18.100A Assignment 1

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February 10, 2023

Problem 1

(a)

Proof. We will show that each set is a subset of the other to prove equality. Let $S = A \cap (B \cup C)$ and $T = (A \cup B) \cap (A \cup C)$.

Let $x \in S$. Then $x \in A$ and $x \in B \cup C$,

$\implies x \in A$ and $x \in B$, or $x \in A$ and $x \in C$

$\implies x \in A \cap B$ or $x \in A \cap C$

$\implies x \in (A \cap B) \cup (A \cap C) = T$.

Thus $x \in S \implies x \in T$, so $S \subseteq T$. Now let $x \in T$. Then $x \in (A \cap B) \cup (A \cap C)$,

$\implies x \in A \cap B$ or $x \in A \cap C$

$\implies x \in A$, and $x \in B$ or C

$\implies x \in A \cap (B \cup C) = S$.

Thus $x \in T \implies x \in S$, so $T \subseteq S$, which means $S = T$.

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

$\square$

(b)

Proof. We proceed as in (a). Let $S = A \cup (B \cap C)$ and $T = (A \cup B) \cap (A \cup C)$.

First let $x \in S$. Then $x \in A$ or $x \in B \cap C$,

$\implies x \in A$ or $x \in B$, and $x \in A$ or $x \in C$

$\implies x \in (A \cup B) \cap (A \cup C) = T$.

Thus $x \in S \implies x \in T$, so $S \subseteq T$. Now let $x \in T$. Then $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then the requirement is satisfied immediately, regardless of whether x is in B or C. Otherwise, if $x \notin A$, then $x \in B$ and $x \in C$ must be true. So $x \in A$, or $x \in B$ and $x \in C$

$\implies x \in A \cup (B \cap C) = S$.

Thus $x \in T \implies x \in S$, so $T \subseteq S$, which means $S = T$.

Hence, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. □

Problem 2

Proof. (By induction).

The inductive hypothesis $P(n)$ is that, for $n \in \mathbb{N}$, $n < 2^n$.

(Base case): $1 < 2^1$, so $P(1)$ is true.

(Inductive step): Assume $P(n)$ is true for $n = m$, i.e. that $m < 2^m$ holds for $m \in \mathbb{N}$. Then

$$\begin{aligned} 2^{m+1} &= 2 \cdot 2^m > 2m && (1) \\ &= m + m > m + 1, \quad \text{since } m > 1 && (2) \\ \implies 2^{m+1} &> m + 1. && (3) \end{aligned}$$

So $P(m) \implies P(m+1)$, which means $P(n)$ is true for all $n \in \mathbb{N}$.

Thus, $\forall n \in \mathbb{N}$, $2^n > n$. □

Problem 3

Proof. Let A be a finite set such that $|A| = n$. We form the power set $\mathcal{P}(A)$ by creating the set of all possible subsets of A. Hence $|\mathcal{P}(A)|$ is equivalent to the number of possible subsets of A, which we compute by summing over the number of combinations that can be created by choosing elements of A, in succession from choosing no elements (the empty set $\emptyset$) to choosing all elements (the full set A). Thus

$$|\mathcal{P}(A)| = \sum_{k=0}^{n} \binom{n}{k}. \tag{4}$$

By the binomial expansion theorem,

$$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p+q)^n. \tag{5}$$

Substituting $p = q = 1$ into (5), we arrive at the desired result,

$$|\mathcal{P}(A)| = (1+1)^n = 2^n. \tag{6}$$

□

Problem 4

Proof. (By induction).

$P(n)$ is the hypothesis that for $n \in \mathbb{N}$, $n^3 + 5n$ is divisible by 6.

(Base case): $1^3 + 5 \cdot 1 = 1 + 5 = 6$ is divisible by 6, so $P(1)$ is true.

(Inductive step): Assume $P(m)$ holds, i.e. 6 divides $m^3 + 5m$. Then

$$
\begin{aligned}
(m+1)^3 + 5m &= m^3 + 3m^2 + 3m + 1 + 5m + 1 && (7)\\
&= m^3 + 5m + 6 + 3m(m+1), && (8)
\end{aligned}
$$

where by the inductive hypothesis $m^3 + 5m + 6$ is divisible by 6 and $3m(m+1)$ is also divisible by 6 because it is divisible by both 3 and 2. So, their sum $(m+1)^3+5(m+1)$ must also be divisible by 6, which means $P(m) \implies P(m+1)$.

Thus, $\forall n \in \mathbb{N}$, $n^3 + 5n$ is divisible by 6. □

Problem 5

Proof. Let $A_1, A_2, A_3, ..., A_n, A_{n+1}, ...$ be finite sets, and let there be infinitely many of them. We wish to find an example where $|\bigcup_{i=1}^{\infty} A_i| = \infty$.

