18.100A Assignment 9

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Problem 1

Proof. We have that $\forall x \in \mathbb{R}$, $|\arctan(x)| < \frac{\pi}{2}$, i.e.

$$\arctan(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{1}$$

which is an open set. So $\forall |y| < \frac{\pi}{2}$, $\exists \epsilon > 0$ such that $(y-\epsilon, y+\epsilon) \subset (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus for every such y, we can always find a $y_0 < y$ and $y_1 > y$ inside this open set. This means that there is no x_1 such that $\arctan(x_1) \geq \arctan(x)$ nor an x_0 such that $\arctan(x_0) \leq \arctan(x)$ $\forall x$.

Hence $f(x) = \arctan(x)$ does not achieve an absolute minimum or maximum. $\square$

Problem 2

Proof. Let $x, y \in (c, \infty)$. Choose $L = \frac{1}{c^2}$. Then

$$\begin{aligned}
|f(y) - f(x)| &= \left|\frac{1}{y} - \frac{1}{x}\right| && (2)\\
&= \frac{|x-y|}{xy} && (3)\\
&< \frac{|x-y|}{c^2} && (4)\\
&= L|x-y|. && (5)
\end{aligned}$$

Therefore $f(x) = \frac{1}{x}$ is Lipschitz continuous. $\square$

Problem 3

Proof. Let $\delta > 0$ and choose $\epsilon_0 = |\sin(\delta)|$. Choose $x = \frac{1}{2\pi k+\delta}$ and $c = \frac{1}{2\pi k}$ for some $k \in \mathbb{N}$. Then

$$
\begin{aligned}
|x - c| &= \left|\frac{1}{2\pi k} - \frac{1}{2\pi k}\right| && (6)\\
&= \left|\frac{2\pi k - (2\pi k + \delta)}{2\pi k(2\pi k + \delta)}\right| && (7)\\
&= \frac{\delta}{4\pi^2 k^2 + 2\pi k\delta} && (8)\\
&< \delta. && (9)
\end{aligned}
$$

We also have

$$
\begin{aligned}
|f(x) - f(c)| &= |\sin(2\pi k + \delta) - \sin(2\pi k)| && (10)\\
&= |\sin(2\pi k)\cos(\delta) + \cos(2\pi k)\sin(\delta) - \sin(2\pi k)| && (11)\\
&= |\sin(\delta)| && (12)\\
&= \epsilon_0. && (13)
\end{aligned}
$$

Hence, $f(x) = \sin\left(\frac{1}{x}\right)$ is not uniformly continuous. □

Problem 4

Proof. Suppose $f : S \to \mathbb{R}$ is Lipschitz continuous on S. Then $\exists L \geq 0$ such that $\forall x, y \in S$, $|f(x) - f(y)| \leq L|x - y|$.

Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{L}$. If $|x - y| < \delta$, then

$$
\begin{aligned}
|f(x) - f(y)| &\leq L|x - y| && (14)\\
&< L\delta && (15)\\
&= \epsilon. && (16)
\end{aligned}
$$

Thus f is uniformly continuous on S. □

Problem 5

(a)

Proof. Let $x, y \in \mathbb{R}$. Choose $L = 1$. Then

$$
\begin{aligned}
|f(x) - f(y)| &= |\cos(x) - \cos(y)| && (17)\\
&= \left|2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right| && (18)\\
&\le 2\left|\sin\left(\frac{x-y}{2}\right)\right| && (19)\\
&\le 2\left|\frac{x-y}{2}\right| && (20)\\
&= |x-y|. && (21)
\end{aligned}
$$

Therefore $f(x) = \cos(x)$ is Lipschitz continuous on $\mathbb{R}$. ☐

(b)

Proof. (1) Let $\epsilon > 0$. Choose $\delta = c^{\frac{2}{3}}\epsilon$. Then $\forall x, c \in [0, 1]$, we have

$$
\begin{aligned}
|f(x) - f(c)| &= |x^{\frac{1}{3}} - c^{\frac{1}{3}}| && (22)\\
&= \frac{|x-c|}{|x^{\frac{2}{3}} + x^{\frac{1}{3}}c^{\frac{1}{3}} + c^{\frac{2}{3}}|} && (23)\\
&< \frac{\delta}{c^{\frac{2}{3}}} && (24)\\
&= \epsilon. && (25)
\end{aligned}
$$

Thus, $f(x) = x^{\frac{1}{3}}$ is uniformly continuous on $[0, 1]$.

(2) (By contradiction.)

Suppose f is Lipschitz continuous on $[0, 1]$. Then $\forall x, y \in [0, 1]$, $\exists L \ge 0$ such that $|x^{\frac{1}{3}} - y^{\frac{1}{3}}| \le L|x - y|$.

Choose $y = 0$. Then $|x^{\frac{1}{3}}| \le L|x|$, i.e. $\frac{1}{x^{\frac{2}{3}}} \le L$. Taking $x \to 0$ on both sides, this implies that $\lim_{x\to 0} \frac{1}{x^{\frac{2}{3}}}$ exists and is finite. But we know that this limit does not exist, so we have arrived at a contradiction.

Therefore $f(x) = x^{\frac{1}{3}}$ is not Lipschitz continuous on $[0, 1]$. ☐