# 18.100A Assignment 8

Octavio Vega

May 20, 2023

## Problem 1

*Proof.* Suppose $\lim_{x\to c} f(x) = \lim_{x\to c} h(x)$. Then since $c$ is a cluster point of $S$ and $\forall x \in S$, $f(x) \le g(x) \le h(x)$, we have

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x) \le \lim_{x\to c} h(x) = \lim_{x\to c} f(x). \tag{1}$$

Thus by the squeeze theorem,

$$\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x), \tag{2}$$

as desired. $\square$

## Problem 2

*Proof.* (1) Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Then if $|x| < \delta$, we have

$$\begin{aligned} |f(x) - f(0)| &= |f(x)| && (3)\\ &\le 2|x| && (4)\\ &< 2\delta && (5)\\ &= \epsilon. && (6) \end{aligned}$$

Therefore, $f$ is continuous at $x = 0$.

(2) Let $\delta > 0$, $\epsilon_0 > 0$. Suppose $|x - 1| < \delta$. Let

$$x_0 = \begin{cases} \epsilon_0\sqrt{2}, & \epsilon_0 \in \mathbb{Q} \\ \epsilon_0, & \epsilon_0 \notin \mathbb{Q}. \end{cases} \tag{7}$$

Then $x_0 \notin \mathbb{Q}\ \forall \epsilon_0 > 0$.

Choose $x = \frac{x_0}{2}$. Then

$$\begin{aligned} |f(x) - f(1)| &= |f(x)| && (8) \\ &= 2|x| && (9) \\ &= x_0 && (10) \\ &\geq \epsilon_0. && (11) \end{aligned}$$

Therefore, $f$ is discontinuous at $x = 1$. $\square$

# Problem 3

*Proof.* Since $f$ is continuous at $c$, then $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$, i.e.

$$f(c) - \epsilon < f(x) < f(c) + \epsilon. \tag{12}$$

Let $\epsilon = \frac{f(c)}{2}$. Then for some $\delta > 0$,

$$0 < \frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}. \tag{13}$$

Simply choose $\alpha = \delta$, and we are done. $\square$