

# 18.100A Midterm

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## Problem 1

(a)

*Proof.* Let  $x \in f^{-1}(C \cap D)$ . Then

$$\implies f(x) \in C \cap D \quad (1)$$

$$\implies f(x) \in C \text{ and } f(x) \in D \quad (2)$$

$$\implies x \in f^{-1}(C) \text{ and } x \in f^{-1}(D) \quad (3)$$

$$\implies x \in f^{-1}(C) \cap f^{-1}(D). \quad (4)$$

Thus,

$$f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D). \quad (5)$$

Now let  $x \in f^{-1}(C) \cap f^{-1}(D)$ . Then

$$\implies f(x) \in C \text{ and } f(x) \in D \quad (6)$$

$$\implies f(x) \in C \cap D \quad (7)$$

$$\implies x \in f^{-1}(C \cap D). \quad (8)$$

Thus,

$$f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D). \quad (9)$$

Therefore by equations (5) and (9),  $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$ .  $\square$

(b)

**Claim:** If  $E \subset \mathbb{R}$  is countable, then the complement  $\mathbb{R} \setminus E$  is always uncountable.

*Proof.* (By contradiction). Suppose  $E^c$  is countable. Then  $E \cup E^c$  is countable as well, since it is the union of two countable sets. But  $E \cup E^c = \mathbb{R}$ , which is uncountable. ( $\Rightarrow \Leftarrow$ ).  $\square$

(c)

By contrast, if  $E \subset \mathbb{R}$  is uncountable, then the complement  $\mathbb{R} \setminus E$  is not always countable. Take for instance,  $E = [0, 1]$ , which is uncountable. Then  $E^c = (-\infty, 0) \cup (1, \infty)$ , which is also uncountable.

## Problem 2

(a)

A set  $U \subset \mathbb{R}$  is *not open* if for every  $\epsilon > 0$ ,  $\exists x \in U$  such that  $(x - \epsilon, x + \epsilon) \not\subset U$ .

(b)

*Proof.* Suppose  $U$  is not open. Let  $\epsilon = \frac{1}{n}$ . Then for every  $n \in \mathbb{N}$ ,  $\exists x \in U$  such that  $(x - \frac{1}{n}, x + \frac{1}{n}) \not\subset U$ . Equivalently, for each  $n \in \mathbb{N}$   $\exists x_n \in U^c$  such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n}. \quad (10)$$

Then we have

$$0 < |x_n - x| < \frac{1}{n}, \quad (11)$$

and taking the limit on all sides gives

$$0 < \lim_{n \rightarrow \infty} |x_n - x| < \lim_{n \rightarrow \infty} \frac{1}{n}. \quad (12)$$

Thus, by the squeeze theorem,  $\lim_{n \rightarrow \infty} x_n = x$ , as desired.  $\square$

(c)

*Proof.* (By contradiction). To show that  $F$  is closed, we must show that  $F^c$  is open. Suppose, toward a contradiction, that  $F^c$  is not open. Then by part (b),  $\exists x \in F^c$  and a sequence  $\{x_n\}_n$  of elements of  $F$  such that  $\lim_{n \rightarrow \infty} x_n = x$ . But by assumption, every convergent sequence of elements of  $F$  has a limit in  $F$ , i.e. we assumed originally that  $x \in F$  ( $\Rightarrow \Leftarrow$ ). Thus,  $F^c$  must be open, so  $F$  is closed.  $\square$

### Problem 3

(a)

*Proof.* Let  $\epsilon > 0$ . Choose  $N = \frac{1}{\sqrt{\epsilon}}$ . Then  $\forall n \geq N$ , we have

$$\left| \frac{10n^2}{n^2 + 16n + 1} - 10 \right| = \left| \frac{-160n - 10}{n^2 + 16n + 1} \right| \quad (13)$$

$$= \left| \frac{160n + 10}{n^2 + 16n + 1} \right| \quad (14)$$

$$< \frac{1}{n^2 + 16n + 1} \quad (15)$$

$$< \frac{1}{n^2} \quad (16)$$

$$< \epsilon. \quad (17)$$

Therefore,  $\lim_{n \rightarrow \infty} \left| \frac{10n^2}{n^2 + 16n + 1} \right| = 10$ .  $\square$

(b)

(i) Let  $x_n = \frac{(-1)^n}{n}$ . Then  $\lim_{n \rightarrow \infty} x_n = 0$ . But  $x_1 = -1 < \frac{1}{2} = x_2$ , whereas  $x_2 = \frac{1}{2} > -\frac{1}{3} = x_3$ .