To satisfy this, choose $A_i = \{i\}$. Then $A_1 \cup A_2 \cup A_3 \cdots = \{1, 2, 3, ...\} = \mathbb{N}$. So their union is not a finite set, as desired. □

Problem 6

(a)

Consider $q = \frac{4}{15}$. Then

$$\frac{4}{15} = \frac{2 \cdot 2}{3 \cdot 5} = \frac{2^2}{3 \cdot 5}. \tag{9}$$

So we compute

$$f\left(\frac{4}{15}\right) = 2^{2 \cdot 2} \cdot 3^{2-1} \cdot 5^{2-1} = 2^4 \cdot 3 \cdot 5. \tag{10}$$

Hence, $f\left(\frac{4}{15}\right) = 240$.

Now suppose $f(q) = 108$. Then we write

$$108 = 2 \cdot 54 = 2 \cdot 2 \cdot 27 = 2^2 \cdot 3^3 = p_1^{2r_1} q_1^{2s_1 - 1}, \tag{11}$$

so we identify $r_1 = 1$ and $s_1 = 2$. Thus,

$$q = \frac{p_1^{r_1}}{q_1^{s_1}} = \frac{2}{3^2} = \frac{2}{9}. \tag{12}$$

(b)

Proof. We first show that f is injective.

Let $f(t_1) = f(t_2)$. Then we consider three cases:

Case 1: $f(t_1) = f(t_2) = 1$,

$\implies t_1 = 1$ and $t_2 = 1$

$\implies t_1 = t_2$.

Case 2: $t \in \mathbb{N}\backslash\{1\}$, $t_1 = p_1^{r_1} \cdots p_N^{r_N}$, $t_2 = q_1^{s_1} \cdots q_N^{s_N}$,

$f(t_1) = f(t_2)$

$\implies p_1^{2r_1} \cdots p_N^{2r_N} = q_1^{2s_1} \cdots q_N^{2s_n}$

$\implies p_1^{r_1} \cdots p_n^{r_N} = q_1^{s_1} \cdots q_N^{s_N}$,

since $t_1, t_2 > 0$. But according to the theorem stated in the problem, these prime numbers and exponents are unique, so if they are equal to the same number, then they must all be equal. Thus $p_i = q_i$ and $r_i = s_i$ $\forall i \in \{1, 2, ..., N\}$, which implies that $t_1 = t_2$.

Case 3: $t_1, t_2 \notin \mathbb{N}$,

$f(t_1) = f(t_2)$

$\implies p_1^{2r_1} \cdots p_N^{2r_N} \cdot q_1^{2s_1-1} \cdots q_M^{2s_M-1} = w_1^{2y_1} \cdots w_N^{2y_N} \cdot x_1^{2z_1-1} \cdots x_M^{2z_M-1}$

$\implies \frac{p_1^{2r_1} \cdots p_N^{2r_N}}{q_1^{1-2s_1} \cdots q_M^{1-2s_M}} = \frac{w_1^{2y_1} \cdots w_N^{2y_N}}{x_1^{1-2z_1} \cdots x_M^{1-2z_M}}$,

but once again the primes and exponents must be the same since they are all unique, so $t_1 = t_2$. Thus, f is injective.

Next, we show that f is surjective.

We want to prove that for any $n \in \mathbb{N}$, $\exists q \in \mathbb{Q}, q > 0$ such that $f(q) = n$. We again consider three cases:

Case 1: $n = 1$, then we simply have $q = 1$ since $f(1) = 1$ by definition.

Case 2: $n = p_1^{2r_1} \cdots p_N^{2r_N}$,

$\implies n = (p_1^{r_1} \cdots p_N^{r_N})^2$

$\implies \sqrt{n} = p_1^{r_1} \cdots p_N^{r_N}$,

since $q > 0$. By part (1) of the theorem, these primes and exponents are unique for $q \in \mathbb{N}$, so $n^2 = q$ for $q > 0$ and $q \in \mathbb{N}$. Then $f(q) = f(p_1^{r_1} \cdots p_N^{r_N}) = n$.

Case 3: $n = p_1^{2r_1} \cdots p_N^{2r_N} \cdot q_1^{2s_1-1} \cdots q_M^{2s_M-1}$,

$\implies n = \frac{p_1^{2r_1} \cdots p_N^{2r_N}}{q_1^{1-2s_1} \cdots q_M^{1-2s_M}}$, so by part (2) of the theorem, this corresponds to a unique $q \in \mathbb{N}$ where $q = \frac{p_1^{r_1} \cdots p_N^{r_N}}{q_1^{s_1} \cdots q_N^{s_N}}$. Therefore, f is surjective as well.

Hence f is both injective and surjective, so we conclude that f is bijective. $\square$