

18.100A Midterm

Octavio Vega

March 22, 2023

Problem 1

(a)

Proof. Let $x \in f^{-1}(C \cap D)$. Then

$$\implies f(x) \in C \cap D \quad (1)$$

$$\implies f(x) \in C \text{ and } f(x) \in D \quad (2)$$

$$\implies x \in f^{-1}(C) \text{ and } x \in f^{-1}(D) \quad (3)$$

$$\implies x \in f^{-1}(C) \cap f^{-1}(D). \quad (4)$$

Thus,

$$f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D). \quad (5)$$

Now let $x \in f^{-1}(C) \cap f^{-1}(D)$. Then

$$\implies f(x) \in C \text{ and } f(x) \in D \quad (6)$$

$$\implies f(x) \in C \cap D \quad (7)$$

$$\implies x \in f^{-1}(C \cap D). \quad (8)$$

Thus,

$$f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D). \quad (9)$$

Therefore by equations (5) and (9), $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. \square

(b)

Claim: If $E \subset \mathbb{R}$ is countable, then the complement $\mathbb{R} \setminus E$ is always uncountable.

Proof. (By contradiction). Suppose E^c is countable. Then $E \cup E^c$ is countable as well, since it is the union of two countable sets. But $E \cup E^c = \mathbb{R}$, which is uncountable. ($\Rightarrow \Leftarrow$). \square

(c)

By contrast, if $E \subset \mathbb{R}$ is uncountable, then the complement $\mathbb{R} \setminus E$ is not always countable. Take for instance, $E = [0, 1]$, which is uncountable. Then $E^c = (-\infty, 0) \cup (1, \infty)$, which is also uncountable.

Problem 2

(a)

A set $U \subset \mathbb{R}$ is *not open* if for every $\epsilon > 0$, $\exists x \in U$ such that $(x - \epsilon, x + \epsilon) \not\subset U$.

(b)

Proof. Suppose U is not open. Let $\epsilon = \frac{1}{n}$. Then for every $n \in \mathbb{N}$, $\exists x \in U$ such that $(x - \frac{1}{n}, x + \frac{1}{n}) \not\subset U$. Equivalently, for each $n \in \mathbb{N}$ $\exists x_n \in U^c$ such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n}. \quad (10)$$

Then we have

$$0 < |x_n - x| < \frac{1}{n}, \quad (11)$$

and taking the limit on all sides gives

$$0 < \lim_{n \rightarrow \infty} |x_n - x| < \lim_{n \rightarrow \infty} \frac{1}{n}. \quad (12)$$

Thus, by the squeeze theorem, $\lim_{n \rightarrow \infty} x_n = x$, as desired. \square

(c)

Proof. (By contradiction). To show that F is closed, we must show that F^c is open. Suppose, toward a contradiction, that F^c is not open. Then by part (b), $\exists x \in F^c$ and a sequence $\{x_n\}_n$ of elements of F such that $\lim_{n \rightarrow \infty} x_n = x$. But by assumption, every convergent sequence of elements of F has a limit in F , i.e. we assumed originally that $x \in F$ ($\Rightarrow \Leftarrow$). Thus, F^c must be open, so F is closed. \square

Problem 3

(a)

Proof. Let $\epsilon > 0$. Choose $N = \frac{1}{\sqrt{\epsilon}}$. Then $\forall n \geq N$, we have

$$\left| \frac{10n^2}{n^2 + 16n + 1} - 10 \right| = \left| \frac{-160n - 10}{n^2 + 16n + 1} \right| \quad (13)$$

$$= \left| \frac{160n + 10}{n^2 + 16n + 1} \right| \quad (14)$$

$$< \frac{1}{n^2 + 16n + 1} \quad (15)$$

$$< \frac{1}{n^2} \quad (16)$$

$$< \epsilon. \quad (17)$$

Therefore, $\lim_{n \rightarrow \infty} \left| \frac{10n^2}{n^2 + 16n + 1} \right| = 10$. \square

(b)

(i) Let $x_n = \frac{(-1)^n}{n}$. Then $\lim_{n \rightarrow \infty} x_n = 0$. But $x_1 = -1 < \frac{1}{2} = x_2$, whereas $x_2 = \frac{1}{2} > -\frac{1}{3} = x_3$.

Therefore $\{x_n\}_n$ converges to 0, but is not monotonic.

