18.100A Midterm

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Problem 1

(a)

Proof. Let $x \in f^{-1}(C \cap D)$. Then

$$\begin{aligned} &\implies f(x) \in C \cap D && (1)\\ &\implies f(x) \in C \text{ and } f(x) \in D && (2)\\ &\implies x \in f^{-1}(C) \text{ and } x \in f^{-1}(D) && (3)\\ &\implies x \in f^{-1}(C) \cap f^{-1}(D). && (4) \end{aligned}$$

Thus,

$$f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D). \tag{5}$$

Now let $x \in f^{-1}(C) \cap f^{-1}(D)$. Then

$$\begin{aligned} &\implies f(x) \in C \text{ and } f(x) \in D && (6)\\ &\implies f(x) \in C \cap D && (7)\\ &\implies x \in f^{-1}(C \cap D). && (8) \end{aligned}$$

Thus,

$$f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D). \tag{9}$$

Therefore by equations (5) and (9), $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. □

(b)

Claim: If $E \subset \mathbb{R}$ is countable, then the complement $\mathbb{R} \backslash E$ is always uncountable.

Proof. (By contradiction). Suppose E^c is countable. Then $E \cup E^c$ is countable as well, since it is the union of two countable sets. But $E \cup E^c = \mathbb{R}$, which is uncountable. ($\Rightarrow\Leftarrow$). □

(c)

By contrast, if $E \subset \mathbb{R}$ is uncountable, then the complement $\mathbb{R} \backslash E$ is not always countable. Take for instance, $E = [0, 1]$, which is uncountable. Then $E^c = (-\infty, 0) \cup (1, \infty)$, which is also uncountable.

Problem 2

(a)

A set $U \subset \mathbb{R}$ is *not open* if for every $\epsilon > 0$, $\exists x \in U$ such that $(x - \epsilon, x + \epsilon) \not\subset \mathbb{R}$.

(b)

Proof. Suppose U is not open. Let $\epsilon = \frac{1}{n}$. Then for every $n \in \mathbb{N}$, $\exists x \in U$ such that $(x - \frac{1}{n}, x + \frac{1}{n}) \not\subset U$. Equivalently, for each $n \in \mathbb{N}$ $\exists x_n \in U^c$ such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n}. \tag{10}$$

Then we have

$$0 < |x_n - x| < \frac{1}{n}, \tag{11}$$

and taking the limit on all sides gives

$$0 < \lim_{n \to \infty} |x_n - x| < \lim_{n \to \infty} \frac{1}{n}. \tag{12}$$

Thus, by the squeeze theorem, $\lim_{n \to \infty} x_n = x$, as desired. □

(c)

Proof. (By contradiction). To show that F is closed, we must show that F^c is open. Suppose, toward a contradiction, that F^c is not open. Then by part **(b)**, $\exists x \in F^c$ and a sequence $\{x_n\}_n$ of elements of F such that $\lim_{n \to \infty} x_n = x$. But by assumption, every convergent sequence of elements of F has a limit in F, i.e. we assumed originally that $x \in F$ ($\Rightarrow\Leftarrow$). Thus, F^c must be open, so F is closed. □

Problem 3

(a)

Proof. Let $\epsilon > 0$. Choose $N = \frac{1}{\sqrt{\epsilon}}$. Then $\forall n \geq N$, we have

$$\begin{aligned}
\left|\frac{10n^2}{n^2+16n+1} - 10\right| &= \left|\frac{-160n-10}{n^2+16n+1}\right| && (13)\\
&= \left|\frac{160n+10}{n^2+16n+1}\right| && (14)\\
&< \frac{1}{n^2+16n+1} && (15)\\
&< \frac{1}{n^2} && (16)\\
&< \epsilon. && (17)
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \left|\frac{10n^2}{n^2+16n+1}\right| = 10$. □

(b)

(i) Let $x_n = \frac{(-1)^n}{n}$. Then $\lim_{n\to\infty} x_n = 0$. But $x_1 = -1 < \frac{1}{2} = x_2$, whereas $x_2 = \frac{1}{2} > -\frac{1}{3} = x_3$.

Therefore $\{x_n\}_n$ converges to 0, but is not monotonic.

(ii) Let

$$x_n = \begin{cases} 0, & n \text{ even} \\ n, & n \text{ odd.} \end{cases} \tag{18}$$

Then $\{x_n\}_n$ is clearly unbounded.

Consider the subsequence $\{x_{n_k}\}_k$ defined by $x_{n_k} = x_{2k}$ for each $k \in \mathbb{N}$. Then $\forall k$, $x_{n_k} = 0$.

Therefore, $\{x_n\}_n$ is unbounded, but $\{x_{n_k}\}_k$ converges to 0.

Problem 4

(a)

(i)

Proof. Suppose $\{x_n\}_n$ and $\{y_n\}_n$ are bounded. Then $\exists B_0 > 0$ and $B_1 > 0$ such that $\forall n \in \mathbb{N}$, $|x_n| \leq B_0$ and $|y_n| \leq B_1$.

Let $B = B_0 + B_1$. Then by the triangle inequality, we have

$$|x_n + y_n| \leq |x_n| + |y_n| \leq B_0 + B_1 = B. \tag{19}$$

Thus, $\forall n \in \mathbb{N}$, $|x_n + y_n| \leq B$.

