18.100A Assignment 2

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Problem 1

Proof. (By contradiction).

Suppose instead that $xy \leq xz$. Then

$\implies xy - xz \leq 0$

$\implies x(y - z) \leq 0.$

Since $x < 0$ by assumption, it must then be true that $y - z \geq 0$. But then

$\implies y \geq z \quad \Rightarrow\Leftarrow,$

which is a contradiction since we assumed that $y < z$. Thus, $xy > xz$. $\square$

Problem 2

(a)

Proof. We want to show that $\exists b \in S$ such that $\forall a \in A$, $a \leq b$.

Since S is ordered, then for every $x, y \in S$, we have that either $x < y$, $x > y$, or $x = y$. But since $A \subset S$, then $\forall a \in A$, $a \in A \implies a \in S$.

$\implies \forall a, b \in A$, either $a < b$, $a > b$, or $a = b$.

So A is also ordered. Since A is finite, then $\exists a_0 \in A$ such that $\forall a \in A$, $a_0 \geq a$.

Thus, A is bounded. $\square$

(b)

Proof. (By contradiction).

Assuming A is finite, suppose instead that there is no maximal element in A. Choose an element $a_1 \in A$. Then, since a_1 is not the maximum, $\exists a_2 \in A$ such that $a_1 < a_2$. But a_2 is also not the maximum of A, so $\exists a_3 \in A$ such that

$a_2 < a_3$. Continuing in this manner, we find an increasing sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements of A, i.e. such that

$$a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots . \tag{1}$$

But because this sequence is infinite and contained in A, this contradicts the assumption that A is finite. Thus, there must exist a maximal element in A.

To show that there exists a minimum element, we recreate the same argument from above where instead, supposing that there is no minimal element, we demonstrate that we can construct an infinite decreasing sequence $\cdots a_n < \cdots < a_2 < a_1$ of elements of A, once again arriving at a contradiction.

Therfore, both $\inf A$ and $\sup A$ exist in A. $\square$