18.100A Midterm

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Problem 1

(a)

Proof. Let $x \in f^{-1}(C \cap D)$. Then

$$\begin{align} &\implies f(x) \in C \cap D \tag{1}\\ &\implies f(x) \in C \text{ and } f(x) \in D \tag{2}\\ &\implies x \in f^{-1}(C) \text{ and } x \in f^{-1}(D) \tag{3}\\ &\implies x \in f^{-1}(C) \cap f^{-1}(D). \tag{4}\end{align}$$

Thus,

$$f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D). \tag{5}$$

Now let $x \in f^{-1}(C) \cap f^{-1}(D)$. Then

$$\begin{align} &\implies f(x) \in C \text{ and } f(x) \in D \tag{6}\\ &\implies f(x) \in C \cap D \tag{7}\\ &\implies x \in f^{-1}(C \cap D). \tag{8}\end{align}$$

Thus,

$$f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D). \tag{9}$$

Therefore by equations (5) and (9), $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. $\square$

(b)

Claim: If $E \subset \mathbb{R}$ is countable, then the complement $\mathbb{R} \backslash E$ is always uncountable.

Proof. (By contradiction). Suppose E^c is countable. Then $E \cup E^c$ is countable as well, since it is the union of two countable sets. But $E \cup E^c = \mathbb{R}$, which is uncountable. ($\Rightarrow\Leftarrow$). $\square$

(c)

By contrast, if $E \subset \mathbb{R}$ is uncountable, then the complement $\mathbb{R} \backslash E$ is not always countable. Take for instance, $E = [0, 1]$, which is uncountable. Then $E^c = (-\infty, 0) \cup (1, \infty)$, which is also uncountable.

Problem 2

(a)

A set $U \subset \mathbb{R}$ is *not open* if for every $\epsilon > 0$, $\exists x \in U$ such that $(x - \epsilon, x + \epsilon) \not\subset \mathbb{R}$.

(b)

Proof. Suppose U is not open. Let $\epsilon = \frac{1}{n}$. Then for every $n \in \mathbb{N}$, $\exists x \in U$ such that $(x - \frac{1}{n}, x + \frac{1}{n}) \not\subset U$. Equivalently, for each $n \in \mathbb{N}$ $\exists x_n \in U^c$ such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n}. \tag{10}$$

Then we have

$$0 < |x_n - x| < \frac{1}{n}, \tag{11}$$

and taking the limit on all sides gives

$$0 < \lim_{n\to\infty} |x_n - x| < \lim_{n\to\infty} \frac{1}{n}. \tag{12}$$

Thus, by the squeeze theorem, $\lim_{n\to\infty} x_n = x$, as desired. $\square$

(c)

Proof. (By contradiction). To show that F is closed, we must show that F^c is open. Suppose, toward a contradiction, that F^c is not open. Then by part **(b)**, $\exists x \in F^c$ and a sequence $\{x_n\}_n$ of elements of F such that $\lim_{n\to\infty} x_n = x$. But by assumption, every convergent sequence of elements of F has a limit in F, i.e. we assumed originally that $x \in F$ ($\Rightarrow\Leftarrow$). Thus, F^c must be open, so F is closed. $\square$