18.100A Assignment 12

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Problem 1

(a)

Proof. Suppose $\exists c \in [a,b]$ such that $f(c) > 0$. Since f is continuous, $\exists \delta > 0$ such that if $|x-c| < \delta$, then $|f(x) - f(x)| < \frac{f(c)}{2}$, i.e. $\frac{f(c)}{2} < f(x)$. We compute

$$\begin{aligned} 0 &> \int_a^b f && (1) \\ &> \int_a^b \frac{f(c)}{2} && (2) \\ &= \frac{f(c)}{2}(b-a) && (3) \\ &> 0, && (4) \end{aligned}$$

i.e. $0 > 0$ ($\Rightarrow\Leftarrow$), which is cleary a contradiction.

Therefore $f(x) = 0 \ \forall x \in [a,b]$. □

(b)

Proof. Let $E = \int_a^b (u')^2 \mathrm{d}x$. Then $E \geq 0$ since $(u')^2 \geq 0$. Using integration by parts, we have

$$\begin{aligned} E &= \int_a^b u'u' \mathrm{d}x && (5) \\ &= uu'\big|_a^b - \int_a^b uu'' \mathrm{d}x && (6) \\ &= u'(b)u(b) - u'(a)u(a) - \int_a^b u(Vu) \mathrm{d}x && (7) \\ &= -\int_a^b Vu^2 \mathrm{d}x. && (8) \end{aligned}$$

But $V(x) \geq 0$ and $u^2 \geq 0$, so $-(Vu^2) \leq 0$, hence $E \leq 0$. Thus $E = 0$, which must mean that

$$\int_a^b (u')^2 \mathrm{d}x = 0, \tag{9}$$

and by part **(a)**, this implies that $(u')^2 = 0 \ \forall x \in [a, b]$. Hence $u'(x) = 0$ for all x, and since $u(a) = 0$, then u remains constant at 0; i.e. $u = 0$ everywhere. □

Problem 2

We compute:

$$\int_{-x}^{x} e^{s^2} \mathrm{d}s = \int_{-x}^{0} e^{s^2} \mathrm{d}s + \int_{0}^{x} e^{s^2} \mathrm{d}s \tag{10}$$

$$= \int_{0}^{x} e^{s^2} \mathrm{d}s - \int_{0}^{-x} e^{s^2} \mathrm{d}s. \tag{11}$$

Differentiating, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-x}^{x} e^{s^2} \mathrm{d}s\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{0}^{x} e^{s^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{0}^{-x} e^{s^2}\right) \tag{12}$$

$$= e^{x^2} + e^{x^2}. \tag{13}$$

Thus $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-x}^{x} e^{s^2} \mathrm{d}s\right) = 2e^{x^2}$.

Problem 3

Proof. Define $G(x) = \int_a^x f(t)\mathrm{d}t$. By the fundamental theorem of calculus, G is continuous on $[a, b]$. Note that $G(x) = 0 \ \forall x \in \mathbb{Q} \cap [a, b]$. We now claim that $G(x) = 0$ on $[a, b]$.

Suppose $\exists c \in [a, b]$ such that $G(c) \neq 0$. For some $x \in [a, b]$, let $\epsilon = \frac{|G(x)|}{2}$ and let $\delta > 0$. Then $\forall x$ such that $|x - c| < \delta$, we have $|G(x) - G(c)| < \epsilon$ since G is continuous. $\exists c \in [a, b] \cap \mathbb{Q}$ such that $|x - c| < \delta$. But then $|G(x) - G(c)| = |G(x)| > \frac{|G(x)|}{2} = \epsilon$, which is a contradiction, since we assumed G to be continuous. Thus $G(x) = 0$ on $[a, b]$, which proves the claim.

Thus, $G(x) = \int_a^x f(t)\mathrm{d}t = 0$ on $[a, b]$. Since G is constant, then $G' = 0$ on $[a, b]$. By the fundamental theorem of calculus, $G'(x) = f(x) = 0 \ \forall x \in [a, b]$, and we are done. □

Problem 4

0.1 (a)

Let $f_n(x) = \frac{e^{\frac{x}{n}}}{n}$ for each $n \in \mathbb{N}$. Then by continuity, we have

$$\begin{align} \lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} e^{\frac{x}{n}} \cdot \lim_{n\to\infty} \frac{1}{n} \tag{14}\\ &= e^{x \lim_{n\to\infty} \frac{1}{n}} \cdot \lim_{n\to\infty} \frac{1}{n} \tag{15}\\ &= e^0 \cdot 0 \tag{16}\\ &= 0. \tag{17} \end{align}$$

Therefore $f_n \to 0$ pointwise.

(b)

Let $M \in \mathbb{N}$. Choose $\epsilon_0 = 1$, $x = n\ln(n)$, and $n = M$. Then

$$\begin{align} |f_n(x) - f(x)| &= \left| \frac{e^{\frac{x}{n}}}{n} \right| \tag{18}\\ &= \frac{e^{\ln(M)}}{M} \tag{19}\\ &= \frac{M}{M} \tag{20}\\ &= 1 \tag{21}\\ &= \epsilon_0. \tag{22} \end{align}$$

Therefore the limit is NOT uniform on $\mathbb{R}$.

(c)

Let $\epsilon > 0$. Choose $M = \frac{1}{\log(\epsilon)}$. Then $\forall x \in [0,1]$ and $\forall n \geq M$, we have

$$\begin{align} |f_n(x) - 0| &= \left| \frac{e^{\frac{x}{n}}}{n} \right| \tag{23}\\ &< \frac{e^{\frac{1}{n}}}{n} \tag{24}\\ &< e^{\frac{1}{n}} \tag{25}\\ &< e^{\log(\epsilon)} \tag{26}\\ &= \epsilon. \tag{27} \end{align}$$

Therefore the limit is uniform on $[0,1]$.

Problem 5

Proof. $\forall \epsilon > 0$, $\exists M_0 \in \mathbb{N}$ such that $\forall n \geq M_0$ and $\forall x \in A$, $|f_n(x) - f(x)| < \frac{\epsilon}{2}$. Similarly, $\forall \epsilon > 0$, $\exists M_1 \in \mathbb{N}$ such that $\forall n \geq M_1$ and $\forall x \in A$, $|g_n(x) - g(x)| < \frac{\epsilon}{2}$.

Let $\epsilon > 0$. Choose $M = \max\{M_0, M_1\}$. Then $\forall n \geq M$,

$$
\begin{aligned}
|f_n(x) + g_n(x) - (f(x) + g(x))| &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| && (28)\\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} && (29)\\
&= \epsilon. && (30)
\end{aligned}
$$

Therefore $f_n + g_n \to f + g$ uniformly on A. $\square$