

# 18.100A Midterm

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March 21, 2023

## Problem 1

(a)

*Proof.* Let  $x \in f^{-1}(C \cap D)$ . Then

$$\implies f(x) \in C \cap D \quad (1)$$

$$\implies f(x) \in C \text{ and } f(x) \in D \quad (2)$$

$$\implies x \in f^{-1}(C) \text{ and } x \in f^{-1}(D) \quad (3)$$

$$\implies x \in f^{-1}(C) \cap f^{-1}(D). \quad (4)$$

Thus,

$$f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D). \quad (5)$$

Now let  $x \in f^{-1}(C) \cap f^{-1}(D)$ . Then

$$\implies f(x) \in C \text{ and } f(x) \in D \quad (6)$$

$$\implies f(x) \in C \cap D \quad (7)$$

$$\implies x \in f^{-1}(C \cap D). \quad (8)$$

Thus,

$$f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D). \quad (9)$$

Therefore by equations (5) and (9),  $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$ .  $\square$

(b)

**Claim:** If  $E \subset \mathbb{R}$  is countable, then the complement  $\mathbb{R} \setminus E$  is always uncountable.

*Proof.* (By contradiction). Suppose  $E^c$  is countable. Then  $E \cup E^c$  is countable as well, since it is the union of two countable sets. But  $E \cup E^c = \mathbb{R}$ , which is uncountable. ( $\Rightarrow \Leftarrow$ ).  $\square$

(c)

By contrast, if  $E \subset \mathbb{R}$  is uncountable, then the complement  $\mathbb{R} \setminus E$  is not always countable. Take for instance,  $E = [0, 1]$ , which is uncountable. Then  $E^c = (-\infty, 0) \cup (1, \infty)$ , which is also uncountable.

## Problem 2

(a)

A set  $U \subset \mathbb{R}$  is *not open* if for every  $\epsilon > 0$ ,  $\exists x \in U$  such that  $(x - \epsilon, x + \epsilon) \not\subset U$ .

(b)

*Proof.* Suppose  $U$  is not open. Let  $\epsilon = \frac{1}{n}$ . Then for every  $n \in \mathbb{N}$ ,  $\exists x \in U$  such that  $(x - \frac{1}{n}, x + \frac{1}{n}) \not\subset U$ . Equivalently, for each  $n \in \mathbb{N}$   $\exists x_n \in U^c$  such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n}. \quad (10)$$

Then we have

$$0 < |x_n - x| < \frac{1}{n}, \quad (11)$$

and taking the limit on all sides gives

$$0 < \lim_{n \rightarrow \infty} |x_n - x| < \lim_{n \rightarrow \infty} \frac{1}{n}. \quad (12)$$

Thus, by the squeeze theorem,  $\lim_{n \rightarrow \infty} x_n = x$ , as desired.  $\square$

(c)

*Proof.* (By contradiction). To show that  $F$  is closed, we must show that  $F^c$  is open. Suppose, toward a contradiction, that  $F^c$  is not open. Then by part (b),  $\exists x \in F^c$  and a sequence  $\{x_n\}_n$  of elements of  $F$  such that  $\lim_{n \rightarrow \infty} x_n = x$ . But by assumption, every convergent sequence of elements of  $F$  has a limit in  $F$ , i.e. we assumed originally that  $x \in F$  ( $\Rightarrow \Leftarrow$ ). Thus,  $F^c$  must be open, so  $F$  is closed.  $\square$

### Problem 3

(a)

*Proof.* Let  $\epsilon > 0$ . Choose  $N = \frac{1}{\sqrt{\epsilon}}$ . Then  $\forall n \geq N$ , we have

$$\left| \frac{10n^2}{n^2 + 16n + 1} - 10 \right| = \left| \frac{-160n - 10}{n^2 + 16n + 1} \right| \quad (13)$$

$$= \left| \frac{160n + 10}{n^2 + 16n + 1} \right| \quad (14)$$

$$< \frac{1}{n^2 + 16n + 1} \quad (15)$$

$$< \frac{1}{n^2} \quad (16)$$

$$< \epsilon. \quad (17)$$

Therefore,  $\lim_{n \rightarrow \infty} \left| \frac{10n^2}{n^2 + 16n + 1} \right| = 10$ .  $\square$