18.100A Assignment 11

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Problem 1

Proof. Let $f(x) = \frac{1}{1121}x^{1121} + \frac{1}{2021}x^{2021} + x + 1$. We compute the derivative:

$$f'(x) = x^{1120} + x^{2020} + 1. \tag{1}$$

Hence $f'(x) > 0\ \forall x \in \mathbb{R}$, so $f(x)$ is increasing.

Suppose $f(x)$ has n real roots, where $n > 1$. Then $\exists x_1, x_2, ..., x_n$ such that $f(x_1) = \cdots = f(x_n) = 0$. Since $f(x)$ is polynomial, then f is continuous $\forall x$ and differentiable on $\mathbb{R}$. By Rolle's theorem, $\exists c \in (x_1, x_2)$ such that $f'(c) = 0$, which is a contradiction since $f'(x) > 0\ \forall x$. Thus f cannot have more than one real root.

Now suppose $f(x)$ has no real roots. Then either $f(x) > 0\ \forall x$ or $f(x) < 0\ \forall x$. Choose $x_0 = 1$. Then $f(x_0) = \frac{1}{1121} + \frac{1}{2021} + 2 > 0$. Choose $x^* = -10$. Then $f(x^*) = -\frac{10^{1121}}{1121} - \frac{10^{2021}}{2021} + 2 < 0$. Then by continuity, $f(x)$ has at least one real root, which is a contradiction. Thus f must have at least one real root.

So we have the number of real roots $1 \leq n \leq 1$, so $n = 1$.

Therefore f has exactly one real root. $\square$