18.100A Final

Octavio Vega

June 21, 2023

Problem 1

We complete the following **negations**:

(i)

Let $S \subset \mathbb{R}$. A function $f : S \to \mathbb{R}$ is **not continuous** at $c \in S$ if $\exists \epsilon_0 > 0$ such that $\forall \delta > 0$ if $|x - c| < \delta$, $|f(x) - f(c)| \geq \epsilon_0$.

(ii)

Let $S \subset \mathbb{R}$. A function $f : S \to \mathbb{R}$ is **not uniformly continuous** on S if $\exists x_0 \in S$ such that $\forall \delta > 0$ $\exists \epsilon_0 > 0$ such that if $|x_0 - x| < \delta$, then $|f(x) - f(x_0)| \geq \epsilon_0$.

(iii)

Let $S \subset \mathbb{R}$. A sequence of functions $f_n : S \to \mathbb{R}$ **does not converge uniformly** to $f : S \to \mathbb{R}$ if $\exists \epsilon_0 > 0$ such that $\forall M \in \mathbb{N}$ $\exists n \geq M$ and $x \in S$ such that $|f_n(x) - f(x)| \geq \epsilon_0$.

Problem 2

(a)

(i)

A continunous function on $(0, 1)$ with neither a global minimum or maximum:

Let $f(x) = 1$ $\forall x \in (0, 1)$. Then $\forall x, y \in (0, 1)$, $f(x) = f(y)$ so f has no absolute maximum or minimum, and f is constant and therefore continuous.

(ii)

A function on $[0, 1]$ with absolute minimum at 0, absolute maximum at 1, and such that $\exists y \in (f(0), f(1))$ not in the range of f:

Define f via

$$f(x) := \begin{cases} x, & x \in \left(0, \frac{1}{2}\right) \cap \left(\frac{1}{2}, 1\right] \\ -1, & x = 0 \\ -\frac{1}{2}, & x = \frac{1}{2}. \end{cases} \tag{1}$$

Then, for example, $-\frac{3}{4} \in (-1, 1) = (f(0), f(1))$, but $-\frac{3}{4}$ is not in the range of f. Also, f has an absolute minimum and maximum at 0 and 1, respectively.

(b)

Proof. Let $\epsilon > 0$. Since f is continuous, then $\exists \delta_0 > 0$ such that if $|x - c| < \delta_0$ then $|f(x) - f(c)| < \frac{\epsilon}{2}$. Similarly, since g is continuous, then $\exists \delta_1 > 0$ such that if $|x - c| < \delta_1$ then $|g(x) - g(c)| < \frac{\epsilon}{2}$.

Choose δ_0, δ_1 such that $|f(x)| + |g(c)| < 2$, and let $\delta = \min\{\delta_0, \delta_1\}$. Then if $|x - c| < \delta$, we have

$$\begin{align} |f(x)g(x) - f(c)g(c)| &= |f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)| \tag{2} \\ &\leq |f(x)||g(x) - g(c)| + |g(c)||f(x) - f(c) \tag{3} \\ &< |f(x)|\frac{\epsilon}{2} + |g(c)|\frac{\epsilon}{2} \tag{4} \\ &< \epsilon. \tag{5} \end{align}$$

Therefore, the product fg is continuous at c. □

Problem 3

(a)

Proof. Let $\{x_n\}_n$ be a sequence of elements in $[a, b]$. Choose $B = \max\{|a|, |b|\}$. Then $\forall n \in \mathbb{N}$, $|x_n| \leq B$, so the sequence $\{x_n\}_n$ is bounded.

By the Bolzano-Weierstrass theorem, $\exists$ a subsequence $\{x_{n_k}\}_k \subset [a, b]$ that converges, i.e. such that $x_{n_k} \to x$ as $k \to \infty$ for some $x \in \mathbb{R}$. Since $[a, b]$ is closed, then $x \in [a, b]$.

Therefore $[a, b]$ is compact. □

(b)

Proof. Let $\{x_n\}_n$ be a sequence of elements in $[a, b]$. Since $[a, b]$ is compact, $\exists \{x_{n_k}\}_k \subset [a, b]$ such that $x_{n_k} \to x$ for some $x \in [a, b]$.

Then $\{f(x_{n_k})\}_k$ is a subsequence in $[a, b]$, and since f is continuous we have

$$\begin{align} \lim_{k \to \infty} f\left(x_{n_k}\right) &= f\left(\lim_{k \to \infty} x_{n_k}\right) \tag{6} \\ &= f(x) \in f\left([a, b]\right), \tag{7} \end{align}$$

so the subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ converges in $[a,b]$.

