18.100A Assignment 8

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Problem 1

Proof. Suppose $\lim_{x\to c} f(x) = \lim_{x\to c} h(x)$. Then since c is a cluster point of S and $\forall x \in S$, $f(x) \le g(x) \le h(x)$, we have

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x) \le \lim_{x\to c} h(x) = \lim_{x\to c} f(x). \tag{1}$$

Thus by the squeeze theorem,

$$\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x), \tag{2}$$

as desired. $\square$

Problem 2

Proof. (1) Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Then if $|x| < \delta$, we have

$$\begin{aligned} |f(x) - f(0)| &= |f(x)| && (3)\\ &\le 2|x| && (4)\\ &< 2\delta && (5)\\ &= \epsilon. && (6) \end{aligned}$$

Therefore, f is continuous at $x = 0$.

(2) Let $\delta > 0$, $\epsilon_0 > 0$. Suppose $|x - 1| < \delta$. Let

$$x_0 = \begin{cases} \epsilon_0\sqrt{2}, & \epsilon_0 \in \mathbb{Q} \\ \epsilon_0, & \epsilon_0 \notin \mathbb{Q}. \end{cases} \tag{7}$$

Then $x_0 \notin \mathbb{Q}$ $\forall \epsilon_0 > 0$.

Choose $x = \frac{x_0}{2}$. Then

$$\begin{aligned} |f(x) - f(1)| &= |f(x)| && (8) \\ &= 2|x| && (9) \\ &= x_0 && (10) \\ &\geq \epsilon_0. && (11) \end{aligned}$$

Therefore, f is discontinuous at $x = 1$. $\square$

Problem 3

Proof. Since f is continuous at c, then $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$, i.e.

$$f(c) - \epsilon < f(x) < f(c) + \epsilon. \tag{12}$$

Let $\epsilon = \frac{f(c)}{2}$. Then for some $\delta > 0$,

$$0 < \frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}. \tag{13}$$

Simply choose $\alpha = \delta$, and we are done. $\square$

Problem 4

Proof. Since $h = f$ on $[-1, 0]$ and $f(x)$ is continuous, then h is continuous on $[-1, 0]$. Similarly, since $h = g$ on $(0, 1]$ and g is continuous, then h is continuous on $(0, 1]$.

It remains only to show that h is continuous at $x = 0$.

Let $\epsilon_0 > 0$. By continuity of f, $\exists \delta_0$ such that if $|x| < \delta_0$, then $|f(x) - f(0)| < \epsilon_0$. Similarly, let $\epsilon_1 > 0$. Then by continuity of g, $\exists \delta_1 > 0$ such that if $|x| < \delta_1$, then $|g(x) - g(0)| < \epsilon_1$.

Let $x \in [-1, 1]$ and let $\epsilon = \min\{\epsilon_0, \epsilon_1\}$. Choose $\delta = \min\{\delta_0, \delta_1\}$. If $|x| < \delta$, then $|x| < \delta_0$ and $|x| < \delta_1$. If $x \leq 0$, then we have

$$|h(x) - h(0)| = |f(x) - f(0)| < \epsilon_0 \leq \epsilon. \tag{14}$$

Similarly if $x > 0$, then we have

$$|h(x) - h(0)| = |g(x) - g(0)| < \epsilon_1 \leq \epsilon. \tag{15}$$

Thus, $h(x)$ is continuous at $x = 0$, so we conclude h is continuous. $\square$

Problem 5

Proof. $(\Rightarrow)$ Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, and let $U \subset \mathbb{R}$ be open.

Let $f(c) \in U$. Then $c = f^{-1}(f(c)) \in f^{-1}(U)$. Since f is continuous at c, then $\forall \epsilon > 0\ \exists \delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$. Since this must hold for every $c \in f^{-1}(U)$, and the subset $(c - \delta, c + \delta)$ is open, then $f^{-1}(U)$ is also open.

$(\Leftarrow)$ Suppose for every $U \subset \mathbb{R}$ open, $f^{-1}(U)$ is also open.

Let $c \in f^{-1}(U)$. Then $f(c) \in U$. Since U is open, then $\exists \epsilon > 0$ such that $(f(c) - \epsilon, f(c) + \epsilon)$. Let $A = (f(c) - \epsilon, f(c) + \epsilon)$. Then since A is open, $f^{-1}(A)$ is open by assumption. Then $\exists \delta > 0$ such that $(c - \delta, c + \delta) \subset f^{-1}(A)$, i.e. if $x \in (c - \delta, c + \delta)$, then $f(x) \in A$.

Thus for every $\epsilon > 0$, $\exists \delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$. Therefore f is continuous at c. This holds for every c in every open subset of $\mathbb{R}$. Hence f is continuous. $\square$