Therefore  $\{x_n\}_n$  converges to 0, but is not monotonic.

(ii) Let

$$x_n = \begin{cases} 0, & n \text{ even} \\ n, & n \text{ odd.} \end{cases} \quad (18)$$

Then  $\{x_n\}_n$  is clearly unbounded.

Consider the subsequence  $\{x_{n_k}\}_k$  defined by  $x_{n_k} = x_{2k}$  for each  $k \in \mathbb{N}$ . Then  $\forall k, x_{n_k} = 0$ .

Therefore,  $\{x_n\}_n$  is unbounded, but  $\{x_{n_k}\}_k$  converges to 0.

### Problem 4

(a)

(i)

*Proof.* Suppose  $\{x_n\}_n$  and  $\{y_n\}_n$  are bounded. Then  $\exists B_0 > 0$  and  $B_1 > 0$  such that  $\forall n \in \mathbb{N}, |x_n| \leq B_0$  and  $|y_n| \leq B_1$ .

Let  $B = B_0 + B_1$ . Then by the triangle inequality, we have

$$|x_n + y_n| \leq |x_n| + |y_n| \leq B_0 + B_1 = B. \quad (19)$$

Thus,  $\forall n \in \mathbb{N}$ ,  $|x_n + y_n| \leq B$ .

Therefore,  $\{x_n + y_n\}_n$  is bounded.  $\square$

(ii)

*Proof.* Let  $a_n = \sup\{x_k \mid k \geq n\}$ ,  $b_n = \sup\{y_k \mid k \geq n\}$ , and  $c_n = \sup\{x_k + y_k \mid k \geq n\}$ .

Then  $\forall n \in \mathbb{N}$ ,  $a_n \geq x_n$  and  $b_n \geq y_n$ , so

$$\implies x_n + y_n \leq a_n + b_n. \quad (20)$$

But  $a_n + b_n$  is also an upper bound for  $\{x_n + y_n\}_n$ , so  $c_n \leq a_n + b_n$ . Then

$$\lim_{n \rightarrow \infty} c_n \leq \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n. \quad (21)$$

Therefore,  $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$ .  $\square$

(b)

*Proof.* Let  $x = 0.1111\dots$ . Then we can write

$$x = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots \quad (22)$$

$$= -1 + \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n \quad (23)$$

$$= -1 + \frac{1}{1 - \frac{1}{10}} \quad (24)$$

$$= -1 + \frac{10}{9} \quad (25)$$

$$= \frac{1}{9}. \quad (26)$$

Let  $\epsilon = \frac{1}{10^n}$  for  $n \in \mathbb{N}$  and let  $x_n = \frac{1}{9} + \frac{1}{10^{n+1}}$ . Then  $x_n \in (\frac{1}{9} - \epsilon, \frac{1}{9} + \epsilon)$ . So

$$\frac{1}{9} - \frac{1}{10^{n+1}} < x_n < \frac{1}{9} + \frac{1}{10^{n+1}}. \quad (27)$$

But  $x_n = \frac{1}{9} + \frac{1}{10^{n+1}} = 0.111\dots11211\dots \in E$ . Thus, for every  $\epsilon > 0$ ,  $\exists x_n \in (\frac{1}{9} - \epsilon, \frac{1}{9} + \epsilon)$  such that  $(\frac{1}{9} - \epsilon, \frac{1}{9} + \epsilon) \cap E \neq \emptyset$ .

Therefore,  $0.1111\dots$  is a cluster point of  $E$ .  $\square$