

# 18.100A Assignment 2

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## Problem 1

*Proof.* (By contradiction).

Suppose instead that  $xy \leq xz$ . Then

$$\implies xy - xz \leq 0$$

$$\implies x(y - z) \leq 0.$$

Since  $x < 0$  by assumption, it must then be true that  $y - z \geq 0$ . But then

$$\implies y \geq z \quad \Rightarrow \Leftarrow,$$

which is a contradiction since we assumed that  $y < z$ . Thus,  $xy > xz$ .  $\square$

## Problem 2

(a)

*Proof.* We want to show that  $\exists b \in S$  such that  $\forall a \in A, a \leq b$ .

Since  $S$  is ordered, then for every  $x, y \in S$ , we have that either  $x < y$ ,  $x > y$ , or  $x = y$ . But since  $A \subset S$ , then  $\forall a \in A, a \in A \implies a \in S$ .

$$\implies \forall a, b \in A, \text{ either } a < b, a > b, \text{ or } a = b.$$

So  $A$  is also ordered. Since  $A$  is finite, then  $\exists a_0 \in A$  such that  $\forall a \in A, a_0 \geq a$ .

Thus,  $A$  is bounded.  $\square$

(b)

*Proof.* (By contradiction).

Assuming  $A$  is finite, suppose instead that there is no maximal element in  $A$ . Choose an element  $a_1 \in A$ . Then, since  $a_1$  is not the maximum,  $\exists a_2 \in A$  such that  $a_1 < a_2$ . But  $a_2$  is also not the maximum of  $A$ , so  $\exists a_3 \in A$  such that

$a_2 < a_3$ . Continuing in this manner, we find an increasing sequence  $\{a_n\}_{n \in \mathbb{N}}$  of elements of  $A$ , i.e. such that

$$a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots . \quad (1)$$

But because this sequence is infinite and contained in  $A$ , this contradicts the assumption that  $A$  is finite. Thus, there must exist a maximal element in  $A$ .

To show that there exists a minimum element, we recreate the same argument from above where instead, supposing that there is no minimal element, we demonstrate that we can construct an infinite decreasing sequence  $\cdots a_n < \cdots < a_2 < a_1$  of elements of  $A$ , once again arriving at a contradiction.

Therefore, both  $\inf A$  and  $\sup A$  exist in  $A$ .  $\square$

### Problem 3

*Proof.* Since  $b$  is an upper bound for  $A$ , then  $\forall a \in A, a \leq b$ .

Suppose  $b \neq \sup A$ . Then  $\exists$  some other element  $c \in A$  such that  $c = \sup A$ , since by problem 2,  $A$  must have a supremum because it is finite and a subset of an ordered set. But since  $b \in A$ , then  $b \leq c$ .

However, we assumed that  $b$  is an upper bound for  $A$ , so since  $c \in A$ , this implies that  $b \geq c$ . Thus we have that  $b \leq c$  and  $b \geq c$ , so it must hold that  $b = c$ .

Therefore  $b = \sup A$ , as desired.  $\square$