ANNALS OF MATHEMATICS
Vol. 40, No. 1, January, 1939

ON UNITARY REPRESENTATIONS OF THE INHOMOGENEOUS LORENTZ GROUP*

BY E. WIGNER

(Received December 22, 1937)

1. ORIGIN AND CHARACTERIZATION OF THE PROBLEM

It is perhaps the most fundamental principle of Quantum Mechanics that the system of states forms a *linear manifold*,[1] in which a unitary *scalar product* is defined.[2] The states are generally represented by wave functions[3] in such a way that φ and constant multiples of φ represent the same physical state. It is possible, therefore, to normalize the wave function, i.e., to multiply it by a constant factor such that its scalar product with itself becomes 1. Then, only a constant factor of modulus 1, the so-called phase, will be left undetermined in the wave function. The linear character of the wave function is called the superposition principle. The square of the modulus of the unitary scalar product (ψ, φ) of two normalized wave functions ψ and φ is called the transition probability from the state ψ into φ, or conversely. This is supposed to give the probability that an experiment performed on a system in the state φ, to see whether or not the state is ψ, gives the result that it is ψ. If there are two or more different experiments to decide this (e.g., essentially the same experiment,

---

* Parts of the present paper were presented at the Pittsburgh Symposium on Group Theory and Quantum Mechanics. Cf. Bull. Amer. Math. Soc., 41, p. 306, 1935.

[1] The possibility of a future non linear character of the quantum mechanics must be admitted, of course. An indication in this direction is given by the theory of the positron, as developed by P. A. M. Dirac (Proc. Camb. Phil. Soc. *30*, 150, 1934, cf. also W. Heisenberg, Zeits. f. Phys. *90*, 209, 1934; *92*, 623, 1934; W. Heisenberg and H. Euler, ibid. *98*, 714, 1936 and R. Serber, Phys. Rev. *48*, 49, 1935; *49*, 545, 1936) which does not use wave functions and is a non linear theory.

[2] Cf. P. A. M. Dirac, *The Principles of Quantum Mechanics*, Oxford 1935, Chapters I and II; J. v. Neumann, *Mathematische Grundlagen der Quantenmechanik*, Berlin 1932, pages 19–24.

[3] The wave functions represent throughout this paper states in the sense of the "Heisenberg picture," i.e. a single wave function represents the state for all past and future. On the other hand, the operator which refers to a measurement at a certain time t contains this t as a parameter. (Cf. e.g. Dirac, l.c. ref. 2, pages 115–123). One obtains the wave function $\varphi_s(t)$ of the Schrödinger picture from the wave function φ_H of the Heisenberg picture by $\varphi_s(t) = \exp(-iHt/\hbar)\varphi_H$. The operator of the Heisenberg picture is $Q(t) = \exp(iHt/\hbar)\, Q \exp(-iHt/\hbar)$, where Q is the operator in the Schrödinger picture which does not depend on time. Cf. also E. Schrödinger, Sitz. d. Kön. Preuss. Akad. p. 418, 1930.

The wave functions are complex quantities and the undetermined factors in them are complex also. Recently attempts have been made toward a theory with real wave functions. Cf. E. Majorana, Nuovo Cim. *14*, 171, 1937 and P. A. M. Dirac, in print.

performed at different times) they are all supposed to give the same result, i.e., the transition probability has an invariant physical sense.

The wave functions form a description of the physical state, not an invariant however, since the same state will be described in different coördinate systems by different wave functions. In order to put this into evidence, we shall affix an index to our wave functions, denoting the Lorentz frame of reference for which the wave function is given. Thus φ_l and $\varphi_{l'}$ represent the same state, but they are different functions. The first is the wave function of the state in the coördinate system l, the second in the coördinate system l'. If $\varphi_l = \psi_{l'}$ the state φ behaves in the coördinate system l exactly as ψ behaves in the coördinate system l'. If φ_l is given, all $\varphi_{l'}$ are determined up to a constant factor. Because of the invariance of the transition probability we have

$$|(\varphi_l, \psi_l)|^2 = |(\varphi_{l'}, \psi_{l'})|^2 \tag{1}$$

and it can be shown[4] that the aforementioned constants in the $\varphi_{l'}$ can be chosen in such a way that the $\varphi_{l'}$ are obtained from the φ_l by a linear unitary operation, depending, of course, on l and l'

$$\varphi_{l'} = D(l', l)\varphi_l . \tag{2}$$

The unitary operators D are determined by the physical content of the theory up to a constant factor again, which can depend on l and l'. Apart from this constant however, the operations $D(l', l)$ and $D(l_1', l_1)$ must be identical if l' arises from l by the same Lorentz transformation, by which l_1' arises from l_1. If this were not true, there would be a real difference between the frames of reference l and l_1. Thus the unitary operator $D(l', l) = D(L)$ is in every Lorentz invariant quantum mechanical theory (apart from the constant factor which has no physical significance) completely determined by the Lorentz transformation L which carries l into $l' = Ll$. One can write, instead of (2)

$$\varphi_{Ll} = D(L)\varphi_l . \tag{2a}$$

By going over from a first system of reference l to a second $l' = L_1 l$ and then to a third $l'' = L_2 L_1 l$ or directly to the third $l'' = (L_2 L_1)l$, one must obtain—apart from the above mentioned constant—the same set of wave functions. Hence from

$$\varphi_{l''} = D(l'', l')D(l', l)\varphi_l$$

$$\varphi_{l''} = D(l'', l)\varphi_l$$

it follows

$$D(l'', l')D(l', l) = \omega D(l'', l) \tag{3}$$

[4] E. Wigner, *Gruppentheorie und ihre Anwendungen auf die Quantenmechanik der Atomspektren.* Braunschweig 1931, pages 251–254.

or

$$D(L_2)D(L_1) = \omega D(L_2 L_1), \tag{3a}$$

where ω is a number of modulus 1 and can depend on L_2 and L_1. Thus the $D(L)$ form, up to a factor, a representation of the inhomogeneous Lorentz group by linear, unitary operators.

We see thus[5] that there corresponds to every invariant quantum mechanical system of equations such a representation of the inhomogeneous Lorentz group. This representation, on the other hand, though not sufficient to replace the quantum mechanical equations entirely, can replace them to a large extent. If we knew, e.g., the operator K corresponding to the measurement of a physical quantity at the time $t = 0$, we could follow up the change of this quantity throughout time. In order to obtain its value for the time $t = t_1$, we could transform the original wave function φ_l by $D(l', l)$ to a coördinate system l' the time scale of which begins a time t_1 later. The measurement of the quantity in question in this coördinate system for the time 0 is given—as in the original one—by the operator K. This measurement is indentical, however, with the measurement of the quantity at time t_1 in the original system. One can say that the representation can replace the equation of motion, it cannot replace, however, connections holding between operators at one instant of time.

It may be mentioned, finally, that these developments apply not only in quantum mechanics, but also to all linear theories, e.g., the Maxwell equations in empty space. The only difference is that there is no arbitrary factor in the description and the ω can be omitted in (3a) and one is led to real representations instead of representations up to a factor. On the other hand, the unitary character of the representation is not a consequence of the basic assumptions.

The increase in generality, obtained by the present calculus, as compared with the usual tensor theory, consists in that no assumptions regarding the field nature of the underlying equations are necessary. Thus more general equations, as far as they exist (e.g., in which the coördinate is quantized, etc.) are also included in the present treatment. It must be realized, however, that some assumptions concerning the continuity of space have been made by assuming Lorentz frames of reference in the classical sense. We should like to mention, on the other hand, that the previous remarks concerning the time-parameter in the observables, have only an explanatory character, and we do not make assumptions of the kind that measurements can be performed instantaneously.

We shall endeavor, in the ensuing sections, to determine all the continuous[6] unitary representations up to a factor of the inhomogeneous Lorentz group, i.e., all continuous systems of linear, unitary operators satisfying (3a).

[5] E. Wigner, l.c. Chapter XX.

[6] The exact definition of the continuous character of a representation up to a factor will be given in Section 5A. The definition of the inhomogeneous Lorentz group is contained in Section 4A.

2. Comparison With Previous Treatments and Some Immediate Simplifications

A. Previous treatments

The representations of the Lorentz group have been investigated repeatedly. The first investigation is due to Majorana,[7] who in fact found all representations of the class to be dealt with in the present work excepting two sets of representations. Dirac[8] and Proca[8] gave more elegant derivations of Majorana's results and brought them into a form which can be handled more easily. Klein's work[9] does not endeavor to derive irreducible representations and seems to be in a less close connection with the present work.

The difference between the present paper and that of Majorana and Dirac lies—apart from the finding of new representations—mainly in its greater mathematical rigor. Majorana and Dirac freely use the notion of infinitesimal operators and a set of functions to all members of which every infinitesimal operator can be applied. This procedure cannot be mathematically justified at present, and no such assumption will be used in the present paper. Also the conditions of reducibility and irreducibility could be, in general, somewhat more complicated than assumed by Majorana and Dirac. Finally, the previous treatments assume from the outset that the space and time coördinates will be continuous variables of the wave function in the usual way. This will not be done, of course, in the present work.

B. Some immediate simplifications

Two representations are *physically equivalent* if there is a one to one correspondence between the states of both which is 1. invariant under Lorentz transformations and 2. of such a character that the transition probabilities between corresponding states are the same.

It follows from the second condition[5] that there either exists a unitary operator S by which the wave functions $\Phi^{(2)}$ of the second representation can be obtained from the corresponding wave functions $\Phi^{(1)}$ of the first representation

$$\Phi^{(2)} = S\Phi^{(1)} \tag{4}$$

or that this is true for the conjugate imaginary of $\Phi^{(2)}$. Although, in the latter case, the two representations are still equivalent physically, we shall, in keeping with the mathematical convention, not call them equivalent.

The first condition now means that if the states $\Phi^{(1)}$, $\Phi^{(2)} = S\Phi^{(1)}$ correspond to each other in one coördinate system, the states $D^{(1)}(L)\Phi^{(1)}$ and $D^{(2)}(L)\Phi^{(2)}$ correspond to each other also. We have then

$$D^{(2)}(L)\Phi^{(2)} = SD^{(1)}(L)\Phi^{(1)} = SD^{(1)}(L)S^{-1}\Phi^{(2)}. \tag{4a}$$

[7] E. Majorana, Nuovo Cim. *9*, 335, 1932.

[8] P. A. M. Dirac, Proc. Roy. Soc. A. *155*, 447, 1936; Al. Proca, J. de Phys. Rad. *7*, 347, 1936.

[9] Klein, Arkiv f. Matem. Astr. och Fysik, *25A*, No. 15, 1936. I am indebted to Mr. Darling for an interesting conversation on this paper.

As this shall hold for every $\Phi^{(2)}$, the existence of a unitary S which transforms $D^{(1)}$ into $D^{(2)}$ is the condition for the equivalence of these two representations. Equivalent representations are not considered to be really different and it will be sufficient to find one sample from every infinite class of equivalent representations.

If there is a closed linear manifold of states which is invariant under all Lorentz transformations, i.e. which contains $D(L)\psi$ if it contains ψ, the linear manifold perpendicular to this one will be invariant also. In fact, if φ belongs to the second manifold, $D(L)\varphi$ will be, on account of the unitary character of $D(L)$, perpendicular to $D(L)\psi'$ if ψ' belongs to the first manifold. However, $D(L^{-1})\psi$ belongs to the first manifold if ψ does and thus $D(L)\varphi$ will be orthogonal to $D(L)D(L^{-1})\psi = \omega\psi$ i.e. to all members of the first manifold and belong itself to the second manifold also. The original representation then "decomposes" into two representations, corresponding to the two linear manifolds. It is clear that, conversely, one can form a representation, by simply "adding" several other representations together, i.e. by considering as states linear combinations of the states of several representations and assume that the states which originate from different representations are perpendicular to each other.

Representations which are equivalent to sums of already known representations are not really new and, in order to master all representations, it will be sufficient to determine those, out of which all others can be obtained by "adding" a finite or infinite number of them together.

Two simple theorems shall be mentioned here which will be proved later (Sections 7A and 8C respectively). The first one refers to unitary representations of any closed group, the second to irreducible unitary representations of any (closed or open) group.

The representations of a closed group by unitary *operators* can be transformed into the sum of unitary representations with matrices of finite dimensions.

Given two non equivalent irreducible unitary representations of an arbitrary group. If the scalar product between the wave functions is invariant under the operations of the group, the wave functions belonging[28] to the first representation are orthogonal to all wave functions belonging to the second representation.

C. Classification of unitary representations according to von Neumann and Murray[10]

Given the operators $D(L)$ of a unitary representations, or a representation up to a factor, one can consider the algebra of these operators, i.e. all linear combinations

$$a_1D(L_1) + a_2D(L_2) + a_3D(L_3) + \cdots$$

of the $D(L)$ and all limits of such linear combinations which are bounded operators. According to the properties of this representation algebra, three classes of unitary representations can be distinguished.

---

[10] F. J. Murray and J. v. Neumann, Ann. of Math. *37*, 116, 1936; J. v. Neumann, to be published soon.

The first class of *irreducible* representations has a representation algebra which contains all bounded operators, i.e. if ψ and φ are two arbitrary states, there is an operator A of the representation algebra for which $A\psi = \varphi$ and $A\psi' = 0$ if ψ' is orthogonal to ψ. It is clear that the center of the algebra contains only the unit operator and multiply thereof. In fact, if C is in the center one can decompose $C\psi = \alpha\psi + \psi'$ so that ψ' shall be orthogonal to ψ. However, ψ' must vanish since otherwise C would not commute with the operator which leaves ψ invariant and transforms every function orthogonal to it into 0. For similar reasons, α must be the same for all ψ. For irreducible representations there is no closed linear manifold of states, (excepting the manifold of all states) which is invariant under all Lorentz transformations. In fact, according to the above definition, a φ' arbitrarily close to any φ can be represented by a finite linear combination

$$a_1D(L_1)\psi + a_2D(L_2)\psi + \cdots + a_nD(L_n)\psi.$$

Hence, a closed linear invariant manifold contains every state if it contains one. This is, in fact, the more customary definition for irreducible representations and the one which will be used subsequently. It is well known that all finite dimensional representations are sums of irreducible representations. This is not true,[10] in general, in an infinite number of dimensions.

The second class of representations will be called *factorial*. For these, the center of the representation algebra still contains only multiples of the unit operator. Clearly, the irreducible representations are all factorial, but not conversely. For finite dimensions, the factorial representations may contain one irreducible representation several times. This is also possible in an infinite number of dimensions, but in addition to this, there are the "continuous" representations of Murray and von Neumann.[10] These are not irreducible as there are invariant linear manifolds of states. On the other hand, it is impossible to carry the decomposition so far as to obtain as parts only irreducible representations. In all the examples known so far, the representations into which these continuous representations can be decomposed, are equivalent to the original representation.

The third class contains all possible unitary representations. In a finite number of dimensions, these can be decomposed first into factorial representations, and these, in turn, in irreducible ones. Von Neumann[10] has shown that the first step still is possible in infinite dimensions. We can assume, therefore, from the outset that we are dealing with factorial representations.

In the theory of representations of finite dimensions, it is sufficient to determine only the irreducible ones, all others are equivalent to sums of these. Here, it will be necessary to determine all factorial representations. Having done that, we shall know from the above theorem of von Neumann, that all representations are equivalent to finite or infinite sums of factorial representations.

It will be one of the results of the detailed investigation that the inhomogeneous Lorentz group has no "continuous" representations, all representations can be decomposed into irreducible ones. Thus the work of Majorana and Dirac appears to be justified from this point of view a posteriori.

D. Classification of unitary representations from the point of view of infinitesimal operators

The existence of an infinitesimal operator of a continuous one parametric (cyclic, abelian) unitary group has been shown by Stone.[11] He proved that the operators of such a group can be written as $\exp(iHt)$ where H is a (bounded or unbounded) hermitean operator and t is the group parameter. However, the Lorentz group has many one parametric subgroups, and the corresponding infinitesimal operators H_1, H_2, $\cdots$ are all unbounded. For every H_i an everywhere dense set of functions φ can be found such that $H_i\varphi$ can be defined. It is not clear, however, that an everywhere dense set can be found, to all members of which every H can be applied. In fact, it is not clear that one such φ can be found.

Indeed, it may be interesting to remark that for an irreducible representation the existence of one function φ to which all infinitesimal operators can be applied, entails the existence of an everywhere dense set of such functions. This again has the consequence that one can operate with infinitesimal operators to a large extent in the usual way.

PROOF: Let $Q(t)$ be a one parametric subgroup such that $Q(t)Q(t') = Q(t + t')$. If the infinitesimal operator of all subgroups can be applied to φ, the

$$\lim_{t=0} t^{-1}(Q(t) - 1)\varphi \tag{5}$$

exists. It follows, then, that the infinitesimal operators can be applied to $R\varphi$ also where R is an arbitrary operator of the representation: Since $R^{-1}Q(t)\,R$ is also a one parametric subgroup

$$\lim_{t=0} t^{-1}(R^{-1}Q(t)R - 1)\varphi = \lim_{t=0} R^{-1}\cdot t^{-1}(Q(t) - 1)R\varphi$$

also exists and hence also (R is unitary)

$$\lim_{t=0} t^{-1}(Q(t) - 1)R\varphi.$$

Every infinitesimal operator can be applied to $R\varphi$ if they all can be applied to φ, and the same holds for sums of the kind

$$a_1R_1\varphi + a_2R_2\varphi + \cdots + a_nR_n\varphi. \tag{6}$$

These form, however, an everywhere dense set of functions if the representation is irreducible.

