18.100A Assignment 10

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Problem 1

(a)

Proof. Suppose $\exists C \geq 0$ such that $\forall x, y \in I$,

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}. \tag{1}$$

Let $\epsilon > 0$. Choose $\delta = \left(\frac{\epsilon}{C}\right)^{\frac{1}{\alpha}}$. Then if $|x - y| < \delta$, we get

$$\begin{aligned} |f(x) - f(y)| &\leq C|x - y|^{\alpha} && (2) \\ &< C\delta^{\alpha} && (3) \\ &= C\frac{\epsilon}{C} && (4) \\ &= \epsilon. && (5) \end{aligned}$$

Therefore f is uniformly continuous on I. □

(b)

Proof. Suppose $\exists C \geq 0$ such that $\forall x, y \in I$, $|f(x) - f(y)| \leq C|x - y|^{\alpha}$.

Since $\alpha > 1$, then $\alpha = 1 + r$ for some $0 < r$, we have

$$\begin{aligned} &\implies 0 \leq |f(x) - f(y)| \leq C|x - y|^{1+r} && (6) \\ &\implies 0 \leq \frac{|f(x) - f(y)|}{|x - y|} \leq C|x - y|^{r} && (7) \\ &\implies \lim_{x \to y} 0 \leq \lim_{x \to y} \frac{|f(x) - f(y)|}{|x - y|} \leq C \lim_{x \to y} |x - y|^{r} && (8) \\ &\implies 0 \leq \lim_{x \to y} \frac{|f(x) - f(y)|}{|x - y|} \leq 0. && (9) \end{aligned}$$

Then by the squeeze theorem, $\lim_{x \to y} \frac{|f(x) - f(y)|}{|x - y|} = 0$. Thus $\forall y \in I$, $f'(y) = 0$.

Therefore f is constant. □

Problem 2

Proof. We compute:

$$
\begin{align}
L &= \lim_{x\to c} \frac{h(x)-h(c)}{x-c} \tag{10}\\
&= \lim_{x\to c} \frac{f(x)g(x)-f(c)g(c)}{x-c} \tag{11}\\
&= \lim_{x\to c} \frac{f(x)g(x)-f(x)g(c)+f(x)g(c)-f(c)g(c)}{x-c} \tag{12}\\
&= \lim_{x\to c} \left[f(x)\left(\frac{g(x)-g(c)}{x-c}\right)\right] + g(x)\lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right) \tag{13}\\
& \tag{14}
\end{align}
$$

Since f is continuous at c, and both f and g are differentiable at c, this gives us

$$L = f(c)g'(c) + g(c)f'(c), \tag{15}$$

which exists.

Therefore $f(x)g(x)$ is differentiable at c. □

Problem 3

Proof. ($\Rightarrow$) Suppose f is Lipschitz. Then $\exists L \geq 0$ such that $\forall x, y \in \mathbb{R}$,

$$\left|\frac{f(x)-f(y)}{x-y}\right| \leq L. \tag{16}$$

Since f is differentiable, we have

$$|f'(y)| = \lim_{x\to y}\left|\frac{f(x)-f(y)}{x-y}\right| \leq \lim_{x\to y} L, \tag{17}$$

so $\forall y \in \mathbb{R}$, $|f'(y)| \leq L$.

Therefore f' is bounded.

($\Leftarrow$) Suppose f' is bounded.

Then $\exists B \geq 0$ such that $|f'(x)| \leq B \ \forall x \in \mathbb{R}$. Let $x, y \in \mathbb{R}$. Then by the mean value theorem, $\exists c \in \mathbb{R}$ such that $f(x) - f(y) = (x-y)f'(c)$, i.e.

$$
\begin{align}
|f(x)-f(y)| &= |x-y||f'(c)| \tag{18}\\
&\leq |x-y|B. \tag{19}
\end{align}
$$

Choose $L = B$, and we see

$$|f(x)-f(y)| \leq L|x-y|. \tag{20}$$

Thus, we conclude f is Lipschitz $\iff$ f' is bounded. □

Problem 4

Proof. Since f, g are differentiable, then f, g are continuous on (a, b). Since $g(c) = 0$ and $g'(x) \neq 0 \; \forall x \in c$, then g is either increasing or decreasing away from 0 when $x \neq c$. Also, since g is continuous, if $g = 0$ for any $x \neq c$, then $\exists c_2 \in (a, b)$ such that $g'(c_2) = 0$, which is a contradiction.

Thus, $g(x) \neq 0 \; \forall x \notin c$. Therefore $\frac{f(x)}{g(x)}$ is continuous on (a, b) except at $x = c$.

The derivatives of f and g at c are:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{f(x)}{x - c} = f'(c), \tag{21}$$

$$\lim_{x \to c} \frac{g(x) - g(c)}{x - c} = \lim_{x \to c} \frac{g(x)}{x - c} = g'(c). \tag{22}$$

Then we compute

$$\begin{aligned}
\lim_{x \to c} \left(\frac{f(x)}{g(x)} \right) &= \lim_{x \to c} \left(\frac{f(x)}{x - c} \frac{x - c}{g(x)} \right) & (23)\\
&= \lim_{x \to c} \left(\frac{f(x)}{x - c} \right) \lim_{x \to c} \left(\frac{x - c}{g(x)} \right) & (24)\\
&= \frac{f'(c)}{g'(c)} & (25)\\
&= \lim_{x \to c} \frac{f'(x)}{g'(x)}, & (26)
\end{aligned}$$

by continuity.

Therefore $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \lim\limits_{x \to c} \frac{f'(x)}{g'(x)}$. □