By assumption, $f'''(c) > 0$, so f'' is increasing at c. Since $f''(c) = 0$, then $f''(x) > 0$ $\forall x > c$. Thus f' is increasing for $x > c$. Since $f'(c) = 0$ also by assumption of local maximum, then $f'(x) > 0$ for $x > c$. Thus f is increasing for $x > c$. This means that $f(x) > f(c)$ for $x > c$, which is a contradiction since we assumed that $f(c)$ is a local maximum.

Thus f has no local maximum at c.

To prove that f cannot have a local minimum, repeat the same proof above with $g(x) = -f(x)$, and we are done. □

Problem 5

(a)

We compute

$$\|\underline{x}^{(r)}\| = \sup_k \left\{ x_k^{(r)} - x_{k-1}^{(r)} \right\} \tag{20}$$

$$= \sup_k \left\{ (b - a)\frac{k}{r} - (b - a)\frac{k - 1}{r} \right\} \tag{21}$$

$$= \sup_k \left\{ \frac{b - a}{r} \right\}. \tag{22}$$

Thus $\|\underline{x}^{(r)}\| = \frac{b-a}{r}$.

(b)

Proof. We write and compute the Riemann sum as

$$
\begin{aligned}
S_f\left(\underline{x}^{(r)}, \underline{\xi}^{(r)}\right) &= \sum_{k=1}^{r} f\left(\xi_k^{(r)}\right)\left(x_k^{(r)} - x_{k-1}^{(r)}\right) && (23)\\
&= \sum_{k=1}^{r} f\left(a + (b-a)\frac{k}{r}\right)\left(\frac{b-a}{r}\right) && (24)\\
&= \sum_{k=1}^{r} \left[\alpha\left(a + (b-a)\frac{k}{r}\right) + \beta\right]\left(\frac{b-a}{r}\right) && (25)\\
&= \sum_{k=1}^{r} \left[\alpha a + \alpha\frac{bk}{r} - \alpha\frac{ak}{r} + \beta\right]\left(\frac{b-a}{r}\right) && (26)\\
&= \sum_{k=1}^{r} \left(\frac{\alpha ab}{r} + \frac{\alpha b^2 k}{r^2} - \frac{\alpha abk}{r^2} + \frac{\beta b}{r} - \frac{\alpha a^2}{r} - \frac{\alpha^2 bka}{r^2} + \frac{\alpha a^2 k}{r^2} - \frac{\beta a}{r}\right) && (27)\\
&= \alpha ab + \frac{\alpha b^2}{r^2}\sum_{k=1}^{r} k - \frac{\alpha ab}{r^2}\sum_{k=1}^{r} k + \beta b - \alpha a^2 - \frac{\alpha^2 ba}{r^2}\sum_{k=1}^{r} k + \frac{\alpha a^2}{r^2}\sum_{k=1}^{2} k - \beta a && (28)\\
&= \alpha ab + \frac{\alpha b^2}{r^2}\frac{r(r+1)}{2} - \frac{\alpha ab}{r^2}\frac{r(r+1)}{2} + \beta b - \alpha a^2 - \frac{\alpha ba}{r^2}\frac{r(r+1)}{2} + \frac{\alpha a^2}{r^2}\frac{r(r+1)}{2} - \beta a && (29)\\
&= \alpha ab + \frac{\alpha b^2}{2} + \frac{\alpha b^2}{2r} - \frac{\alpha ab}{2} - \frac{\alpha ab}{2r} + \beta b - \alpha a^2 - \frac{\alpha^2 ba}{2} - \frac{\alpha^2 ba}{2r} + \frac{\alpha a^2}{2} + \frac{\alpha a^2}{2r} - \beta a. && (30)
\end{aligned}
$$

Now sending $r \to \infty$, we find

$$
\lim_{r\to\infty} S_f\left(\underline{x}^{(r)}, \underline{\xi}^{(r)}\right) = \alpha\left(\frac{b^2 - a^2}{2}\right) + \beta(b-a), \qquad (31)
$$

as desired. ∎