18.100A Assignment 4

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Problem 1

(a)

Proof. Define the complement of $[a, b]$ via

$$[a, b]^c := \{x \in \mathbb{R} \mid x < a,\ x > b\}. \tag{1}$$

We can write this complement as the union of two sets:

$$\begin{aligned} [a, b]^c &= \{x \in \mathbb{R} \mid x < a\} \cup \{x \in \mathbb{R} \mid x > b\} && (2) \\ &= (-\infty, a) \cup (b, \infty). && (3) \end{aligned}$$

Both the sets $(-\infty, a)$ and (b, ∞) are open, as proved in PS3.5a. We also proved that the union of open sets is open. Thus, $[a, b]^c$ is open.

Therefore, we conclude that $[a, b]$ is closed. □

(b)

Claim: The set $\mathbb{Z} \subset \mathbb{R}$ is closed.

Proof. Consider the complement of the integers in the real numbers, $\mathbb{Z}^c = \mathbb{R} \backslash \mathbb{Z}$. We may write this complement as a union of open sets, where each of the open sets represents the set of numbers between (but not including) consecutive integers:

$$\mathbb{Z}^c = \bigcup_{n \in \mathbb{Z}} (n, n+1). \tag{4}$$

Since the sets being unioned are all open, then so is the union, i.e. $\mathbb{Z}^c$ is open.

Thus, $\mathbb{Z}$ is closed. □

(c)

Claim: The set of rationals $\mathbb{Q} \subset \mathbb{R}$ is not closed.

Proof. The complement of the rational numbers $\mathbb{Q}$ in the reals is the set of irrationals:

$$\mathbb{Q}^c = \mathbb{R} \backslash \mathbb{Q}. \tag{5}$$

Let $i \in \mathbb{Q}^c$. In class, we proved the density of $\mathbb{Q}$ in $\mathbb{R}$. Additionally, in PS3.1, we proved the density of the irrationals in $\mathbb{R}$. It follows that $\exists q, r \in \mathbb{Q}$ such that $q < i < r$.

Let $\epsilon > 0$. Since $i - \epsilon, i + \epsilon \in \mathbb{R}$, then $\exists p \in \mathbb{Q}$ such that $i - \epsilon < p < i + \epsilon$. This implies that $p \in (q, r)$, but $p \in \mathbb{Q} \implies p \notin \mathbb{Q}^c$, so $\mathbb{Q}^c$ is not open.

Therefore, $\mathbb{Q}$ is not closed. $\square$

Problem 2

(a)

Proof. Let $x \notin \bigcap_{\lambda \in \Lambda} F_\lambda$. Then $x \in \left(\bigcap_{\lambda \in \Lambda} F_\lambda\right)^c$, so

$$x \in \bigcup_{\lambda \in \Lambda} F_\lambda^c. \tag{6}$$

So for at least one $\lambda \in \Lambda$, $x \in F_\lambda^c$.

Since F_λ is closed, then F_λ^c is open. Thus $\exists \epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset F_\lambda^c$. But since this must hold for arbitrary x, then it holds for every $x \in \left(\bigcap_{\lambda \in \Lambda} F_\lambda\right)^c$.

Hence, $\left(\bigcap_{\lambda \in \Lambda} F_\lambda\right)^c$ is open $\implies \bigcap_{\lambda \in \Lambda} F_\lambda$ is closed. $\square$

(b)

Proof. Let $x \notin \bigcup_{m=1}^n F_m$. Then

$$x \in \left(\bigcup_{m=1}^{n} F_m\right)^c \implies x \in \bigcap_{m=1}^{n} F_m^c. \tag{7}$$

So $\forall m \in \{1, ..., n\}$, $x \in F_m^c$. Similarly,

$$x \in \bigcap_{m=1}^{n} F_m^c \implies x \in \left(\bigcup_{m=1}^{n} F_m\right)^c. \tag{8}$$

[*Intuition*: If x is in the complement of the union of several sets, then it can't be in any of them individually (it must simultaneously be in none of them), which means that it must be in the intersection of the complements of each of the sets. Likewise, if x is in the intersection of the complements of several sets, then it must be in none of the individual sets, so it has to be outside the union of all the sets.]

Equations (7) and (8) imply $\left(\bigcup_{m=1}^{n} F_m\right)^c \subseteq \bigcap_{m=1}^{n} F_m^c$ and $\bigcap_{m=1}^{n} F_m^c \subseteq \left(\bigcup_{m=1}^{n} F_m\right)^c$, respectively. Then

$$\left(\bigcup_{m=1}^{n} F_m\right)^c = \bigcap_{m=1}^{n} F_m^c. \tag{9}$$

Since F_m is closed, then F_m^c is open. Then the intersection $\bigcap_{m=1}^{n} F_m^c$ must be open, which implies by (9) that $\left(\bigcup_{m=1}^{n} F_m\right)^c$ is open.

