18.100A Assignment 3

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February 21, 2023

Problem 1

Proof. Let $x, y \in \mathbb{R}$ with $x < y$. By the density of $\mathbb{Q}$, we have that $\exists r \in \mathbb{Q}$ such that $x < r < y$.

Then $x + \sqrt{2} < y + \sqrt{2}$. Then $\exists r \in \mathbb{Q}$ such that

$$x + \sqrt{2} < r < y + \sqrt{2} \tag{1}$$

$$\implies x < r - \sqrt{2} < y. \tag{2}$$

But since $r \in \mathbb{Q}$ and $\sqrt{2} \notin \mathbb{Q}$, then the number $i := r - \sqrt{2} \notin \mathbb{Q}$.

So $x < i < y$ with $i \in \mathbb{R} \backslash \mathbb{Q}$, as desired. $\square$

Problem 2

Proof. Define the function $f : E \to \wp(\mathbb{N})$ such that if $x = 0.d_{-1}d_{-2}...$, then

$$f(x) = \{j \in \mathbb{N} \mid d_{-j} = 2\}. \tag{3}$$

We want to show that f is a bijection. First, we show that f is injective.

Let $x_1 = 0.d_{-1}^{(1)}d_{-2}^{(1)}...$ and $x_2 = 0.d_{-1}^{(2)}d_{-2}^{(2)}...$ for $x_1, x_2 \in E$. Suppose $f(x_1) = f(x_2)$. Then

$$\{j \in \mathbb{N} \mid d_{-j}^{(1)} = 2\} = \{k \in \mathbb{N} \mid d_{-k}^{(2)} = 2\}. \tag{4}$$

Since each digit $d_{-j} \in \{1, 2\}$, then the sets of digits must be the same:

$$\{d_{-j}^{(1)} \mid j \in \mathbb{N}\} = \{d_{-k}^{(2)} \mid k \in \mathbb{N}\}. \tag{5}$$

But by the theorem from class, we know that for every set of digits $\exists! x \in [0, 1]$ such that $x = 0.d_{-1}d_{-2}....$ So if all of the digits are the same, then the numbers must be the same, i.e.

$$f(x_1) = f(x_2) \implies x_1 = x_2. \tag{6}$$

Thus f is injective.

Next, we show that f is surjective.

Let $S \in \wp(\mathbb{N})$ with

$$S := \{j \in \mathbb{N} \mid d_{-j} = 2\}. \tag{7}$$

Since this corresponds to the indices for a set of digits, then by the theorem from class $\exists x \in [0, 1]$ such that $x = 0.d_{-1}d_{-2}...$; i.e. for any $S \in \wp(\mathbb{N})$, $\exists x \in E$ such that $f(x) = S$.

Hence, f is also surjective, which means that it is bijective.

Therefore we conclude that $|E| = |\wp(\mathbb{N})|$. □

Problem 3

(a)

Proof. We want to show that there exists a bijection $h : A \cup B \to \mathbb{N}$.

Recall that we can construct the sets of even and odd natural numbers as follows:

$$\mathcal{O} := \{2n + 1 \mid n \in \mathbb{N}\}, \text{ and } \mathcal{E} := \{2n \mid n \in \mathbb{N}\}. \tag{8}$$

We also know that $|\mathcal{O}| = |\mathcal{E}| = |\mathbb{N}|$ because the functions defined via $f_e(n) = 2n$ and $f_o(n) = 2n+1$, mapping $\mathcal{E}$ to $\mathbb{N}$ and $\mathcal{O}$ to $\mathbb{N}$, respectively, are both bijective. Finally, we note that $\mathcal{O} \cup \mathcal{E} = \mathbb{N}$.

Since A and B are both countably infinite, i.e. $|A| = |B| = |\mathbb{N}|$, then $\exists$ bijections f, g such that

$$f : a \to \mathbb{N}, \text{ and } g : B \to \mathbb{N}. \tag{9}$$

Then $\forall a \in A$, $2f(a)$ is even, and $\forall b \in B$, $2g(b) + 1$ is odd. So we define the even and odd sets

$$\mathcal{E} := \{2f(a) \mid a \in A\}, \text{ and } \mathcal{O} := \{2g(b) + 1 \mid b \in B\}. \tag{10}$$

Then we construct the function h defined via

$$h(x) = \begin{cases} 2f(x) & \text{if } x \in A \\ 2g(x) + 1 & \text{if } x \in B. \end{cases} \tag{11}$$

First, we show that h is injective.