Therefore $f([a,b])$ is compact. $\square$

Problem 4

(a)

(i)

Proof. Since f is differentiable at c. then the limit

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \tag{8}$$

exists. Then we have

$$\begin{aligned}
\lim_{x\to c}(f(x)-f(c)) &= \lim_{x\to c}\left[\left(\frac{f(x)-f(c)}{x-c}\right)(x-c)\right] &(9)\\
&= \lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right)\cdot \lim_{x\to c}(x-c) &(10)\\
&= f'(c)\lim_{x\to c}(x-c) &(11)\\
&= 0. &(12)
\end{aligned}$$

Therefore $\lim_{x\to c} f(x) = f(c)$, so f is continuous at c. $\square$

(ii)

Consider the following example disproving the converse of part **(i)**:

$f(x) = |x|$ is continuous but not differentiable at 0. Checking this is simple: we simply compute the left and right limits of the difference quotient and find that they do not agree, signifying that the limit (derivative) does not exist at 0.

(b)

Proof. Using the limit definition of the derivative, we have

$$\begin{aligned}
f'(0) &= \lim_{x\to 0}\frac{x^2\sin\left(\frac{1}{x}\right)}{x} &(13)\\
&= \lim_{x\to 0}\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} &(14)\\
&= 0, &(15)
\end{aligned}$$

so the limit exists.

Therefore f is differentiable at 0. $\square$

Problem 5

(a)

Proof. Let $f(x) = e^x$. By Taylor's theorem, $\forall x \in [-R, R]$ $\exists c \in (0, x)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(0) x^k + \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \tag{16}$$

, which is equivalent to

$$e^x - \sum_{k=0}^{n} \frac{x^k}{k!} = \frac{e^c}{(n+1)!} x^{n+1} \tag{17}$$

But $|x| \leq R$ and $c < R$, therefore

$$e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \leq \frac{e^R}{(n+1)!} R^{n+1}, \tag{18}$$

as desired. □

(b)

Proof. Using integration by parts, we have

$$\int_c^x f''(t)(x-t)\mathrm{d}t = (x-t)f'(t)\big|_c^x + \int_c^x f'(t)\mathrm{d}t \tag{19}$$

$$= -(x-c)f'(c) + f(x) - f(c), \tag{20}$$

i.e.

$$f(x) = f(c) + f'(c)(x-c) + \int_c^x f''(t)(x-c)\mathrm{d}t, \tag{21}$$

as desired. □

Problem 6

(a)

We compute using integration by parts:

$$\int_a^1 x\log(x)\mathrm{d}x = \frac{x^2}{2}\log(x)\big|_a^1 - \int_a^1 \frac{x}{2}\mathrm{d}x \tag{22}$$

$$= \frac{1}{2}\log(1) - \frac{a^2}{2}\log(a) - \frac{x^2}{4}\big|_a^1 \tag{23}$$

$$= -\frac{a^2}{2}\log(a) - \frac{1}{4} + \frac{a^2}{4}. \tag{24}$$

Taking the limit, we have

$$\int_0^1 x\log(x)\mathrm{d}x := \lim_{a\to 0^+}\int_a^1 x\log(x)\mathrm{d}x \tag{25}$$

$$= -\frac{1}{4} + \lim_{a\to 0^+}\left[\frac{a^2}{4} - \frac{a^2}{2}\log(a)\right] \tag{26}$$

$$= -\frac{1}{4} - \frac{1}{2}\lim_{a\to 0^+}\left[a^2\log(a)\right]. \tag{27}$$

Let $f(a) = \log(a)$ and $g(a) = \frac{1}{a^2}$. Then for $a \in (0,1)$, $g(a) \neq 0$, $f(a) \to \infty$ as $a \to 0^+$, and $g(a) \to \infty$ as $a \to 0^+$. Computing the derivatives, we find $f'(a) = \frac{1}{a}$ and $g'(a) = -\frac{2}{a^3}$.

