

# 18.100A Final

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June 21, 2023

## Problem 1

We complete the following **negations**:

(i)

Let  $S \subset \mathbb{R}$ . A function  $f : S \rightarrow \mathbb{R}$  is **not continuous** at  $c \in S$  if  $\exists \epsilon_0 > 0$  such that  $\forall \delta > 0$  if  $|x - c| < \delta$ ,  $|f(x) - f(c)| \geq \epsilon_0$ .

(ii)

Let  $S \subset \mathbb{R}$ . A function  $f : S \rightarrow \mathbb{R}$  is **not uniformly continuous** on  $S$  if  $\exists x_0 \in S$  such that  $\forall \delta > 0 \exists \epsilon_0 > 0$  such that if  $|x_0 - x| < \delta$ , then  $|f(x) - f(x_0)| \geq \epsilon_0$ .

(iii)

Let  $S \subset \mathbb{R}$ . A sequence of functions  $f_n : S \rightarrow \mathbb{R}$  **does not converge uniformly** to  $f : S \rightarrow \mathbb{R}$  if  $\exists \epsilon_0 > 0$  such that  $\forall M \in \mathbb{N} \exists n \geq M$  and  $x \in S$  such that  $|f_n(x) - f(x)| \geq \epsilon_0$ .

## Problem 2

(a)

(i)

A continuous function on  $(0, 1)$  with neither a global minimum or maximum:

Let  $f(x) = 1 \forall x \in (0, 1)$ . Then  $\forall x, y \in (0, 1)$ ,  $f(x) = f(y)$  so  $f$  has no absolute maximum or minimum, and  $f$  is constant and therefore continuous.

(ii)

A function on  $[0, 1]$  with absolute minimum at 0, absolute maximum at 1, and such that  $\exists y \in (f(0), f(1))$  not in the range of  $f$ :

Define  $f$  via

$$f(x) := \begin{cases} x, & x \in (0, \frac{1}{2}) \cap (\frac{1}{2}, 1] \\ -1, & x = 0 \\ -\frac{1}{2}, & x = \frac{1}{2}. \end{cases} \quad (1)$$

Then, for example,  $-\frac{3}{4} \in (-1, 1) = (f(0), f(1))$ , but  $-\frac{3}{4}$  is not in the range of  $f$ . Also,  $f$  has an absolute minimum and maximum at 0 and 1, respectively.

(b)

*Proof.* Let  $\epsilon > 0$ . Since  $f$  is continuous, then  $\exists \delta_0 > 0$  such that if  $|x - c| < \delta_0$  then  $|f(x) - f(c)| < \frac{\epsilon}{2}$ . Similarly, since  $g$  is continuous, then  $\exists \delta_1 > 0$  such that if  $|x - c| < \delta_1$  then  $|g(x) - g(c)| < \frac{\epsilon}{2}$ .

Choose  $\delta_0, \delta_1$  such that  $|f(x)| + |g(c)| < 2$ , and let  $\delta = \min\{\delta_0, \delta_1\}$ . Then if  $|x - c| < \delta$ , we have

$$|f(x)g(x) - f(c)g(c)| = |f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)| \quad (2)$$

$$\leq |f(x)||g(x) - g(c)| + |g(c)||f(x) - f(c)| \quad (3)$$

$$< |f(x)|\frac{\epsilon}{2} + |g(c)|\frac{\epsilon}{2} \quad (4)$$

$$< \epsilon. \quad (5)$$

Therefore, the product  $fg$  is continuous at  $c$ .  $\square$

### Problem 3

(a)

*Proof.* Let  $\{x_n\}_n$  be a sequence of elements in  $[a, b]$ . Choose  $B = \max\{|a|, |b|\}$ . Then  $\forall n \in \mathbb{N}$ ,  $|x_n| \leq B$ , so the sequence  $\{x_n\}_n$  is bounded.

By the Bolzano-Weierstrass theorem,  $\exists$  a subsequence  $\{x_{n_k}\}_k \subset [a, b]$  that converges, i.e. such that  $x_{n_k} \rightarrow x$  as  $k \rightarrow \infty$  for some  $x \in \mathbb{R}$ . Since  $[a, b]$  is closed, then  $x \in [a, b]$ .

Therefore  $[a, b]$  is compact.  $\square$

(b)

*Proof.* Let  $\{x_n\}_n$  be a sequence of elements in  $[a, b]$ . Since  $[a, b]$  is compact,  $\exists \{x_{n_k}\}_k \subset [a, b]$  such that  $x_{n_k} \rightarrow x$  for some  $x \in [a, b]$ .

Then  $\{f(x_{n_k})\}_k$  is a subsequence in  $[a, b]$ , and since  $f$  is continuous we have

$$\lim_{k \rightarrow \infty} f(x_{n_k}) = f\left(\lim_{k \rightarrow \infty} x_{n_k}\right) \quad (6)$$

$$= f(x) \in f([a, b]), \quad (7)$$

so the subsequence  $\{f_{n_k}\}_k$  of  $\{f_n\}_n$  converges in  $[a, b]$ .

Therefore  $f([a, b])$  is compact.  $\square$

## Problem 4

(a)

(i)

*Proof.* Since  $f$  is differentiable at  $c$ , then the limit

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \quad (8)$$

exists. Then we have

$$\lim_{x \rightarrow c} (f(x) - f(c)) = \lim_{x \rightarrow c} \left[ \left( \frac{f(x) - f(c)}{x - c} \right) (x - c) \right] \quad (9)$$

$$= \lim_{x \rightarrow c} \left( \frac{f(x) - f(c)}{x - c} \right) \cdot \lim_{x \rightarrow c} (x - c) \quad (10)$$

$$= f'(c) \lim_{x \rightarrow c} (x - c) \quad (11)$$

$$= 0. \quad (12)$$

Therefore  $\lim_{x \rightarrow c} f(x) = f(c)$ , so  $f$  is continuous at  $c$ .  $\square$

(ii)

Consider the following example disproving the converse of part (i):

$f(x) = |x|$  is continuous but not differentiable at 0. Checking this is simple: we simply compute the left and right limits of the difference quotient and find that they do not agree, signifying that the limit (derivative) does not exist at 0.

(b)

*Proof.* Using the limit definition of the derivative, we have

$$f'(0) = \lim_{x \rightarrow 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} \quad (13)$$

$$= \lim_{x \rightarrow 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \quad (14)$$

$$= 0, \quad (15)$$

so the limit exists.

Therefore  $f$  is differentiable at 0.  $\square$