18.100A Assignment 9

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Problem 1

Proof. We have that $\forall x \in \mathbb{R}$, $|\arctan(x)| < \frac{\pi}{2}$, i.e.

$$\arctan(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{1}$$

which is an open set. So $\forall |y| < \frac{\pi}{2}$, $\exists \epsilon > 0$ such that $(y-\epsilon, y+\epsilon) \subset (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus for every such y, we can always find a $y_0 < y$ and $y_1 > y$ inside this open set. This means that there is no x_1 such that $\arctan(x_1) \geq \arctan(x)$ nor an x_0 such that $\arctan(x_0) \leq \arctan(x)$ $\forall x$.

Hence $f(x) = \arctan(x)$ does not achieve an absolute minimum or maximum. $\square$

Problem 2

Proof. Let $x, y \in (c, \infty)$. Choose $L = \frac{1}{c^2}$. Then

$$\begin{aligned} |f(y) - f(x)| &= \left|\frac{1}{y} - \frac{1}{x}\right| \qquad (2) \\ &= \frac{|x-y|}{xy} \qquad (3) \\ &< \frac{|x-y|}{c^2} \qquad (4) \\ &= L|x-y|. \qquad (5) \end{aligned}$$

Therefore $f(x) = \frac{1}{x}$ is Lipschitz continuous. $\square$

Problem 3

Proof. Let $\delta > 0$ and choose $\epsilon_0 = |\sin(\delta)|$. Choose $x = \frac{1}{2\pi k+\delta}$ and $c = \frac{1}{2\pi k}$ for some $k \in \mathbb{N}$. Then

$$|x - c| = \left|\frac{1}{2\pi k} - \frac{1}{2\pi k}\right| \tag{6}$$
$$= \left|\frac{2\pi k - (2\pi k + \delta)}{2\pi k(2\pi k + \delta)}\right| \tag{7}$$
$$= \frac{\delta}{4\pi^2 k^2 + 2\pi k\delta} \tag{8}$$
$$< \delta. \tag{9}$$

We also have

$$|f(x) - f(c)| = |\sin(2\pi k + \delta) - \sin(2\pi k)| \tag{10}$$
$$= |\sin(2\pi k)\cos(\delta) + \cos(2\pi k)\sin(\delta) - \sin(2\pi k)| \tag{11}$$
$$= |\sin(\delta)| \tag{12}$$
$$= \epsilon_0. \tag{13}$$

Hence, $f(x) = \sin\left(\frac{1}{x}\right)$ is not uniformly continuous. $\square$

Problem 4

Proof. Suppose $f : S \to \mathbb{R}$ is Lipschitz continuous on S. Then $\exists L \geq 0$ such that $\forall x, y \in S$, $|f(x) - f(y)| \leq L|x - y|$.

Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{L}$. If $|x - y| < \delta$, then

$$|f(x) - f(y)| \leq L|x - y| \tag{14}$$
$$< L\delta \tag{15}$$
$$= \epsilon. \tag{16}$$

Thus f is uniformly continuous on S. $\square$

Problem 5

(a)

Proof. Let $x, y \in \mathbb{R}$. Choose $L = 1$. Then

$$
\begin{aligned}
|f(x) - f(y)| &= |\cos(x) - \cos(y)| && (17)\\
&= \left|2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right| && (18)\\
&\leq 2\left|\sin\left(\frac{x-y}{2}\right)\right| && (19)\\
&\leq 2\left|\frac{x-y}{2}\right| && (20)\\
&= |x-y|. && (21)
\end{aligned}
$$

Therefore $f(x) = \cos(x)$ is Lipschitz continuous on $\mathbb{R}$. $\square$

(b)

Proof. (1) Let $\epsilon > 0$. Choose $\delta = c^{\frac{2}{3}}\epsilon$. Then $\forall x, c \in [0,1]$, we have

$$
\begin{aligned}
|f(x) - f(c)| &= |x^{\frac{1}{3}} - c^{\frac{1}{3}}| && (22)\\
&= \frac{|x-c|}{|x^{\frac{2}{3}} + x^{\frac{1}{3}}c^{\frac{1}{3}} + c^{\frac{2}{3}}|} && (23)\\
&< \frac{\delta}{c^{\frac{2}{3}}} && (24)\\
&= \epsilon. && (25)
\end{aligned}
$$

Thus, $f(x) = x^{\frac{1}{3}}$ is uniformly continuous on $[0,1]$.

(2) (By contradiction.)

Suppose f is Lipschitz continuous on $[0,1]$. Then $\forall x, y \in [0,1]$, $\exists L \geq 0$ such that $|x^{\frac{1}{3}} - y^{\frac{1}{3}}| \leq L|x-y|$.

Choose $y = 0$. Then $|x^{\frac{1}{3}}| \leq L|x|$, i.e. $\frac{1}{x^{\frac{2}{3}}} \leq L$. Taking $x \to 0$ on both sides, this implies that $\lim_{x\to 0} \frac{1}{x^{\frac{2}{3}}}$ exists and is finite. But we know that this limit does not exist, so we have arrived at a contradiction.

Therefore $f(x) = x^{\frac{1}{3}}$ is not Lipschitz continuous on $[0,1]$. $\square$

Problem 6

(a)

Proof. Choose $M = \frac{1}{\sqrt{\epsilon}}$. Then $\forall x \geq M$,

$$\begin{align}
|f(x) - L| &= \left|\frac{x^2}{x^2+1} - 1\right| \tag{26}\\
&= \left|\frac{x^2 - x^2 - 1}{x^2+1}\right| \tag{27}\\
&= \frac{1}{x^2+1} \tag{28}\\
&< \frac{1}{x^2} \tag{29}\\
&= \epsilon. \tag{30}
\end{align}$$

Therefore $\lim\limits_{x\to\infty} \frac{x^2}{x^2+1} = 1$. □

(b)

Proof. (By contradiction.)

Suppose $L = \lim\limits_{x\to\infty} \sin(x)$ exists. Let $M \in \mathbb{R}$ and choose $x = \pi$. Let $\epsilon_0 = L$. Then

$$\begin{align}
|\sin(x) - L| &= |\sin(\pi) - L| \tag{31}\\
&= |-1 - L| \tag{32}\\
&= 1 + L \tag{33}\\
&> L \tag{34}\\
&= \epsilon_0. \quad (\Rightarrow\Leftarrow) \tag{35}
\end{align}$$

Therefore $\lim\limits_{x\to\infty} \sin(x)$ does not exist. □