18.100A Midterm

Octavio Vega

March 20, 2023

Problem 1

(a)

Proof. Let $x \in f^{-1}(C \cap D)$. Then

$$\begin{aligned}
&\implies f(x) \in C \cap D && (1)\\
&\implies f(x) \in C \text{ and } f(x) \in D && (2)\\
&\implies x \in f^{-1}(C) \text{ and } x \in f^{-1}(D) && (3)\\
&\implies x \in f^{-1}(C) \cap f^{-1}(D). && (4)
\end{aligned}$$

Thus,

$$f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D). \tag{5}$$

Now let $x \in f^{-1}(C) \cap f^{-1}(D)$. Then

$$\begin{aligned}
&\implies f(x) \in C \text{ and } f(x) \in D && (6)\\
&\implies f(x) \in C \cap D && (7)\\
&\implies x \in f^{-1}(C \cap D). && (8)
\end{aligned}$$

Thus,

$$f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D). \tag{9}$$

Therefore by equations (5) and (9), $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. □

(b)

Claim: If $E \subset \mathbb{R}$ is countable, then the complement $\mathbb{R} \backslash E$ is always uncountable.

Proof. (By contradiction). Suppose E^c is countable. Then $E \cup E^c$ is countable as well, since it is the union of two countable sets. But $E \cup E^c = \mathbb{R}$, which is uncountable. ($\Rightarrow\Leftarrow$). □

(c)

By contrast, if $E \subset \mathbb{R}$ is uncountable, then the complement $\mathbb{R} \backslash E$ is not always countable. Take for instance, $E = [0, 1]$, which is uncountable. Then $E^c = (-\infty, 0) \cup (0, \infty)$, which is also uncountable.

Problem 2

(a)

A set $U \subset \mathbb{R}$ is *not open* if for every $\epsilon > 0$, $\exists x \in U$ such that $(x - \epsilon, x + \epsilon) \not\subset \mathbb{R}$.