18.100A Assignment 10

Octavio Vega

June 6, 2023

Problem 1

(a)

Proof. Suppose $\exists C \geq 0$ such that $\forall x, y \in I$,

$$|f(x) - f(y)| \leq C|x - y|^\alpha. \tag{1}$$

Let $\epsilon > 0$. Choose $\delta = \left(\frac{\epsilon}{C}\right)^{\frac{1}{\alpha}}$. Then if $|x - y| < \delta$, we get

$$\begin{aligned} |f(x) - f(y)| &\leq C|x - y|^\alpha && (2) \\ &< C\delta^\alpha && (3) \\ &= C\frac{\epsilon}{C} && (4) \\ &= \epsilon. && (5) \end{aligned}$$

Therefore f is uniformly continuous on I. □

(b)

Proof. Suppose $\exists C \geq 0$ such that $\forall x, y \in I$, $|f(x) - f(y)| \leq C|x - y|^\alpha$.

Since $\alpha > 1$, then $\alpha = 1 + r$ for some $0 < r$, we have

$$\begin{aligned} &\implies 0 \leq |f(x) - f(y)| \leq C|x - y|^{1+r} && (6) \\ &\implies 0 \leq \frac{|f(x) - f(y)|}{|x - y|} \leq C|x - y|^r && (7) \\ &\implies \lim_{x \to y} 0 \leq \lim_{x \to y} \frac{|f(x) - f(y)|}{|x - y|} \leq C \lim_{x \to y} |x - y|^r && (8) \\ &\implies 0 \leq \lim_{x \to y} \frac{|f(x) - f(y)|}{|x - y|} \leq 0. && (9) \end{aligned}$$

Then by the squeeze theorem, $\lim_{x \to y} \frac{|f(x)-f(y)|}{|x-y|} = 0$. Thus $\forall y \in I$, $f'(y) = 0$.

Therefore f is constant. □

Problem 2

Proof. We compute:

$$\begin{aligned}
L &= \lim_{x\to c} \frac{h(x)-h(c)}{x-c} && (10)\\
&= \lim_{x\to c} \frac{f(x)g(x)-f(c)g(c)}{x-c} && (11)\\
&= \lim_{x\to c} \frac{f(x)g(x)-f(x)g(c)+f(x)g(c)-f(c)g(c)}{x-c} && (12)\\
&= \lim_{x\to c} \left[f(x)\left(\frac{g(x)-g(c)}{x-c}\right)\right] + g(x)\lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right) && (13)\\
& && (14)
\end{aligned}$$

Since f is continuous at c, and both f and g are differentiable at c, this gives us

$$L = f(c)g'(c) + g(c)f'(c), \tag{15}$$

which exists.

Therefore $f(x)g(x)$ is differentiable at c. □