(ii) Let

$$x_n = \begin{cases} 0, & n \text{ even} \\ n, & n \text{ odd.} \end{cases} \quad (18)$$

Then $\{x_n\}_n$ is clearly unbounded.

Consider the subsequence $\{x_{n_k}\}_k$ defined by $x_{n_k} = x_{2k}$ for each $k \in \mathbb{N}$. Then $\forall k, x_{n_k} = 0$.

Therefore, $\{x_n\}_n$ is unbounded, but $\{x_{n_k}\}_k$ converges to 0.

Problem 4

(a)

(i)

Proof. Suppose $\{x_n\}_n$ and $\{y_n\}_n$ are bounded. Then $\exists B_0 > 0$ and $B_1 > 0$ such that $\forall n \in \mathbb{N}, |x_n| \leq B_0$ and $|y_n| \leq B_1$.

Let $B = B_0 + B_1$. Then by the triangle inequality, we have

$$|x_n + y_n| \leq |x_n| + |y_n| \leq B_0 + B_1 = B. \quad (19)$$

Thus, $\forall n \in \mathbb{N}$, $|x_n + y_n| \leq B$.

Therefore, $\{x_n + y_n\}_n$ is bounded. \square

(ii)

Proof. Let $a_n = \sup\{x_k \mid k \geq n\}$, $b_n = \sup\{y_k \mid k \geq n\}$, and $c_n = \sup\{x_k + y_k \mid k \geq n\}$.

Then $\forall n \in \mathbb{N}$, $a_n \geq x_n$ and $b_n \geq y_n$, so

$$\implies x_n + y_n \leq a_n + b_n. \quad (20)$$

But $a_n + b_n$ is also an upper bound for $\{x_n + y_n\}_n$, so $c_n \leq a_n + b_n$. Then

$$\lim_{n \rightarrow \infty} c_n \leq \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n. \quad (21)$$

Therefore, $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$. \square

(b)

Proof. Let $x = 0.1111\dots$. Then we can write

$$x = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots \quad (22)$$

$$= -1 + \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n \quad (23)$$

$$= -1 + \frac{1}{1 - \frac{1}{10}} \quad (24)$$

$$= -1 + \frac{10}{9} \quad (25)$$

$$= \frac{1}{9}. \quad (26)$$

Let $\epsilon = \frac{1}{10^n}$ for $n \in \mathbb{N}$ and let $x_n = \frac{1}{9} + \frac{1}{10^{n+1}}$. Then $x_n \in (\frac{1}{9} - \epsilon, \frac{1}{9} + \epsilon)$. So

$$\frac{1}{9} - \frac{1}{10^{n+1}} < x_n < \frac{1}{9} + \frac{1}{10^{n+1}}. \quad (27)$$

But $x_n = \frac{1}{9} + \frac{1}{10^{n+1}} = 0.111\dots11211\dots \in E$. Thus, for every $\epsilon > 0$, $\exists x_n \in (\frac{1}{9} - \epsilon, \frac{1}{9} + \epsilon)$ such that $(\frac{1}{9} - \epsilon, \frac{1}{9} + \epsilon) \cap E \neq \emptyset$.

Therefore, $0.1111\dots$ is a cluster point of E . \square

Problem 5

(a)

Proof. (\Rightarrow) Suppose $\sum_n a_n$ converges.

Since $\forall n \in \mathbb{N}$, $a_n > 0$, $b_n > 0$ and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$, then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\frac{a_n}{b_n} > L - 1 \quad (28)$$

$$\implies a_n > (L - 1)b_n \quad (29)$$

$$\implies (L - 1) \sum_n b_n < \sum_n a_n. \quad (30)$$

But $\sum_n a_n$ converges, so $(L - 1) \sum_n b_n$ converges by comparison.

Thus, $\sum_n b_n$ converges.

(\Leftarrow) Suppose $\sum_n b_n$ converges.

Then similarly, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\frac{a_n}{b_n} < L + 1 \quad (31)$$

$$\implies a_n < (L + 1)b_n \quad (32)$$

$$\implies \frac{1}{L + 1} a_n < b_n. \quad (33)$$

But $\sum_n b_n$ converges, so $\frac{1}{L + 1} \sum_n a_n$ converges by comparison.

Therefore, $\sum_n a_n$ converges. \square

(b)

TODO TODO TODO