

18.100A Assignment 5

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Problem 1

Proof. We have that

$$L = \lim_{n \rightarrow \infty} \frac{|x_{n+1} - x|}{|x_n - x|} < 1. \quad (1)$$

Then for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\left| \frac{x_{n+1} - x}{x_n - x} \right| - 1 < \epsilon. \quad (2)$$

Rearranging gives

$$|x_{n+1} - x| < (1 + \epsilon)|x_n - x|. \quad (3)$$

By taking ϵ to be arbitrarily small, we have, $\forall n \geq N$,

$$\xrightarrow{\epsilon \rightarrow 0} |x_{n+1} - x| < |x_n - x|. \quad (4)$$

Define $y_n := |x_n - x|$. Then $\forall n \geq N$,

$$0 \leq y_{n+1} < y_n, \quad (5)$$

so $\{y_n\}_n$ is a decreasing sequence bounded below by 0. Hence,

$$\implies y_n \rightarrow 0 \quad (6)$$

$$\implies |x_n - x| \rightarrow 0 \quad (7)$$

$$\implies x_n \rightarrow x. \quad (8)$$

Therefore, $\{x_n\}_n$ converges to x . \square

Problem 2

(a)

Let $x_n = \frac{(-1)^n}{n}$. Then $\forall n \in \mathbb{N}$,

$$-\frac{1}{n} \leq x_n \leq \frac{1}{n}. \quad (9)$$

Allowing $n \rightarrow \infty$ on all sides of the inequality gives

$$0 = \lim_{n \rightarrow \infty} \left(-\frac{1}{n} \right) \leq \lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) = 0. \quad (10)$$

So by the Squeeze Theorem, $\lim_{n \rightarrow \infty} x_n = 0$. Finally, by the theorem from lecture 9, we conclude that $\liminf x_n = \limsup x_n = 0$.

(b)

Let $x_n = (-1)^n \frac{(n-1)}{n}$. Define

$$a_n := \sup\{x_k \mid k \geq n\}, \text{ and } b_n := \sup\{x_k \mid k \geq n\}. \quad (11)$$

Then $\forall n \in \mathbb{N}$, we have

$$|x_n| = \left| \frac{n-1}{n} \right| \cdot |(-1)^n| \quad (12)$$

$$= \frac{n-1}{n} \quad (13)$$

$$= 1 - \frac{1}{n} \quad (14)$$

$$\leq 1. \quad (15)$$

Thus, x_n is bounded and $-1 \leq x_n \leq 1$.

Let $n_k = 2k$ and $m_k = 2k-1$ for $k \in \mathbb{N}$. Then we construct two subsequences $\{x_{n_k}\}_k$ and $\{x_{m_k}\}_k$ of $\{x_n\}_n$ via

$$x_{n_k} := \frac{2k-1}{2k} (-1)^{2k} = \frac{2k-1}{2k}, \quad (16)$$

and

$$x_{m_k} := \frac{2k-2}{2k-1} (-1)^{2k-1} = \frac{2-2k}{2k-1}. \quad (17)$$

Taking the limit of the first subsequence gives

$$\lim_{k \rightarrow \infty} x_{n_k} = \lim_{k \rightarrow \infty} \left(1 - \frac{1}{2k} \right) = 1, \quad (18)$$

and for the second subsequence we get

$$\lim_{k \rightarrow \infty} x_{m_k} = \lim_{k \rightarrow \infty} \left(\frac{1}{2k-1} - 1 \right) = -1. \quad (19)$$

Thus, $\{x_{n_k}\}_k$ converges to 1, and $\{x_{m_k}\}_k$ converges to -1. Then $\sup\{x_k \mid k \geq n\} \geq 1$. But $-1 \leq x_n \leq 1$, so it must also be true that

$$\sup\{x_k \mid k \geq n\} \leq \sup x_n = 1. \quad (20)$$

Thus, $\sup\{x_k \mid k \geq n\} = 1$, and we can conclude that $\limsup x_n = 1$.