Therefore, $\bigcup_{m=1}^{n} F_m$ is closed. □

Problem 3

Proof. (By contradiction). Suppose instead that $x \in F^c$. Since F is closed, then F^c is open. Then $\exists \epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset F^c$.

Since $\{x_n\}$ converges to x, then $\lim_{n\to\infty} x_n = x$. So for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $|x_n - x| < \epsilon$. Then for $n > N$,

$$x - \epsilon < x_n < x + \epsilon. \tag{10}$$

But then $\forall n > N$, $x_n \in (x - \epsilon, x + \epsilon) \subset F^c$, i.e. $x_n \in F^c$ ($\Rightarrow\Leftarrow$). This is a contradiction because we assumed that the elements of $\{x_n\}_n$ are in F.

Therefore, $x \in F$. □

Problem 4

Proof. (By induction on k).

<u>Base Case</u>: $k = 1$.

$$\lim_{n\to\infty} x_n^1 = \left(\lim_{n\to\infty} x_n\right)^1 \tag{11}$$

So, the inductive hypothesis is true for $k = 1$.

<u>Inductive Step</u>: Suppose the hypothesis is true for $k = m$, i.e.

$$\lim_{n\to\infty} x_n^m = \left(\lim_{n\to\infty} x_n\right)^m. \tag{12}$$

Then we have

$$\begin{aligned}
\lim_{n\to\infty} x_n^{m+1} &= \lim_{n\to\infty} (x_n^m \cdot x_n) && (13)\\
&= \lim_{n\to\infty} (x_n^m) \cdot \lim_{n\to\infty} x_n && (14)\\
&= \left(\lim_{n\to\infty} x_n\right)^m \cdot \lim_{n\to\infty} x_n && (15)\\
&= \left(\lim_{n\to\infty} x_n\right)^{m+1}. && (16)
\end{aligned}$$

Hence, the hypothesis is true for $k = m + 1$.

Therefore, we conclude that $\forall k \in \mathbb{N}$, $\lim_{n\to\infty} x_n^k = \left(\lim_{n\to\infty} x_n\right)^k$. □

Problem 5

Proof. We have that $x_n = 1 - \frac{\cos n}{n}$. Then

$$|x_n| = |1 - \frac{\cos n}{n}| \tag{17}$$
$$\leq |1| + \frac{|\cos n|}{n} \tag{18}$$
$$\leq 1 + \frac{1}{n}. \tag{19}$$

Also, $x_n = 1 - \frac{\cos n}{n} \geq 1 - \frac{1}{n}$, so

$$1 - \frac{1}{n} \leq x_n \leq 1 + \frac{1}{n}. \tag{20}$$

Note that $\lim_{n\to\infty}(1 - \frac{1}{n}) = \lim_{n\to\infty}(1 + \frac{1}{n}) = 1$. So by the squeeze theorem,

$$\lim_{n\to\infty} x_n = 1. \tag{21}$$

Thus, we conclude that $x_n \to 1$. □

Problem 6

Proof. ($\Rightarrow$) Suppose $a_0 = \sup A$. By PS3.4, then $\forall n \in \mathbb{N}\ \exists a_n \in A$ such that

$$a_0 - \frac{1}{n} < a_n \leq a_0. \tag{22}$$

Rearranging, we have

$$-\frac{1}{n} < a_n - a_0 \leq 0. \tag{23}$$

But $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = 0$, so by the Squeeze Theorem, $a_n - a_0 \to 0$. Then,

$$a_n \to a_0. \tag{24}$$

($\Leftarrow$) Let a_0 be an upper bound for A. Then $\forall n \in \mathbb{N}$, $a_n \leq a_0$.

Suppose $\exists$ a sequence $\{a_n\}_n$ of elements of A such that $\lim_{n\to\infty} a_n = a_0$. Then for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - a_0| < \epsilon$.

Suppose $\exists$ another upper bound a_1 for A such that $a_1 < a_0$. Then $\forall n \in \mathbb{N}$,

$$a_n \leq a_1 < a_0, \tag{25}$$

which gives

$$0 \leq a_1 - a_n < a_0 - a_n. \tag{26}$$

But since $a_n \to a_0$, then by the Squeeze Theorem, $a_n \to a_1$. But $\{a_n\}_n$ can only converge to one value, so $a_1 = a_0$ ($\Rightarrow\Leftarrow$). This is a contradiction to the initial assumption that $a_0 \neq a_1$.

Therefore, we conclude $a_0 = \sup A$. □