18.100A Assignment 10

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Problem 1

(a)

Proof. Suppose $\exists C \geq 0$ such that $\forall x, y \in I$,

$$|f(x) - f(y)| \leq C|x-y|^\alpha. \tag{1}$$

Let $\epsilon > 0$. Choose $\delta = \left(\frac{\epsilon}{C}\right)^{\frac{1}{\alpha}}$. Then if $|x-y| < \delta$, we get

$$\begin{aligned} |f(x) - f(y)| &\leq C|x-y|^\alpha & (2)\\ &< C\delta^\alpha & (3)\\ &= C\frac{\epsilon}{C} & (4)\\ &= \epsilon. & (5) \end{aligned}$$

Therefore f is uniformly continuous on I. □

(b)

Proof. Suppose $\exists C \geq 0$ such that $\forall x, y \in I$, $|f(x) - f(y)| \leq C|x-y|^\alpha$.

Since $\alpha > 1$, then $\alpha = 1 + r$ for some $0 < r$, we have

$$\begin{aligned} &\implies 0 \leq |f(x) - f(y)| \leq C|x-y|^{1+r} & (6)\\ &\implies 0 \leq \frac{|f(x) - f(y)|}{|x-y|} \leq C|x-y|^r & (7)\\ &\implies \lim_{x \to y} 0 \leq \lim_{x \to y} \frac{|f(x) - f(y)|}{|x-y|} \leq C \lim_{x \to y} |x-y|^r & (8)\\ &\implies 0 \leq \lim_{x \to y} \frac{|f(x) - f(y)|}{|x-y|} \leq 0. & (9) \end{aligned}$$

Then by the squeeze theorem, $\lim\limits_{x \to y} \frac{|f(x)-f(y)|}{|x-y|} = 0$. Thus $\forall y \in I$, $f'(y) = 0$.

Therefore f is constant. □