If the representation is not irreducible, one can consider the set N_0 of such wave functions to which every infinitesimal operator can be applied. This set is

[11] M. H. Stone, Proc. Nat. Acad. *16*, 173, 1930, Ann. of Math. *33*, 643, 1932, also J. v. Neumann, ibid, *33*, 567, 1932.

clearly linear and, according to the previous paragraph, invariant under the operations of the group (i.e. contains every $R\varphi$ if it contains φ). The same holds for the closed set N generated by N_0 and also of the set P of functions which are perpendicular to all functions of N. In fact, if φ_p is perpendicular to all φ_n of N, it is perpendicular also to all $R^{-1}\varphi_n$ and, for the unitary character of R, the $R\varphi_p$ is perpendicular to all φ_n, i.e. is also contained in the set P.

We can decompose thus, by a unitary transformation, every unitary representation into a "normal" and a "pathological" part. For the former, there is an everywhere dense set of functions, to which all infinitesimal operators can be applied. There is no single wave functions to which all infinitesimal operators of a "pathological" representation could be applied.

According to Murray and von Neumann, if the original representation was factorial, all representations into which it can be decomposed will be factorial also. Thus every representation is equivalent to a sum of factorial representations, part of which is "normal," the other part "pathological."

It will turn out again that the inhomogeneous Lorentz group has no pathological representations. Thus this assumption of Majorana and Dirac also will be justified a posteriori. Every unitary representation of the inhomogenous Lorentz group can be decomposed into normal irreducible representations. It should be stated, however, that the representations in which the unit operator corresponds to every translation have not been determined to date (cf. also section 3, end). Hence, the above statements are not proved for these representations, which are, however, more truly representations of the homogeneous Lorentz group, than of the inhomogeneous group.

While all these points may be of interest to the mathematician only, the new representation of the Lorentz group which will be described in section 7 may interest the physicist also. It describes a particle with a continuous spin.

Acknowledgement. The subject of this paper was suggested to me as early as 1928 by P. A. M. Dirac who realised even at that date the connection of representations with quantum mechanical equations. I am greatly indebted to him also for many fruitful conversations about this subject, especially during the years 1934/35, the outgrowth of which the present paper is.

I am indebted also to J. v. Neumann for his help and friendly advice.

3. Summary of Ensuing Sections

Section 4 will be devoted to the definition of the inhomogeneous Lorentz group and the theory of characteristic values and characteristic vectors of a homogeneous (ordinary) Lorentz transformation. The discussion will follow very closely the corresponding, well-known theory of the group of motions in ordinary space and the theory of characteristic values of orthogonal transformations.[12] It will contain only a straightforward generalization of the methods usually applied in those discussions.

[12] Cf. e.g. E. Wigner, l.c. Chapter III. O. Veblen and J. W. Young, *Projective Geometry*, Boston 1917. Vol. 2, especially Chapter VII.

In Section 5, it will be proved that one can determine the physically meaningless constants in the $D(L)$ in such a way that instead of (3a) the more special equation

$$D(L_1)D(L_2) = \pm D(L_1L_2) \tag{7}$$

will be valid. This means that instead of a representation up to a factor, we can consider representations up to the sign. For the case that either L_1 or L_2 is a pure translation, Dirac[13] has given a proof of (7) using infinitesimal operators. A consideration very similar to his can be carried out, however, also using only finite transformations.

For representations with a finite number of dimensions (corresponding to an only finite number of linearly independent states), (7) could be proved also if both L_1 and L_2 are homogeneous Lorentz transformations, by a straightforward application of the method of Weyl and Schreier.[14] However, the Lorentz group has no finite dimensional representation (apart from the trivial one in which the unit operation corresponds to every L). Thus the method of Weyl and Schreier cannot be applied. Its first step is to normalize the indeterminate constants in every matrix $D(L)$ in such a way that the determinant of $D(L)$ becomes 1. No determinant can be defined for general unitary operators.

The method to be employed here will be to decompose every L into a product of two involutions $L = MN$ with $M^2 = N^2 = 1$. Then $D(M)$ and $D(N)$ will be normalized so that their squares become unity and $D(L) = D(M)D(N)$ set. It will be possible, then, to prove (7) without going back to the topology of the group.

Sections 6, 7, and 8 will contain the determination of the representations. The pure translations form an invariant subgroup of the whole inhomogeneous Lorentz group and Frobenius' method[15] will be applied in Section 6 to build up the representations of the whole group out of representations of the subgroup, by means of a "little group." In Section 6, it will be shown on the basis of an as yet unpublished work[24] of J. v. Neumann that there is a characteristic (invariant) set of "momentum vectors" for every irreducible representation. The irreducible representations of the Lorentz group will be divided into four classes. The momentum vectors of the

1*st class* are time-like,
2*nd class* are null-vectors, but not all their components will be zero,
3*rd class* vanish (i.e., all their components will be zero),
4*th class* are space-like.

Only the first two cases will be considered in Section 7, although the last case

---

[13] P. A. M. Dirac, mimeographed notes of lectures delivered at Princeton University, 1934/35, page 5a.

[14] H. Weyl, Mathem. Zeits. *23*, 271; *24*, 328, 377, 789, 1925; O. Schreier, Abhandl. Mathem. Seminar Hamburg, *4*, 15, 1926; *5*, 233, 1927.

[15] G. Frobenius, Sitz. d. Kön. Preuss. Akad. p. 501, 1898, I. Schur, ibid, p. 164, 1906; F. Seitz, Ann. of Math. *37*, 17, 1936.

may be the most interesting from the mathematical point of view. I hope to return to it in another paper. I did not succeed so far in giving a complete discussion of the 3rd class. (All these restrictions appear in the previous treatments also.)

In Section 7, we shall find again all known representations of the inhomogeneous Lorentz group (i.e., all known Lorentz invariant equations) and two new sets.

Sections 5, 6, 7 will deal with the "restricted Lorentz group" only, i.e. Lorentz transformations with determinant 1 which do not reverse the direction of the time axis. In section 8, the representations of the extended Lorentz group will be considered, the transformations of which are not subject to these conditions.

4. Description of the Inhomogeneous Lorentz Group

A.

An inhomogeneous Lorentz transformation $L = (a, \Lambda)$ is the product of a translation by a real vector a

$$x_i' = x_i + a_i \qquad (i = 1, 2, 3, 4) \tag{8}$$

and a homogeneous Lorentz transformation Λ with real coefficients

$$x_i' = \sum_{k=1}^{4} \Lambda_{ik} x_k . \tag{9}$$

The translation shall be performed after the homogeneous transformation. The coefficients of the homogeneous transformation satisfy three conditions: (1) They are real and Λ leaves the indefinite quadratic form $-x_1^2 - x_2^2 - x_3^2 + x_4^2$ invariant:

$$\Lambda F \Lambda' = F \tag{10}$$

where the prime denotes the interchange of rows and columns and F is the diagonal matrix with the diagonal elements $-1, -1, -1, +1$.—(2) The determinant $| \Lambda_{ik} | = 1$ and—(3) $\Lambda_{44} > 0$.

We shall denote the Lorentz-hermitean product of two vectors x and y by

$$\{x, y\} = -x_1^* y_1 - x_2^* y_2 - x_3^* y_3 + x_4^* y_4 . \tag{11}$$

(The star denotes the conjugate imaginary.) If $\{x, x\} < 0$ the vector x is called space-like, if $\{x, x\} = 0$, it is a null vector, if $\{x, x\} > 0$, it is called time-like. A real time-like vector lies in the positive light cone if $x_4 > 0$; it lies in the negative light cone if $x_4 < 0$. Two vectors x and y are called orthogonal if $\{x, y\} = 0$.

On account of its linear character a homogeneous Lorentz transformation is completely defined if Λv is given for four linearly independent vectors $v^{(1)}$, $v^{(2)}$, $v^{(3)}$, $v^{(4)}$.

From (11) and (10) it follows that $\{v, w\} = \{\Lambda v, \Lambda w\}$ for every pair of vectors v, w. This will be satisfied for every pair if it is satisfied for all pairs $v^{(i)}$, $v^{(k)}$

of four linearly independent vectors. The reality condition is satisfied if $(\Lambda v^{(i)})^* = \Lambda(v^{(i)*})$ holds for four such vectors.

The scalar product of two vectors x and y is positive if both lie in the positive light cone or both in the negative light cone. It is negative if one lies in the positive, the other in the negative light cone. Since both x and y are time-like $|x_4|^2 > |x_1|^2 + |x_2|^2 + |x_3|^2$; $|y_4|^2 > |y_1|^2 + |y_2|^2 + |y_3|^2$. Hence, by Schwarz's inequality $|\overset{*}{x}_4 y_4| > |\overset{*}{x}_1 y_1 + \overset{*}{x}_2 y_2 + \overset{*}{x}_3 y_3|$ and the sign of the scalar product of two real time-like vectors is determined by the product of their time components.

A time-like vector is transformed by a Lorentz transformation into a time-like vector. Furthermore, on account of the condition $\Lambda_{44} > 0$, the vector $v^{(0)}$ with the components 0, 0, 0, 1 remains in the positive light cone, since the fourth component of $\Lambda v^{(0)}$ is Λ_{44}. If $v^{(1)}$ is another vector[16] in the positive light cone $\{v^{(1)}, v^{(0)}\} > 0$ and hence also $\{\Lambda v^{(1)}, \Lambda v^{(0)}\} > 0$ and $\Lambda v^{(1)}$ is in the positive light cone also. The third condition for a Lorentz transformation can be formulated also as the requirement that every vector in (or on) the positive light cone shall remain in (or, respectively, on) the positive light cone.

This formulation of the third condition shows that the third condition holds for the product of two homogeneous Lorentz transformations if it holds for both factors. The same is evident for the first two conditions.

From $\Lambda F \Lambda' = F$ one obtains by multiplying with Λ^{-1} from the left and $\Lambda'^{-1} = (\Lambda^{-1})'$ from the right $F = \Lambda^{-1}F(\Lambda^{-1})'$ so that the reciprocal of a homogeneous Lorentz transformation is again such a transformation. The homogeneous Lorentz transformations form a group, therefore.

One easily calculates that the product of two inhomogeneous Lorentz transformations (b, M) and (c, N) is again an inhomogeneous Lorentz transformation (a, Λ)

$$(b, M)(c, N) = (a, \Lambda) \tag{12}$$

where

$$\Lambda_{ik} = \sum_j M_{ij}N_{jk}; \qquad a_i = b_i + \sum_j M_{ij}c_j, \tag{12a}$$

or, somewhat shorter

$$\Lambda = MN; \qquad a = b + Mc. \tag{12b}$$

B. Theory of characteristic values and characteristic vectors of a homogeneous Lorentz transformation

Linear homogeneous transformations are most simply described by their characteristic values and vectors. Before doing this for the homogeneous Lorentz group, however, we shall need two rules about orthogonal vectors.

---

[16] Wherever a confusion between vectors and vector components appears to be possible, upper indices will be used for distinguishing different vectors and lower indices for denoting the components of a vector.

[1] *If* $\{v, w\} = 0$ *and* $\{v, v\} > 0$, *then* $\{w, w\} < 0$; *if* $\{v, w\} = 0$, $\{v, v\} = 0$, *then* w *is either space-like, or parallel to* v (*either* $\{w, w\} < 0$, *or* $w = cv$).

PROOF:

$$v_4^* w_4 = v_1^* w_1 + v_2^* w_2 + v_3^* w_3 \,. \tag{13}$$

By Schwarz's inequality, then

$$|v_4|^2 \, |w_4|^2 \leqq (|v_1|^2 + |v_2|^2 + |v_3|^2)(|w_1|^2 + |w_2|^2 + |w_3|^2). \tag{14}$$

For $|v_4|^2 > |v_1|^2 + |v_2|^2 + |v_3|^2$ it follows that $|w_4|^2 < |w_1|^2 + |w_2|^2 + |w_3|^2$. If $|v_4|^2 = |v_1|^2 + |v_2|^2 + |v_3|^2$ the second inequality still follows if the inequality sign holds in (14). The equality sign can hold only, however, if the first three components of the vectors v and w are proportional. Then, on account of (13) and both being null vectors, the fourth components are in the same ratio also.

[2] *If four vectors* $v^{(1)}$, $v^{(2)}$, $v^{(3)}$, $v^{(4)}$ *are mutually orthogonal and linearly independent, one of them is time-like, three are space-like.*

PROOF: It follows from the previous paragraph that only one of four mutually orthogonal, linearly independent vectors can be time-like or a null vector. It remains to be shown therefore only that one of them *is* time-like. Since they are linearly independent, it is possible to express by them any time-like vector

$$v^{(t)} = \sum_{k=1}^{4} \alpha_k v^{(k)}.$$

The scalar product of the left side of this equation with itself is positive and therefore

$$\left\{\sum_k \alpha_k v^{(k)}, \quad \sum_k \alpha_k v^{(k)}\right\} > 0$$

or

$$\sum_k |\alpha_k|^2 \{v^{(k)}, v^{(k)}\} > 0 \tag{15}$$

si

and one $\{v^{(k)}, v^{(k)}\}$ must be positive. Four mutually orthogonal vectors are not necessarily linearly independent, because a null vector is perpendicular to itself. The linear independence follows, however, if none of the four is a null vector.

We go over now to the characteristic values λ of Λ. These make the determinant $|\Lambda - \lambda 1|$ of the matrix $\Lambda - \lambda 1$ vanish.

[3] *If* λ *is a characteristic value,* λ^*, λ^{-1} *and* λ^{*-1} *are characteristic values also.*

PROOF: For λ^* this follows from the fact that Λ is real. Furthermore, from $|\Lambda - \lambda 1| = 0$ also $|\Lambda' - \lambda 1| = 0$ follows, and this multiplied by the determinants of ΛF and F^{-1} gives

$$|\Lambda F| \cdot |\Lambda' - \lambda 1| \cdot |F|^{-1} = |\Lambda F \Lambda' F^{-1} - \lambda \Lambda| = |1 - \lambda\Lambda| = 0,$$

so that λ^{-1} is a characteristic value also.

[4] *The characteristic vectors* v_1 *and* v_2 *belonging to two characteristic values* λ_1 *and* λ_2 *are orthogonal if* $\lambda_1^* \lambda_2 \neq 1$.

PROOF:

$$\{v_1, v_2\} = \{\Lambda v_1, \Lambda v_2\} = \{\lambda_1 v_1, \lambda_2 v_2\} = \lambda_1^* \lambda_2 \{v_1, v_2\}.$$

Thus if $\{v_1, v_2\} \neq 0$, $\lambda_1^* \lambda_2 = 1$.

[5] *If the modulus of a characteristic value* λ *is* $|\lambda| \neq 1$, *the corresponding characteristic vector* v *is a null vector and* λ *itself real and positive.*

From $\{v, v\} = \{\Lambda v, \Lambda v\} = |\lambda|^2 \{v, v\}$ the $\{v, v\} = 0$ follows immediately for $|\lambda| \neq 1$. If λ were complex, λ^* would be a characteristic value also. The characteristic vectors of λ and λ^* would be two different null vectors and, because of [4], orthogonal to each other. This is impossible on account of [1]. Thus λ is real and v a real null vector. Then, on account of the third condition for a homogeneous Lorentz transformation, λ must be positive.

[6] *The characteristic value* λ *of a characteristic vector* v *of length null is real and positive.*

If λ were not real, λ^* would be a characteristic value also. The corresponding characteristic vector v^* would be different from v, a null vector also, and perpendicular to v on account of [4]. This is impossible because of [1].

[7] *The characteristic vector* v *of a complex characteristic value* λ *(the modulus of which is* 1 *on account of* [5]) *is space-like*: $\{v, v\} < 0$.

PROOF: λ^* is a characteristic value also, the corresponding characteristic vector is v^*. Since $(\lambda^*)^*\lambda = \lambda^2 \neq 1$, $\{v^*, v\} = 0$. Since they are different, at least one is space-like. On account of $\{v, v\} = \{v^*, v^*\}$ both are space-like. If all four characteristic values were complex and the corresponding characteristic vectors linearly independent (which is true except if Λ has elementary divisors) we should have four space-like, mutually orthogonal vectors. This is impossible, on account of [2]. Hence

[8] *There is not more than one pair of conjugate complex characteristic values, if* Λ *has no elementary divisors. Similarly, under the same condition, there is not more than one pair* λ, λ^{-1} *of characteristic values whose modulus is different from* 1. Otherwise their characteristic vectors would be orthogonal, which they cannot be, being null vectors.

For homogeneous Lorentz transformations which do not have elementary divisors, the following possibilities remain:

(a) There is a pair of complex characteristic values, their modulus is 1, on account of [5]

$$\lambda_1 = \lambda_2^* = \lambda_2^{-1}; \qquad |\lambda_1| = |\lambda_2| = 1, \tag{16}$$

and also a pair of characteristic values λ_3, λ_4, the modulus of which is not 1· These must be real and positive:

$$\lambda_4 = \lambda_3^{-1}; \qquad \lambda_3 = \lambda_3^* > 0. \tag{16a}$$

The characteristic vectors of the conjugate complex characteristic values are conjugate complex, perpendicular to each other and space-like so that they can be normalized to -1

$$\begin{gathered} v_1 = v_2^*; \qquad \{v_1, v_2\} = \{v_1, v_1^*\} = 0 \\ \{v_1, v_1\} = \{v_2, v_2\} = -1 \end{gathered} \tag{17}$$

those of the real characteristic values are real null vectors, their scalar product can be normalized to 1

(17a)
$$v_3 = v_3^* \qquad v_4 = v_4^* \qquad \{v_3, v_4\} = 1$$
$$\{v_3, v_3\} = \{v_4, v_4\} = 0.$$

Finally, the former pair of characteristic vectors is perpendicular to the latter kind

(17b)
$$\{v_1, v_3\} = \{v_1, v_4\} = \{v_2, v_3\} = \{v_2, v_4\} = 0.$$

It will turn out that all the other cases in which Λ has no elementary divisor are special cases of (a).

