18.100A Assignment 10

Octavio Vega

June 6, 2023

Problem 1

(a)

Proof. Suppose $\exists C \geq 0$ such that $\forall x, y \in I$,

$$|f(x) - f(y)| \leq C|x-y|^{\alpha}. \tag{1}$$

Let $\epsilon > 0$. Choose $\delta = \left(\frac{\epsilon}{C}\right)^{\frac{1}{\alpha}}$. Then if $|x-y| < \delta$, we get

$$\begin{aligned} |f(x) - f(y)| &\leq C|x-y|^{\alpha} && (2)\\ &< C\delta^{\alpha} && (3)\\ &= C\frac{\epsilon}{C} && (4)\\ &= \epsilon. && (5) \end{aligned}$$

Therefore f is uniformly continuous on I. $\square$

(b)

Proof. Suppose $\exists C \geq 0$ such that $\forall x, y \in I$, $|f(x) - f(y)| \leq C|x-y|^{\alpha}$.

Since $\alpha > 1$, then $\alpha = 1 + r$ for some $0 < r$, we have

$$\begin{aligned} &\implies 0 \leq |f(x) - f(y)| \leq C|x-y|^{1+r} && (6)\\ &\implies 0 \leq \frac{|f(x) - f(y)|}{|x-y|} \leq C|x-y|^{r} && (7)\\ &\implies \lim_{x\to y} 0 \leq \lim_{x\to y} \frac{|f(x) - f(y)|}{|x-y|} \leq C \lim_{x\to y} |x-y|^{r} && (8)\\ &\implies 0 \leq \lim_{x\to y} \frac{|f(x) - f(y)|}{|x-y|} \leq 0. && (9) \end{aligned}$$

Then by the squeeze theorem, $\lim\limits_{x\to y} \frac{|f(x)-f(y)|}{|x-y|} = 0$. Thus $\forall y \in I$, $f'(y) = 0$.

Therefore f is constant. $\square$

Problem 2

Proof. We compute:

$$
\begin{align}
L &= \lim_{x\to c} \frac{h(x)-h(c)}{x-c} \tag{10}\\
&= \lim_{x\to c} \frac{f(x)g(x)-f(c)g(c)}{x-c} \tag{11}\\
&= \lim_{x\to c} \frac{f(x)g(x)-f(x)g(c)+f(x)g(c)-f(c)g(c)}{x-c} \tag{12}\\
&= \lim_{x\to c} \left[f(x) \left(\frac{g(x)-g(c)}{x-c} \right) \right] + g(x) \lim_{x\to c} \left(\frac{f(x)-f(c)}{x-c} \right) \tag{13}\\
& \tag{14}
\end{align}
$$

Since f is continuous at c, and both f and g are differentiable at c, this gives us

$$L = f(c)g'(c) + g(c)f'(c), \tag{15}$$

which exists.

Therefore $f(x)g(x)$ is differentiable at c. ☐

Problem 3

Proof. ($\Rightarrow$) Suppose f is Lipschitz. Then $\exists L \geq 0$ such that $\forall x, y \in \mathbb{R}$,

$$\left| \frac{f(x)-f(y)}{x-y} \right| \leq L. \tag{16}$$

Since f is differentiable, we have

$$|f'(y)| = \lim_{x\to y} \left| \frac{f(x)-f(y)}{x-y} \right| \leq \lim_{x\to y} L, \tag{17}$$

so $\forall y \in \mathbb{R}$, $|f'(y)| \leq L$.

Therefore f' is bounded.

($\Leftarrow$) Suppose f' is bounded.

Then $\exists B \geq 0$ such that $|f'(x)| \leq B$ $\forall x \in \mathbb{R}$. Let $x, y \in \mathbb{R}$. Then by the mean value theorem, $\exists c \in \mathbb{R}$ such that $f(x) - f(y) = (x-y)f'(c)$, i.e.

$$
\begin{align}
|f(x)-f(y)| &= |x-y||f'(c)| \tag{18}\\
&\leq |x-y|B. \tag{19}
\end{align}
$$

Choose $L = B$, and we see

$$|f(x)-f(y)| \leq L|x-y|. \tag{20}$$

Thus, we conclude f is Lipschitz $\iff$ f' is bounded. ☐

Problem 4

Proof. Since f, g are differentiable, then f, g are continuous on (a, b). Since $g(c) = 0$ and $g'(x) \neq 0\ \forall x \in c$, then g is either increasing or decreasing away from 0 when $x \neq c$. Also, since g is continuous, if $g = 0$ for any $x \neq c$, then $\exists c_2 \in (a, b)$ such that $g'(c_2) = 0$, which is a contradiction.

Thus, $g(x) \neq 0\ \forall x \notin c$. Therefore $\frac{f(x)}{g(x)}$ is continuous on (a, b) except at $x = c$.

The derivatives of f and g at c are:

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c} \frac{f(x)}{x - c} = f'(c), \tag{21}$$

$$\lim_{x\to c} \frac{g(x) - g(c)}{x - c} = \lim_{x\to c} \frac{g(x)}{x - c} = g'(c). \tag{22}$$

Then we compute

$$\begin{aligned}
\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) &= \lim_{x\to c} \left(\frac{f(x)}{x - c} \frac{x - c}{g(x)}\right) && (23)\\
&= \lim_{x\to c} \left(\frac{f(x)}{x - c}\right) \lim_{x\to c} \left(\frac{x - c}{g(x)}\right) && (24)\\
&= \frac{f'(c)}{g'(c)} && (25)\\
&= \lim_{x\to c} \frac{f'(x)}{g'(x)}, && (26)
\end{aligned}$$

by continuity.

Therefore $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = \lim\limits_{x\to c} \frac{f'(x)}{g'(x)}$. □

Problem 5

(a.i)

Proof. Define $g(x) := f(x) - f(a)$ and $h(x) := x - a$. Then both g and h are continuous, and $g(a) = f(a) - f(a) = 0$, $h(a) = a - a = 0$. Also, $h'(x) = 1 \neq 0$ $\forall x$. Then by problem **(4)**:

$$\begin{aligned}
f'(a) &= \lim x \to a \frac{f(x) - f(a)}{x - a} && (27)\\
&= \lim_{x\to a} \frac{g(x)}{h(x)} && (28)\\
&= \lim_{x\to a} \frac{g'(x)}{h'(x)} && (29)\\
&= \lim_{x\to a} f'(x) && (30)\\
&= L. && (31)
\end{aligned}$$

Therefore $f'(a) = L$. $\square$

(a.ii)

Proof. $f'(b) = \lim_{x \to b} \frac{f(x)-f(b)}{x-b}$. As in part **(a.i)**, define $g(x) := f(x) - f(b)$, and $h(x) := x - b$, and proceed with L'Hopital's rule.

Then $f'(b) = L$. $\square$

(b)

Proof. The function $f(x)$ is continuous (a, b) and differentiable on $(a, c) \cup (c, b)$. Since f is continuous on $[c, b)$ and differentiable on (c, b), then by part **(a.i)** f is differentiable at c and $\lim_{x \to c^+} f'(x) = L$. So $f'(x) = L$. Similarly, f is continuous on $(a, c]$ and differentiable on (a, c) so by part **(a.ii)** f is differentiable at c and $\lim_{x \to c^-} f'(x) = L$. So $f'(x) = L$.

Therefore $f'(c) = L$. $\square$