

# 18.100A Assignment 3

Octavio Vega

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## Problem 1

*Proof.* Let  $x, y \in \mathbb{R}$  with  $x < y$ . By the density of  $\mathbb{Q}$ , we have that  $\exists r \in \mathbb{Q}$  such that  $x < r < y$ .

Then  $x + \sqrt{2} < y + \sqrt{2}$ . Then  $\exists r \in \mathbb{Q}$  such that

$$x + \sqrt{2} < r < y + \sqrt{2} \quad (1)$$

$$\implies x < r - \sqrt{2} < y. \quad (2)$$

But since  $r \in \mathbb{Q}$  and  $\sqrt{2} \notin \mathbb{Q}$ , then the number  $i := r - \sqrt{2} \notin \mathbb{Q}$ .

So  $x < i < y$  with  $i \in \mathbb{R} \setminus \mathbb{Q}$ , as desired.  $\square$

## Problem 2

*Proof.* Define the function  $f : E \rightarrow \wp(\mathbb{N})$  such that if  $x = 0.d_{-1}d_{-2}\dots$ , then

$$f(x) = \{j \in \mathbb{N} \mid d_{-j} = 2\}. \quad (3)$$

We want to show that  $f$  is a bijection. First, we show that  $f$  is injective.

Let  $x_1 = 0.d_{-1}^{(1)}d_{-2}^{(1)}\dots$  and  $x_2 = 0.d_{-1}^{(2)}d_{-2}^{(2)}\dots$  for  $x_1, x_2 \in E$ . Suppose  $f(x_1) = f(x_2)$ . Then

$$\{j \in \mathbb{N} \mid d_{-j}^{(1)} = 2\} = \{k \in \mathbb{N} \mid d_{-k}^{(2)} = 2\}. \quad (4)$$

Since each digit  $d_{-j} \in \{1, 2\}$ , then the sets of digits must be the same:

$$\{d_{-j}^{(1)} \mid j \in \mathbb{N}\} = \{d_{-k}^{(2)} \mid k \in \mathbb{N}\}. \quad (5)$$

But by the theorem from class, we know that for every set of digits  $\exists! x \in [0, 1]$  such that  $x = 0.d_{-1}d_{-2}\dots$ . So if all of the digits are the same, then the numbers must be the same, i.e.

$$f(x_1) = f(x_2) \implies x_1 = x_2. \quad (6)$$

Thus  $f$  is injective.

Next, we show that  $f$  is surjective.

Let  $S \in \wp(\mathbb{N})$  with

$$S := \{j \in \mathbb{N} \mid d_{-j} = 2\}. \quad (7)$$

Since this corresponds to the indices for a set of digits, then by the theorem from class  $\exists x \in [0, 1]$  such that  $x = 0.d_{-1}d_{-2}\dots$ ; i.e. for any  $S \in \wp(\mathbb{N})$ ,  $\exists x \in E$  such that  $f(x) = S$ .

Hence,  $f$  is also surjective, which means that it is bijective.

Therefore we conclude that  $|E| = |\wp(\mathbb{N})|$ .  $\square$

### Problem 3

(a)

*Proof.* We want to show that there exists a bijection  $h : A \cup B \rightarrow \mathbb{N}$ .

Recall that we can construct the sets of even and odd natural numbers as follows:

$$\mathcal{O} := \{2n + 1 \mid n \in \mathbb{N}\}, \text{ and } \mathcal{E} := \{2n \mid n \in \mathbb{N}\}. \quad (8)$$

We also know that  $|\mathcal{O}| = |\mathcal{E}| = |\mathbb{N}|$  because the functions defined via  $f_e(n) = 2n$  and  $f_o(n) = 2n + 1$ , mapping  $\mathcal{E}$  to  $\mathbb{N}$  and  $\mathcal{O}$  to  $\mathbb{N}$ , respectively, are both bijective. Finally, we note that  $\mathcal{O} \cup \mathcal{E} = \mathbb{N}$ .