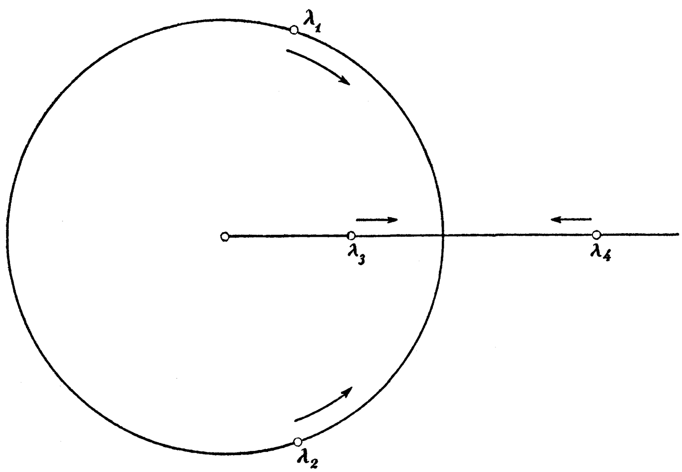

FIG. 1. Position of the characteristic values for the general case a) in the complex plane. In case b), λ_3 and λ_4 coincide and are equal 1; in case c), λ_1 and λ_2 coincide and are either $+1$ or -1. In case d) both pairs $\lambda_3 = \lambda_4 = 1$ and $\lambda_1 = \lambda_2 = \pm 1$ coincide.

(b) There is a pair of complex characteristic values λ_1, $\lambda_2 = \lambda_1^{-1} = \lambda_1^*$, $\lambda_1 \neq \lambda_1^*$, $|\lambda_1| = |\lambda_2| = 1$. No pair with $|\lambda_3| \neq 1$, however. Then on account of [8], still $\lambda_3 = \lambda_3^*$ which gives with $|\lambda_3| = 1$, $\lambda_3 = \pm 1$. Since the product $\lambda_1\lambda_2\lambda_3\lambda_4 = 1$, on account of the second condition for homogeneous Lorentz transformations, also $\lambda_4 = \lambda_3 = \pm 1$. The double characteristic value ± 1 has two linearly independent characteristic vectors v_3 and v_4 which can be assumed to be perpendicular to each other, $\{v_3, v_4\} = 0$. According to [2], one of the four characteristic vectors must be time-like and since those of λ_1 and λ_2 are space-like, the time-like one must belong to ± 1. This must be positive, therefore $\lambda_3 = \lambda_4 = 1$. Out of the time-like and space-like vectors $\{v_3, v_3\} = -1$ and $\{v_4, v_4\} = 1$, one can build two null vectors $v_4 + v_3$ and $v_4 - v_3$. Doing this, case (b) becomes the special case of (a) in which the real positive characteristic values become equal $\lambda_3 = \lambda_4^{-1} = 1$.

(c) All characteristic values are real; there is however one pair $\lambda_3 = \lambda_3^*$,

$\lambda_4 = \lambda_3^{-1}$, the modulus of which is not unity. Then $\{v_3, v_3\} = \{v_4, v_4\} = 0$ and $\lambda_3 > 0$ and one can conclude for λ_1 and λ_2, as before for λ_3 and λ_4 that $\lambda_1 = \lambda_2 = \pm 1$. This again is a special case of (a); here the two characteristic values of modulus 1 become equal.

(d) All characteristic values are real and of modulus 1. If all of them are $+1$, we have the unit matrix which clearly can be considered as a special case of (a). The other case is $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = \lambda_4 = +1$. The characteristic vectors of λ_1 and λ_2 must be space-like, on account of the third condition for a homogeneous Lorentz transformation; they can be assumed to be orthogonal and normalized to -1. This is then a special case of (b) and hence of (a) also. The cases (a), (b), (c), (d) are illustrated in Fig. 1.

The cases remain to be considered in which Λ has an elementary divisor. We set therefore

$$(18) \qquad \Lambda_e v_e = \lambda_e v_e ; \qquad \Lambda_e w_e = \lambda_e w_e + v_e .$$

It follows from [5] that either $|\lambda_e| = 1$, or $\{v_e, v_e\} = 0$. We have $\{v_e, w_e\} = \{\Lambda_e v_e, \Lambda_e w_e\} = |\lambda_e|^2\{v_e, w_e\} + \{v_e, v_e\}$. From this equation

$$(19) \qquad \{v_e, v_e\} = 0$$

follows for $|\lambda_e| = 1$, so that (19) holds in any case. It follows then from [6] that λ_e is real, positive and v_e, w_e can be assumed to be real also. The last equation now becomes $\{v_e, w_e\} = \lambda_e^2\{v_e, w_e\}$ so that either $\lambda_e = 1$ or $\{v_e, w_e\} = 0$. Finally, we have

$$\{w_e, w_e\} = \{\Lambda_e w_e, \Lambda_e w_e\} = \lambda_e^2\{w_e, w_e\} + 2\lambda_e\{w_e, v_e\} + \{v_e, v_e\}.$$

This equation now shows that

$$(19a) \qquad \{w_e, v_e\} = 0$$

even if $\lambda_e = 1$. From (19), (19a) it follows that w_e is space-like and can be normalized to

$$(19b) \qquad \{w_e, w_e\} = -1.$$

Inserting (19a) into the preceding equation we finally obtain

$$(19c) \qquad \lambda_e = 1.$$

[9] *If* Λ_e *has an elementary divisor, all its characteristic roots are* 1.

From (19c) we see that the root of the elementary divisor is 1 and this is at least a double root. If Λ had a pair of characteristic values $\lambda_1 \neq 1$, $\lambda_2 = \lambda_1^{-1}$, the corresponding characteristic vectors v_1 and v_2 would be orthogonal to v_e and therefore space-like. On account of [5], then $|\lambda_1| = |\lambda_2| = 1$ and $\{v_1, v_2\} = 0$. Furthermore, from $\{w_e, v_1\} = \{\Lambda_e w_e, \Lambda_e v_1\} = \lambda_1\{w_e, v_1\} + \lambda_1\{v_e, v_1\}$ and from $\{v_e, v_1\} = 0$ also $\{w_e, v_1\} = 0$ follows. Thus all the four vectors v_1, v_2, v_e, w_e would be mutually orthogonal. This is excluded by [2] and (19).

Two cases are conceivable now. Either the fourfold characteristic root has only one characteristic vector, or there is in addition to v_e (at least) another characteristic vector v_1. In the former case four linearly independent vectors v_e, w_e, z_e, x_e could be found such that

$$\Lambda_e v_e = v_e \qquad \Lambda_e w_e = w_e + v_e$$
$$\Lambda_e z_e = z_e + w_e \qquad \Lambda_e x_e = x_e + z_e.$$

However $\{v_e, x_e\} = \{\Lambda_e v_e, \Lambda_e x_e\} = \{v_e, x_e\} + \{v_e, z_e\}$ from which $\{v_e, z_e\} = 0$ follows. On the other hand

$$\{w_e, z_e\} = \{\Lambda_e w_e, \Lambda_e z_e\} = \{w_e, z_e\} + \{w_e, w_e\} + \{v_e, z_e\} + \{v_e, w_e\}.$$

This gives with (19a) and (19b) $\{v_e, z_e\} = 1$ so that this case must be excluded.

(e) There is thus a vector v_1 so that in addition to (18)

$$\Lambda_e v_1 = v_1 \tag{18a}$$

holds. From $\{w_e, v_1\} = \{\Lambda_e w_e, \Lambda_e v_1\} = \{w_e, v_1\} + \{v_e, v_1\}$ follows

$$\{v_e, v_1\} = 0. \tag{19d}$$

The equations (18), (18a) will remain unchanged if we add to w_e and v_1 a multiple of v_e. We can achieve in this way that the fourth components of both w_e and v_1 vanish. Furthermore, v_1 can be normalized to -1 and added to w_e also with an arbitrary coefficient, to make it orthogonal to v_1. Hence, we can assume that

$$v_{14} = w_{e4} = 0; \qquad \{v_1, v_1\} = -1; \qquad \{w_e, v_1\} = 0. \tag{19e}$$

We can finally define the null vector z_e to be orthogonal to w_e and v_1 and have a scalar product 1 with v_e

$$\{z_e, z_e\} = \{z_e, w_e\} = \{z_e, v_1\} = 0; \qquad \{z_e, v_e\} = 1. \tag{19f}$$

Then the null vectors v_e and z_e represent the momenta of two light beams in opposite directions. If we set $\Lambda_e z_e = av_e + bw_e + cz_e + dv_1$ the conditions $\{z_e, v\} = \{\Lambda_e z_e, \Lambda_e v\}$ give, if we set for v the vectors v_e, w_e, z_e, v_1 the conditions $c = 1; b = c; 2ac - b^2 - d^2 = 0; d = 0$. Hence

$$\begin{aligned} \Lambda_e v_e &= v_e \qquad & \Lambda_e w_e &= w_e + v_e \\ \Lambda_e v_1 &= v_1 \qquad & \Lambda_e z_e &= z_e + w_e + \tfrac{1}{2} v_e. \end{aligned} \tag{20}$$

A Lorentz transformation with an elementary divisor can be best characterized by the null vector v_e which is invariant under it and the space part of which forms with the two other vectors w_e and v_1 three mutually orthogonal vectors in ordinary space. The two vectors w_e and v_1 are normalized, v_1 is invariant under Λ_e while the vector v_e is added to w_e upon application of Λ_e. The result of the application of Λ_e to a vector which is linearly independent of v_e, w_e and v_1 is, as we saw, already determined by the expressions for $\Lambda_e v_e$, $\Lambda_e w_e$ and $\Lambda_e v_1$.

The $\Lambda_e(\gamma)$ which have the invariant null vector v_e and also w_e (and hence also

v_1) in common and differ only by adding to w_e different multiples γv_e of v_e, form a cyclic group with $\gamma = 0$, the unit transformation as unity:

$$\Lambda_e(\gamma)\Lambda_e(\gamma') = \Lambda_e(\gamma + \gamma').$$

The Lorentz transformation $M(\alpha)$ which leaves v_1 and w_e invariant but replaces v_e by αv_e (and z_e by $\alpha^{-1}z_e$) has the property of transforming $\Lambda_e(\gamma)$ into

$$M(\alpha)\Lambda_e(\gamma)M(\alpha)^{-1} = \Lambda_e(\alpha\gamma). \tag{+}$$

An example of $\Lambda_e(\gamma)$ and $M(\alpha)$ is

$$\Lambda_e(\gamma) = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \gamma & \gamma \\ 0 & -\gamma & 1 - \frac{1}{2}\gamma^2 & -\frac{1}{2}\gamma^2 \\ 0 & \gamma & \frac{1}{2}\gamma^2 & 1 + \frac{1}{2}\gamma^2 \end{Vmatrix};$$

$$M(\alpha) = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(\alpha + \alpha^{-1}) & \frac{1}{2}(\alpha - \alpha^{-1}) \\ 0 & 0 & \frac{1}{2}(\alpha - \alpha^{-1}) & \frac{1}{2}(\alpha + \alpha^{-1}) \end{Vmatrix}.$$

These Lorentz transformations play an important rôle in the representations with space like momentum vectors.

A behavior like (+) is impossible for finite unitary matrices because the characteristic values of $M(\alpha)^{-1}\Lambda_e(\gamma)M(\alpha)$ and $\Lambda_e(\gamma)$ are the same—those of $\Lambda_e(\gamma\alpha) = \Lambda_e(\gamma)^{\alpha}$ the α^{th} powers of those of $\Lambda_e(\gamma)$. This shows very simply that the Lorentz group has no true unitary representation in a finite number of dimensions.

C. Decomposition of a homogeneous Lorentz transformation into rotations and an acceleration in a given direction

The homogeneous Lorentz transformation is, from the point of view of the physicist, a transformation to a uniformly moving coördinate system, the origin of which coincided at $t = 0$ with the origin of the first coördinate system. One can, therefore, first perform a rotation which brings the direction of motion of the second system into a given direction—say the direction of the third axis—and impart it a velocity in this direction, which will bring it to rest. After this, the two coördinate systems can differ only in a rotation. This means that every homogeneous Lorentz transformation can be decomposed in the following way[17]

$$\Lambda = RZS \tag{21}$$

[17] Cf. e.g. L. Silberstein, *The Theory of Relativity*, London 1924, p. 142.

where R and S are pure rotations, (i.e. $R_{i4} = R_{4i} = S_{i4} = S_{4i} = 0$ for $i \neq 4$ and $R_{44} = S_{44} = 1$, also $R' = R^{-1}$, $S' = S^{-1}$) and Z is an acceleration in the direction of the third axis, i.e.

$$Z = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & a \end{Vmatrix}$$

with $a^2 - b^2 = 1$, $a > b > 0$. The decomposition (21) is clearly not unique. It will be shown, however, that Z is uniquely determined, i.e. the same in every decomposition of the form (21).

In order to prove this mathematically, we chose R so that in $R^{-1}\Lambda = I$ the first two components in the fourth column $I_{14} = I_{24} = 0$ become zero: R^{-1} shall bring the vector with the components Λ_{14}, Λ_{24}, Λ_{34} into the third axis. Then we take $I_{34} = (\Lambda_{14}^2 + \Lambda_{24}^2 + \Lambda_{34}^2)^{\frac{1}{2}}$ and $I_{44} = \Lambda_{44}$ for b and a to form Z; they satisfy the equation $I_{44}^2 - I_{34}^2 = 1$. Hence, the first three components of the fourth column of $J = Z^{-1}I = Z^{-1}R^{-1}\Lambda$ will become zero and $J_{44} = 1$, because of $J_{44}^2 - J_{14}^2 - J_{24}^2 - J_{34}^2 = 1$. Furthermore, the first three components of the fourth row of J will vanish also, on account of $J_{44}^2 - J_{41}^2 - J_{42}^2 - J_{43}^2 = 1$, i.e. $J = S = Z^{-1}R^{-1}\Lambda$ is a pure rotation. This proves the possibility of the decomposition (21).

The trace of $\Lambda\Lambda' = RZ^2R^{-1}$ is equal to the trace of Z^2, i.e. equal to $2a^2 + 2b^2 + 2 = 4a^2 = 4b^2 + 4$ which shows that the a and b of Z are uniquely determined. In particular $a = 1$, $b = 0$ and Z the unit matrix if $\Lambda\Lambda' = 1$, i.e. Λ a pure rotation.

It is easy to show now that the group space of the homogeneous Lorentz transformations is only doubly connected. If a continuous series $\Lambda(t)$ of homogeneous Lorentz transformations is given, which is unity both for $t = 0$ and for $t = 1$, we can decompose it according to (21)

$$\Lambda(t) = R(t)Z(t)S(t). \tag{21a}$$

It is also clear from the foregoing, that $R(t)$ can be assumed to be continuous in t, except for values of t, for which $\Lambda_{14} = \Lambda_{24} = \Lambda_{34} = 0$, i.e. for which Λ is a pure rotation. Similarly, $Z(t)$ will be continuous in t and this will hold even where $\Lambda(t)$ is a pure rotation. Finally, $S = Z^{-1}R^{-1}\Lambda$ will be continuous also, except where $\Lambda(t)$ is a pure rotation.

Let us consider now the series of Lorentz transformations

$$\Lambda_s(t) = R(t)Z(t)^sS(t) \tag{21b}$$

where the b of $Z(t)^s$ is s times the b of $Z(t)$. By decreasing s from 1 to 0 we continuously deform the set $\Lambda_1(t) = \Lambda(t)$ of Lorentz transformations into a set of rotations $\Lambda_0(t) = R(t)S(t)$. Both the beginning $\Lambda_0(0) = 1$ and the end $\Lambda_s(1) = 1$ of the set remain the unit matrix and the sets $\Lambda_s(t)$ remain continuous in t for all values of s. This last fact is evident for such t for which $\Lambda(t)$ is not a rotation: for such t all factors of (21b) are continuous. But it is true also for t_0 for which $\Lambda(t_0)$ is a rotation, and for which, hence $Z(t_0) = 1$ and $\Lambda_s(t_0) = \Lambda_1(t_0) = \Lambda(t_0)$. As $Z(t)$ is everywhere continuous, there will be a neighborhood of t_0 in which $Z(t)$ and hence also $Z(t)^s$ is arbitrarily close to the unit matrix. In this neighborhood $\Lambda_s(t) = \Lambda(t)$. $S(t)^{-1}Z(t)^{-1}Z(t)^s\ S(t)$ is arbitrarily close to $\Lambda(t)$; and, if the neighborhood is small enough, this is arbitrarily close to $\Lambda(t_0) = \Lambda_s(t_0)$.

Thus (21b) replaces the continuous set $\Lambda(t)$ of Lorentz transformations by a continuous set of rotations. Since these form an only doubly connected manifold, the manifold of Lorentz transformations can not be more than doubly connected. The existence of a two valued representation[18] shows that it is actually doubly and not simply connected.

