18.100A Assignment 8

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Problem 1

Proof. Suppose $\lim_{x\to c} f(x) = \lim_{x\to c} h(x)$. Then since c is a cluster point of S and $\forall x \in S$, $f(x) \le g(x) \le h(x)$, we have

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x) \le \lim_{x\to c} h(x) = \lim_{x\to c} f(x). \tag{1}$$

Thus by the squeeze theorem,

$$\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x), \tag{2}$$

as desired. ☐

Problem 2

Proof. (1) Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Then if $|x| < \delta$, we have

$$\begin{aligned} |f(x) - f(0)| &= |f(x)| && (3)\\ &\le 2|x| && (4)\\ &< 2\delta && (5)\\ &= \epsilon. && (6) \end{aligned}$$

Therefore, f is continuous at $x = 0$.

(2) Let $\delta > 0$, $\epsilon_0 > 0$. Suppose $|x - 1| < \delta$. Let

$$x_0 = \begin{cases} \epsilon_0\sqrt{2}, & \epsilon_0 \in \mathbb{Q} \\ \epsilon_0, & \epsilon_0 \notin \mathbb{Q}. \end{cases} \tag{7}$$

Then $x_0 \notin \mathbb{Q}$ $\forall \epsilon_0 > 0$.

Choose $x = \frac{x_0}{2}$. Then

$$
\begin{aligned}
|f(x) - f(1)| &= |f(x)| && (8)\\
&= 2|x| && (9)\\
&= x_0 && (10)\\
&\geq \epsilon_0. && (11)
\end{aligned}
$$

Therefore, f is discontinuous at $x = 1$. $\square$