Since  $A$  and  $B$  are both countably infinite, i.e.  $|A| = |B| = |\mathbb{N}|$ , then  $\exists$  bijections  $f, g$  such that

$$f : A \rightarrow \mathbb{N}, \text{ and } g : B \rightarrow \mathbb{N}. \quad (9)$$

Then  $\forall a \in A$ ,  $2f(a)$  is even, and  $\forall b \in B$ ,  $2g(b) + 1$  is odd. So we define the even and odd sets

$$\mathcal{E} := \{2f(a) \mid a \in A\}, \text{ and } \mathcal{O} := \{2g(b) + 1 \mid b \in B\}. \quad (10)$$

Then we construct the function  $h$  defined via

$$h(x) = \begin{cases} 2f(x) & \text{if } x \in A \\ 2g(x) + 1 & \text{if } x \in B. \end{cases} \quad (11)$$

First, we show that  $h$  is injective.

Case 1:  $x, y \in A$ . Then

$$h(x) = h(y) \quad (12)$$

$$\implies 2f(x) = 2f(y) \quad (13)$$

$$\implies f(x) = f(y) \quad (14)$$

$$\implies x = y, \quad (15)$$

Since  $f$  is injective.

Case 2:  $x, y \in B$ . Then

$$h(x) = h(y) \quad (16)$$

$$\implies 2g(x) + 1 = 2g(y) + 1 \quad (17)$$

$$\implies g(x) = g(y) \quad (18)$$

$$\implies x = y, \quad (19)$$

since  $g$  is injective.

Case 3:  $x \in A, y \in B$  WOLOG, and  $h(x) = h(y)$ . This case is vacuous because  $\mathcal{E}$  and  $\mathcal{O}$  are disjoint, whereas  $2f(x) \in \mathcal{E}$  while  $2g(x) + 1 \in \mathcal{O}$ .

Thus, according to cases 1 and 2,  $h$  is injective.

Now we show that  $h$  is surjective. Let  $n \in \mathbb{N}$ .

Case 1:  $n$  is even. Then by surjectivity of  $f$ ,  $\exists a \in A$  such that  $f(a) = \frac{n}{2}$ , i.e.  $h(a) = n$ .

Case 2:  $n$  is odd. Then by surjectivity of  $g$ ,  $\exists b \in B$  such that  $g(b) = \frac{n-1}{2}$ , i.e.  $h(b) = n$ .

Thus,  $h$  is surjective as well.

Therefore we conclude that  $h$  is a bijection, hence  $|A \cup B| = |\mathbb{N}|$ .  $\square$

**(b)**

*Proof.* (By contradiction). Suppose instead that  $\mathbb{R} \setminus \mathbb{Q}$  is countable. We write:

$$\mathbb{R} = \mathbb{R} \setminus \mathbb{Q} \cup \mathbb{Q} \quad (20)$$

$$\implies |\mathbb{R}| = |\mathbb{R} \setminus \mathbb{Q} \cup \mathbb{Q}|. \quad (21)$$

We know that  $\mathbb{Q}$  is countably infinite, so  $|\mathbb{Q}| = |\mathbb{N}|$ . We also assumed that  $\mathbb{R} \setminus \mathbb{Q}$  is countable, so  $|\mathbb{R} \setminus \mathbb{Q}| = |\mathbb{N}|$ . Then by part **(a)**, we have that  $|\mathbb{R}| = |\mathbb{N}|$ , but the reals are uncountable, so contradiction ( $\Rightarrow \Leftarrow$ ).

Hence  $\mathbb{R} \setminus \mathbb{Q}$  must be uncountable.  $\square$

## Problem 4

*Proof.* ( $\Rightarrow$ ) Suppose  $a_0 = \sup A$ . Then for any other upper bound  $b \in \mathbb{R}$  of  $A$ ,  $a_0 \leq b$ . Also, for any  $a \in A$ ,  $a \leq a_0$ .

Let  $\epsilon > 0$ . Then  $a_0 + \epsilon > a_0 \implies a_0 - \epsilon < a_0$ , but  $a_0 = \sup A$  so  $a_0 - \epsilon \neq \sup A$ . Then  $\exists a \in A$  such that  $a > a_0 - \epsilon$ .