We can form a new group[14] from the Lorentz group, the elements of which are the elements of the Lorentz group, together with a way $\Lambda(t)$, connecting $\Lambda(1) = \Lambda$ with the unity $\Lambda(0) = E$. However, two ways which can be continuously deformed into each other are not considered different. The product of the element "Λ with the way $\Lambda(t)$" with the element "I with the way $I(t)$" is the element ΛI with the way which goes from E along $\Lambda(t)$ to Λ and hence along $\Lambda I(t)$ to ΛI. Clearly, the Lorentz group is isomorphic with this group and two elements (corresponding to the two essentially different ways to Λ) of this group correspond to one element of the Lorentz group. It is well known,[18] that this group is holomorphic with the group of unimodular complex two dimensional transformations.

Every continuous representation of the Lorentz group "up to the sign" is a singlevalued, continuous representation of this group. The transformation which corresponds to "Λ with the way $\Lambda(t)$" is that $d(\Lambda)$ which is obtained by going over from $d(E) = d(\Lambda(0)) = 1$ continuously along $d(\Lambda(t))$ to $d(\Lambda(1)) = d(\Lambda)$.

D. The homogeneous Lorentz group is simple

It will be shown, first, that an invariant subgroup of the homogeneous Lorentz group contains a rotation (i.e. a transformation which leaves x_4 invariant).— We can write an arbitrary element of the invariant subgroup in the form RZS of (21). From its presence in the invariant subgroup follows that of $S\cdot RZS\cdot S^{-1} = SRZ = TZ$. If X_π is the rotation by π about the first axis, $X_\pi Z X_\pi = Z^{-1}$ and $X_\pi TZX_\pi^{-1} = X_\pi TX_\pi X_\pi ZX_\pi = X_\pi TX_\pi Z^{-1}$ is contained in the invariant subgroup also and thus the transform of this with Z, i.e. $Z^{-1}X_\pi TX_\pi$ also. The product of this with TZ is $TX_\pi TX_\pi$ which leaves x_4 invariant. If $TX_\pi TX_\pi = 1$ we can take $TY_\pi TY_\pi$. If this is the unity also, $TX_\pi TX_\pi = TY_\pi TY_\pi$ and T commutes with $X_\pi Y_\pi$, i.e. is a rotation about the third axis. In this case the

[18] Cf. H. Weyl, *Gruppentheorie und Quantenmechanik,* 1st. ed. Leipzig 1928, pages 110–114, 2nd ed. Leipzig 1931, pages 130–133. It may be interesting to remark that essentially the same isomorphism has been recognized already by L. Silberstein, l.c. pages 148–157.

space like (complex) characteristic vectors of TZ lie in the plane of the first two coördinate axes. Transforming TZ by an acceleration in the direction of the first coördinate axis we obtain a new element of the invariant subgroup for which the space like characteristic vector will have a not vanishing fourth component. Taking this for RZS we can transform it with S again to obtain a new $SRZ = TZ$. However, since S leaves x_4 invariant, the fourth component of the space like characteristic vectors of this TZ will not vanish and we can obtain from it by the procedure just described a rotation which must be contained in the invariant subgroup.

It remains to be shown that an invariant subgroup which contains a rotation, contains the whole homogeneous Lorentz group. Since the three-dimensional rotation group is simple, all rotations must be contained in the invariant subgroup. Thus the rotation by π around the first axis X_π and also its transform with Z and also

$$ZX_\pi Z^{-1} \cdot X_\pi = Z \cdot X_\pi Z^{-1} X_\pi = Z^2$$

is contained in the invariant subgroup. However, the general acceleration in the direction of the third axis can be written in this form. As all rotations are contained in the invariant subgroup also, (21) shows that this holds for all elements of the homogeneous Lorentz group.

It follows from this that the homogeneous Lorentz group has apart from the representation with unit matrices only true representations. It follows then from the remark at the end of part B, that these have all infinite dimensions. This holds even for the two-valued representations to which we shall be led in Section 5 equ. (52D), as the group elements to which the positive or negative unit matrix corresponds must form an invariant subgroup also, and because the argument at the end of part B holds for two-valued representations also. One easily sees furthermore from the equations (52B), (52C) that it holds for the inhomogeneous Lorentz group equally well.

5. Reduction of Representations Up to a Factor to Two-Valued Representations

The reduction will be effected by giving each unitary transformation, which is defined by the physical content of the theory and the consideration of reference only up to a factor of modulus unity, a "phase," which will leave only the sign of the representation operators undetermined. The unitary operator corresponding to the translation a will be denoted by $T(a)$, that to the homogeneous Lorentz transformation Λ by $d(\Lambda)$. To the general inhomogeneous Lorentz transformation then $D(a, \Lambda) = T(a)d(\Lambda)$ will correspond. Instead of the relations (12), we shall use the following ones.

(22B) $$T(a)T(b) = \omega(a, b)T(a + b)$$

(22C) $$d(\Lambda)T(a) = \omega(\Lambda, a)T(\Lambda a)d(\Lambda)$$

(22D) $$d(\Lambda)d(I) = \omega(\Lambda, I)d(\Lambda I).$$

The ω are numbers of modulus 1. They enter because the multiplication rules (12) hold for the representatives only up to a factor. Otherwise, the relations (22) are consequences of (12) and can in their return replace (12). We shall replace the $T(a)$, $d(\Lambda)$ by $\Omega(a)T(a)$ and $\Omega(\Lambda)d(\Lambda)$ respectively, for which equations similar to (22) hold, however with

$$\omega(a, b) = 1; \quad \omega(\Lambda, a) = 1; \quad \omega(\Lambda, I) = \pm 1. \tag{22'}$$

A.

It is necessary, first, to show that the undetermined factors in the representation $D(L)$ can be assumed in such a way that the $\omega(a, b)$, $\omega(\Lambda, a)$, $\omega(\Lambda, I)$ become —apart from regions of lower dimensionality—continuous functions of their arguments. This is a consequence of the continuous character of the representation and shall be discussed first.

(a) From the point of view of the physicist, the natural definition of the continuity of a representation up to a factor is as follows. The neighborhood δ of a Lorentz transformation $L_0 = (b, I)$ shall contain all the transformations $L = (a, \Lambda)$ for which $| a_\kappa - b_\kappa | < \delta$ and $| \Lambda_{ik} - I_{ik} | < \delta$. The representation up to a factor $D(L)$ is continuous if there is to every positive number ϵ, every normalized wave function φ and every Lorentz transformation L_0 such a neighborhood δ of L_0 that for every L of this neighborhood one can find an Ω of modulus 1 (the Ω depending on L and φ) such that $(u_\varphi, u_\varphi) < \epsilon$ where

$$u_\varphi = (D(L_0) - \Omega D(L))\varphi. \tag{23}$$

Let us now take a point L_0 in the group space and find a normalized wave function φ for which $| (\varphi, D(L_0)\varphi) | > 1/6$. There always exists a φ with this property, if $|(\varphi, D(L_0)\varphi)| < 1/6$ then $\psi = \alpha\varphi + \beta D(L_0)\varphi$ with suitably chosen α and β will be normalized and $| (\psi, D(L_0)\psi) | > 1/6$. We consider then such a neighborhood $\mathfrak{N}$ of L_0 for all L of which $| (\varphi, D(L)\varphi) | > 1/12$. It is well known[19] that the whole group space can be covered with such neighborhoods. We want to show now that the $D(L)\varphi$ can be multiplied with such phase factors (depending on L) of modulus unity that it becomes strongly continuous in the region $\mathfrak{N}$.

We shall chose that phase factor so that $(\varphi, D(L)\varphi)$ becomes real and positive. Denoting then

$$(D(L_1) - D(L))\varphi = U_\varphi, \tag{23'}$$

the (U_φ, U_φ) can be made arbitrarily small by letting L approach sufficiently near to L_1, if L_1 is in $\mathfrak{N}$. Indeed, on account of the continuity, as defined above, there is an $\Omega = e^{i\kappa}$ such that $(u, u) < \epsilon$ if L is sufficiently near to L_1 where

$$u = (D(L_1) - e^{i\kappa}D(L))\varphi.$$

[19] This condition is the "separability" of the group. Cf. e.g. A. Haar, Ann. of Math., 34, 147, 1933.

Taking the absolute value of the scalar product of u with φ one obtains

$$|(\varphi, D(L_1)\varphi) - \cos\kappa(\varphi, D(L)\varphi) - i\sin\kappa(\varphi, D(L)\varphi)| = |(\varphi, u)| \leqq \sqrt{\epsilon},$$

because of Schwartz's inequality. If only $\sqrt{\epsilon} < 1/12$, the κ must be smaller than $\pi/2$ because the absolute value is certainly greater than the real part, and both $(\varphi, D(L_1)\varphi)$ and $(\varphi, D(L)\varphi)$ are real and greater than $1/12$.

As the absolute value is also greater than the imaginary part, we

$$\sin\kappa < 12\sqrt{\epsilon}.$$

On the other hand,

$$U_\varphi = u + (e^{i\kappa} - 1)D(L)\varphi,$$

and thus

$$(U_\varphi, U_\varphi)^{\frac{1}{2}} \leqq (u, u)^{\frac{1}{2}} + |e^{i\kappa} - 1| \leqq \sqrt{\epsilon} + 2\sin\kappa/2$$

$$(U_\varphi, U_\varphi) \leqq 625\,\epsilon.$$

(b) It shall be shown next that if $D(L)\varphi$ is strongly continuous in a region and $D(L)$ is continuous in the sense defined at the beginning of this section, then $D(L)\psi$ with an arbitrary ψ is (strongly) continuous in that region also. We shall see, hence, that the $D(L)$, with any normalization which makes a $D(L)\varphi$ strongly continuous, is continuous in the ordinary sense: There is to every L_1, ϵ and *every* ψ a δ so that $(U_\psi, U_\psi) < \epsilon$ where

$$U_\psi = (D(L_1) - D(L))\psi$$

if L is in the neighborhood δ of L_1.

It is sufficient to show the continuity of $D(L)\psi$ where ψ is orthogonal to φ. Indeed, every ψ' can be decomposed into two terms, $\psi' = \alpha\varphi + \beta\psi$ the one of which is parallel, the other perpendicular to φ. Since $D(L)\varphi$ is continuous, according to supposition, $D(L)\psi' = \alpha D(L)\varphi + \beta D(L)\psi$ will be continuous also if $D(L)\psi$ is continuous.

The continuity of the representation up to a factor requires that it is possible to achieve that $(u_\psi, u_\psi) < \epsilon$ and $(u_{\psi+\varphi}, u_{\psi+\varphi}) < \epsilon$ where

$$u_\psi = (D(L_1) - \Omega_\psi D(L))\psi, \tag{23a}$$

$$u_{\psi+\varphi} = (D(L_1) - \Omega_{\psi+\varphi} D(L))(\psi + \varphi), \tag{23b}$$

with suitably chosen Ω's. According to the foregoing, it also is possible to choose L and L_1 so close that $(U_\varphi, U_\varphi) < \epsilon$.

Subtracting (23′) and (23a) from (23b) and applying $D(L)^{-1}$ on both sides gives

$$(\Omega_\psi - \Omega_{\psi+\varphi})\psi + (1 - \Omega_{\psi+\varphi})\varphi = D(L)^{-1}(u_{\psi+\varphi} - u_\psi - U_\varphi)$$

The scalar product of the right side with itself is less than 9ϵ. Hence both $|\Omega_\psi - \Omega_{\psi+\varphi}| < 3\epsilon^{\frac{1}{2}}$ and $|1 - \Omega_{\psi+\varphi}| < 3\epsilon^{\frac{1}{2}}$ or $|1 - \Omega_\psi| < 6\epsilon^{\frac{1}{2}}$. Because of $U_\psi = u_\psi - (1 - \Omega_\psi)D(L)\psi$, the $(U_\psi, U_\psi)^{\frac{1}{2}} < (u_\psi, u_\psi)^{\frac{1}{2}} + |1 - \Omega_\psi|$ and thus $(U_\psi, U_\psi) < 49\epsilon$.

This completes the proof of the theorem stated under (b). It also shows that not only the continuity of $D(L)\varphi$ has been achieved in the neighborhood of L_0 by the normalization used in (a) but also that of $D(L)\psi$ with every ψ, i.e., the continuity of $D(L)$.

It is clear also that every finite part of the group space can be covered by a finite number of neighborhoods in which $D(L)$ can be made continuous. It is easy to see that the ω of (22) will be also continuous in these neighborhoods so that is is possible to make them continuous, apart from regions of lower dimensionality than their variables have. In the following only the fact will be used that they can be made continuous in the neighborhood of any a, b and Λ.

B.

(a) We want to show next that all $T(a)$ commute. From (22B) we have

$$T(a)T(b)T(a)^{-1} = c(a, b)T(b) \tag{24}$$

where $c(a, b) = \omega(a, b)/\omega(b, a)$ and hence

$$c(a, b) = c(b, a)^{-1}. \tag{24a}$$

Transforming (24) with $T(a')$ one obtains

$$T(a')T(a)T(b)T(a)^{-1}T(a')^{-1} = c(a, b)T(a')T(b)T(a')^{-1}$$

or $$\omega(a', a)T(a' + a)T(b)\omega(a', a)^{-1}T(a' + a)^{-1} = c(a, b)c(a', b)T(b)$$

or

$$c(a, b)c(a', b) = c(a + a', b). \tag{25}$$

It follows[20] from (25) and the partial continuity of $c(a, b)$ that

$$c(a, b) = \exp\left(2\pi i \sum_{\kappa=1}^{4} a_\kappa f_\kappa(b)\right) \tag{26}$$

and, since this is equal to $c(b, a)^{-1} = \exp(-2\pi i \sum b_\kappa f_\kappa(a))$

$$\sum_{\kappa=1}^{4} (a_\kappa f_\kappa(b) + b_\kappa f_\kappa(a)) = n(a, b) \tag{27}$$

where $n(a, b)$ is an integer. Setting in (27) for b the vector $e^{(\lambda)}$ the λ component of which is 1, all the others zero and for $f_\kappa(e^{(\lambda)}) = -f_{\kappa\lambda}$

$$f_\lambda(a) = n(a, e^{(\lambda)}) + \sum_\kappa a_\kappa f_{\kappa\lambda},$$

and putting this back into (27) we obtain

$$\sum_{\kappa,\lambda=1}^{4} f_{\kappa\lambda}(a_\lambda b_\kappa + b_\lambda a_\kappa) + \sum_{\kappa=1}^{4} a_\kappa n(b, e^{(\kappa)}) + b_\kappa n(a, e^{(\kappa)}) = n(a, b). \tag{28}$$

---

[20] G. Hamel, Math. Ann. 60, 460, 1905, quoted from H. Hahn, *Theorie der reellen Funktionen.* Berlin 1921, pages 581–583.

Assuming for the components of a and b such values which are transcendental both with respect to each other and the $f_{\kappa\lambda}$ (which are fixed numbers), one sees that (28) cannot hold except if the coefficient of every one vanishes

$$f_{\kappa\lambda} + f_{\lambda\kappa} = 0; \qquad n(b, e^{(\kappa)}) = 0, \tag{29}$$

so that (26) becomes

$$c(a, b) = \exp\left(2\pi i \sum_{\kappa,\lambda=1}^{4} f_{\kappa\lambda} a_\lambda b_\kappa\right). \tag{30}$$

It is necessary now to consider the existence of an operator $d(\Lambda)$ satisfying (22C). Transforming this equation with the similar equation containing b instead of a

$$d(\Lambda)T(b)d(\Lambda)^{-1}d(\Lambda)T(a)d(\Lambda)^{-1}d(\Lambda)T(b)^{-1}d(\Lambda)^{-1}$$
$$= \omega(\Lambda, b)T(\Lambda b)\omega(\Lambda, a)T(\Lambda a)\omega(\Lambda, b)^{-1}T(\Lambda b)^{-1} = \omega(\Lambda, a)c(\Lambda b, \Lambda a)T(\Lambda a),$$

while the first line is clearly $d(\Lambda)c(b, a)T(a)d(\Lambda)^{-1} = \omega(\Lambda, a)c(b, a)T(\Lambda a)$ whence

$$c(b, a) = c(\Lambda b, \Lambda a) \tag{31}$$

holds for every Lorentz transformation Λ. Combined with (30) this gives

$$\sum_{\kappa\lambda}\left(f_{\kappa\lambda} a_\kappa b_\lambda - \sum_{\nu\mu} f_{\nu\mu}\Lambda_{\nu\kappa}\Lambda_{\mu\lambda} a_\kappa b_\lambda\right) = n'(a, b),$$

where $n'(a, b)$ is again an integer. As this equation holds for every a, b

$$f_{\kappa\lambda} = \sum_{\nu\mu} f_{\nu\mu}\Lambda_{\nu\kappa}\Lambda_{\mu\lambda}; \qquad f = \Lambda' f \Lambda$$

must hold also, for every Lorentz transformation. However, the only form invariant under all Lorentz transformations are multiples of the F of (10). Actually, because of (29), f must vanish and $c(a, b) = 1$, all the operators corresponding to translations commute

$$T(a)T(b) = T(b)T(a). \tag{32}$$

It is well to remember that it was necessary for obtaining this result to use the existence of $d(\Lambda)$ satisfying (22C).

