18.100A Assignment 12

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Problem 1

(a)

Proof. Suppose $\exists c \in [a,b]$ such that $f(c) > 0$. Since f is continuous, $\exists \delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(x)| < \frac{f(c)}{2}$, i.e. $\frac{f(c)}{2} < f(x)$. We compute

$$0 > \int_a^b f \tag{1}$$

$$> \int_a^b \frac{f(c)}{2} \tag{2}$$

$$= \frac{f(c)}{2}(b - a) \tag{3}$$

$$> 0, \tag{4}$$

i.e. $0 > 0$ ($\Rightarrow\Leftarrow$), which is cleary a contradiction.

Therefore $f(x) = 0 \ \forall x \in [a,b]$. $\qquad\square$

(b)

Proof. Let $E = \int_a^b (u')^2 \mathrm{d}x$. Then $E \geq 0$ since $(u')^2 \geq 0$. Using integration by parts, we have

$$E = \int_a^b u'u' \mathrm{d}x \tag{5}$$

$$= uu'\big|_a^b - \int_a^b uu'' \mathrm{d}x \tag{6}$$

$$= u'(b)u(b) - u'(a)u(a) - \int_a^b u(Vu)\mathrm{d}x \tag{7}$$

$$= -\int_a^b Vu^2 \mathrm{d}x. \tag{8}$$

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But $V(x) \geq 0$ and $u^2 \geq 0$, so $-(Vu^2) \leq 0$, hence $E \leq 0$. Thus $E = 0$, which must mean that

$$\int_a^b (u')^2 \mathrm{d}x = 0, \tag{9}$$

and by part **(a)**, this implies that $(u')^2 = 0 \ \forall x \in [a, b]$. Hence $u'(x) = 0$ for all x, and since $u(a) = 0$, then u remains constant at 0; i.e. $u = 0$ everywhere. $\square$

Problem 2

We compute:

$$\int_{-x}^{x} e^{s^2} \mathrm{d}s = \int_{-x}^{0} e^{s^2} \mathrm{d}s + \int_{0}^{x} e^{s^2} \mathrm{d}s \tag{10}$$

$$= \int_{0}^{x} e^{s^2} \mathrm{d}s - \int_{0}^{-x} e^{s^2} \mathrm{d}s. \tag{11}$$

Differentiating, we get

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\int_{-x}^{x} e^{s^2} \mathrm{d}s \right) = \frac{\mathrm{d}}{\mathrm{d}x} \left(\int_{0}^{x} e^{s^2} \right) - \frac{\mathrm{d}}{\mathrm{d}x} \left(\int_{0}^{-x} e^{s^2} \right) \tag{12}$$

$$= e^{x^2} + e^{x^2}. \tag{13}$$

Thus $\frac{\mathrm{d}}{\mathrm{d}x} \left(\int_{-x}^{x} e^{s^2} \mathrm{d}s \right) = 2e^{x^2}$.