Therefore, $\{x_n + y_n\}_n$ is bounded. $\square$

(ii)

Proof. Let $a_n = \sup\{x_k \mid k \geq n\}$, $b_n = \sup\{y_k \mid k \geq n\}$, and $c_n = \sup\{x_k + y_k \mid k \geq n\}$.

Then $\forall n \in \mathbb{N}$, $a_n \geq x_n$ and $b_n \geq y_n$, so

$$\implies x_n + y_n \leq a_n + b_n. \tag{20}$$

But $a_n + b_n$ is also an upper bound for $\{x_n + y_n\}_n$, so $c_n \leq a_n + b_n$. Then

$$\lim_{n\to\infty} c_n \leq \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n. \tag{21}$$

Therefore, $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$. $\square$

(b)

Proof. Let $x = 0.1111....$ Then we can write

$$\begin{aligned} x &= \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots && (22)\\ &= -1 + \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n && (23)\\ &= -1 + \frac{1}{1 - \frac{1}{10}} && (24)\\ &= -1 + \frac{10}{9} && (25)\\ &= \frac{1}{9}. && (26) \end{aligned}$$

Let $\epsilon = \frac{1}{10^n}$ for $n \in \mathbb{N}$ and let $x_n = \frac{1}{9} + \frac{1}{10^{n+1}}$. Then $x_n \in \left(\frac{1}{9} - \epsilon, \frac{1}{9} + \epsilon\right)$. So

$$\frac{1}{9} - \frac{1}{10^{n+1}} < x_n < \frac{1}{9} + \frac{1}{10^{n+1}}. \tag{27}$$

But $x_n = \frac{1}{9} + \frac{1}{10^{n+1}} = 0.111...11211... \in E$. Thus, for every $\epsilon > 0$, $\exists x_n \in \left(\frac{1}{9} - \epsilon, \frac{1}{9} + \epsilon\right)$ such that $\left(\frac{1}{9} - \epsilon, \frac{1}{9} + \epsilon\right) \cap E \neq \varnothing$.

Therefore, 0.1111... is a cluster point of E. $\square$

Problem 5

(a)

Proof. ($\Rightarrow$) Suppose $\sum_n a_n$ converges.

Since $\forall n \in \mathbb{N}$, $a_n > 0$, $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\frac{a_n}{b_n} > L - 1 \tag{28}$$
$$\implies a_n > (L-1)b_n \tag{29}$$
$$\implies (L-1)\sum_n b_n < \sum_n a_n. \tag{30}$$

But $\sum_n a_n$ converges, so $(L-1)\sum_n b_n$ converges by comparison.

Thus, $\sum_n b_n$ converges.

($\Leftarrow$) Suppose $\sum_n b_n$ converges.

Then similarly, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\frac{a_n}{b_n} < L + 1 \tag{31}$$
$$\implies a_n < (L+1)b_n \tag{32}$$
$$\implies \frac{1}{L+1} a_n < b_n. \tag{33}$$

But $\sum_n b_n$ converges, so $\frac{1}{L+1}\sum_n a_n$ converges by comparison.

Therefore, $\sum_n a_n$ converges. $\square$

(b)

(i) Let $a_n(x) = \frac{(-1)^n}{2020n}(x-10)^n$ for each $n \in \mathbb{N}$. Then $|a_n(x)| = \frac{|x-10|^n}{2020n}$. Using the ratio test, in order for the series to converge absolutely we have

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \tag{34}$$
$$= \lim_{n\to\infty}\left|\frac{(x-10)^{n+1}}{2020(n+1)}\right| \cdot \left|\frac{2020n}{(x-10)^n}\right| \tag{35}$$
$$= \lim_{n\to\infty}\left|\frac{n(x-10)}{n+1}\right| \tag{36}$$
$$= |x-10| < 1. \tag{37}$$

Thus, $\sum_n a_n(x)$ converges absolutely for all $9 < x < 11$.

Since this series is alternating, then for it to converge we must have that $\frac{(x-10)^n}{n}$ is monotonically decreasing and converges to zero. In order for this to be true, x can be no smaller than 9 and no larger than 11.

Therefore, $\sum_n \frac{(-1)^n}{2020n}(x-10)^n$ converges for all $9 \leq x \leq 11$.

(ii) Let $a_n(x) = n!x^{n!}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!x^{(n+1)!}}{n!x^{n!}}\right| \tag{38}$$

$$= (n+1)\left|\frac{x^{(n+1)!}}{x^{n!}}\right| \tag{39}$$

$$= (n+1)\left|\left(\frac{x^{n+1}}{x}\right)^{n!}\right| \tag{40}$$

$$= (n+1)\left|x^{n\cdot n!}\right|. \tag{41}$$

Then by the ratio test, in order for $\sum_n a_n$ to converge, we must have

$$\lim_{n\to\infty}(n+1)\left|x^{n\cdot n!}\right| < 1. \tag{42}$$

This holds when $|x| < 1$; otherwise the terms diverge.

Therefore, $\sum_n n!x^{n!}$ converges for all $-1 < x < 1$, and the same holds for absolute convergence.