18.100A Assignment 2

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Problem 1

Proof. (By contradiction).

Suppose instead that $xy \leq xz$. Then

$\implies xy - xz \leq 0$

$\implies x(y-z) \leq 0.$

Since $x < 0$ by assumption, it must then be true that $y - z \geq 0$. But then

$\implies y \geq z \quad \Rightarrow\Leftarrow,$

which is a contradiction since we assumed that $y < z$. Thus, $xy > xz$. $\square$

Problem 2

(a)

Proof. We want to show that $\exists b \in S$ such that $\forall a \in A$, $a \leq b$.

Since S is ordered, then for every $x, y \in S$, we have that either $x < y$, $x > y$, or $x = y$. But since $A \subset S$, then $\forall a \in A$, $a \in A \implies a \in S$.

$\implies \forall a, b \in A$, either $a < b$, $a > b$, or $a = b$.

So A is also ordered. Since A is finite, then $\exists a_0 \in A$ such that $\forall a \in A$, $a_0 \geq a$.

Thus, A is bounded. $\square$

(b)

Proof. (By contradiction).

Assuming A is finite, suppose instead that there is no maximal element in A. Choose an element $a_1 \in A$. Then, since a_1 is not the maximum, $\exists a_2 \in A$ such that $a_1 < a_2$. But a_2 is also not the maximum of A, so $\exists a_3 \in A$ such that

$a_2 < a_3$. Continuing in this manner, we find an increasing sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements of A, i.e. such that

$$a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots . \tag{1}$$

But because this sequence is infinite and contained in A, this contradicts the assumption that A is finite. Thus, there must exist a maximal element in A.

To show that there exists a minimum element, we recreate the same argument from above where instead, supposing that there is no minimal element, we demonstrate that we can construct an infinite decreasing sequence $\cdots a_n < \cdots < a_2 < a_1$ of elements of A, once again arriving at a contradiction.

Therfore, both $\inf A$ and $\sup A$ exist in A. □

Problem 3

Proof. Since b is an upper bound for A, then $\forall a \in A$, $a \leq b$.

Suppose $b \neq \sup A$. Then $\exists$ some other element $c \in A$ such that $c = \sup A$, since by problem 2, A must have a supremum because it is finite and a subset of an ordered set. But since $b \in A$, then $b \leq c$.

However, we assumed that b is an upper bound for A, so since $c \in A$, this imples that $b \geq c$. Thus we have that $b \leq c$ and $b \geq c$, so it must hold that $b = c$.

Therefore $b = \sup A$, as desired. □

Problem 4

Proof. Suppose $\sup A \notin A$, and let $x_0 \in A$. Towards a contradiction, suppose that $\forall x \in A$, $x \leq x_0$. Then x_0 is an upper bound for A.

Since $x_0 \in A$, then by problem 3, $x_0 = \sup A$. But we assumed $\sup A \notin A$, so contradiction.

So for any $x_0 \in A$, $\exists x_1$ such that $x_1 > x_0$. We can repeat the logic above to show that this holds for arbitrary $x_i \in A$, since no x_i can both be an upper bound for A while also being contained in A. Thus $\forall x_i \in A$, $\exists x_{i+1}$ such that $x_i < x_{i+1}$.

So we obtain an infinite decreasing sequence $\{x_i\}_{i\in\mathbb{N}}$ of elements of A. Hence, these elements form a countably infinite subset of A, since $|\{x_i | i \in \mathbb{N}\}| = |\mathbb{N}|$.

Therefore, A does indeed contain a countably infinite subset. □

Problem 5

(a)

Proof. (Arithmetic Mean - Geometric Mean Inequality)

Consider the difference $\sqrt{x} - \sqrt{y}$. We compute:

$$0 \leq \left(\sqrt{x} - \sqrt{y}\right)^2 = x + y - 2\sqrt{xy} \tag{2}$$
$$\implies 2\sqrt{xy} \leq x + y \tag{3}$$

Dividing both sides of (3) by 2, we obtain the desired result: $\sqrt{xy} \leq \frac{x+y}{2}$. $\square$

(b)

Proof. ($\Rightarrow$) Suppose $\sqrt{xy} = \frac{x+y}{2}$. Then

$$\implies 2\sqrt{xy} - x - y = 0 \tag{4}$$
$$\implies x + y - 2\sqrt{xy} = 0 \tag{5}$$
$$\implies \left(\sqrt{x} - \sqrt{y}\right)^2 = 0 \tag{6}$$
$$\implies \sqrt{x} - \sqrt{y} = 0 \tag{7}$$
$$\implies \sqrt{x} = \sqrt{y} \tag{8}$$
$$\implies x = y. \tag{9}$$

($\Leftarrow$) Suppose $x = y$. Then

$$\frac{x+y}{2} = \frac{x+x}{2} = x \tag{10}$$
$$= \sqrt{x} \cdot \sqrt{x} = \sqrt{x} \cdot \sqrt{y} \tag{11}$$
$$= \sqrt{xy}. \tag{12}$$

Thus, $\frac{x+y}{2} = \sqrt{xy} \iff x = y$. $\square$

Problem 6

(a)

Proof. Since A is bounded, then $\exists a_0$ such that $\forall a \in A$, $a \geq a_0$. Similarly, since B is bounded, then $\exists b_0$ such that $\forall b \in B$, $b \geq b_0$.

