

18.100A Assignment 1

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Problem 1

(a)

Proof. We will show that each set is a subset of the other to prove equality. Let $S = A \cap (B \cup C)$ and $T = (A \cup B) \cap (A \cup C)$.

Let $x \in S$. Then $x \in A$ and $x \in B \cup C$,

$$\implies x \in A \text{ and } x \in B, \text{ or } x \in A \text{ and } x \in C$$

$$\implies x \in A \cap B \text{ or } x \in A \cap C$$

$$\implies x \in (A \cap B) \cup (A \cap C) = T.$$

Thus $x \in S \implies x \in T$, so $S \subseteq T$. Now let $x \in T$. Then $x \in (A \cap B) \cup (A \cap C)$,

$$\implies x \in A \cap B \text{ or } x \in A \cap C$$

$$\implies x \in A, \text{ and } x \in B \text{ or } C$$

$$\implies x \in A \cap (B \cup C) = S.$$

Thus $x \in T \implies x \in S$, so $T \subseteq S$, which means $S = T$.

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

□

(b)

Proof. We proceed as in (a). Let $S = A \cup (B \cap C)$ and $T = (A \cup B) \cap (A \cup C)$.

First let $x \in S$. Then $x \in A$ or $x \in B \cap C$,

$$\implies x \in A \text{ or } x \in B, \text{ and } x \in A \text{ or } x \in C$$

$$\implies x \in (A \cup B) \cap (A \cup C) = T.$$

Thus $x \in S \implies x \in T$, so $S \subseteq T$. Now let $x \in T$. Then $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then the requirement is satisfied immediately, regardless

of whether x is in B or C . Otherwise, if $x \notin A$, then $x \in B$ and $x \in C$ must be true. So $x \in A$, or $x \in B$ and $x \in C$

$$\implies x \in A \cup (B \cap C) = S.$$

Thus $x \in T \implies x \in S$, so $T \subseteq S$, which means $S = T$.

Hence, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. □

Problem 2

Proof. (By induction).

The inductive hypothesis $P(n)$ is that, for $n \in \mathbb{N}$, $n < 2^n$.

(Base case): $1 < 2^1$, so $P(1)$ is true.

(Inductive step): Assume $P(n)$ is true for $n = m$, i.e. that $m < 2^m$ holds for $m \in \mathbb{N}$. Then

$$2^{m+1} = 2 \cdot 2^m > 2m \tag{1}$$

$$= m + m > m + 1, \quad \text{since } m > 1 \tag{2}$$

$$\implies 2^{m+1} > m + 1. \tag{3}$$

So $P(m) \implies P(m+1)$, which means $P(n)$ is true for all $n \in \mathbb{N}$.

Thus, $\forall n \in \mathbb{N}$, $2^n > n$. □

Problem 3

Proof. Let A be a finite set such that $|A| = n$. We form the power set $\mathcal{P}(A)$ by creating the set of all possible subsets of A . Hence $|\mathcal{P}(A)|$ is equivalent to the number of possible subsets of A , which we compute by summing over the number of combinations that can be created by choosing elements of A , in succession from choosing no elements (the empty set \emptyset) to choosing all elements (the full set A). Thus

$$|\mathcal{P}(A)| = \sum_{k=0}^n \binom{n}{k}. \tag{4}$$

By the binomial expansion theorem,

$$\sum_{k=0}^n \binom{n}{k} p^k q^{n-k} = (p+q)^n. \tag{5}$$

Substituting $p = q = 1$ into (5), we arrive at the desired result,

$$|\mathcal{P}(A)| = (1+1)^n = 2^n. \tag{6}$$

□

Problem 4

Proof. (By induction).

$P(n)$ is the hypothesis that for $n \in \mathbb{N}$, $n^3 + 5n$ is divisible by 6.

(Base case): $1^3 + 5 \cdot 1 = 1 + 5 = 6$ is divisible by 6, so $P(1)$ is true.

(Inductive step): Assume $P(m)$ holds, i.e. 6 divides $m^3 + 5m$. Then

$$(m+1)^3 + 5m = m^3 + 3m^2 + 3m + 1 + 5m + 1 \quad (7)$$

$$= m^3 + 5m + 6 + 3m(m+1), \quad (8)$$

where by the inductive hypothesis $m^3 + 5m + 6$ is divisible by 6 and $3m(m+1)$ is also divisible by 6 because it is divisible by both 3 and 2. So, their sum $(m+1)^3 + 5(m+1)$ must also be divisible by 6, which means $P(m) \implies P(m+1)$.

Thus, $\forall n \in \mathbb{N}$, $n^3 + 5n$ is divisible by 6. \square

Problem 5

Proof. Let $A_1, A_2, A_3, \dots, A_n, A_{n+1}, \dots$ be finite sets, and let there be infinitely many of them. We wish to find an example where $|\bigcup_{i=1}^{\infty} A_i| = \infty$.

To satisfy this, choose $A_i = \{i\}$. Then $A_1 \cup A_2 \cup A_3 \cdots = \{1, 2, 3, \dots\} = \mathbb{N}$. So their union is not a finite set, as desired. \square