

# 18.100A Assignment 8

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## Problem 1

*Proof.* Suppose  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x)$ . Then since  $c$  is a cluster point of  $S$  and  $\forall x \in S, f(x) \leq g(x) \leq h(x)$ , we have

$$\lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x) \leq \lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} f(x). \quad (1)$$

Thus by the squeeze theorem,

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x), \quad (2)$$

as desired.  $\square$

## Problem 2

*Proof.* (1) Let  $\epsilon > 0$ . Choose  $\delta = \frac{\epsilon}{2}$ . Then if  $|x| < \delta$ , we have

$$|f(x) - f(0)| = |f(x)| \quad (3)$$

$$\leq 2|x| \quad (4)$$

$$< 2\delta \quad (5)$$

$$= \epsilon. \quad (6)$$

Therefore,  $f$  is continuous at  $x = 0$ .

(2) Let  $\delta > 0, \epsilon_0 > 0$ . Suppose  $|x - 1| < \delta$ . Let

$$x_0 = \begin{cases} \epsilon_0 \sqrt{2}, & \epsilon_0 \in \mathbb{Q} \\ \epsilon_0, & \epsilon_0 \notin \mathbb{Q}. \end{cases} \quad (7)$$

Then  $x_0 \notin \mathbb{Q} \forall \epsilon_0 > 0$ .

Choose  $x = \frac{x_0}{2}$ . Then

$$|f(x) - f(1)| = |f(x)| \quad (8)$$

$$= 2|x| \quad (9)$$

$$= x_0 \quad (10)$$

$$\geq \epsilon_0. \quad (11)$$

Therefore,  $f$  is discontinuous at  $x = 1$ .  $\square$

### Problem 3

*Proof.* Since  $f$  is continuous at  $c$ , then  $\forall \epsilon > 0$ ,  $\exists \delta > 0$  such that if  $|x - c| < \delta$ , then  $|f(x) - f(c)| < \epsilon$ , i.e.

$$f(c) - \epsilon < f(x) < f(c) + \epsilon. \quad (12)$$

Let  $\epsilon = \frac{f(c)}{2}$ . Then for some  $\delta > 0$ ,

$$0 < \frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}. \quad (13)$$

Simply choose  $\alpha = \delta$ , and we are done.  $\square$

### Problem 4

*Proof.* Since  $h = f$  on  $[-1, 0]$  and  $f(x)$  is continuous, then  $h$  is continuous on  $[-1, 0]$ . Similarly, since  $h = g$  on  $(0, 1]$  and  $g$  is continuous, then  $h$  is continuous on  $(0, 1]$ .

It remains only to show that  $h$  is continuous at  $x = 0$ .

Let  $\epsilon_0 > 0$ . By continuity of  $f$   $\exists \delta_0$  such that if  $|x| < \delta_0$ , then  $|f(x) - f(0)| < \epsilon_0$ . Similarly, let  $\epsilon_1 > 0$ . Then by continuity of  $g$ ,  $\exists \delta_1 > 0$  such that if  $|x| < \delta_1$ , then  $|g(x) - g(0)| < \epsilon_1$ .

Let  $x \in [-1, 1]$  and let  $\epsilon = \min\{\epsilon_0, \epsilon_1\}$ . Choose  $\delta = \min\{\delta_0, \delta_1\}$ . If  $|x| < \delta$ , then  $|x| < \delta_0$  and  $|x| < \delta_1$ . If  $x \leq 0$ , then we have

$$|h(x) - h(0)| = |f(x) - f(0)| < \epsilon_0 \leq \epsilon. \quad (14)$$

Similarly if  $x > 0$ , then we have

$$|h(x) - h(0)| = |g(x) - g(0)| < \epsilon_1 \leq \epsilon. \quad (15)$$

Thus,  $h(x)$  is continuous at  $x = 0$ , so we conclude  $h$  is continuous.  $\square$