18.100A Assignment 6

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March 6, 2023

Problem 1

(a)

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{3}{9n+1} &= 3\sum_{n=1}^{\infty} \frac{1}{9n+1} && (1)\\
&= \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n+\frac{1}{9}} && (2)\\
&= \frac{1}{3}\sum_{n=2}^{\infty} \frac{1}{(n-1)+\frac{1}{9}} && (3)\\
&= \frac{1}{3}\sum_{n=2}^{\infty} \frac{1}{n-\frac{8}{9}} && (4)\\
&> \sum_{n=2}^{\infty} \frac{1}{n}. && (5)
\end{aligned}$$

But the Harmonic series, $\sum_n \frac{1}{n}$, diverges.

Therefore, we conclude by comparison that the series $\sum_n \frac{3}{9n+1}$ diverges.

(b)

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n-\frac{1}{2}} > \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}. \qquad (6)$$

Therefore, by comparison, $\sum_n \frac{1}{2n-1}$ diverges.

(c)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} - \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2}, \qquad (7)$$

which is the difference of two convergent series.

Therefore, $\sum_n \frac{(-1)^n}{n^2}$ converges.

(d)

We can express the series $\sum_n \frac{1}{n(n+1)}$ as a telescoping sum:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1} && (8)\\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots && (9)\\
&= 1. && (10)
\end{aligned}$$

Therefore, $\sum_n \frac{1}{n(n+1)}$ converges to 1.

(e)

We note that $\forall n \in \mathbb{N}$, $e^{n^2} \geq n^3$. Then we have

$$\sum_{n=1}^{\infty} \frac{n}{e^{n^2}} \leq \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}, \tag{11}$$

which converges.

Therefore, $\sum_n ne^{-n^2}$ converges.