(b) Equation (32) is clearly independent of the normalization of the $T(a)$. If we could fix the translation operators in four linearly independent directions $e^{(1)}, e^{(2)}, e^{(3)}, e^{(4)}$ so that for each of these directions

$$T(ae^{(\kappa)})T(be^{(\kappa)}) = T((a + b)e^{(\kappa)}) \tag{33}$$

be valid for every pair of numbers a, b, then the normalization

$$T(a_1e^{(1)} + a_2e^{(2)} + a_3e^{(3)} + a_4e^{(4)}) = T(a_1e^{(1)})T(a_2e^{(2)})T(a_3e^{(3)})T(a_4e^{(4)}) \tag{33a}$$

and (32) would ensure the general validity of

$$T(a)T(b) = T(a + b). \tag{34}$$

As the four linearly independent directions $e^{(1)}, \cdots, e^{(4)}$ we shall take four null vectors. If e is a null vector, there is, according to section 3, a homogeneous Lorentz transformation[21] Λ_e such that $\Lambda_e e = 2e$.

We normalize $T(e)$ so that

$$d(\Lambda_e)T(e)d(\Lambda_e)^{-1} = T(e)^2. \tag{35}$$

This is clearly independent of the normalization of $d(\Lambda_e)$. We further normalize for all (positive and negative) integers n

$$d(\Lambda_e)^n T(e) d(\Lambda_e)^{-n} = T(2^n e). \tag{35a}$$

It follows from this equation also that

$$T(2^n e)^2 = d(\Lambda_e)^n T(e)^2 d(\Lambda_e)^{-n} = d(\Lambda_e)^n d(\Lambda_e) T(e) d(\Lambda_e)^{-1} d(\Lambda_e)^{-n} = T(2^{n+1} e). \tag{36}$$

This allows us to normalize for every positive integer k

$$T(k \cdot 2^{-n} e) = T(2^{-n} e)^k \tag{35b}$$

in such a way that the normalization remains the same if we replace k by $2^m k$ and n by $n + m$. This ensures, together with (36), the validity of

$$\begin{aligned} T(\nu e)T(\mu e) &= T((\nu + \mu)e) \\ d(\Lambda_e)T(\nu e)d(\Lambda_e)^{-1} &= T(2\nu e) \end{aligned} \tag{36a}$$

for all dyadic fractions ν and μ.

It must be shown that if $\nu_1, \nu_2, \nu_3, \cdots$ is a sequence of dyadic fractions, converging to 0, $\lim T(\nu_i e) = 1$. From $T(a) \cdot T(0) = \omega(a, 0)T(a)$ it follows that $T(0)$ is a constant. According to the theorem of part (A)(b), the $T(\nu e)$, if multiplied by proper constants Ω_ν will converge to 1, i.e., by choosing an arbitrary φ, it is possible to make both $(1 - \Omega_\nu T(\nu e))\varphi = u$ and $(1 - \Omega_\nu T(\nu e)) \cdot d(\Lambda_e)^{-1}\varphi = u'$ arbitrarily small, by making ν small. Applying $d(\Lambda_e)$ to the second expression, one obtains, for (36a), that $(1 - \Omega_\nu T(2\nu e))\varphi = d(\Lambda_e)u'$ is also small. On the other hand, applying $T(\nu e)$ to the first expression one sees that $(T(\nu e) - \Omega_\nu T(2\nu e))\varphi = T(\nu e)u$ approaches zero also. Hence, the difference of these two quantities $(1 - T(\nu e))\varphi$ goes to zero, i.e. $T(\nu_i e)\varphi$ converges to φ if $\nu_1, \nu_2, \nu_3, \cdots$ is a sequence of dyadic fractions approaching 0.

Now $\nu_1, \nu_2, \nu_3, \cdots$ be a sequence of dyadic fractions converging to an arbitrary number a. It will be shown then that $T(\nu_i e)$ converges to a multiple of $T(ae)$ and this multiple of $T(ae)$ will be the normalized $T(ae)$. Again, it follows from the continuity that there are such Ω_i that $\Omega_i T(\nu_i e)\varphi$ converges to $T(ae)\varphi$. The $\Omega_j^{-1} T(\nu_j e)^{-1} \Omega_i T(\nu_i e)\varphi$ will converge to φ, therefore, as both i and j tend to infinity. However, according to the previous paragraph, $T((\nu_i - \nu_j)e)\varphi$ tends to φ and thus $\Omega_j^{-1}\Omega_i$ tends to 1. It follows that Ω_i^{-1} converges to a definite number Ω. Hence $\Omega_i^{-1} \cdot \Omega_i T(\nu_i e)\varphi$ converges to $\Omega T(ae)\varphi$ which will be denoted, henceforth, by $T(ae)$. For the $T(ae)$, normalized in this way, (33) will hold,

---

[21] The index e denotes here the vector e for which $\Lambda_e e = 2e$; this Λ_e has no elementary divisor.

since if $\mu_1, \mu_2, \mu_3, \cdots$ are dyadic fractions converging to b, we obtain, with the help (36a)

$$T(ae)T(be)\varphi = \lim_{i,j=\infty} T((\nu_i + \mu_j)e)\varphi = T((a+b)e)\varphi.$$

This argument not only shows that it is possible to normalize the $T(ae^{(k)})$ and hence by (33a) the $T(a)$ so that (34) holds for them but, in addition to this, that these $T(a)$ will be continuous in the ordinary sense.

C.

It is clear that (34) will remain valid if one replaces $T(a)$ by $\exp(2\pi i\{a, c\})T(a)$ where c is an arbitrary vector. This remaining freedom in the normalization of $T(a)$ will be used to eliminate the $\omega(\Lambda, a)$ from (22C).

Transforming (22C) $d(\Lambda)T(a)d(\Lambda)^{-1} = \omega(\Lambda, a)T(\Lambda a)$ with $d(M)$ one obtains on the left side $\omega(M, \Lambda)d(M\Lambda)T(a)\omega(M, \Lambda)^{-1} d(M\Lambda)^{-1} = \omega(M\Lambda, a)T(M\Lambda a)$ while the right side becomes $\omega(\Lambda, a)\omega(M, \Lambda a)T(M\Lambda a)$. Hence

$$\omega(M\Lambda, a) = \omega(M, \Lambda a)\omega(\Lambda, a). \tag{37}$$

On the other hand, the product of two equations (22C) with the same Λ but with a and b respectively, instead of a yields with the help of (34)

$$\omega(\Lambda, a)\omega(\Lambda, b) = \omega(\Lambda(a + b)).$$

Hence

$$\omega(\Lambda, a) = \exp(2\pi i\{a, f(\Lambda)\}),$$

where $f(\Lambda)$ is a vector which can depend on Λ. Inserting this back into (37) one obtains

$$\{a, f(M\Lambda)\} = \{\Lambda a, f(M)\} + \{a, f(\Lambda)\} + n,$$

$$\{a, f(M\Lambda) - \Lambda^{-1}f(M) - f(\Lambda)\} = n,$$

where n is an integer which must vanish since it is a linear function of a. Hence

$$f(M\Lambda) = \Lambda^{-1}f(M) + f(\Lambda). \tag{38}$$

If we can show that the most general solution of the equation is

$$f(\Lambda) = (\Lambda^{-1} - 1)v_0, \tag{39}$$

where v_0 is a vector independent of Λ, the $\omega(\Lambda, a)$ will become $\omega(\Lambda, a) = \exp(2\pi i\{(\Lambda - 1)a, v_0\})$. Then $\omega(\Lambda, a)$ in (22C) will disappear if we replace $T(a)$ by $\exp(2\pi i\{a, v_0\})T(a)$.

The proof that (39) is a consequence of (38) is somewhat laborious. One can first consider the following homogeneous Lorentz transformations

$$X(\alpha_1, \gamma_1) = \begin{Vmatrix} C_1 & 0 & 0 & S_1 \\ 0 & c_1 & s_1 & 0 \\ 0 & -s_1 & c_1 & 0 \\ S_1 & 0 & 0 & C_1 \end{Vmatrix}; \qquad Y(\alpha_2, \gamma_2) = \begin{Vmatrix} c_2 & 0 & -s_2 & 0 \\ 0 & C_2 & 0 & S_2 \\ s_2 & 0 & c_2 & 0 \\ 0 & S_2 & 0 & C_2 \end{Vmatrix}$$

(40)

$$Z(\alpha_3, \gamma_3) = \begin{Vmatrix} c_3 & s_3 & 0 & 0 \\ -s_3 & c_3 & 0 & 0 \\ 0 & 0 & C_3 & S_3 \\ 0 & 0 & S_3 & C_3 \end{Vmatrix}$$

where $c_i = \cos \alpha_i$; $s_i = \sin \alpha_i$; $C_i = \mathrm{Ch}\gamma_i$; $S_i = \mathrm{Sh}\gamma_i$. All the $X(\alpha, \gamma)$ commute. Let us choose, therefore, two angles α_1, γ_1 for which $1 - X(\alpha_1, \gamma_1)^{-1}$ has a reciprocal. It follows then from (38)

$$X(\alpha, \gamma)^{-1}f(X(\alpha_1, \gamma_1)) + f(X(\alpha, \gamma)) = X(\alpha_1, \gamma_1)^{-1}f(X(\alpha, \gamma)) + f(X(\alpha_1, \gamma_1))$$

(41) or $$f(X(\alpha, \gamma)) = [1 - X(\alpha_1, \gamma_1)^{-1}]^{-1}[1 - X(\alpha, \gamma)^{-1}]f(X(\alpha_1, \gamma_1))$$
$$f(X(\alpha, \gamma)) = (1 - X(\alpha, \gamma)^{-1})v_X,$$

where v_X is independent of α, γ. Similar equations hold for the $f(Y(\alpha, \gamma))$ and $f(Z(\alpha, \gamma))$. Let us denote now $X(\pi, 0) = X$; $Y(\pi, 0) = Y$; $Z(\pi, 0) = Z$. These anticommute in the following sense with the transformations (40):

(42) $$YX(\alpha, \gamma)Y = ZX(\alpha, \gamma)Z = X(\alpha, \gamma)^{-1}.$$

From (38) one easily calculates

$$f(YX(\alpha, \gamma)Y) = (YX(\alpha, \gamma)^{-1} + 1)f(Y) + Yf(X(\alpha, \gamma)),$$

or, because of (41) and (42), after some trivial transformations

(43) $$(1 - X(\alpha, \gamma))(1 - Y)(v_X - v_Y) = 0.$$

As α, γ can be taken arbitrarily, the first factor can be dropped. This leaves $(1 - Y)(v_X - v_Y) = 0$, or that the first and third components of v_X and v_Y are equal. One similarly concludes, however, that $(1 - X)(v_Y - v_X) = 0$ and thus that the first three components of v_X, v_Y and also of v_Z are equal.

For $\gamma_1 = \gamma_2 = \gamma_3 = 0$ the transformations (40) are the generators of all rotations, i.e. all Lorentz transformations R not affecting the fourth coördinate. As the 4–4 matrix element of these transformations is 1, the expression $(1 - R^{-1})v$ is independent of the fourth component of v and $(1 - R^{-1})v_X = (1 - R^{-1})v_Y = (1 - R^{-1})v_Z$. It follows from (38) that if $f(R) = (1 - R^{-1})v_X$ and $f(S) = (1 - S^{-1})v_X$, then $f(SR) = (1 - R^{-1}S^{-1})v_X$. Thus $f(R) = (1 - R^{-1})v_X$ is valid with the same v_X for all rotations.

Now

$$f(X(\alpha, \gamma)R) = R^{-1}(1 - X(\alpha, \gamma)^{-1})v_X + (1 - R^{-1})v_X = (1 - (X(\alpha, \gamma)R)^{-1})v_X.$$

One easily concludes from (38) that the $f(E)$ corresponding to the unit operation vanishes and $f(\Lambda^{-1}) = -\Lambda f(\Lambda)$. Hence $f(R^{-1}X(\alpha, \gamma)^{-1}) = (1 - X(\alpha, \gamma)R)v_X$; and one concludes further that for all Lorentz transformations $\Lambda = RX(\alpha, \gamma)S$, (39) holds with $v_0 = -v_X$ if R and S are rotations. However, every homogeneous Lorentz transformation can be brought into this form (Section 4C). This completes the proof of (39) and thus of $\omega(\Lambda, a) = 1$.

D.

The quantities $\omega(a, b)$ and $\omega(\Lambda, a)$ for which it has just been shown that they can be assumed to be 1, are independent from the normalization of $d(\Lambda)$. We can affix therefore an arbitrary factor of modulus 1 to all the $d(\Lambda)$, without interfering with the normalizations so far accomplished. In consequence hereof, the ensuing discussion will be simply a discussion of the normalization

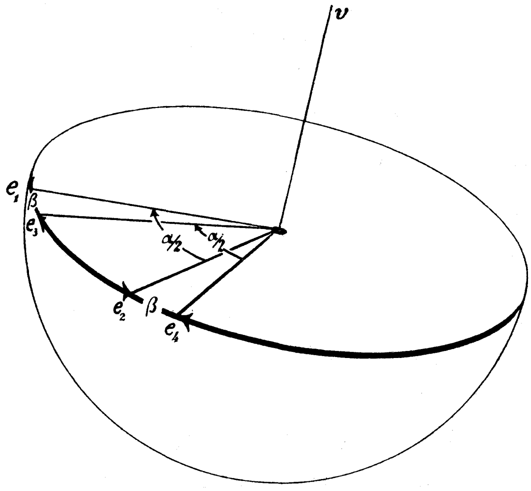

Fig. 2

of the operators for the homogeneous Lorentz group and the result to be obtained will be valid for that group also.

Partly because the representations up to a factor of the three dimensional rotation group may be interesting in themselves, but more particularly because the procedure to be followed for the Lorentz group can be especially simply demonstrated for this group, the three dimensional rotation group shall be taken up first.

It is well known that the normalization cannot be carried so far that $\omega(\Lambda, I) = 1$ in (22D) and there are well known representations for which $\omega(\Lambda, I) = \pm 1$. We shall allow this ambiguity therefore from the outset.

One can observe, first, that the operator corresponding to the unity of the group is a constant. This follows simply from $d(\Lambda)d(E) = \omega(\Lambda, E)d(\Lambda)$. The square of an operator corresponding to an involution is a constant, therefore.

The operator corresponding to the rotation about the axis e by the angle π, normalized so that its square be actually 1, will be denoted by $\tilde{e}$; $\tilde{e}^2 = 1$. The $\tilde{e}$ are—apart from the sign—uniquely defined.

A rotation R about v by the angle α is the product of two rotations by π about e_1 and e_2 where e_1 and e_2 are perpendicular to v and e_2 arises from e_1 by rotation about v with $\alpha/2$. Choosing for every v an arbitrary e_1 perpendicular to v, we can normalize, therefore

$$d(R) = \pm\tilde{e}_1\tilde{e}_2\,. \tag{44}$$

Now $d(R)$ commutes with every $d(S)$ if S is also a rotation about v. This is proved in equations (24)–(30). The f_{11} in (30) must vanish on account of (29).

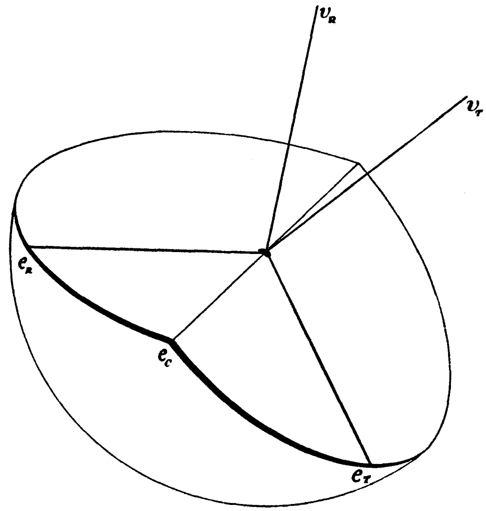

FIG. 3

(Also, both R and S can be arbitrarily accurately represented as powers of a very small rotation about v). Hence, transforming (44) by $d(S)$ one obtains

$$d(R) = \pm d(S)\tilde{e}_1 d(S)^{-1}\cdot d(S)\tilde{e}_2 d(S)^{-1}. \tag{44a}$$

Now $d(S)\tilde{e}_1 d(S)^{-1}$ corresponds to a rotation by π about an axis, perpendicular to v and enclosing an angle β with e_1, where β is the angle of rotation of S. Since the square of $d(S)\tilde{e}_1 d(S)^{-1}$ is also 1, (44a) is simply another way of writing $d(R) = \tilde{e}_3\tilde{e}_4$ as a product of two $\tilde{e}$ and we see that the normalization (44) is independent of the choice of the axis e_1 (Cf. Fig. 2).

For computing $d(R)d(T)$ we can draw the planes perpendicular to the axes of rotation of R and T and use for $d(R) = \tilde{e}_R\tilde{e}_C$ such a development that the axis e_C of the second involution coincide with the intersection line of the above-mentioned planes, while for $d(T) = \tilde{e}_C\tilde{e}_T$ we choose the first involution to be a rotation about this intersection line (Fig. 3). Then, the product

$$d(R)d(T) = \pm\tilde{e}_R\tilde{e}_C\tilde{e}_C\tilde{e}_T = \pm\tilde{e}_R\tilde{e}_T \tag{45}$$

will automatically have the normalization corresponding to (44). This shows that the operators normalized in (44) give a representation up to the sign.

For the Lorentz group, the proof can be performed along the same line, only the underlying geometrical facts are less obvious. Let Λ be a Lorentz transformation without elementary divisors with the characteristic values $e^{2i\gamma}$, $e^{-2i\gamma}$, $e^{2\chi}$, $e^{-2\chi}$ and the characteristic vectors v_1, $v_2 = v_1^*$, v_3, v_4, as described in section 4B.