Let $L = \lim\limits_{a\to 0^+}\frac{\frac{1}{a}}{-\frac{2}{a^3}} = 0$. Then by L'Hopital's rule, we find

$$\lim_{a\to 0^+}\frac{f(a)}{g(a)} = \lim_{a\to 0^+} a^2\log(a) = 0. \tag{28}$$

Therefore, substituting into (27) gives

$$\int_0^1 x\log(x)\mathrm{d}x = -\frac{1}{4}. \tag{29}$$

(b)

Proof. Let $f_n(x) = x^n\sin(x)$. Then since $\sin(x) \leq x$, we have

$$f_n(x) \leq x^{n+1}. \tag{30}$$

This gives

$$\int_0^1 f_n(x)\mathrm{d}x \leq \int_0^1 x^{n+1}\mathrm{d}x \tag{31}$$

$$= \frac{x^{n+2}}{n+2}\Big|_0^1 \tag{32}$$

$$= \frac{1}{n+2}. \tag{33}$$

So,

$$\lim_{n\to\infty}\int_0^1 f_n(x)\mathrm{d}x \leq \lim_{n\to\infty}\frac{1}{n+2} = 0. \tag{34}$$

Also, $f_n(x) \geq 0\ \forall x \in [0,1]$. Therefore the squeeze theorem implies that

$$\lim_{n\to\infty}\int_0^1 x^n\sin(x)\mathrm{d}x = 0, \tag{35}$$

as desired. □

(c)

Proof. Using integration by parts, we have

$$\int_{-\pi}^{\pi} \sin(nx)f(x)\mathrm{d}x = -\frac{1}{n}f(x)\cos(nx)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{1}{n}\cos(nx)f'(x)\mathrm{d}x \tag{36}$$

$$= \frac{1}{n}\cos(n\pi)\left[f(-\pi) - f(\pi)\right] + \frac{1}{n}\int_{-\pi}^{\pi} \cos(nx)f'(x)\mathrm{d}x. \tag{37}$$

Since $|\cos(n\pi)| \le 1$ and f is continuous on $[-\pi, \pi]$, we have that

$$\lim_{n\to\infty} \frac{1}{n}\cos(n\pi)\left[f(-\pi) - f(\pi)\right] = 0. \tag{38}$$

Also,

$$\frac{1}{n}\int_{-\pi}^{\pi} \cos(nx)f'(x)\mathrm{d}x \le \frac{1}{n}\int_{-\pi}^{\pi} f'(x)\mathrm{d}x \tag{39}$$

$$= \frac{1}{n}\left[f(\pi) - f(-\pi)\right]. \tag{40}$$

Thus, sending $n \to \infty$ gives us

$$\lim_{n\to\infty} \frac{1}{n}\int_{-\pi}^{\pi} \cos(nx)f'(x)\mathrm{d}x = 0. \tag{41}$$

Hence, putting everything together, we get

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \sin(nx)f(x)\mathrm{d}x = 0, \tag{42}$$

and we are done. □

Problem 7

(a)

We wish to fins a sequence $\{f_n\}_n$ of continuous functions $f_n : (0,1) \to \mathbb{R}$ such that $f_n \to f$ pointwise, but not uniformly.

Consider $f_n(x) := nxe^{-nx}$. Then the pointwise limit is given by

$$\lim_{n\to\infty} f_n(x) = x \lim_{n\to\infty} \frac{n}{e^{nx}} = 0, \tag{43}$$

so $f_n \to 0$ pointwise.

Let $M \in \mathbb{N}$. Then $\forall n \ge M$, let $x = \frac{1}{n}$ and $\epsilon_0 = e^{-1}$. Then

$$|f_n(x)| = \left|nxe^{-nx}\right| \tag{44}$$

$$= e^{-1} \tag{45}$$

$$= \epsilon_0. \tag{46}$$

Hence f_n does not converge uniformly to 0 on $(0,1)$.

Therefore $f_n(x) = nxe^{-nx}$ does the job.

(b)

(i)

Proof. Since f is differentiable on $\mathbb{R}$, then f is continuous on $\mathbb{R}$. Let $x, y \in \mathbb{R}$ with $x < y$. Then by the mean value theorem, $\exists c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x)$, i.e.

$$\begin{aligned} |f(y) - f(x)| &= |f'(c)||y - x| && (47) \\ &\leq L|y - x|, && (48) \end{aligned}$$

where (48) follows from the assumption.

Therefore f is Lipschitz continuous on $\mathbb{R}$. $\square$

(ii)

Proof. Let $\epsilon > 0$. Choose $M - \frac{L}{\epsilon}$. Then $\forall x \in \mathbb{R}$ and $n \geq M$, we have (by Lipschitz continuity)

$$\begin{aligned} |f_n(x) - f(x)| &= |f(x + \frac{1}{n}) - f(x)| && (49) \\ &\leq L|x + \frac{1}{n} - x| && (50) \\ &= L\frac{1}{n} && (51) \\ &< L\frac{\epsilon}{L} && (52) \\ &= \epsilon. && (53) \end{aligned}$$

Therefore $f_n \to f$ uniformly on $\mathbb{R}$. $\square$