18.100A Assignment 7

Octavio Vega

April 17, 2023

Problem 1

Proof. Since $\sum_n a_n$ and $\sum_n b_n$ converge absolutely, suppose that $\sum_n |a_n| < M$ and $\sum_n |b_n| < N$. Then

$$\begin{aligned}
\sum_{n=0}^{m} |c_n| &= \sum_{n=0}^{m} \left| \sum_{k=0}^{n} a_k b_{n-k} \right| && (1) \\
&\leq \sum_{n=0}^{m} \sum_{k=0}^{n} |a_k b_{n-k}| && (2) \\
&= |a_0 b_0| + (|a_0 b_1| + |a_1 b_0|) + \cdots + \\
&(|a_0 b_m| + |a_1 b_{m-1}| + \cdots + |a_m b_0|) && (3) \\
&= \sum_{n=0}^{m} |a_n| \sum_{k=0}^{m-n} |b_k| && (4) \\
&< MN. && (5)
\end{aligned}$$

Thus $\sum_n |c_n|$ is bounded above and monotone, so it converges. $\square$

Problem 2

(a)

Let $a_n = 2^n x^n$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}x^{n+1}}{2^n x^n}\right| = 2|x|$.

By the ratio test, we must have

$$L = \lim_{n\to\infty} 2|x| < 1. \tag{6}$$

Thus, $\sum_{n=0}^{\infty} 2^n x^n$ converges for all $|x| < \frac{1}{2}$.

(b)

We have $a_n = nx^n$, so $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)x^{n+1}}{nx^n}\right| = \frac{n+1}{n}|x|$.

Thus, we require

$$\lim_{n\to\infty} \frac{n+1}{n}|x| < 1. \tag{7}$$

Therefore, $\sum_n nx^n$ converges for all $|x| < 1$.

(c)

Proceeding with the ratio test, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-10)^{n+1}(2n)!}{(2n+2)!(x-10)^n}\right| = \left|\frac{x-10}{(2n+2)(2n+1)}\right| \tag{8}$$

Then, we require

$$\lim_{n\to\infty} \left|\frac{x-10}{4n^2+6n+2}\right| = 0 < 1, \tag{9}$$

which is always satisfied. Thus, $\sum_n \frac{1}{(2n)!}(x-10)^n$ converges $\forall x \in \mathbb{R}$.

(d)

Letting $a_n = n!x^n$, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x|. \tag{10}$$

Thus we must have

$$\lim_{n\to\infty} (n+1)|x| < 1, \tag{11}$$

which is only satisfied for $x = 0$. Thus, $\sum_n n!x^n$ converges only for $x = 0$.