<u>Case 1</u>: $x, y \in A$. Then

$$\begin{aligned} & h(x) = h(y) && (12) \\ \implies & 2f(x) = 2f(y) && (13) \\ \implies & f(x) = f(y) && (14) \\ \implies & x = y, && (15) \end{aligned}$$

Since f is injective.

Case 2: $x, y \in B$. Then

$$\begin{aligned}
& h(x) = h(y) && (16)\\
\implies & 2g(x) + 1 = 2g(y) + 1 && (17)\\
\implies & g(x) = g(y) && (18)\\
\implies & x = y, && (19)
\end{aligned}$$

since g is injective.

Case 3: $x \in A$, $y \in B$ WOLOG, and $h(x) = h(y)$. This case is vacuous because $\mathcal{E}$ and $\mathcal{O}$ are disjoint, whereas $2f(x) \in \mathcal{E}$ while $2g(x) + 1 \in \mathcal{O}$.

Thus, according to cases 1 and 2, h is injective.

Now we show that h is surjective. Let $n \in \mathbb{N}$.

Case 1: n is even. Then by surjectivity of f, $\exists a \in A$ such that $f(a) = \frac{n}{2}$, i.e. $h(a) = n$.

Case 2: n is odd. Then by surjectivity of g, $\exists b \in B$ such that $g(b) = \frac{n-1}{2}$, i.e. $h(b) = n$.

Thus, h is surjective as well.

Therefore we conclude that h is a bijection, hence $|A \cup B| = |\mathbb{N}|$. $\square$

(b)

Proof. (By contradiction). Suppose instead that $\mathbb{R} \backslash \mathbb{Q}$ is countable. We write:

$$\begin{aligned}
& \mathbb{R} = \mathbb{R} \backslash \mathbb{Q} \cup \mathbb{Q} && (20)\\
\implies & |\mathbb{R}| = |\mathbb{R} \backslash \mathbb{Q} \cup \mathbb{Q}|. && (21)
\end{aligned}$$

We know that $\mathbb{Q}$ is countably infinite, so $|\mathbb{Q}| = |\mathbb{N}|$. We also assumed that $\mathbb{R} \backslash \mathbb{Q}$ is countable, so $|\mathbb{R} \backslash \mathbb{Q}| = |\mathbb{N}|$. Then by part **(a)**, we have that $|\mathbb{R}| = |\mathbb{N}|$, but the reals are uncountable, so contradiction ($\Rightarrow\Leftarrow$).

Hence $\mathbb{R} \backslash \mathbb{Q}$ must be uncountable. $\square$

Problem 4

Proof. ($\Rightarrow$) Suppose $a_0 = \sup A$. Then for any other upper bound $b \in \mathbb{R}$ of A, $a_0 \leq b$. Also, for any $a \in A$, $a \leq a_0$.

Let $\epsilon > 0$. Then $a_0 + \epsilon > a_0 \implies a_0 - \epsilon < a_0$, but $a_0 = \sup A$ so $a_0 - \epsilon \neq \sup A$. Then $\exists a \in A$ such that $a > a_0 - \epsilon$.

($\Leftarrow$) Suppose a_0 is an upper bound for $A \subset \mathbb{R}$, and that for every $\epsilon > 0$ $\exists a \in A$ such that $a_0 - \epsilon < a$. Since A is bounded above by a_0, then $\forall a \in A$, $a \leq a_0$. Then for every $\epsilon > 0$,

$$\implies a - \epsilon \leq a_0 - \epsilon < a \tag{22}$$
$$\implies a \leq a_0 < a + \epsilon. \tag{23}$$

But a_0 is an upper bound for A, so we have shown that for any $\epsilon > 0$, $\exists a \in A$ such that $a + \epsilon \notin A$ and $a_0 < a + \epsilon$. Thus a_0 is the smallest upper bound for A, which means that $a_0 = \sup A$ by definition. $\square$

Problem 5

(a)

Proof. (i) Let $\epsilon > 0$. Then $a - \epsilon < a \implies a - \epsilon \in (-\infty, a)$. Since $(-\infty, a)$ is not bounded below, $\exists x \in (-\infty, a)$ such that $-\infty < x < a - \epsilon$, i.e. $-\infty < x + \epsilon < a$.

