18.100A Midterm

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Problem 1

(a)

Proof. Let $x \in f^{-1}(C \cap D)$. Then

$$\begin{align} &\implies f(x) \in C \cap D \tag{1}\\ &\implies f(x) \in C \text{ and } f(x) \in D \tag{2}\\ &\implies x \in f^{-1}(C) \text{ and } x \in f^{-1}(D) \tag{3}\\ &\implies x \in f^{-1}(C) \cap f^{-1}(D). \tag{4}\end{align}$$

Thus,

$$f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D). \tag{5}$$

Now let $x \in f^{-1}(C) \cap f^{-1}(D)$. Then

$$\begin{align} &\implies f(x) \in C \text{ and } f(x) \in D \tag{6}\\ &\implies f(x) \in C \cap D \tag{7}\\ &\implies x \in f^{-1}(C \cap D). \tag{8}\end{align}$$

Thus,

$$f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D). \tag{9}$$

Therefore by equations (5) and (9), $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. □

(b)

Claim: If $E \subset \mathbb{R}$ is countable, then the complement $\mathbb{R} \backslash E$ is always uncountable.

Proof. (By contradiction). Suppose E^c is countable. Then $E \cup E^c$ is countable as well, since it is the union of two countable sets. But $E \cup E^c = \mathbb{R}$, which is uncountable. ($\Rightarrow\Leftarrow$). □

(c)

By contrast, if $E \subset \mathbb{R}$ is uncountable, then the complement $\mathbb{R} \backslash E$ is not always countable. Take for instance, $E = [0, 1]$, which is uncountable. Then $E^c = (-\infty, 0) \cup (1, \infty)$, which is also uncountable.

Problem 2

(a)

A set $U \subset \mathbb{R}$ is *not open* if for every $\epsilon > 0$, $\exists x \in U$ such that $(x - \epsilon, x + \epsilon) \not\subset \mathbb{R}$.

(b)

Proof. Suppose U is not open. Let $\epsilon = \frac{1}{n}$. Then for every $n \in \mathbb{N}$, $\exists x \in U$ such that $(x - \frac{1}{n}, x + \frac{1}{n}) \not\subset U$. Equivalently, for each $n \in \mathbb{N}$ $\exists x_n \in U^c$ such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n}. \tag{10}$$

Then we have

$$0 < |x_n - x| < \frac{1}{n}, \tag{11}$$

and taking the limit on all sides gives

$$0 < \lim_{n\to\infty} |x_n - x| < \lim_{n\to\infty} \frac{1}{n}. \tag{12}$$

Thus, by the squeeze theorem, $\lim_{n\to\infty} x_n = x$, as desired. □

(c)

Proof. (By contradiction). To show that F is closed, we must show that F^c is open. Suppose, toward a contradiction, that F^c is not open. Then by part **(b)**, $\exists x \in F^c$ and a sequence $\{x_n\}_n$ of elements of F such that $\lim_{n\to\infty} x_n = x$. But by assumption, every convergent sequence of elements of F has a limit in F, i.e. we assumed originally that $x \in F$ ($\Rightarrow\Leftarrow$). Thus, F^c must be open, so F is closed. □

Problem 3

(a)

Proof. Let $\epsilon > 0$. Choose $N = \frac{1}{\sqrt{\epsilon}}$. Then $\forall n \geq N$, we have

$$\begin{aligned}\left|\frac{10n^2}{n^2+16n+1} - 10\right| &= \left|\frac{-160n-10}{n^2+16n+1}\right| &(13)\\ &= \left|\frac{160n+10}{n^2+16n+1}\right| &(14)\\ &< \frac{1}{n^2+16n+1} &(15)\\ &< \frac{1}{n^2} &(16)\\ &< \epsilon. &(17)\end{aligned}$$

Therefore, $\lim_{n\to\infty} \left|\frac{10n^2}{n^2+16n+1}\right| = 10.$ □

(b)

(i) Let $x_n = \frac{(-1)^n}{n}$. Then $\lim_{n\to\infty} x_n = 0$. But $x_1 = -1 < \frac{1}{2} = x_2$, whereas $x_2 = \frac{1}{2} > -\frac{1}{3} = x_3$.

Therefore $\{x_n\}_n$ converges to 0, but is not monotonic.

(ii) Let

$$x_n = \begin{cases} 0, & n \text{ even} \\ n, & n \text{ odd.} \end{cases} \tag{18}$$

Then $\{x_n\}_n$ is clearly unbounded.

Consider the subsequence $\{x_{n_k}\}_k$ defined by $x_{n_k} = x_{2k}$ for each $k \in \mathbb{N}$. Then $\forall k$, $x_{n_k} = 0$.

Therefore, $\{x_n\}_n$ is unbounded, but $\{x_{n_k}\}_k$ converges to 0.

Problem 4

(a)

(i)

Proof. Suppose $\{x_n\}_n$ and $\{y_n\}_n$ are bounded. Then $\exists B_0 > 0$ and $B_1 > 0$ such that $\forall n \in \mathbb{N}$, $|x_n| \leq B_0$ and $|y_n| \leq B_1$.

Let $B = B_0 + B_1$. Then by the triangle inequality, we have

$$|x_n + y_n| \leq |x_n| + |y_n| \leq B_0 + B_1 = B. \tag{19}$$

Thus, $\forall n \in \mathbb{N}$, $|x_n + y_n| \leq B$.

Therefore, $\{x_n + y_n\}_n$ is bounded. $\square$

(ii)

Proof. Let $a_n = \sup\{x_k \mid k \geq n\}$, $b_n = \sup\{y_k \mid k \geq n\}$, and $c_n = \sup\{x_k + y_k \mid k \geq n\}$.

Then $\forall n \in \mathbb{N}$, $a_n \geq x_n$ and $b_n \geq y_n$, so

$$\implies x_n + y_n \leq a_n + b_n. \tag{20}$$

But $a_n + b_n$ is also an upper bound for $\{x_n + y_n\}_n$, so $c_n \leq a_n + b_n$. Then

$$\lim_{n\to\infty} c_n \leq \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n. \tag{21}$$

Therefore, $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$. $\square$