

18.100A Assignment 11

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Problem 1

Proof. Let $f(x) = \frac{1}{1121}x^{1121} + \frac{1}{2021}x^{2021} + x + 1$. We compute the derivative:

$$f'(x) = x^{1120} + x^{2020} + 1. \quad (1)$$

Hence $f'(x) > 0 \forall x \in \mathbb{R}$, so $f(x)$ is increasing.

Suppose $f(x)$ has n real roots, where $n > 1$. Then $\exists x_1, x_2, \dots, x_n$ such that $f(x_1) = \dots = f(x_n) = 0$. Since $f(x)$ is polynomial, then f is continuous $\forall x$ and differentiable on \mathbb{R} . By Rolle's theorem, $\exists c \in (x_1, x_2)$ such that $f'(c) = 0$, which is a contradiction since $f'(x) > 0 \forall x$. Thus f cannot have more than one real root.

Now suppose $f(x)$ has no real roots. Then either $f(x) > 0 \forall x$ or $f(x) < 0 \forall x$. Choose $x_0 = 1$. Then $f(x_0) = \frac{1}{1121} + \frac{1}{2021} + 2 > 0$. Choose $x^* = -10$. Then $f(x^*) = -\frac{10^{1121}}{1121} - \frac{10^{2021}}{2021} + 2 < 0$. Then by continuity, $f(x)$ has at least one real root, which is a contradiction. Thus f must have at least one real root.

So we have the number of real roots $1 \leq n \leq 1$, so $n = 1$.

Therefore f has exactly one real root. □

Problem 2

(a)

Let $f(x) = \sin(x)$ and $x_0 = 0$. We compute

$$P_4(x) = \sum_{k=0}^4 \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k. \quad (2)$$

The derivatives are

$$f'(x) = \cos(x) \implies f'(x_0) = \cos(0) = 1 \quad (3)$$

$$f''(x) = -\sin(x) \implies f''(x_0) = -\sin(0) = 0 \quad (4)$$

$$f'''(x) = -\cos(x) \implies f'''(x_0) = -\cos(0) = -1 \quad (5)$$

$$f^{(4)}(x) = \sin(x) \implies f^{(4)}(x_0) = \sin(0) = 0. \quad (6)$$

Therefore, the fourth Taylor polynomial is

$$P_4(x) = x - \frac{1}{3!}x^3. \quad (7)$$

(b)

Let $f(x) = \frac{1}{1-x}$ and $x_0 = -1$. The derivatives are

$$f'(x) = \frac{1}{(1-x)^2} \implies f'(x_0) = \frac{1}{4} \quad (8)$$

$$f''(x) = \frac{2}{(1-x)^3} \implies f''(x_0) = \frac{1}{4} \quad (9)$$

$$f'''(x) = \frac{6}{(1-x)^4} \implies f'''(x_0) = \frac{3}{8} \quad (10)$$

$$f^{(4)}(x) = \frac{24}{(1-x)^5} \implies f^{(4)}(x_0) = \frac{3}{4}. \quad (11)$$

Therefore, the fourth Taylor polynomial is

$$P_4(x) = \frac{1}{2} + \frac{1}{4}(x+1) + \frac{1}{8}(x+1)^2 + \frac{1}{16}(x+1)^3 + \frac{1}{32}(x+1)^4. \quad (12)$$

Problem 3

(a)

We compute using L'Hopital:

$$\lim_{x \rightarrow 0} \frac{x - \sin(x)}{x^3} = \lim_{x \rightarrow 0} \frac{1 - \cos(x)}{3x^2} \quad (13)$$

$$= \lim_{x \rightarrow 0} \frac{\sin(x)}{6x} \quad (14)$$

$$= \lim_{x \rightarrow 0} \frac{\cos(x)}{6} \quad (15)$$

$$= \frac{1}{6}. \quad (16)$$

(b)

We proceed again using L'Hopital's rule:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin(x)}{\left(x - \frac{\pi}{2}\right)^2} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{-\cos(x)}{2\left(x - \frac{\pi}{2}\right)} \quad (17)$$

$$= -\frac{1}{2} \lim_{x \rightarrow \frac{\pi}{2}} \sin(x) \quad (18)$$

$$= -\frac{1}{2}. \quad (19)$$

Problem 4

Proof. (By contradiction.) Assume without loss of generality that f has a local maximum at c . Then $\exists \delta > 0$ such that $\forall x \in (a, b)$, if $|x - c| < \delta$ then $f(c) \geq f(x)$.

Case 1: f is constant around c . Then $\forall |x - c| < \delta$, we have $f'(x) = 0$. By differentiating twice, then we must have $f'''(c) = 0$, which is a contradiction since we assumed f''' to be positive.

So f cannot have a local max at c .

Case 2: f is not constant around c . Then $\forall |x - c| < \delta$, $f(c) > f(x)$. By assumption, $f'''(c) > 0$, so f'' is increasing at c . Since $f''(c) = 0$, then $f''(x) > 0 \forall x > c$. Thus f' is increasing for $x > c$. Since $f'(c) = 0$ also by assumption of local maximum, then $f'(x) > 0$ for $x > c$. Thus f is increasing for $x > c$. This means that $f(x) > f(c)$ for $x > c$, which is a contradiction since we assumed that $f(c)$ is a local maximum.

Thus f has no local maximum at c .

To prove that f cannot have a local minimum, repeat the same proof above with $g(x) = -f(x)$, and we are done. \square