18.100A Final

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Problem 1

We complete the following **negations**:

(i)

Let $S \subset \mathbb{R}$. A function $f : S \to \mathbb{R}$ is **not continuous** at $c \in S$ if $\exists \epsilon_0 > 0$ such that $\forall \delta > 0$ if $|x - c| < \delta$, $|f(x) - f(c)| \geq \epsilon_0$.

(ii)

Let $S \subset \mathbb{R}$. A function $f : S \to \mathbb{R}$ is **not uniformly continuous** on S if $\exists x_0 \in S$ such that $\forall \delta > 0 \; \exists \epsilon_0 > 0$ such that if $|x_0 - x| < \delta$, then $|f(x) - f(x_0)| \geq \epsilon_0$.

(iii)

Let $S \subset \mathbb{R}$. A sequence of functions $f_n : S \to \mathbb{R}$ **does not converge uniformly** to $f : S \to \mathbb{R}$ if $\exists \epsilon_0 > 0$ such that $\forall M \in \mathbb{N} \; \exists n \geq M$ and $x \in S$ such that $|f_n(x) - f(x)| \geq \epsilon_0$.

Problem 2

(a)

(i)

A continunous function on $(0, 1)$ with neither a global minimum or maximum:

Let $f(x) = 1 \; \forall x \in (0, 1)$. Then $\forall x, y \in (0, 1)$, $f(x) = f(y)$ so f has no absolute maximum or minimum, and f is constant and therefore continuous.

(ii)

A function on $[0, 1]$ with absolute minimum at 0, absolute maximum at 1, and such that $\exists y \in (f(0), f(1))$ not in the range of f:

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Define f via

$$
f(x) := \begin{cases} x, & x \in \left(0, \frac{1}{2}\right) \cap \left(\frac{1}{2}, 1\right] \\ -1, & x = 0 \\ -\frac{1}{2}, & x = \frac{1}{2}. \end{cases} \tag{1}
$$

Then, for example, $-\frac{3}{4} \in (-1, 1) = (f(0), f(1))$, but $-\frac{3}{4}$ is not in the range of f. Also, f has an absolute minimum and maximum at 0 and 1, respectively.

(b)

Proof. Let $\epsilon > 0$. Since f is continuous, then $\exists \delta_0 > 0$ such that if $|x - c| < \delta_0$ then $|f(x) - f(c)| < \frac{\epsilon}{2}$. Similarly, since g is continuous, then $\exists \delta_1 > 0$ such that if $|x - c| < \delta_1$ then $|g(x) - g(c)| < \frac{\epsilon}{2}$.

Choose δ_0, δ_1 such that $|f(x)| + |g(c)| < 2$, and let $\delta = \min\{\delta_0, \delta_1\}$. Then if $|x - c| < \delta$, we have

$$
\begin{align}
|f(x)g(x) - f(c)g(c)| &= |f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)| \tag{2} \\
&\leq |f(x)||g(x) - g(c)| + |g(c)||f(x) - f(c) \tag{3} \\
&< |f(x)|\frac{\epsilon}{2} + |g(c)|\frac{\epsilon}{2} \tag{4} \\
&< \epsilon. \tag{5}
\end{align}
$$

Therefore, the product fg is continuous at c. $\qquad\square$