18.100A Assignment 6

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Problem 1

(a)

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{3}{9n+1} &= 3\sum_{n=1}^{\infty} \frac{1}{9n+1} && (1)\\ &= \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n+\frac{1}{9}} && (2)\\ &= \frac{1}{3}\sum_{n=2}^{\infty} \frac{1}{(n-1)+\frac{1}{9}} && (3)\\ &= \frac{1}{3}\sum_{n=2}^{\infty} \frac{1}{n-\frac{8}{9}} && (4)\\ &> \sum_{n=2}^{\infty} \frac{1}{n}. && (5)\end{aligned}$$

But the Harmonic series, $\sum_n \frac{1}{n}$, diverges.

Therefore, we conclude by comparison that the series $\sum_n \frac{3}{9n+1}$ diverges.

(b)

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n-\frac{1}{2}} > \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}. \tag{6}$$

Therefore, by comparison, $\sum_n \frac{1}{2n-1}$ diverges.

(c)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} - \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2}, \tag{7}$$

which is the difference of two convergent series.

Therefore, $\sum_n \frac{(-1)^n}{n^2}$ converges.

(d)

We can express the series $\sum_n \frac{1}{n(n+1)}$ as a telescoping sum:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1} && (8) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots && (9) \\ &= 1. && (10)\end{aligned}$$

Therefore, $\sum_n \frac{1}{n(n+1)}$ converges to 1.

(e)

We note that $\forall n \in \mathbb{N}$, $e^{n^2} \geq n^3$. Then we have

$$\sum_{n=1}^{\infty} \frac{n}{e^{n^2}} \leq \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}, \tag{11}$$

which converges.

Therefore, $\sum_n ne^{-n^2}$ converges.

Problem 2

(a)

Proof. Suppose $\sum_n x_n$ converges.

Then the sequence of partial sums $\{s_m\}_m = \{x_1 + \cdots x_m\}_{m=1}^{\infty}$ also converges.

We construct two subsequences of $\{s_m\}_m$, defined via

$$\{s_{m_{2k}}\}_k = \{x_2, x_2 + x_4, x_2 + x_4 + x_6, \cdots\} = \{x_2 + \cdots x_{2k}\}_k, \tag{12}$$

and

$$\{s_{m_{2k+1}}\}_k = \{x_1, x_1 + x_3, x_1 + x_3 + x_7, \cdots\} = \{x_1 + \cdots x_{2k+1}\}_k. \tag{13}$$

Note that for each $m \in \mathbb{N}$, if m is even then $m = 2k$ for some $k \in \mathbb{N}$. Then

$$s_m = s_{m_{2k-1}} + s_{m_{2k}}. \tag{14}$$

Likewise if m is odd, then

$$s_m = s_{m_{2k}} + s_{m_{2k+1}}. \tag{15}$$

In both cases, the partial sum s_m must converge because the series $\sum_n x_n$ converges. Thus, the sum of the two partial sums must also converge, i.e. $\{s_{m_{2k}}\}_k$ and $\{s_{m_{2k+1}}\}_k$ both converge.

Then both series $\sum_n x_{2n}$ and $\sum_n x_{2n+1}$ must converge, and so does their sum:

$$\sum_n x_{2n} + \sum_n x_{2n+1} < \infty. \tag{16}$$

Therefore, $\sum_n (x_{2n} + x_{2n+1})$ converges. □

(b)

Consider the following counterexample to the converse of the theorem proven in problem **(a)**:

Let $x_n = (-1)^n$. Then

$$\sum_{n=1}^{\infty} (x_{2n} + x_{2n+1}) = \sum_{n=1}^{\infty} \left[(-1)^{2n} + (-1)^{2n+1}\right] \tag{17}$$

$$= \sum_{n=1}^{\infty} (1^n - 1^n) \tag{18}$$

$$= 0. \tag{19}$$

But $\sum_n (-1)^n$ does not converge.

Therefore, the converse to **(a)** does not always hold.

Problem 3

Proof. Suppose $\sum_n x_n$ converges absolutely.

Since the series $\sum_n |x_n|$ converges, then $\sum_n x_n$ converges by comparison, because $x_n \leq |x_n|$.

Let $N \in \mathbb{N}$. Then the triangle inequality gives

$$|x_1 + x_2 + \cdots + x_N| \leq |x_1| + |x_2| + \cdots + |x_N| = \sum_{n=1}^{N} |x_n|. \tag{20}$$

Because $\sum_n |x_n|$ converges, we can now take $N \to \infty$, which gives

$$\left|\sum_{n=1}^{\infty} x_n\right| \leq \sum_{n=1}^{\infty} |x_n|, \tag{21}$$

as desired. □

Problem 4

(a)

Using the sum of a geometric sequence, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{2^{2n+1}} &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^{2n}} && (22)\\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n && (23)\\
&= \frac{1}{2} \cdot \frac{1}{1-\frac{1}{4}} && (24)\\
&= \frac{1}{2} \cdot \frac{4}{3}. && (25)
\end{aligned}$$

Therefore, $\sum_n \left(\frac{1}{2}\right)^{2n+1}$ converges to $\frac{2}{3}$.

(b)

Let $x_n = \frac{(-1)^n(n-1)}{n}$. Then we easily check that $\lim_{n\to\infty} x_n \neq 0$, since the non-alternating component approaches 1, but then alternates between -1 and 1; i.e. the limit does not exist.

Therefore, $\sum_n \frac{(-1)^n(n-1)}{n}$ diverges.

(c)

Note that since $n^{\frac{1}{10}} \leq n \; \forall n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{10}}} \geq \sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \tag{26}$$

which diverges.

Therefore, $\sum_n \frac{(-1)^n}{n^{\frac{1}{10}}}$ diverges by comparison.

(d)

We have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n^n}{(n+1)^{2n}} &= \sum_{n=1}^{\infty} \frac{n^n}{(n^2+2n+1)^n} && (27)\\
&= \sum_{n=1}^{\infty} \left(\frac{n}{n^2+2n+1}\right)^n. && (28)
\end{aligned}$$

We also have that $\forall n \in \mathbb{N}$,

$$\left| \frac{n}{n^2 + 2n + 1} \right| < \frac{1}{4} < 1. \tag{29}$$

Then the series is upper bounded by a geometric series:

$$\sum_{n=1}^{\infty} \frac{n^n}{(n+1)^{2n}} < \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{4}{3}. \tag{30}$$

Therefore, $\sum_n \frac{n^n}{(n+1)^{2n}}$ converges.