Similarly, $\inf\{x_k \mid k \geq n\} \leq -1$, but $-1 \leq x_n \leq 1$, so we must have

$$\inf\{x_k \mid k \geq n\} \geq \inf x_n = -1, \quad (21)$$

so $\inf\{x_k \mid k \geq n\} = -1$. Therefore, we conclude also that $\liminf x_n = -1$.

Problem 3

Proof. (i) Since $x_n \leq y_n \forall n \in \mathbb{N}$, then for each n , we have

$$\sup\{x_k \mid k \geq n\} \leq \sup\{y_k \mid k \geq n\}. \quad (22)$$

Taking the limit on both sides of (22) gives

$$\xrightarrow{n \rightarrow \infty} \limsup x_n \leq \limsup y_n, \quad (23)$$

as desired.

(ii) Since $x_n \leq y_n \forall n \in \mathbb{N}$, then for each n , we have

$$\inf\{x_k \mid k \geq n\} \leq \inf\{y_k \mid k \geq n\}. \quad (24)$$

Taking the limit in (24) yields

$$\xrightarrow{n \rightarrow \infty} \liminf x_n \leq \liminf y_n, \quad (25)$$

and we are done. \square

Problem 4

(a)

Proof. Since $\{x_n\}_n$ and $\{y_n\}_n$ are bounded sequences, then $\exists B_0 \geq 0$ and $B_1 \geq 0$ such that $\forall n \in \mathbb{N}$, $|x_n| \leq B_0$ and $|y_n| \leq B_1$.

Choose $B = B_0 + B_1$. Then by the triangle inequality,

$$|x_n + y_n| \leq |x_n| + |y_n| \leq B_0 + B_1 = B. \quad (26)$$

Therefore, the sequence $\{x_n + y_n\}_n$ is bounded. \square

(b)

Proof. Let $\{x_{n_k}\}_k$ be a subsequence of $\{x_n\}_n$. Since the sequence $\{x_n\}_n$ is bounded, then $\{x_{n_k}\}_k$ must also be bounded. Then by the Bolzano-Weierstrass

theorem, \exists a (sub) subsequence $\{x_{n_{k_i}}\}_i$ of the subsequence $\{x_{n_k}\}_k$ such that $\{x_{n_{k_i}}\}_i$ converges; i.e. that $\lim_{i \rightarrow \infty} x_{n_{k_i}}$ exists. Then

$$\liminf x_n \leq \lim_{i \rightarrow \infty} x_{n_{k_i}}. \quad (27)$$

By the same logic as above, \exists a (sub-sub) subsequence $\{y_{n_{k_{i_l}}}\}_l$ of $\{y_{n_{k_i}}\}_i$ such that $\lim_{l \rightarrow \infty} y_{n_{k_{i_l}}}$ exists. Then

$$\liminf y_n \leq \lim_{l \rightarrow \infty} y_{n_{k_{i_l}}}. \quad (28)$$

Thus the corresponding subsequence $\{x_{n_{k_{i_l}}} + y_{n_{k_{i_l}}}\}_l$ of $\{x_{n_k} + y_{n_k}\}_k$ is convergent, and

$$\lim_{l \rightarrow \infty} (x_{n_{k_{i_l}}} + y_{n_{k_{i_l}}}) = \lim_{k \rightarrow \infty} x_{n_k} + \lim_{k \rightarrow \infty} y_{n_k}. \quad (29)$$

But we also have

$$\lim_{l \rightarrow \infty} (x_{n_{k_{i_l}}} + y_{n_{k_{i_l}}}) = \lim_{l \rightarrow \infty} x_{n_{k_{i_l}}} + \lim_{l \rightarrow \infty} y_{n_{k_{i_l}}} \quad (30)$$

$$= \lim_{i \rightarrow \infty} x_{n_{k_i}} + \lim_{l \rightarrow \infty} y_{n_{k_{i_l}}} \quad (31)$$

$$\geq \liminf x_n + \liminf y_n. \quad (32)$$

Additionally,

$$\liminf(x_n + y_n) \geq \lim_{l \rightarrow \infty} (x_{n_{k_{i_l}}} + y_{n_{k_{i_l}}}). \quad (33)$$

Therefore, $\liminf x_n + \liminf y_n \leq \liminf(x_n + y_n)$. \square