

# 18.100A Assignment 6

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March 7, 2023

## Problem 1

(a)

$$\sum_{n=1}^{\infty} \frac{3}{9n+1} = 3 \sum_{n=1}^{\infty} \frac{1}{9n+1} \quad (1)$$

$$= \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n + \frac{1}{9}} \quad (2)$$

$$= \frac{1}{3} \sum_{n=2}^{\infty} \frac{1}{(n-1) + \frac{1}{9}} \quad (3)$$

$$= \frac{1}{3} \sum_{n=2}^{\infty} \frac{1}{n - \frac{8}{9}} \quad (4)$$

$$> \sum_{n=2}^{\infty} \frac{1}{n}. \quad (5)$$

But the Harmonic series,  $\sum_n \frac{1}{n}$ , diverges.

Therefore, we conclude by comparison that the series  $\sum_n \frac{3}{9n+1}$  diverges.

(b)

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n - \frac{1}{2}} > \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}. \quad (6)$$

Therefore, by comparison,  $\sum_n \frac{1}{2n-1}$  diverges.

(c)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} - \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2}, \quad (7)$$

which is the difference of two convergent series.

Therefore,  $\sum_n \frac{(-1)^n}{n^2}$  converges.

(d)

We can express the series  $\sum_n \frac{1}{n(n+1)}$  as a telescoping sum:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1} \quad (8)$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \quad (9)$$

$$= 1. \quad (10)$$

Therefore,  $\sum_n \frac{1}{n(n+1)}$  converges to 1.

(e)

We note that  $\forall n \in \mathbb{N}, e^{n^2} \geq n^3$ . Then we have

$$\sum_{n=1}^{\infty} \frac{n}{e^{n^2}} \leq \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad (11)$$

which converges.

Therefore,  $\sum_n n e^{-n^2}$  converges.

## Problem 2

(a)

*Proof.* Suppose  $\sum_n x_n$  converges.

Then the sequence of partial sums  $\{s_m\}_m = \{x_1 + \cdots + x_m\}_{m=1}^{\infty}$  also converges.

We construct two subsequences of  $\{s_m\}_m$ , defined via

$$\{s_{m_{2k}}\}_k = \{x_2, x_2 + x_4, x_2 + x_4 + x_6, \cdots\} = \{x_2 + \cdots + x_{2k}\}_k, \quad (12)$$

and

$$\{s_{m_{2k+1}}\}_k = \{x_1, x_1 + x_3, x_1 + x_3 + x_5, \cdots\} = \{x_1 + \cdots + x_{2k+1}\}_k. \quad (13)$$

Note that for each  $m \in \mathbb{N}$ , if  $m$  is even then  $m = 2k$  for some  $k \in \mathbb{N}$ . Then

$$s_m = s_{m_{2k-1}} + s_{m_{2k}}. \quad (14)$$

Likewise if  $m$  is odd, then

$$s_m = s_{m_{2k}} + s_{m_{2k+1}}. \quad (15)$$

In both cases, the partial sum  $s_m$  must converge because the series  $\sum_n x_n$  converges. Thus, the sum of the two partial sums must also converge, i.e.  $\{s_{m_{2k}}\}_k$  and  $\{s_{m_{2k+1}}\}_k$  both converge.

Then both series  $\sum_n x_{2n}$  and  $\sum_n x_{2n+1}$  must converge, and so does their sum:

$$\sum_n x_{2n} + \sum_n x_{2n+1} < \infty. \quad (16)$$

Therefore,  $\sum_n (x_{2n} + x_{2n+1})$  converges.  $\square$

**(b)**

Consider the following counterexample to the converse of the theorem proven in problem **(a)**:

Let  $x_n = (-1)^n$ . Then

$$\sum_{n=1}^{\infty} (x_{2n} + x_{2n+1}) = \sum_{n=1}^{\infty} [(-1)^{2n} + (-1)^{2n+1}] \quad (17)$$

$$= \sum_{n=1}^{\infty} (1^n - 1^n) \quad (18)$$

$$= 0. \quad (19)$$

But  $\sum_n (-1)^n$  does not converge.

Therefore, the converse to **(a)** does not always hold.

### Problem 3

*Proof.* Suppose  $\sum_n x_n$  converges absolutely.

Since the series  $\sum_n |x_n|$  converges, then  $\sum_n x_n$  converges by comparison, because  $x_n \leq |x_n|$ .

Let  $N \in \mathbb{N}$ . Then the triangle inequality gives

$$|x_1 + x_2 + \cdots + x_N| \leq |x_1| + |x_2| + \cdots + |x_N| = \sum_{n=1}^N |x_n|. \quad (20)$$

Because  $\sum_n |x_n|$  converges, we can now take  $N \rightarrow \infty$ , which gives

$$\left| \sum_{n=1}^{\infty} x_n \right| \leq \sum_{n=1}^{\infty} |x_n|, \quad (21)$$

as desired.  $\square$

## Problem 4

(a)

Using the sum of a geometric sequence, we have

$$\sum_{n=1}^{\infty} \frac{1}{2^{2n+1}} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \quad (22)$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n \quad (23)$$

$$= \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{4}} \quad (24)$$

$$= \frac{1}{2} \cdot \frac{4}{3}. \quad (25)$$

Therefore,  $\sum_n \left(\frac{1}{2}\right)^{2n+1}$  converges to  $\frac{2}{3}$ .

(b)

Let  $x_n = \frac{(-1)^n(n-1)}{n}$ . Then we easily check that  $\lim_{n \rightarrow \infty} x_n \neq 0$ , since the non-alternating component approaches 1, but then alternates between  $-1$  and  $1$ ; i.e. the limit does not exist.

Therefore,  $\sum_n \frac{(-1)^n(n-1)}{n}$  diverges.

(c)

Note that since  $n^{\frac{1}{10}} \leq n \ \forall n \in \mathbb{N}$ , we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{10}}} \geq \sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \quad (26)$$

which diverges.

Therefore,  $\sum_n \frac{(-1)^n}{n^{\frac{1}{10}}}$  diverges by comparison.

(d)

We have

$$\sum_{n=1}^{\infty} \frac{n^n}{(n+1)^{2n}} = \sum_{n=1}^{\infty} \frac{n^n}{(n^2 + 2n + 1)^n} \quad (27)$$

$$= \sum_{n=1}^{\infty} \left( \frac{n}{n^2 + 2n + 1} \right)^n. \quad (28)$$

We also have that  $\forall n \in \mathbb{N}$ ,

$$\left| \frac{n}{n^2 + 2n + 1} \right| < \frac{1}{4} < 1. \quad (29)$$

Then the series is upper bounded by a geometric series:

$$\sum_{n=1}^{\infty} \frac{n^n}{(n+1)^{2n}} < \sum_{n=1}^{\infty} \left( \frac{1}{4} \right)^n = \frac{4}{3}. \quad (30)$$

Therefore,  $\sum_n \frac{n^n}{(n+1)^{2n}}$  converges.

## Problem 5

Let  $x_n = \frac{(-1)^n}{\sqrt{n}}$ . Then  $x_n^2 = \frac{1}{n}$ .

Since  $\{\frac{1}{n}\}_n$  is decreasing and  $\lim_{n \rightarrow \infty} x_n = 0$ , then  $\sum_n \frac{(-1)^n}{\sqrt{n}}$  converges. But the harmonic series  $\sum_n \frac{1}{n}$  diverges.

Therefore,  $\sum_n x_n$  converges, but  $\sum_n x_n^2$  diverges.