

18.100A Midterm

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Problem 1

(a)

Proof. Let $x \in f^{-1}(C \cap D)$. Then

$$\implies f(x) \in C \cap D \quad (1)$$

$$\implies f(x) \in C \text{ and } f(x) \in D \quad (2)$$

$$\implies x \in f^{-1}(C) \text{ and } x \in f^{-1}(D) \quad (3)$$

$$\implies x \in f^{-1}(C) \cap f^{-1}(D). \quad (4)$$

Thus,

$$f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D). \quad (5)$$

Now let $x \in f^{-1}(C) \cap f^{-1}(D)$. Then

$$\implies f(x) \in C \text{ and } f(x) \in D \quad (6)$$

$$\implies f(x) \in C \cap D \quad (7)$$

$$\implies x \in f^{-1}(C \cap D). \quad (8)$$

Thus,

$$f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D). \quad (9)$$

Therefore by equations (5) and (9), $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. \square

(b)

Claim: If $E \subset \mathbb{R}$ is countable, then the complement $\mathbb{R} \setminus E$ is always uncountable.

Proof. (By contradiction). Suppose E^c is countable. Then $E \cup E^c$ is countable as well, since it is the union of two countable sets. But $E \cup E^c = \mathbb{R}$, which is uncountable. ($\Rightarrow \Leftarrow$). \square

(c)

By contrast, if $E \subset \mathbb{R}$ is uncountable, then the complement $\mathbb{R} \setminus E$ is not always countable. Take for instance, $E = [0, 1]$, which is uncountable. Then $E^c = (-\infty, 0) \cup (1, \infty)$, which is also uncountable.

Problem 2

(a)

A set $U \subset \mathbb{R}$ is *not open* if for every $\epsilon > 0$, $\exists x \in U$ such that $(x - \epsilon, x + \epsilon) \not\subset U$.

(b)

Proof. Suppose U is not open. Let $\epsilon = \frac{1}{n}$. Then for every $n \in \mathbb{N}$, $\exists x \in U$ such that $(x - \frac{1}{n}, x + \frac{1}{n}) \not\subset U$. Equivalently, for each $n \in \mathbb{N}$ $\exists x_n \in U^c$ such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n}. \quad (10)$$

Then we have

$$0 < |x_n - x| < \frac{1}{n}, \quad (11)$$

and taking the limit on all sides gives

$$0 < \lim_{n \rightarrow \infty} |x_n - x| < \lim_{n \rightarrow \infty} \frac{1}{n}. \quad (12)$$

Thus, by the squeeze theorem, $\lim_{n \rightarrow \infty} x_n = x$, as desired. \square

(c)

Proof. (By contradiction). To show that F is closed, we must show that F^c is open. Suppose, toward a contradiction, that F^c is not open. Then by part (b), $\exists x \in F^c$ and a sequence $\{x_n\}_n$ of elements of F such that $\lim_{n \rightarrow \infty} x_n = x$. But by assumption, every convergent sequence of elements of F has a limit in F , i.e. we assumed originally that $x \in F$ ($\Rightarrow \Leftarrow$). Thus, F^c must be open, so F is closed. \square

Problem 3

(a)

Proof. Let $\epsilon > 0$. Choose $N = \frac{1}{\sqrt{\epsilon}}$. Then $\forall n \geq N$, we have

$$\left| \frac{10n^2}{n^2 + 16n + 1} - 10 \right| = \left| \frac{-160n - 10}{n^2 + 16n + 1} \right| \quad (13)$$

$$= \left| \frac{160n + 10}{n^2 + 16n + 1} \right| \quad (14)$$

$$< \frac{1}{n^2 + 16n + 1} \quad (15)$$

$$< \frac{1}{n^2} \quad (16)$$

$$< \epsilon. \quad (17)$$

Therefore, $\lim_{n \rightarrow \infty} \left| \frac{10n^2}{n^2 + 16n + 1} \right| = 10$. \square

(b)

(i) Let $x_n = \frac{(-1)^n}{n}$. Then $\lim_{n \rightarrow \infty} x_n = 0$. But $x_1 = -1 < \frac{1}{2} = x_2$, whereas $x_2 = \frac{1}{2} > -\frac{1}{3} = x_3$.

Therefore $\{x_n\}_n$ converges to 0, but is not monotonic.

(ii) Let

$$x_n = \begin{cases} 0, & n \text{ even} \\ n, & n \text{ odd} \end{cases}. \quad (18)$$

Then $\{x_n\}_n$ is clearly unbounded.

Consider the subsequence $\{x_{n_k}\}_k$ defined by $x_{n_k} = x_{2k}$ for each $k \in \mathbb{N}$. Then $\forall k, x_{n_k} = 0$.

Therefore, $\{x_n\}_n$ is unbounded, but $\{x_{n_k}\}_k$ converges to 0.

Problem 4

(a)

TODO TODO