Also, for $\epsilon > 0$ and $x \in (-\infty, a)$, it holds that $-\infty < x - \epsilon < a$. So

$$(x - \epsilon, x + \epsilon) \subset (-\infty, a). \tag{24}$$

Therefore, $(-\infty, a)$ is open.

(ii) $\forall x \in (a, b)$, we have that $b - x > 0$ and $x - a > 0$. Let $\epsilon = \frac{1}{2}\min\{x - a, b - x\} > 0$, and let $y \in \mathbb{R}$.

If $y \in (x - \epsilon, x + \epsilon)$, then $-\epsilon < y - x < \epsilon$. But $\epsilon < b - x \implies y - x < b - x$, i.e. $y < b$.

Also, $\epsilon < x - a \implies -\epsilon > a - x$, so $y - x > a - x$, i.e. $y > a$. Thus, $a < y < b$ $\forall y \in (x - \epsilon, x + \epsilon)$.

Therefore, (a, b) is open.

(iii) Let $x \in (b, \infty)$. Then $x > b$. Let $\epsilon = \frac{x-b}{2} > 0$, and let $y \in \mathbb{R}$.

If $y \in (x - \epsilon, x + \epsilon)$, then $-\epsilon < y - x < \epsilon$. Since $\epsilon < x - b \implies -\epsilon > b - x$,

$$\implies b - x < y - x < x - b \tag{25}$$
$$\implies b < y < 2x - b < \infty \tag{26}$$
$$\implies y \in (b, \infty). \tag{27}$$

Therefore, (b, ∞) is closed. $\square$

(b)

Proof. Suppose $U_\lambda \subset \mathbb{R}$ is open $\forall \lambda \in \Lambda$.

Take any $x \in \bigcup_{\lambda \in \Lambda} U_\lambda$. Then $x \in U_\lambda$ for at least one $\lambda \in \Lambda$. But since the U_λ are open, then $\exists \epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U_\lambda$.

But this must hold for every $x \in \bigcup_{\lambda\in\Lambda} U_\lambda$, i.e. for every such x, $\exists\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subset \bigcup_{\lambda\in\Lambda} U_\lambda. \tag{28}$$

Therefore, the union must also be open. □

(c)

Proof. Suppose U_m is open $\forall m \in \{1, ..., n\}$. Choose any $x \in \bigcap_{m=1}^n U_m$. Then if x is in the intersection of the sets, it must be in each of the sets, i.e.
$x \in U_1, U_2, ..., U_n$.

Since the U_m are open, then $\exists\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset U_m$ $\forall m \in \{1, ..., m\}$. But this must hold for every x in the intersection, i.e. $\forall x \in \bigcap_{m=1}^n U_m$, $\exists\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subset \bigcap_{m=1}^{n} U_m. \tag{29}$$

Therefore, the intersection must also be open. □

(d)

Claim: The set of rationals $\mathbb{Q} \subset \mathbb{R}$ is NOT open.

Proof. (By contradiction). Suppose instead that $\mathbb{Q}$ is open.

Then for every $q \in \mathbb{Q}$, $\exists\epsilon > 0$ such that

$$(q - \epsilon, q + \epsilon) \subset \mathbb{Q}. \tag{30}$$

Since $q - \epsilon, q + \epsilon \in \mathbb{R}$ and $q - \epsilon < q + \epsilon$, then $\exists i \in \mathbb{R}\backslash\mathbb{Q}$ such that

$$q - \epsilon < i < q + \epsilon, \tag{31}$$

as proven in problem 1. Then since there is an irrational number $i \in (q-\epsilon, q+\epsilon)$, this implies

$$(q - \epsilon, q + \epsilon) \not\subset \mathbb{Q}, \quad (\Rightarrow\Leftarrow) \tag{32}$$

which is a contradiction to the initial assumption.

Therefore, $\mathbb{Q}$ is not open. □