

18.100A Midterm

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Problem 1

(a)

Proof. Let $x \in f^{-1}(C \cap D)$. Then

$$\implies f(x) \in C \cap D \quad (1)$$

$$\implies f(x) \in C \text{ and } f(x) \in D \quad (2)$$

$$\implies x \in f^{-1}(C) \text{ and } x \in f^{-1}(D) \quad (3)$$

$$\implies x \in f^{-1}(C) \cap f^{-1}(D). \quad (4)$$

Thus,

$$f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D). \quad (5)$$

Now let $x \in f^{-1}(C) \cap f^{-1}(D)$. Then

$$\implies f(x) \in C \text{ and } f(x) \in D \quad (6)$$

$$\implies f(x) \in C \cap D \quad (7)$$

$$\implies x \in f^{-1}(C \cap D). \quad (8)$$

Thus,

$$f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D). \quad (9)$$

Therefore by equations (5) and (9), $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. \square

(b)

Claim: If $E \subset \mathbb{R}$ is countable, then the complement $\mathbb{R} \setminus E$ is always uncountable.

Proof. (By contradiction). Suppose E^c is countable. Then $E \cup E^c$ is countable as well, since it is the union of two countable sets. But $E \cup E^c = \mathbb{R}$, which is uncountable. ($\Rightarrow \Leftarrow$). \square

(c)

By contrast, if $E \subset \mathbb{R}$ is uncountable, then the complement $\mathbb{R} \setminus E$ is not always countable. Take for instance, $E = [0, 1]$, which is uncountable. Then $E^c = (-\infty, 0) \cup (1, \infty)$, which is also uncountable.