18.100A Assignment 9

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Problem 1

Proof. We have that $\forall x \in \mathbb{R}$, $|\arctan(x)| < \frac{\pi}{2}$, i.e.

$$\arctan(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{1}$$

which is an open set. So $\forall |y| < \frac{\pi}{2}$, $\exists \epsilon > 0$ such that $(y - \epsilon, y + \epsilon) \subset (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus for every such y, we can always find a $y_0 < y$ and $y_1 > y$ inside this open set. This means that there is no x_1 such that $\arctan(x_1) \geq \arctan(x)$ nor an x_0 such that $\arctan(x_0) \leq \arctan(x)$ $\forall x$.

Hence $f(x) = \arctan(x)$ does not achieve an absolute minimum or maximum. $\square$

Problem 2

Proof. Let $x, y \in (c, \infty)$. Choose $L = \frac{1}{c^2}$. Then

$$\begin{aligned} |f(y) - f(x)| &= \left|\frac{1}{y} - \frac{1}{x}\right| && (2) \\ &= \frac{|x-y|}{xy} && (3) \\ &< \frac{|x-y|}{c^2} && (4) \\ &= L|x-y|. && (5) \end{aligned}$$

Therefore $f(x) = \frac{1}{x}$ is Lipschitz continuous. $\square$

Problem 3

Proof. Let $\delta > 0$ and choose $\epsilon_0 = |\sin(\delta)|$. Choose $x = \frac{1}{2\pi k+\delta}$ and $c = \frac{1}{2\pi k}$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned} |x - c| &= \left|\frac{1}{2\pi k} - \frac{1}{2\pi k}\right| &(6)\\ &= \left|\frac{2\pi k - (2\pi k + \delta)}{2\pi k(2\pi k + \delta)}\right| &(7)\\ &= \frac{\delta}{4\pi^2 k^2 + 2\pi k\delta} &(8)\\ &< \delta. &(9)\end{aligned}$$

We also have

$$\begin{aligned} |f(x) - f(c)| &= |\sin(2\pi k + \delta) - \sin(2\pi k)| &(10)\\ &= |\sin(2\pi k)\cos(\delta) + \cos(2\pi k)\sin(\delta) - \sin(2\pi k)| &(11)\\ &= |\sin(\delta)| &(12)\\ &= \epsilon_0. &(13)\end{aligned}$$

Hence, $f(x) = \sin\left(\frac{1}{x}\right)$ is not uniformly continuous. $\square$

Problem 4

Proof. Suppose $f : S \to \mathbb{R}$ is Lipschitz continuous on S. Then $\exists L \geq 0$ such that $\forall x, y \in S$, $|f(x) - f(y)| \leq L|x - y|$.

Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{L}$. If $|x - y| < \delta$, then

$$\begin{aligned} |f(x) - f(y)| &\leq L|x - y| &(14)\\ &< L\delta &(15)\\ &= \epsilon. &(16)\end{aligned}$$

Thus f is uniformly continuous on S. $\square$