We want to make $\Lambda = MN$ with $M^2 = N^2 = 1$. For $\Lambda N = M$, we have $\Lambda N \Lambda N = 1$ and thus $\Lambda N \Lambda = N$. Setting $Nv_i = \sum_k \alpha_{ik} v_k$, we obtain $\Lambda N \Lambda v_i = \sum \lambda_k \alpha_{ik} \lambda_i v_i = \sum \alpha_{ik} v_k$. Because of the linear independence of the v_k this amounts to $\lambda_i \lambda_k \alpha_{ik} = \alpha_{ik}$: all α_{ik} are zero, except those for which $\lambda_i \lambda_k = 1$. As in none of the cases (a), (b), (c), (d) of section 4B is λ_1 or λ_2 reciprocal to one of the last two λ, the vectors v_1 and v_2 will be transformed by N into a linear combination of v_1 and v_2 again, and the same holds for v_3 and v_4. This means that N can be considered as the product of two transformations $N = N_s N_t$, the first in the $v_1 v_2$ plane, the second in the $v_3 v_4$ plane. (Instead of $v_1 v_2$ plane one really should say $v_1 + v_2$, $iv_1 - iv_2$ plane, as v_1 and v_2 are complex themselves. This will be meant always by $v_1 v_2$ plane, etc.). The same holds for M also.

Both N_s and N_t must satisfy the first and third condition for Lorentz transformations (cf. 4A) and both determinants must be either 1, or -1. Furthermore, the square of both of them must be unity.

If both determinants were $+1$, the N_t had to be unity itself, while N_s could be the unity or a rotation by π in the $v_1 v_2$ plane. Thus v_1, v_2, v_3, v_4 would be characteristic vectors of N itself.

If both determinants are -1 (this will turn out to be the case), N_s is a reflection on a line in the $v_1 v_2$ plane and N_t a reflection in the $v_3 v_4$ plane, interchanging v_3 and v_4. In this case v_1, v_2, v_3, v_4 would not all be characteristic vectors of N.

If v_1, v_2, v_3, v_4 are characteristic vectors of N, they are characteristic vectors of $M = \Lambda N$ also. Then both M and N would be either unity, or a rotation by π in the $v_1 v_2$ plane. If both of them were rotations in the $v_1 v_2$ plane, their product Λ would be the unity which we want to exclude for the present. We can exclude the remaining cases in which the determinants of N_s and N_t are $+1$ by further stipulating that neither M nor N shall be the unity in the decomposition $\Lambda = MN$.

Hence N is the product of a reflection in the $v_1 v_2$ plane

$$Ns_\nu' = s_\nu' ; \qquad Ns_\nu = -s_\nu \tag{46a}$$

where s_ν and s_ν' are two perpendicular real vectors in the $v_1 v_2$ plane

$$s_\nu' = e^{i\nu} v_1 + e^{-i\nu} v_2 ; \qquad s_\nu = i(e^{i\nu} v_1 - e^{-i\nu} v_2), \tag{46b}$$

and of a reflection in the $v_3 v_4$ plane

$$Nt_\mu' = t_\mu' ; \qquad Nt_\mu = -t_\mu , \tag{46c}$$

where again t_μ, t'_μ are real vectors in the v_3v_4 plane, perpendicular to each other, t_μ being space-like, t'_μ time-like:

$$t'_\mu = e^{\mu}v_3 + e^{-\mu}v_4\,; \qquad t_\mu = e^{\mu}v_3 - e^{-\mu}v_4\,. \tag{46d}$$

Thus N becomes a rotation by π in the purely space like $s_\nu t_\mu$ plane. The M can be calculated from $M = \Lambda N$

$$\begin{aligned} Ms'_\nu &= \Lambda N s'_\nu = \Lambda s'_\nu = e^{i\nu+2i\gamma}v_1 + e^{-i\nu-2i\gamma}v_2 \\ &= \tfrac{1}{2}e^{2i\gamma}(s'_\nu - is_\nu) + \tfrac{1}{2}e^{-2i\gamma}(s'_\nu + is_\nu) = \cos 2\gamma\cdot s'_\nu + \sin 2\gamma\cdot s_\nu \\ Ms_\nu &= \sin 2\gamma\cdot s'_\nu - \cos 2\gamma\cdot s_\nu \\ Mt'_\mu &= \Lambda N t'_\mu = \Lambda t'_\mu = e^{\mu+2\chi}v_3 + e^{-\mu-2\chi}v_4 \\ &= \tfrac{1}{2}e^{2\chi}(t'_\mu + t_\mu) + \tfrac{1}{2}e^{-2\chi}(t'_\mu - t_\mu) = \mathrm{Ch}\ 2\chi\cdot t'_\mu + \mathrm{Sh}\ 2\chi\cdot t_\mu \\ Mt_\mu &= -\mathrm{Sh}\ 2\chi\cdot t'_\mu - \mathrm{Ch}\ 2\chi\cdot t_\mu\,. \end{aligned} \tag{46e}$$

Thus M also becomes a product of two reflections, one in the $v_1v_2 = s'_\nu s_\nu$ the other in the $v_3v_4 = t'_\mu t_\mu$ plane. This completes the decomposition of Λ into two involutions. One of the involutions can be taken to be a rotation by π in an arbitrary space like plane, intersecting both the v_1v_2 and the v_3v_4 planes, as the freedom in choosing ν and μ allows us to fix the lines s_ν and t_μ arbitrarily in those planes. The involution characterized by (46) will be called $N_{\nu\mu}$ henceforth. The other involution M is then a similar rotation, in a plane, however, which is completely determined once the $s_\nu t_\mu$ plane is fixed. It will be denoted by $M_{\nu\mu}$ (it is, in fact $M_{\nu\mu} = N_{\nu+\gamma\ \mu+\chi}$). One sees the complete analogy to the three dimensional case if one remembers that γ and χ are the half angles of rotation.

The $d(M)$ and $d(N)$ so normalized that their squares be 1 shall be denoted by $d_1(M_{\nu\mu})$ and $d_1(N_{\nu\mu})$. We must show that the normalization for

$$d(\Lambda) = \pm d_1(M_{\nu\mu})d_1(N_{\nu\mu}) \tag{47}$$

is independent of ν and μ. For this purpose, we transform

$$d(\Lambda) = \pm d_1(M_{00})d_1(N_{00}) \tag{47a}$$

with $d(\Lambda_1)$ where Λ_1 has the same characteristic vectors as Λ but different characteristic values, namely $e^{i\nu}$, $e^{-i\nu}$, e^{μ} and $e^{-\mu}$. Since $\Lambda_1 M_{00}\Lambda_1^{-1} = M_{\nu\mu}$ and $\Lambda_1 N_{00}\Lambda_1^{-1} = N_{\nu\mu}$ we have $d(\Lambda_1)d_1(M_{00})d(\Lambda_1)^{-1} = \omega d_1(M_{\nu\mu})$ where $\omega = \pm 1$, as the squares of both sides are 1. Hence, (47a) becomes if transformed with $d(\Lambda_1)$ just

$$d(\Lambda_1)d(\Lambda)d(\Lambda_1)^{-1} = \pm d_1(M_{\nu\mu})d_1(N_{\nu\mu}). \tag{47b}$$

The normalization (47) would be clearly independent of ν and μ if $d(\Lambda_1)$ commuted with $d(\Lambda)$.

Again, the argument contained in equations (24)-(30) can be applied and shows that

$$d(\Lambda_1)d(\Lambda)d(\Lambda_1)^{-1} = \exp\,(2\pi i f(2\gamma\mu - 2\chi\nu))d(\Lambda) \tag{48}$$

holds for every γ, χ, ν, μ. However, the exponential in (48) must be 1 if $\gamma = 0$; $\nu = 2\pi/n$; $\chi = \frac{1}{2}n\mu$ since in this case $\Lambda = \Lambda_1^n$. Thus exp $(-4\pi^2 if\mu) = 1$ for every μ and $f = 0$ and the left side of (47b) can be replaced by $d(\Lambda)$; the normalization in (47) is independent of ν and μ.

In order to have the analogue of (45), we must show that, having two Lorentz transformations $\Lambda = M_{\nu\mu}N_{\nu\mu}$ and $I = P_{\alpha\beta}Q_{\alpha\beta}$ we can chose ν, μ and α, β so that $N_{\nu\mu} = P_{\alpha\beta}$ i.e. that the plane of rotation $s_\nu t_\mu$ of $N_{\nu\mu}$ coincide with the plane of rotation of $P_{\alpha\beta}$. As the latter plane can be made to an arbitrary space-like plane intersecting both the w_1w_2 and the w_3w_4 planes (where w_1, w_2, w_3, w_4 are the characteristic vectors of I), we must show the existence of a space like plane, intersecting all four planes v_1v_2, v_3v_4, w_1w_2, w_3w_4. Both the first and the second pair of planes are orthogonal.

One can show[22] that if Λ and I have no common null vector as characteristic

[22] We first suppose the existence of a real plane p intersecting all four planes v_1v_2, v_3v_4, w_1w_2, w_3w_4. If p intersects v_1v_2 the plane q perpendicular to p will intersect the plane v_3v_4 perpendicular to v_1v_2. Indeed, the line which is perpendicular to both p and v_1v_2 (there is such a line as p and v_1v_2 intersect) is contained in both q and v_3v_4. This shows that if there is a plane intersecting all four planes, the plane perpendicular to this will have this property also.

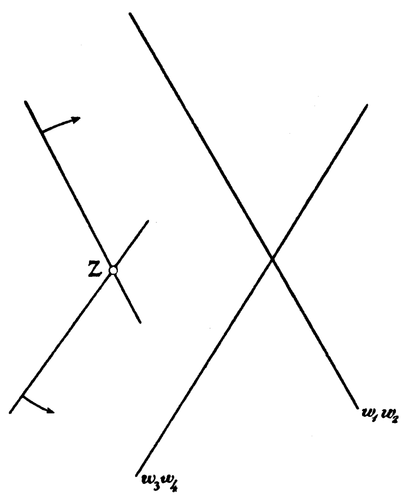

Fig. 4 gives a projection of all lines into the x_1x_2 plane. One sees that there are, in general, two intersecting planes, only in exceptional cases is there only one.

If the plane p—the existence of which we suppose for the time being—contains a time-like vector, q will be space-like (Section 4B, [1]). Both in this case and if p contains only space-like vectors, the theorem in the text is valid. There is a last possibility, that p is tangent to the light cone, i.e. contains only space like vectors and a null vector v. The space-like vectors of p are all orthogonal to v, otherwise p would contain time-like vectors also. In this case the plane q, perpendicular to p will contain v also. The line in which v_1v_2 intersects p is space-like and orthogonal to the vector in which v_3v_4 intersects p. The latter intersection must coincide with v, therefore, as no other vector of p is orthogonal to vector, there are always two planes, perpendicular to each other which intersect four such planes. One of these is always space like. It is possible to assume, therefore, that both $N_{\nu\mu}$ and $P_{\alpha\beta}$ are the rotation by π in this plane. Thus

$$\begin{aligned} d(\Lambda)d(I) &= \pm d_1(M_{\nu\mu})d_1(N_{\nu\mu})d_1(P_{\alpha\beta})d_1(Q_{\alpha\beta}) \\ &= \pm d_1(M_{\nu\mu})d_1(Q_{\alpha\beta}), \end{aligned} \tag{49}$$

and $d(\Lambda)d(I)$ has the normalization corresponding to the product of two involutions, neither of which is unity. This is, however, also the normalization adopted for $d(\Lambda I)$. Hence

$$d(\Lambda)d(I) = \pm d(\Lambda I) \tag{49a}$$

holds if Λ, I and ΛI are Lorentz transformations corresponding to one of the cases (a), (b), (c) or (d) of section 4B and if Λ and I have no common characteristic null vector. In addition to this (49a) holds also, assuming $d(E) = \pm 1$, if any of the transformations Λ, I, ΛI is unity, or if both characteristic null vectors of Λ and I are equal, as in this case the planes v_3v_4 and w_3w_4 and also v_1v_2 and w_1w_2 coincide and there are many space like planes intersecting all.

If Λ and I have one common characteristic null vector, $v_3 = w_3$, the others, v_4 and w_4 respectively, being different, one can use an aritifice to prove (49a) which will be used in later parts of this section extensively. One can find a Lorentz transformation J so that none of the pairs $I - J$; $\Lambda - IJ$; $\Lambda IJ - J^{-1}$ has a common characteristic null vector. This will be true, e.g. if the characteristic null vectors of J are v_4 and another null vector, different from v_3, w_4 and the characteristic vectors of ΛI. Then (49a) will hold for all the above pairs and

$$\begin{aligned} d(\Lambda)d(I) &= \pm d(\Lambda)d(I)d(J)d(J^{-1}) = \pm d(\Lambda)d(IJ)d(J^{-1}) \\ &= \pm d(\Lambda IJ)d(J^{-1}) = \pm d(\Lambda I). \end{aligned}$$

---

any space-like vector in it. Hence, v is the intersection of p and v_3v_4 and is either v_3 or v_4. One can conclude in the same way that v coincides with either w_3 or w_4 also and we see that if p is tangent to the light cone the two transformations Λ and I have a common null vector as characteristic vector. Thus the theorem in the text is correct if we can show the existence of an arbitrary real plane p intersecting all four planes v_1v_2, v_3v_4, w_1w_2, w_3w_4.

Let us draw a coördinate system in our four dimensional space, the x_1x_2 plane of which is the v_1v_2 plane, the x_3 and x_4 axes having the directions of the vectors $v_3 - v_4$ and $v_3 + v_4$, respectively. The three dimensional manifold M characterized by $x_4 = 1$ intersects all planes in a line, the v_1v_2 plane in the line at infinity of the x_1x_2 plane, the v_3v_4 plane in the x_3 axis. The intersection of M with the w_1w_2 and w_3w_4 planes will be lines in M with directions perpendicular to each other. They will have a common normal through the origin of M, intersecting it at reciprocal distances. This follows from their orthogonality in the four dimensional space.

A plane intersecting v_1v_2 and v_3v_4 will be a line parallel to x_1x_2 through the x_3 axis. If we draw such lines through all points of the line corresponding to w_1w_2, the direction of this line will turn by π if we go from one end of this line to the other. Similarly, the lines going through the line corresponding to w_3w_4 will turn by π in the *opposite* direction. Thus the first set of lines will have at least one line in common with the second set and this line will correspond to a real plane intersecting all four planes v_1v_2, v_3v_4, w_1w_2, w_3w_4. This completes the proof of the theorem referred to in the text.

This completes the proof of (49a) for all cases in which Λ, I and ΛI have no elementary divisors. It is evident also that we can replace in the normalization (47) the d_1 by d. One also concludes easily that $d(M)^2$ is in the same representation either $+1$ for all involutions M, or -1 for every involution. The former ones will give real representations, the latter ones representations up to the sign.

If Λ has an elementary divisor, it can be expressed in the v_e, w_e, z_e, v_1 scheme as the matrix (Cf. equ. (20))

$$\Lambda_e = \begin{Vmatrix} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

and can be written, in the same scheme, as the product of two Lorentz transformations with the square 1

$$\Lambda_e = M_0 N_0 = \begin{Vmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix} \cdot \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}.$$

We can normalize therefore $d(\Lambda_e) = \pm d(M_0)d(N_0)$. If Λ can be written as the product of two other involutions also $\Lambda_e = M_1N_1$ the corresponding normalization will be identical with the original one. In order to prove this, let us consider a Lorentz transformation J such that neither of the Lorentz transformations J, N_0J, N_1J, $\Lambda_eJ = M_0N_0J = M_1N_1J$ have an elementary divisor. Since the number of free parameters is only 4 in case (e), while 6 for case (a), this is always possible. Then, for (45a)

$$\begin{aligned} d(M_0)d(N_0)d(J) &= \pm d(M_0)d(N_0J) = \pm d(M_0N_0J) \\ &= \pm d(M_1N_1J) = \pm d(M_1)d(N_1J) = \pm d(M_1)d(N_1)d(J) \end{aligned}$$

and thus $d(M_0)d(N_0) = \pm d(M_1)d(N_1)$. This shows also that even if ΛI is in case (e), $\omega(\Lambda, I) = \pm 1$, since (49) leads to the correct normalization.

If $\Lambda = MN$ has an elementary divisor, I not, $d(\Lambda)d(I)$ still will have the normalization corresponding to the product of two involutions. One can find again a J such that neither of the transformations J, J^{-1}, IJ, NIJ, $MNIJ$, have an elementary divisor. Then

$$\begin{aligned} d(M)d(N)d(I) &= \pm d(M)d(N)d(I)d(J)d(J)^{-1} \\ &= \pm d(M)d(N)d(IJ)d(J)^{-1} = \pm d(M)d(NIJ)d(J)^{-1} \\ &= d(\Lambda IJ)d(J^{-1}). \end{aligned}$$

The last product has, however, the normalization corresponding to two involutions, as was shown in (49a), since neither ΛIJ, nor J^{-1} is in case (e).

Lastly, we must consider the case when both Λ and I may have an elementary divisor. In this case, we need a J such that neither of J, J^{-1}, IJ have one. Then, because of the generalization of (49a) just proved, in which the first factor is in case (e)

$$d(\Lambda)d(I) = \pm d(\Lambda)d(I)d(J)d(J^{-1}) = \pm d(\Lambda)d(IJ)d(J^{-1})$$
$$= \pm d(\Lambda IJ)d(J^{-1})$$

which has the right normalization.

This completes the proof of

$$\omega(\Lambda, I) = \pm 1 \tag{50}$$

for all possible cases, and the normalization of all $D(L)$ of a representation of the inhomogeneous Lorentz group up to a factor, is carried out in such a way that the normalized operators give a representation up to the sign. It is even carried so far that in the first two of equations (22) $\omega = 1$ can be set. We shall consider henceforth systems of operators satisfying (7), or, more specifically, (22B) and (22C) with $\omega(a, b) = \omega(\Lambda, a) = 1$ and (22D) with $\omega(\Lambda, I) = \pm 1$.