Let $c \in C$, i.e. let $a \in A$, $b \in B$, and take $c = a + b$. Then for all such c,

$$c = a + b \geq a_0 + b \geq a_0 + b_0 \tag{13}$$

Hence, the set C is bounded below. By the same logic, we can show that $\forall c \in C$, $c < a_1 + b_1$ where a_1 and b_1 are upper bounds for A and B, respectively. So C is also bounded above.

Therefore, C is bounded. $\square$

(b)

Proof. Let $\epsilon > 0$. By definition of supremum, $\exists x \in A$, $y \in B$ such that $x + \frac{\epsilon}{2} > \sup A$ and $y + \frac{\epsilon}{2} > \sup B$. Then

$$x + y + \epsilon = \left(x + \frac{\epsilon}{2}\right) + \left(y + \frac{\epsilon}{2}\right) \tag{14}$$
$$> \sup A + \sup B. \tag{15}$$

So for every $\epsilon > 0$, $\exists c = x + y \in C$ such that

$$c + \epsilon > \sup A + \sup B. \tag{16}$$

Then we conclude that $\sup C = \sup A + \sup B$. $\square$

(c)

Proof. Let $\epsilon > 0$. By definition of infimum, $\exists x \in A$, $y \in B$ such that $x - \frac{\epsilon}{2} < \inf A$ and $y - \frac{\epsilon}{2} < \inf B$. Then

$$x + y - \epsilon = \left(x - \frac{\epsilon}{2}\right) + \left(y - \frac{\epsilon}{2}\right) \tag{17}$$
$$< \inf A + \inf B. \tag{18}$$

So for every $\epsilon > 0$, $\exists c = x + y \in C$ such that

$$c - \epsilon < \inf A + \inf B. \tag{19}$$

Then we conclude that $\inf C = \inf A + \inf B$. $\square$

Problem 7

(a)

Proof. Let $E = \{x \in \mathbb{R} \mid x > 0 \text{ and } x^3 < 2\}$. So $\forall x \in E$, $x^3 < 2$. Then

$$\implies x^3 - 2 < 0 \tag{20}$$
$$\implies \left(x - 2^{\frac{1}{3}}\right)\left(x^2 + 2^{\frac{1}{3}}x + 2^{\frac{2}{3}}\right) < 0. \tag{21}$$

But $x > 0 \implies x^2 + 2^{\frac{1}{3}}x + 2^{\frac{2}{3}} > 0$, so $\forall x \in E$,

$$\implies x - 2^{\frac{1}{3}} < 0 \tag{22}$$
$$\implies x < \sqrt[3]{2}. \tag{23}$$

Thus, E is bounded above. $\square$

(b)

Proof. Let $r = \sup E$, which exists by part (a) and problem 2 (b). We first show that $r > 0$.

By definition of the supremum and the set E, $\forall x \in E$, $x \leq r$ and $x > 0$. Thus $0 < x \leq r$, so r is indeed positive.

Next we show that $r^3 = 2$.

Suppose, toward a contradiction, that $r > \sqrt[3]{2}$. Since $r = \sup E$, then $x \leq r$ for any $x \in E$. But since $\sqrt[3]{2}$ is an upper bound for E, then $x < \sqrt[3]{2}$ $\forall x \in E$. So

$$\implies x < \sqrt[3]{2} < r \tag{24}$$

$$\implies r \neq \sup E, \quad \Rightarrow\Leftarrow . \tag{25}$$

This is a contradiction, since we assumed that $r = \sup E$. Thus $r \leq \sqrt[3]{2} \implies r^3 \leq 2$.

Now suppose instead, toward another contradiction, that $r < \sqrt[3]{2}$. Then

$$r^3 < \left(\sqrt[3]{2}\right)^3 = 2. \tag{26}$$

But then $r \in E$, and $\sqrt[3]{2} > r$ which implies that $r \neq \sup E$, another contradiction. Thus $r \geq \sqrt[3]{2}$. So we have shown that both $r \leq \sqrt[3]{2}$ and $r \geq \sqrt[3]{2}$. Then we conclude $r = \sqrt[3]{2}$.

Hence, $r^3 = 2$. $\square$