( $\Leftarrow$ ) Suppose  $a_0$  is an upper bound for  $A \subset \mathbb{R}$ , and that for every  $\epsilon > 0$   $\exists a \in A$  such that  $a_0 - \epsilon < a$ . Since  $A$  is bounded above by  $a_0$ , then  $\forall a \in A$ ,  $a \leq a_0$ . Then for every  $\epsilon > 0$ ,

$$\implies a - \epsilon \leq a_0 - \epsilon < a \quad (22)$$

$$\implies a \leq a_0 < a + \epsilon. \quad (23)$$

But  $a_0$  is an upper bound for  $A$ , so we have shown that for any  $\epsilon > 0$ ,  $\exists a \in A$  such that  $a + \epsilon \notin A$  and  $a_0 < a + \epsilon$ . Thus  $a_0$  is the smallest upper bound for  $A$ , which means that  $a_0 = \sup A$  by definition.  $\square$

## Problem 5

(a)

*Proof.* (i) Let  $\epsilon > 0$ . Then  $a - \epsilon < a \implies a - \epsilon \in (-\infty, a)$ . Since  $(-\infty, a)$  is not bounded below,  $\exists x \in (-\infty, a)$  such that  $-\infty < x < a - \epsilon$ , i.e.  $-\infty < x + \epsilon < a$ .

Also, for  $\epsilon > 0$  and  $x \in (-\infty, a)$ , it holds that  $-\infty < x - \epsilon < a$ . So

$$(x - \epsilon, x + \epsilon) \subset (-\infty, a). \quad (24)$$

Therefore,  $(-\infty, a)$  is open.

(ii)  $\forall x \in (a, b)$ , we have that  $b - x > 0$  and  $x - a > 0$ . Let  $\epsilon = \frac{1}{2} \min\{x - a, b - x\} > 0$ , and let  $y \in \mathbb{R}$ .

If  $y \in (x - \epsilon, x + \epsilon)$ , then  $-\epsilon < y - x < \epsilon$ . But  $\epsilon < b - x \implies y - x < b - x$ , i.e.  $y < b$ .

Also,  $\epsilon < x - a \implies -\epsilon > a - x$ , so  $y - x > a - x$ , i.e.  $y > a$ . Thus,  $a < y < b$   $\forall y \in (x - \epsilon, x + \epsilon)$ .

Therefore,  $(a, b)$  is open.

(iii) Let  $x \in (b, \infty)$ . Then  $x > b$ . Let  $\epsilon = \frac{x - b}{2} > 0$ , and let  $y \in \mathbb{R}$ .

If  $y \in (x - \epsilon, x + \epsilon)$ , then  $-\epsilon < y - x < \epsilon$ . Since  $\epsilon < x - b \implies -\epsilon > b - x$ ,

$$\implies b - x < y - x < x - b \quad (25)$$

$$\implies b < y < 2x - b < \infty \quad (26)$$

$$\implies y \in (b, \infty). \quad (27)$$

Therefore,  $(b, \infty)$  is closed.  $\square$

(b)

*Proof.* Suppose  $U_\lambda \subset \mathbb{R}$  is open  $\forall \lambda \in \Lambda$ .

Take any  $x \in \bigcup_{\lambda \in \Lambda} U_\lambda$ . Then  $x \in U_\lambda$  for at least one  $\lambda \in \Lambda$ . But since the  $U_\lambda$  are open, then  $\exists \epsilon > 0$  such that  $(x - \epsilon, x + \epsilon) \subset U_\lambda$ .

But this must hold for every  $x \in \bigcup_{\lambda \in \Lambda} U_\lambda$ , i.e. for every such  $x$ ,  $\exists \epsilon > 0$  such that

$$(x - \epsilon, x + \epsilon) \subset \bigcup_{\lambda \in \Lambda} U_\lambda. \quad (28)$$

Therefore, the union must also be open.  $\square$

**(c)**

*Proof.* Suppose  $U_m$  is open  $\forall m \in \{1, \dots, n\}$ . Choose any  $x \in \bigcap_{m=1}^n U_m$ . Then if  $x$  is in the intersection of the sets, it must be in each of the sets, i.e.  $x \in U_1, U_2, \dots, U_n$ .

Since the  $U_m$  are open, then  $\exists \epsilon > 0$  such that  $(x - \epsilon, x + \epsilon) \subset U_m \forall m \in \{1, \dots, n\}$ . But this must hold for every  $x$  in the intersection, i.e.  $\forall x \in \bigcap_{m=1}^n U_m$ ,  $\exists \epsilon > 0$  such that

$$(x - \epsilon, x + \epsilon) \subset \bigcap_{m=1}^n U_m. \quad (29)$$

Therefore, the intersection must also be open.  $\square$