E.

Lastly, it shall be shown that the renormalization not only did not spoil the partly continuous character of the representation, attained at the first normalization in part (A) of this section, but that the same holds now *everywhere*, in the ordinary sense for $T(a)$ and, apart from the ambiguity of sign, also for $d(\Lambda)$. For $T(a)$ this was proved in part $(B)(b)$ of this section, for $d(\Lambda)$ it means that to every Λ_1, ϵ and φ there is such a δ that *one* of the two quantities

$$((d(\Lambda_1) \mp d(\Lambda))\varphi, (d(\Lambda_1) \mp d(\Lambda))\varphi) < \epsilon \tag{51}$$

if Λ is in the neighborhood δ of Λ_1. The inequality (51) is equivalent to

$$((1 \mp d(\Lambda_0))\varphi, (1 \mp d(\Lambda_0))\varphi) < \epsilon, \tag{51a}$$

where $\Lambda_0 = \Lambda_1^{-1}\Lambda$ now can be assumed to be in the neighborhood of the unity. Thus, the continuity of $d(\Lambda)$ at $\Lambda = E$ entails the continuity everywhere.[23] In fact, it would be sufficient to show that the $d(X)$, $d(Y)$ and $d(Z)$ corresponding to the transformations (40) converge to ± 1, as α, γ approach 0, since one can write every transformation in the neighborhood of the unit element as a product $\Lambda = Z(0, \gamma_3)Y(0, \gamma_2)X(0, \gamma_1)X(\alpha_1, 0)Y(\alpha_2, 0)Z(\alpha_3, 0)$ and the parameters $\alpha_1, \cdots, \gamma_3$ will converge to 0 as Λ converges to 1. However, we shall carry out the proof for an arbitrary Λ without an elementary divisor.

For $d(\Lambda)$, equations (46) show that as Λ approaches E (i.e., as γ and χ approach zero) both M_{00} and N_{00} approach the same involution, which we shall call K. Let us now consider a wave function $\psi = \varphi + d_1(K)\varphi$ or, if this vanishes $\psi = \varphi - d_1(K)\varphi$. We have $d_1(K)\psi = \pm\psi$. If Λ is sufficiently near to unity,

[23] J. von Neumann, Sitz. d. kön. Preuss. Akad. p. 76, 1927.

$d_1(N_{00})\psi$ will be sufficiently near to $\Omega d_1(K)\psi = \pm\Omega\psi$ and all we have to show is that Ω approaches ± 1. The same thing will hold for $d_1(M_{00})$. Indeed from $d_1(N_{00})\psi - \Omega\psi = u$ it follows by applying $d_1(N_{00})$ on both sides $\psi - \Omega^2\psi = (d_1(N_{00}) + \Omega)u$. As (u, u) goes to zero, Ω must go to ± 1, and consequently, also $d_1(N_{00})\psi$ goes to ψ or to $-\psi$. Applying $d_1(M_{00})$ to this, one sees that $d_1(M_{00})d_1(N_{00})\psi = d(\Lambda)\psi$ goes to $\pm\psi$ as Λ goes to unity. The argument given in (A)(b) shows that this holds not only for ψ but for every other function also, i.e. $d(\Lambda)$ converges to $\pm 1 = d(E)$ as Λ approaches E. Thus $d(\Lambda)$ is continuous in the neighborhood of E and hence everywhere.

According to the last remark in part 4, the operators $\pm d(\Lambda)$ form a single valued representation of the group of complex unimodular two dimensional matrices C. Let us denote the homogeneous Lorentz transformation which corresponds in the isomorphism to C by $\tilde{C}$. Our task of solving the equs. (22) has been reduced to finding all single valued unitary representation of the group with the elements $[a, C] = [a, 1]\,[0, C]$, the multiplication rule of which is $[a, C_1]\,[b, C_2] = [a + \tilde{C}_1 b, C_1C_2]$. For the representations of this group $D[a, C] = T(a)d[C]$ we had

$$\begin{aligned} T(a)T(b) &= T(a + b) \\ d[C]T(a) &= T(\tilde{C}a)\,d[C] \\ d[C_1]\,d[C_2] &= d[C_1C_2]. \end{aligned} \tag{52a}$$

It would be more natural, perhaps, from the mathematical point of view, to use henceforth this new notation for the representations and let the d depend on the C rather than on the $\tilde{C}$ or Λ. However, in order to be reminded on the geometrical significance of the group elements, it appeared to me to be better to keep the old notation. Instead of the equations (22B), (22C), (22D) we have, then

$$T(a)T(b) = T(a + b) \tag{52B}$$

$$d(\Lambda)T(a) = T(\Lambda a)\,d(\Lambda) \tag{52C}$$

$$d(\Lambda)\,d(I) = \pm d(\Lambda I). \tag{52D}$$

6. Reduction of the Representations of the Inhomogeneous Lorentz Group to Representations of a "Little Group"

This section, unlike the other ones, will often make use of methods, which though commonly accepted in physics, must be further justified from a rigorous mathematical point of view. This has been done, in the meanwhile, by J. von Neumann in an as yet unpublished article and I am much indebted to him for his coöperation in this respect and for his readiness in communicating his results to me. A reference to his paper[24] will be made whenever his work is necessary for making inexact considerations of this section rigorous.

[24] J. von Neumann, Ann. of Math. to appear shortly.

A.

Since the translation operators all commute, it is possible[24] to introduce such a coördinate system in Hilbert space that the wave functions $\varphi(p, \zeta)$ contain momentum variables p_1, p_2, p_3, p_4 and a discrete variable ζ so that

$$T(a)\varphi(p, \zeta) = e^{i\{p,a\}}\varphi(p, \zeta). \tag{53}$$

p will stand for the four variables p_1, p_2, p_3, p_4.

Of course, the fact that the Lorentzian scalar product enters in the exponent, rather than the ordinary, is entirely arbitrary and could be changed by changing the signs of p_1, p_2, p_3.

The unitary scalar product of two wave functions is not yet completely defined by the requirements so far made on the coördinate system. It can be a summation over ζ and an arbitrary Stieltjes integral over the components of p:

$$(\psi, \varphi) = \sum_{\zeta} \int \psi(p, \zeta)^* \varphi(p, \zeta)\, df(p, \zeta). \tag{54}$$

The importance of introducing a weight factor, depending on p, for the scalar product lies not so much in the possibility of giving finite but different weights to different regions in p space. Such a weight distribution $g(p, \zeta)$ always could be absorbed into the wave functions, replacing all $\varphi(p, \zeta)$ by $\sqrt{g(p, \zeta)} \cdot \varphi(p, \zeta)$. The necessity of introducing the $f(p, \zeta)$ lies rather in the possibility of some regions of p having zero weight while, on the other hand, at other places points may have finite weights. On account of the definite metric in Hilbert space, the integral $\int df(p, \zeta)$ over any region r, for any ζ, is either positive, or zero, since it is the scalar product of that function with itself, which is 1 in the region r of p and the value ζ of the discrete variable, zero otherwise.

Let us now define the operators

$$P(\Lambda)\varphi(p, \zeta) = \varphi(\Lambda^{-1}p, \zeta). \tag{55}$$

This equation defines the function $P(\Lambda)\varphi$, which is, at the point p, ζ, as great as the function φ at the point $\Lambda^{-1}p$, ζ. The operator $P(\Lambda)$ is not necessarily unitary, on account of the weight factor in (54). We can easily calculate

$$P(\Lambda)T(a)\varphi(p, \zeta) = T(a)\varphi(\Lambda^{-1}p, \zeta) = e^{i\{\Lambda^{-1}p,a\}}\varphi(\Lambda^{-1}p, \zeta),$$

$$T(\Lambda a)P(\Lambda)\varphi(p, \zeta) = e^{i\{p,\Lambda a\}}P(\Lambda)\varphi(p, \zeta) = e^{i\{p,\Lambda a\}}\varphi(\Lambda^{-1}p, \zeta),$$

so that, for $\{\Lambda^{-1}p, a\} = \{p, \Lambda a\}$, we have

$$P(\Lambda)T(a) = T(\Lambda a)P(\Lambda). \tag{56}$$

This, together with (52C), shows that $d(\Lambda)P(\Lambda)^{-1} = Q(\Lambda)$ commutes with all $T(a)$ and, therefore, with the multiplication with every function of p, since the exponentials form a complete set of functions of p_1, p_2, p_3, p_4. Thus

$$d(\Lambda) = Q(\Lambda)P(\Lambda), \tag{57}$$

where $Q(\Lambda)$ is an operator in the space of the ζ alone[24] which can depend, however, on the particular value of p in the underlying space:

$$Q(\Lambda)\varphi(p, \zeta) = \sum_\eta Q(p, \Lambda)_{\zeta\eta}\varphi(p, \eta). \tag{57a}$$

Here, $Q(p, \Lambda)_{\zeta\eta}$ are the components of an ordinary (finite or infinite) matrix, depending on p and Λ. From (57), we obtain

$$\begin{aligned} d(\Lambda)\varphi(p, \zeta) &= \sum_\eta Q(p, \Lambda)_{\zeta\eta}P(\Lambda)\varphi(p, \eta) \\ &= \sum_\eta Q(p, \Lambda)_{\zeta\eta}\varphi(\Lambda^{-1}p, \eta). \end{aligned} \tag{57b}$$

As the exponentials form a complete set of functions, we can approximate the operation of multiplication with any function of p_1, p_2, p_3, p_4 by a linear combination

$$f(p)\varphi = \sum_n c_n T(a_n)\varphi. \tag{58}$$

If we choose $f(p)$ to be such a function that

$$f(p) = f(\Lambda p) \tag{58a}$$

the operation of multiplication with $f(p)$ will commute with all operations of the group. It commutes evidently with the $T(a)$ and the $Q(p, \Lambda)$, and on account of (56) and (58), (58a) also with $P(\Lambda)$. Thus the operation of (58) belongs to the centrum of the algebra of our representation. Since, however, we assume that the representation is factorial (cf. 2), the centrum contains only multiples of the unity and

$$f(p)\varphi(p, \zeta) = c\varphi(p, \zeta). \tag{58b}$$

This can be true only if φ is different from zero only for such momenta p which can be obtained from each other by homogeneous Lorentz transformations, because $f(p)$ needs to be equal to $f(p')$ only if there is a Λ which brings them into each other.

It will be sufficient, henceforth, to consider only such representations, the wave functions of which vanish except for such momenta which can be obtained from one by homogeneous Lorentz transformations. One can restrict, then, the definition domain of the φ to these momenta.

These representations can now naturally be divided into the four classes enumerated in section 3, and two classes contain two subclasses. There will be representations, the wave functions of which are defined for such p that

(1) $\{p, p\} = P > 0$ (3) $p = 0$

(2) $\{p, p\} = P = 0; p \neq 0$ (4) $\{p, p\} = P < 0$.

The classes 1 and 2 contain two sub-classes each. In the positive subclasses P_+ and 0_+ the time components of all momenta are $p_4 > 0$, in the negative subclasses P_- and 0_- the fourth components of the momenta are negative. Class 3 will be denoted by 0_0. If P is negative, it has no index.

From the condition that $d(\Lambda)$ shall be a unitary operator, it is possible to infer[24] that one can introduce a coördinate system in Hilbert space in such a way that

$$\int_r df(p, \zeta) = \int_{\Lambda r} df(p, \eta) \tag{59}$$

if $Q(p, \Lambda)_{\zeta\eta} \neq 0$ for the p of the domain r. Otherwise, r is an arbitrary domain in the space of p_1, p_2, p_3, p_4 and Λr is the domain which contains Λp if r contains p. Equation (59) holds for all ζ, η, except for such pairs for which $Q(p, \Lambda)_{\zeta\eta} = 0$. It is possible, hence, to decompose the original representation in such a way that (59) holds within every reduced part. Neither $T(a)$ nor $d(\Lambda)$ can have matrix elements between such η and ζ for which (59) does not hold.

In the third class of representations, the variable p can be dropped entirely, and $T(a)\varphi(\zeta) = \varphi(\zeta)$, i.e., all wave functions are invariant under the operations of the invariant subgroup, formed by the translations. The equation $T(a)\varphi(\zeta) = \varphi(\zeta)$ is an invariant characterization of the representations of the third class, i.e., a characterization which is not affected by a similarity transformation. Hence, the reduced parts of a representation of class 3 also belong to this class.

Since no wave function of the other classes can remain invariant under all translations, no representation of the third class can be contained in any representation of one of the other classes. In the other classes, the variability domain of p remains three dimensional. It is possible, therefore, to introduce instead of p_1, p_2, p_3, p_4 three independent variables. In the cases 1 and 2 with which we shall be concerned most, p_1, p_2, p_3 can be kept for these three variables. On account of (59), the Stieltjes integral can be replaced by an ordinary integral[24] over these variables, the weight factor being $|p_4|^{-1} = (P + p_1^2 + p_2^2 + p_3^2)^{-\frac{1}{2}}$

$$\{\psi, \varphi\} = \sum_\zeta \int \int \int_{-\infty}^{\infty} \psi(p, \zeta)^* \varphi(p, \zeta) \, |p_4|^{-1} dp_1 \, dp_2 \, dp_3. \tag{59a}$$

In fact, with the weight factor $|p_4|^{-1}$ the weight of the domain r i.e., $W_r = \int\int\int_r |p_4|^{-1} dp_1 \, dp_2 \, dp_3$ is equal to the weight of the domain $W_{\Lambda r}$ as required[25] by (59). Having the scalar product fixed in this way, $P(\Lambda)$ becomes a unitary operator and, hence, $Q(\Lambda)$ will be unitary also.

We want to give next a characterization of the representations with a given P, which is independent of the coördinate system in Hilbert space. It follows from

---

[25] The invariance of integrals of the character of (59a) is frequently made use of in relativity theory. One can prove it by calculating the Jacobian of the transformation

$$p_i' = \Lambda_{i1}p_1 + \Lambda_{i2}p_2 + \Lambda_{i3}p_3 + (P + p_1^2 + p_2^2 + p_3^2)^{\frac{1}{2}} \qquad (i = 1, 2, 3)$$

which comes out to be $(P + p_1^2 + p_2^2 + p_3^2)^{\frac{1}{2}}(P + p_1'^2 + p_2'^2 + p_3'^2)^{-\frac{1}{2}}$. Equ. (59a) will not be used in later parts of this paper.

(53), that in a representation with a given P the wave functions $\psi_1, \psi_2, \cdots$ which are different from zero only in a finite domain of p, form an everywhere dense set, to all elements of which the infinitesimal operators of translation can be applied arbitrarily often

$$\lim_{h=0} h^{-n}(T(he) - 1)^n\psi = \lim_{h=0} h^{-n}(e^{ih\{p,e\}} - 1)^n\psi = i^n\{p, e\}^n\psi, \tag{60}$$

where e will be a unit vector in the direction of a coördinate axis or oppositely directed to it. Hence for all members ψ of this everywhere dense set

$$\lim_{h=0} \sum_k \pm h^{-2}(T(2he_k) - 2T(he_k) + 1)\psi = (p_1^2 + p_2^2 + p_3^2 - p_4^2)\psi = -P\psi, \tag{61}$$

where e_k is a unit vector in (or opposite) the k^{th} coördinate axis and the $\pm$ is $+$ for $k = 4$, and $-$ for $k = 1, 2, 3$.

On the other hand, there is no φ for which

$$\lim_{h=0} \sum_k \pm h^{-2}(T(2he_k) - 2T(he_k) + 1)\varphi \tag{61a}$$

if it exists, would be different from $-P\varphi$. Suppose the limit in (61a) exists and is $-P\varphi + \varphi'$. Let us choose then a normalized ψ, from the above set, such that $(\psi, \varphi') = \delta$ with $\delta > 0$ and an h so that the expression after the lim sign in (61a) assumes the value $-P\varphi + \varphi' + u$ with $(u, u) < \delta/3$ and also the expression after the lim sign in (61), with oppositely directed e_k becomes $-P\psi + u'$ with $(u', u') < \delta/3$. Then, on account of the unitary character of $T(a)$ and because of $T(-a) = T(a)^{-1}$

$$\left(\psi, \sum_k \pm h^{-2}(T(2he_k) - 2T(he_k) + 1)\varphi\right),$$
$$= \left(\sum_k \pm h^{-2}(T(-2he_k) - 2T(-he_k) + 1)\psi, \varphi\right),$$

or

$$- P(\psi, \varphi) + (\psi, \varphi') + (\psi, u) = -P(\psi, \varphi) + (u', \varphi),$$

which is clearly impossible.

Thus if the lim in (61a) exists, it is $-P\varphi$ and this constitutes a characterization of the representation which is independent of similarity transformations. Since, according to the foregoing, it is always possible to find wave functions for a representation, to which (61a) can be applied, every reduced part of a representation with a given P must have this same P and no representation with one P can be contained in a representation with an other P. The same argument can be applied evidently to the positive and negative sub-classes of class 1 and 2.

B.

Every automorphism $L \to L^\circ$ of the group allows us to construct from one representation $D(L)$ another representation

$$D^\circ(L) = D(L^\circ). \tag{62}$$

This principle will allow us to restrict ourselves, for representations with finite, positive or negative P, to one value of P which can be taken respectively, to be $+1_+$ and -1. It will also allow in cases 1 and 2 to construct the representations of the negative sub-classes out of representations of the positive sub-classes.

