18.100A Assignment 5

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Problem 1

Proof. We have that

$$L = \lim_{n\to\infty} \frac{|x_{n+1} - x|}{|x_n - x|} < 1. \tag{1}$$

Then for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\left| \frac{x_{n+1} - x}{x_n - x} \right| - 1 < \epsilon. \tag{2}$$

Rearranging gives

$$|x_{n+1} - x| < (1 + \epsilon)|x_n - x|. \tag{3}$$

By taking ϵ to be arbitrarily small, we have, $\forall n \geq N$,

$$\xRightarrow{\epsilon\to 0} |x_{n+1} - x| < |x_n - x|. \tag{4}$$

Define $y_n := |x_n - x|$. Then $\forall n \geq N$,

$$0 \leq y_{n+1} < y_n, \tag{5}$$

so $\{y_n\}_n$ is a decreasing sequence bounded below by 0. Hence,

$$\implies y_n \to 0 \tag{6}$$

$$\implies |x_n - x| \to 0 \tag{7}$$

$$\implies x_n \to x. \tag{8}$$

Therefore, $\{x_n\}_n$ converges to x. $\square$

Problem 2

(a)

Let $x_n = \frac{(-1)^n}{n}$. Then $\forall n \in \mathbb{N}$,

$$-\frac{1}{n} \leq x_n \leq \frac{1}{n}. \tag{9}$$

Allowing $n \to \infty$ on all sides of the inequality gives

$$0 = \lim_{n\to\infty} \left(-\frac{1}{n}\right) \leq \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0. \tag{10}$$

So by the Squeeze Theorem, $\lim_{n\to\infty} x_n = 0$. Finally, by the theorem from lecture 9, we conclude that $\liminf x_n = \limsup x_n = 0$.