18.100A Assignment 3

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Problem 1

Proof. Let $x, y \in \mathbb{R}$. By the density of $\mathbb{Q}$, we have that $\exists r \in \mathbb{Q}$ such that $x < r < y$.

Then $x + \sqrt{2} < y + \sqrt{2}$. Then $\exists r \in \mathbb{Q}$ such that

$$x + \sqrt{2} < r < y + \sqrt{2} \tag{1}$$
$$\implies x < r - \sqrt{2} < y. \tag{2}$$

But since $r \in \mathbb{Q}$ and $\sqrt{2} \notin \mathbb{Q}$, then the number $i := r - \sqrt{2} \notin \mathbb{Q}$.

So $x < i < y$ with $i \in \mathbb{R} \backslash \mathbb{Q}$, as desired. $\square$

Problem 2

Proof. Define the function $f : E \longrightarrow \wp(\mathbb{N})$ such that if $x = 0.d_{-1}d_{-2}...$, then

$$f(x) = \{j \in \mathbb{N} \mid d_{-j} = 2\}. \tag{3}$$

We want to show that f is a bijection. First, we show that f is injective.

Let $x_1 = 0.d_{-1}^{(1)}d_{-2}^{(1)}...$ and $x_2 = 0.d_{-1}^{(2)}d_{-2}^{(2)}...$ for $x_1, x_2 \in E$. Suppose $f(x_1) = f(x_2)$. Then

$$\{j \in \mathbb{N} \mid d_{-j}^{(1)} = 2\} = \{k \in \mathbb{N} \mid d_{-k}^{(2)} = 2\}. \tag{4}$$

Since each digit $d_{-j} \in \{1, 2\}$, then the sets of digits must be the same:

$$\{d_{-j}^{(1)} \mid j \in \mathbb{N}\} = \{d_{-k}^{(2)} \mid k \in \mathbb{N}\}. \tag{5}$$

But by the theorem from class, we know that for every set of digits $\exists! x \in [0, 1]$ such that $x = 0.d_{-1}d_{-2}....$ So if all of the digits are the same, then the numbers must be the same, i.e.

$$f(x_1) = f(x_2) \implies x_1 = x_2. \tag{6}$$

Thus f is injective.

Next, we show that f is surjective.

Let $S \in \wp(\mathbb{N})$ with

$$S := \{j \in \mathbb{N} \mid d_{-j} = 2\}. \tag{7}$$

Since this is a set of digits, then by the theorem from class $\exists x \in [0, 1]$ such that $x = 0.d_{-1}d_{-2}...$; i.e. for any $S \in \wp(\mathbb{N})$, $\exists x \in E$ such that $f(x) = S$.

Hence, f is also surjective, which means that it is bijective.

Therefore we conclude that $|E| = |\wp(\mathbb{N})|$. $\square$