

18.100A Assignment 2

Octavio Vega

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Problem 1

Proof. (By contradiction).

Suppose instead that $xy \leq xz$. Then

$$\implies xy - xz \leq 0$$

$$\implies x(y - z) \leq 0.$$

Since $x < 0$ by assumption, it must then be true that $y - z \geq 0$. But then

$$\implies y \geq z \quad \Rightarrow \Leftarrow,$$

which is a contradiction since we assumed that $y < z$. Thus, $xy > xz$. \square

Problem 2

(a)

Proof. We want to show that $\exists b \in S$ such that $\forall a \in A, a \leq b$.

Since S is ordered, then for every $x, y \in S$, we have that either $x < y$, $x > y$, or $x = y$. But since $A \subset S$, then $\forall a \in A, a \in A \implies a \in S$.

$$\implies \forall a, b \in A, \text{ either } a < b, a > b, \text{ or } a = b.$$

So A is also ordered. Since A is finite, then $\exists a_0 \in A$ such that $\forall a \in A, a_0 \geq a$.

Thus, A is bounded. \square

(b)

Proof. (By contradiction).

Assuming A is finite, suppose instead that there is no maximal element in A . Choose an element $a_1 \in A$. Then, since a_1 is not the maximum, $\exists a_2 \in A$ such that $a_1 < a_2$. But a_2 is also not the maximum of A , so $\exists a_3 \in A$ such that

$a_2 < a_3$. Continuing in this manner, we find an increasing sequence $\{a_n\}_{n \in \mathbb{N}}$ of elements of A , i.e. such that

$$a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots. \quad (1)$$

But because this sequence is infinite and contained in A , this contradicts the assumption that A is finite. Thus, there must exist a maximal element in A .

To show that there exists a minimum element, we recreate the same argument from above where instead, supposing that there is no minimal element, we demonstrate that we can construct an infinite decreasing sequence $\cdots a_n < \cdots < a_2 < a_1$ of elements of A , once again arriving at a contradiction.

Therefore, both $\inf A$ and $\sup A$ exist in A . \square

Problem 3

Proof. Since b is an upper bound for A , then $\forall a \in A, a \leq b$.

Suppose $b \neq \sup A$. Then \exists some other element $c \in A$ such that $c = \sup A$, since by problem 2, A must have a supremum because it is finite and a subset of an ordered set. But since $b \in A$, then $b \leq c$.

However, we assumed that b is an upper bound for A , so since $c \in A$, this implies that $b \geq c$. Thus we have that $b \leq c$ and $b \geq c$, so it must hold that $b = c$.

Therefore $b = \sup A$, as desired. \square

Problem 4

Proof. Suppose $\sup A \notin A$, and let $x_0 \in A$. Towards a contradiction, suppose that $\forall x \in A, x \leq x_0$. Then x_0 is an upper bound for A .

Since $x_0 \in A$, then by problem 3, $x_0 = \sup A$. But we assumed $\sup A \notin A$, so contradiction.

So for any $x_0 \in A$, $\exists x_1$ such that $x_1 > x_0$. We can repeat the logic above to show that this holds for arbitrary $x_i \in A$, since no x_i can both be an upper bound for A while also being contained in A . Thus $\forall x_i \in A, \exists x_{i+1}$ such that $x_i < x_{i+1}$.

So we obtain an infinite decreasing sequence $\{x_i\}_{i \in \mathbb{N}}$ of elements of A . Hence, these elements form a countably infinite subset of A , since $|\{x_i | i \in \mathbb{N}\}| = |\mathbb{N}|$.

Therefore, A does indeed contain a countably infinite subset. \square

Problem 5

(a)

Proof. (Arithmetic Mean - Geometric Mean Inequality)

Consider the difference $\sqrt{x} - \sqrt{y}$. We compute:

$$0 \leq (\sqrt{x} - \sqrt{y})^2 = x + y - 2\sqrt{xy} \quad (2)$$

$$\implies 2\sqrt{xy} \leq x + y \quad (3)$$

Dividing both sides of (3) by 2, we obtain the desired result: $\sqrt{xy} \leq \frac{x+y}{2}$. \square

(b)

Proof. (\Rightarrow) Suppose $\sqrt{xy} = \frac{x+y}{2}$. Then

$$\implies 2\sqrt{xy} - x - y = 0 \quad (4)$$

$$\implies x + y - 2\sqrt{xy} = 0 \quad (5)$$

$$\implies (\sqrt{x} - \sqrt{y})^2 = 0 \quad (6)$$

$$\implies \sqrt{x} - \sqrt{y} = 0 \quad (7)$$

$$\implies \sqrt{x} = \sqrt{y} \quad (8)$$

$$\implies x = y. \quad (9)$$

(\Leftarrow) Suppose $x = y$. Then

$$\frac{x+y}{2} = \frac{x+x}{2} = x \quad (10)$$

$$= \sqrt{x} \cdot \sqrt{x} = \sqrt{x} \cdot \sqrt{y} \quad (11)$$

$$= \sqrt{xy}. \quad (12)$$

Thus, $\frac{x+y}{2} = \sqrt{xy} \iff x = y$. \square

Problem 6

TODO TODO TODO TODO TODO

Problem 7

(a)

Proof. Let $E = \{x \in \mathbb{R} \mid x > 0 \text{ and } x^3 < 2\}$. So $\forall x \in E, x^3 < 2$. Then

$$\implies x^3 - 2 < 0 \quad (13)$$

$$\implies \left(x - 2^{\frac{1}{3}}\right) \left(x^2 + 2^{\frac{1}{3}}x + 2^{\frac{2}{3}}\right) < 0. \quad (14)$$

But $x > 0 \implies x^2 + 2^{\frac{1}{3}}x + 2^{\frac{2}{3}} > 0$, so $\forall x \in E$,

$$\implies x - 2^{\frac{1}{3}} < 0 \quad (15)$$

$$\implies x < \sqrt[3]{2}. \quad (16)$$

Thus, E is bounded above. \square

(b)

Proof. Let $r = \sup E$, which exists by part (a) and problem 2 (b). We first show that $r > 0$.

By definition of the supremum and the set E , $\forall x \in E$, $x \leq r$ and $x > 0$. Thus $0 < x \leq r$, so r is indeed positive.

Next we show that $r^3 = 2$.

Suppose, toward a contradiction, that $r > \sqrt[3]{2}$. Since $r = \sup E$, then $x \leq r$ for any $x \in E$. But since $\sqrt[3]{2}$ is an upper bound for E , then $x < \sqrt[3]{2} \forall x \in E$. So

$$\implies x < \sqrt[3]{2} < r \quad (17)$$

$$\implies r \neq \sup E, \implies \Leftarrow. \quad (18)$$

This is a contradiction, since we assumed that $r = \sup E$. Thus $r \leq \sqrt[3]{2} \implies r^3 \leq 2$.

Now suppose instead, toward another contradiction, that $r < \sqrt[3]{2}$. Then

$$r^3 < \left(\sqrt[3]{2}\right)^3 = 2. \quad (19)$$

But then $r \in E$, and $\sqrt[3]{2} > r$ which implies that $r \neq \sup E$, another contradiction. Thus $r \geq \sqrt[3]{2}$. So we have shown that both $r \leq \sqrt[3]{2}$ and $r \geq \sqrt[3]{2}$. Then we conclude $r = \sqrt[3]{2}$.

Hence, $r^3 = 2$. \square