

# 18.100A Assignment 1

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## Problem 1

(a)

*Proof.* We will show that each set is a subset of the other to prove equality. Let  $S = A \cap (B \cup C)$  and  $T = (A \cup B) \cap (A \cup C)$ .

Let  $x \in S$ . Then  $x \in A$  and  $x \in B \cup C$ ,

$$\implies x \in A \text{ and } x \in B, \text{ or } x \in A \text{ and } x \in C$$

$$\implies x \in A \cap B \text{ or } x \in A \cap C$$

$$\implies x \in (A \cap B) \cup (A \cap C) = T.$$

Thus  $x \in S \implies x \in T$ , so  $S \subseteq T$ . Now let  $x \in T$ . Then  $x \in (A \cap B) \cup (A \cap C)$ ,

$$\implies x \in A \cap B \text{ or } x \in A \cap C$$

$$\implies x \in A, \text{ and } x \in B \text{ or } C$$

$$\implies x \in A \cap (B \cup C) = S.$$

Thus  $x \in T \implies x \in S$ , so  $T \subseteq S$ , which means  $S = T$ .

Hence,  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ .

□

(b)

*Proof.* We proceed as in (a). Let  $S = A \cup (B \cap C)$  and  $T = (A \cup B) \cap (A \cup C)$ .

First let  $x \in S$ . Then  $x \in A$  or  $x \in B \cap C$ ,

$$\implies x \in A \text{ or } x \in B, \text{ and } x \in A \text{ or } x \in C$$

$$\implies x \in (A \cup B) \cap (A \cup C) = T.$$

Thus  $x \in S \implies x \in T$ , so  $S \subseteq T$ . Now let  $x \in T$ . Then  $x \in A \cup B$  and  $x \in A \cup C$ . If  $x \in A$ , then the requirement is satisfied immediately, regardless

of whether  $x$  is in  $B$  or  $C$ . Otherwise, if  $x \notin A$ , then  $x \in B$  and  $x \in C$  must be true. So  $x \in A$ , or  $x \in B$  and  $x \in C$

$$\implies x \in A \cup (B \cap C) = S.$$

Thus  $x \in T \implies x \in S$ , so  $T \subseteq S$ , which means  $S = T$ .

Hence,  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ . □

## Problem 2

*Proof.* (By induction).

The inductive hypothesis  $P(n)$  is that, for  $n \in \mathbb{N}$ ,  $n < 2^n$ .

(Base case):  $1 < 2^1$ , so  $P(1)$  is true.

(Inductive step): Assume  $P(n)$  is true for  $n = m$ , i.e. that  $m < 2^m$  holds for  $m \in \mathbb{N}$ . Then

$$2^{m+1} = 2 \cdot 2^m > 2m \tag{1}$$

$$= m + m > m + 1, \quad \text{since } m > 1 \tag{2}$$

$$\implies 2^{m+1} > m + 1. \tag{3}$$

So  $P(m) \implies P(m + 1)$ , which means  $P(n)$  is true for all  $n \in \mathbb{N}$ .

Thus,  $\forall n \in \mathbb{N}$ ,  $2^n > n$ . □

## Problem 3

*Proof.* Let  $A$  be a finite set such that  $|A| = n$ . We form the power set  $\mathcal{P}(A)$  by creating the set of all possible subsets of  $A$ . Hence  $|\mathcal{P}(A)|$  is equivalent to the number of possible subsets of  $A$ , which we compute by summing over the number of combinations that can be created by choosing elements of  $A$ , in succession from choosing no elements (the empty set  $\emptyset$ ) to choosing all elements (the full set  $A$ ). Thus

$$|\mathcal{P}(A)| = \sum_{k=0}^n \binom{n}{k}. \tag{4}$$

By the binomial expansion theorem,

$$\sum_{k=0}^n \binom{n}{k} p^k q^{n-k} = (p + q)^n. \tag{5}$$

Substituting  $p = q = 1$  into (5), we arrive at the desired result,

$$|\mathcal{P}(A)| = (1 + 1)^n = 2^n. \tag{6}$$

□

## Problem 4

*Proof.* (By induction).

$P(n)$  is the hypothesis that for  $n \in \mathbb{N}$ ,  $n^3 + 5n$  is divisible by 6.

(Base case):  $1^3 + 5 \cdot 1 = 1 + 5 = 6$  is divisible by 6, so  $P(1)$  is true.

(Inductive step): Assume  $P(m)$  holds, i.e. 6 divides  $m^3 + 5m$ . Then

$$(m+1)^3 + 5m = m^3 + 3m^2 + 3m + 1 + 5m + 1 \quad (7)$$

$$= m^3 + 5m + 6 + 3m(m+1), \quad (8)$$

where by the inductive hypothesis  $m^3 + 5m + 6$  is divisible by 6 and  $3m(m+1)$  is also divisible by 6 because it is divisible by both 3 and 2. So, their sum  $(m+1)^3 + 5(m+1)$  must also be divisible by 6, which means  $P(m) \implies P(m+1)$ .

Thus,  $\forall n \in \mathbb{N}$ ,  $n^3 + 5n$  is divisible by 6.  $\square$

## Problem 5

*Proof.* Let  $A_1, A_2, A_3, \dots, A_n, A_{n+1}, \dots$  be finite sets, and let there be infinitely many of them. We wish to find an example where  $|\bigcup_{i=1}^{\infty} A_i| = \infty$ .

To satisfy this, choose  $A_i = \{i\}$ . Then  $A_1 \cup A_2 \cup A_3 \cdots = \{1, 2, 3, \dots\} = \mathbb{N}$ . So their union is not a finite set, as desired.  $\square$

## Problem 6

(a)

Consider  $q = \frac{4}{15}$ . Then

$$\frac{4}{15} = \frac{2 \cdot 2}{3 \cdot 5} = \frac{2^2}{3 \cdot 5}. \quad (9)$$

So we compute

$$f\left(\frac{4}{15}\right) = 2^{2 \cdot 2} \cdot 3^{2-1} \cdot 5^{2-1} = 2^4 \cdot 3 \cdot 5. \quad (10)$$

Hence,  $f\left(\frac{4}{15}\right) = 240$ .

Now suppose  $f(q) = 108$ . Then we write

$$108 = 2 \cdot 54 = 2 \cdot 2 \cdot 27 = 2^2 \cdot 3^3 = p_1^{2r_1} q_1^{2s_1-1}, \quad (11)$$

so we identify  $r_1 = 1$  and  $s_1 = 2$ . Thus,

$$q = \frac{p_1^{r_1}}{q_1^{s_1}} = \frac{2}{3^2} = \frac{2}{9}. \quad (12)$$