The first automorphism is $a^\circ = \alpha a$, $\Lambda^\circ = \Lambda$. Evidently Equs. (12) are invariant under this transformation. If we set, however,

$$T^\circ(a)\varphi = T(\alpha a)\varphi; \qquad d^\circ(\Lambda)\varphi = d(\Lambda)\varphi,$$

then the occurring p

$$T^\circ(a)\varphi = T(\alpha a)\varphi = e^{i\{p,\alpha a\}}\varphi = e^{i\{\alpha p, a\}}\varphi,$$

will be the p occurring for the unprimed representation, multiplied by α. This allows, with a real positive α, to construct all representations with all possible numerical values of P, from all representation with one numerical value of P. If we take α negative, the representations of the negative sub-classes are obtained from the representations of the positive sub-class.

In case $P = 0_0$ evidently all representations go over into themselves by the transformation (62). In case $P = 0_+$ and $P = 0_-$ it will turn out that for positive α, (62) carries every representation into an equivalent one.

C.

On account of (53) and (56), (57), the equs. (52B) and (52C) are automatically satisfied and the $Q(p, \Lambda)_{\zeta\eta}$ must be determined by (52D). This gives

$$\sum_{\eta\vartheta} Q(p, \Lambda)_{\zeta\eta} Q(\Lambda^{-1}p, I)_{\eta\vartheta}\,\varphi(I^{-1}\Lambda^{-1}p, \vartheta) = \pm \sum_{\vartheta} Q(p, \Lambda I)_{\zeta\vartheta}\,\varphi(I^{-1}\Lambda^{-1}p, \vartheta). \tag{63}$$

Since this must hold for every φ, one would conclude

$$\sum_{\eta} Q(p, \Lambda)_{\zeta\eta} Q(\Lambda^{-1}p, I)_{\eta\vartheta} = \pm Q(p, \Lambda I)_{\zeta\vartheta}. \tag{63a}$$

Actually, this conclusion is not justified, since two wave functions must be considered to be equal even if they are different on a set of measure zero. Thus one cannot conclude, without further consideration, that the two sides of (63a) are equal at every point p. On the other hand,[24] the value of $Q(p, \Lambda)_{\zeta\eta}$ can be changed on a set of measure zero and one can make it continuous in the neighborhood of every point, if the representation is continuous. This allows then, to justify (63a). It follows from (63a) that $Q(p, 1)_{\zeta\eta} = \delta_{\zeta\eta}$.

Let us choose[15] now a basic p_0 arbitrarily. We can consider then the subgroup of all homogeneous Lorentz transformations which leave this p_0 unchanged. For all elements λ, ι of this "little group," we have

$$\sum_\eta Q(p_0, \lambda)_{\zeta\eta} Q(p_0, \iota)_{\eta\vartheta} = \pm Q(p_0, \lambda\iota)_{\zeta\vartheta} \tag{64}$$
$$q(\lambda)q(\iota) = \pm q(\lambda\iota),$$

where $g(\lambda)$ is the matrix $q(\lambda)_{\zeta\eta} = Q(p_0, \lambda)_{\zeta\eta}$. Because of the unitary character of $Q(\Lambda)$, the $Q(p_0, \Lambda)_{\zeta\eta}$ is unitary matrix and $q(\lambda)$ is unitary also.

If we consider, according to the last paragraph of Section 5, the group formed out of the translations and unimodular two-dimensional matrices, rather than Lorentz transformations, the $\pm$ sign in (64) can be replaced by a $+$ sign. In this case, λ and ι are unimodular two-dimensional matrices and the little group is formed by those matrices, the corresponding Lorentz transformations $\tilde{\lambda}$, $\tilde{\iota}$ to which leave p_0 unchanged $\tilde{\lambda}p_0 = \tilde{\iota}p_0 = p_0$.

Adopting this interpretation of (64), one can also see, conversely, that the representation $q(\lambda)$ of the little group, together with the class and P of the representation of the whole group, determines the latter representation, apart from a similarity transformation. In order to prove this, let us define for every p a two-dimensional unimodular matrix $\alpha(p)$ in such a way that the corresponding Lorentz transformation

$$\tilde{\alpha}(p)p_0 = p \tag{65}$$

brings p_0 into p. The $\alpha(p)$ can be quite arbitrary except of being an almost everywhere continuous function of p, especially continuous for $p = p_0$ and $\alpha(p_0) = 1$. Then, we can set

$$d(\alpha(p)^{-1})\varphi(p_0, \zeta) = \varphi(p, \zeta), \tag{66}$$
$$d(\alpha(p))\varphi(p, \zeta) = \varphi(p_0, \zeta).$$

This is equivalent to setting in (58)

$$Q(p, \alpha(p)) = 1 \tag{66a}$$

and can be achieved by a similarity transformation which replaces $\varphi(p, \zeta)$ by $\sum_\eta Q(p_0, \alpha(p)^{-1})_{\zeta\eta}\varphi(p, \eta)$. As the matrix $Q(p_0, \alpha(p)^{-1})$ is unitary, this is a unitary transformation. It does not affect, furthermore, (53) since it contains p only as a parameter.

Assuming this transformation to be carried out, (66) will be valid and will define, together with the $d(\lambda)$, all the remaining $Q(p, \Lambda)$ uniquely. In fact, calculating $d(\Lambda)\varphi(p, \zeta)$, we can decompose Λ into three factors

$$\Lambda = \alpha(p). \quad \alpha(p)^{-1}\Lambda\alpha(\tilde{\Lambda}^{-1}p). \quad \alpha(\tilde{\Lambda}^{-1}p)^{-1} \tag{67}$$

The second factor $\beta = \alpha(p)^{-1}\Lambda\alpha(\tilde{\Lambda}^{-1}p)$ belongs into the little group: $\tilde{\alpha}(p)^{-1}$ $\tilde{\Lambda}\tilde{\alpha}(\tilde{\Lambda}^{-1}p)p_0 = \tilde{\alpha}(p)^{-1}\tilde{\Lambda}\cdot\tilde{\Lambda}^{-1}p = \tilde{\alpha}(p)^{-1}p = p_0$. We can write, therefore $(\tilde{\Lambda}^{-1}p = p')$

$$
\begin{aligned}
d(\Lambda)\varphi(p, \zeta) &= d(\alpha(p))d(\beta)d(\alpha(p'))^{-1}\varphi(p, \zeta) \\
&= d(\beta)d(\alpha(p'))^{-1}\varphi(p_0, \zeta) \qquad (67a) \\
&= \sum_\eta q(\beta)_{\zeta\eta}d(\alpha(p')^{-1})\varphi(p_0, \eta) = \sum_\eta q(\beta)_{\zeta\eta}\varphi(p', \eta).
\end{aligned}
$$

This shows that all representations of the whole inhomogeneous Lorentz group are equivalent which have the same P and the same representation of the little group. Further than this, the same holds even if the representations of the little group are not the same for the two representations but only equivalent to each other. Let us assume $q_1(\Lambda) = sq_2(\Lambda)s^{-1}$. Then by replacing $\varphi(p, \zeta)$ by $\sum_\eta s(\zeta, \eta)\varphi(p, \eta)$ we obtain a new form of the representation for which (53) still holds but $q_2(\beta)$ for the little group is replaced by $q_1(\beta)$. Then, by the transformation just described (Eq. (66)), we can bring $d(\Lambda)$ for both into the form (67a). The equivalence of two representations of the little group must be defined as the existence of a *unitary* transformation which transforms them into each other. (Only unitary transformations are used for the whole group, also).

On the other hand, if the representations of the whole group are equivalent, the representations of the little group are equivalent also: the representation of the whole group determines the representation of the little group up to a similarity transformation uniquely.

The representation of the little group was defined as the set of matrices $Q(p_0, \lambda)_{\zeta\eta}$ if the representation is so transformed that (53) and (66a) hold. Having two equivalent representations D and $SDS^{-1} = D^\circ$ for both of which (53) and (66a) holds, the unitary transformation S bringing the first into the second must leave all displacement operators invariant. Hence, it must have the form (57a), i.e., operate on the ζ only and depend on p only as on a parameter.

$$
S\varphi(p, \zeta) = \sum_\eta S(p)_{\zeta\eta}\varphi(p, \eta). \qquad (68)
$$

Denoting the matrix Q for the two representations by Q and Q°, the condition $SD(\Lambda) = D^\circ(\Lambda)S$ gives that

$$
\sum_\eta S(p)_{\zeta\eta}Q(p, \Lambda)_{\eta\vartheta} = \sum_\eta Q^\circ(p, \Lambda)_{\zeta\eta}S(\Lambda^{-1}p)_{\eta\vartheta} \qquad (68a)
$$

holds, for every Λ, for almost every p. Setting $\Lambda = \alpha(p_1)$ we can let p approach p_1 in such a way that (68a) remains valid. Since Q is a continuous function of p both $Q(p, \Lambda)$ and $Q^\circ(p, \Lambda)$ will approach their limiting value 1. It follows that there is no domain in which

$$
S(p_1) = S(\alpha(p_1)^{-1}p_1) = S(p_0) \qquad (69)
$$

would not hold, i.e., that (69) holds for almost every p_1. Since all our equations must hold only for almost every p, the $S(p)_{\zeta\eta}$ can be assumed to be independent of p and (68a) then to hold for every p also. It then follows that the representations of the little group in D and D° are transformed into each other by $S_{\zeta\eta}$.

The definition of the little group involved an arbitrarily chosen momentum vector p_0. It is clear, however, that the little groups corresponding to two different momentum vectors p_0 and p are holomorphic. In fact they can be transformed into each other by $\alpha(p)$: If Λ is an element of the little group leaving p invariant then $\alpha(p)^{-1}\Lambda\alpha(p) = \beta$ is an element of the little group which leaves p_0 invariant. We can see furthermore from (67a) that if Λ is in the little group corresponding to p, i.e. $\Lambda p = p$ then the representation matrix $q(\beta)$ of the little group of p_0, corresponding to β, is identical with the representation matrix of the little group of p, corresponding to $\Lambda = \alpha(p)\beta\alpha(p)^{-1}$. Thus when characterizing a representation of the whole inhomogeneous Lorentz group by P and the representation of the little group, it is not necessary to say which p_0 is left invariant by the little group.

D.

Lastly we shall determine the constitution of the little group in the different cases.

1_+. In case 1_+ we can take for p_0 the vector with the components 0, 0, 0, 1. The little group which leaves this invariant obviously contains all rotations in the space of the first three coördinates. This holds for the little group of all representations of the first class.

0_0. In case 0_0, the little group is the whole homogeneous Lorentz group.

-1. In case $P = -1$ the p_0 can be assumed to have the components 1, 0, 0, 0. The little group then contains all transformations which leave the form $-x_2^2 - x_3^2 + x_4^2$ invariant, i.e., is the $2 + 1$ dimensional homogeneous Lorentz group. The same holds for all representations with $P < 0$.

0_+. The determination of the little group for $P = 0_+$ is somewhat more complicated. It can be done, however, rather simply, for the group of unimodular two dimensional matrices. The Lorentz transformation corresponding to the matrix $\begin{Vmatrix} a & b \\ c & d \end{Vmatrix}$ with $ad - bc = 1$ brings the vector with the components x_1, x_2, x_3, x_4, into the vector with the components x_1', x_2', x_3', x_4', where[18]

$$\begin{Vmatrix} a & b \\ c & d \end{Vmatrix} \begin{Vmatrix} x_4 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_4 - x_3 \end{Vmatrix} \begin{Vmatrix} a^* & c^* \\ b^* & d^* \end{Vmatrix} = \begin{Vmatrix} x_4' + x_3' & x_1' + ix_2' \\ x_1' - ix_2' & x_4' - x_3' \end{Vmatrix}. \tag{70}$$

The condition that a null-vector p_0, say with the components 0, 0, 1, 1 be invariant is easily found to be $|a|^2 = 1, c = 0$. Hence the most general element of the little group can be written

$$\begin{Vmatrix} e^{-i\beta/2} & (x + iy)e^{i\beta/2} \\ 0 & e^{i\beta/2} \end{Vmatrix}, \tag{71}$$

with real x, y, β and $0 \leqq \beta < 4\pi$. The general element (71) can be written as $t(x, y)\ \delta(\beta)$ where

$$(71a) \qquad t(x, y) = \begin{Vmatrix} 1 & x + iy \\ 0 & 1 \end{Vmatrix}; \qquad \delta(\beta) = \begin{Vmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{Vmatrix}.$$

The multiplication rules for these are

$$(71b) \qquad t(x, y)t(x', y') = t(x + x', y + y'),$$

$$(71c) \qquad \delta(\beta)t(x, y) = t(x \cos \beta + y \sin \beta, -x \sin \beta + y \cos \beta)\delta(\beta),$$

$$(71d) \qquad \delta(\beta)\delta(\beta') = \delta(\beta + \beta').$$

One could restrict the variability domain of β in $\delta(\beta)$ from 0 to 2π. As $\delta(2\pi)$ commutes with all elements of the little group, it will be a constant and from $\delta(2\pi)^2 = \delta(4\pi) = 1$ it can be $\delta(2\pi) = \pm 1$. Hence $\delta(\beta + 2\pi) = \pm\delta(\beta)$ and inserting a $\pm$ into equation (71d) one could restrict β to $0 \leqq \beta < 2\pi$.

These equations are analogous to the equations (52)–(52D) and show that the little group is, in this case, isomorphic with the inhomogeneous rotation group of two dimensions, i.e. the two dimensional Euclidean group.

It may be mentioned that the Lorentz transformations corresponding to $t(x, y)$ have elementary divisors, and constitute all transformations of class e) in 4B, for which $v_e = p_0$. The transformations $\delta(\beta)$ can be considered to be rotations in the ordinary three dimensional space, about the direction of the space part of the vector p_0. It is possible, then, to prove equations (71) also directly.

7. The Representations of the Little Groups

A. Representations of the three dimensional rotation group by unitary transformations.

The representations of the three dimensional rotation group in a space with a finite member of dimensions are well known. There is one irreducible representation with the dimensions 1, 2, 3, 4, $\cdots$ each, the representations with an odd number of dimensions are single valued, those with an even number of dimensions are two-valued. These representations will be denoted by $D^{(j)}(R)$ where the dimension is $2j + 1$. Thus for single valued representations j is an integer, for double valued representations a half integer. Every finite dimensional representation can be decomposed into these irreducible representations. Consequently those representations of the Lorentz group with positive P in which the representation of the little group—as defined by (64)—has a finite number of dimensions, can be decomposed into such representations in which the representation of the little group is one of the well known irreducible representations of the rotation group. This result will hold for all representations of the inhomogeneous Lorentz group with positive P, since we shall show that even the infinite dimensional representations of the rotation group can be decomposed into the same, finite, irreducible representations.

In the following, it is more appropriate to consider the subgroup of the two dimensional unimodular group which corresponds to rotations, than the rotation group itself, as we can restrict ourselves to single valued representations in this case (cf. equations (52)). From (70), one easily sees[18] that the condition for $\begin{Vmatrix} a & b \\ c & d \end{Vmatrix}$ to leave the vector with the components 0, 0, 0, 1 invariant is that it shall be unitary. It is, therefore, the two dimensional unimodular unitary group the representations of which we shall consider, instead of the representations of the rotation group.

Let us introduce a discrete coördinate system in the representation space and denote the coefficients of the unitary representation by $q(R)_{\kappa\lambda}$ where R is a two dimensional unitary transformation. The condition for the unitary character of the representation $q(R)$ gives

$$(72) \qquad \sum_{\kappa} q(R)^*_{\kappa\lambda} q(R)_{\kappa\mu} = \delta_{\lambda\mu}; \qquad \sum_{\lambda} q(R)^*_{\kappa\lambda} q(R)_{\nu\lambda} = \delta_{\kappa\nu},$$

$$(72a) \qquad \sum_{\kappa} | q(R)_{\kappa\lambda} |^2 = 1; \qquad \sum_{\lambda} | q(R)_{\kappa\lambda} |^2 = 1.$$

This show also that $| q(R)_{\kappa\lambda} | \leqq 1$ and the $q(R)_{\kappa\lambda}$ are therefore, as functions of R, square integrable:

$$\int | q(R)_{\kappa\lambda} |^2 dR$$

exists if $\int \cdots dR$ is the well known invariant integral in group space. Since this is finite for the rotation group (or the unimodular unitary group), it can be normalized to 1. We then have

$$(73) \qquad \sum_{\kappa} \int | q(R)_{\kappa\lambda} |^2 dR = \sum_{\lambda} \int | q(R)_{\kappa\lambda} |^2 dR = 1.$$

The $(2j + 1)^{\frac{1}{2}} D^{(j)}(R)_{kl}$ form,[26] a complete set of normalized orthogonal functions for R. We set

$$(74) \qquad q(R)_{\kappa\lambda} = \sum_{jkl} C^{\kappa\lambda}_{jkl} D^{(j)}(R)_{kl}.$$

We shall calculate now the integral over group space of the product of $D^{(j)}(R)^*_{kl}$ and

$$(75) \qquad q(RS)_{\kappa\mu} = \sum_{\lambda} q(R)_{\kappa\lambda} q(S)_{\lambda\mu}.$$

The sum on the right converges uniformly, as for (72a)

$$\sum_{\lambda=N}^{\infty} | q(R)_{\kappa\lambda} q(S)_{\lambda\mu} | \leqq \left(\sum_{\lambda=N}^{\infty} | q(R)_{\kappa\lambda} |^2 \sum_{\lambda=N}^{\infty} | q(S)_{\lambda\mu} |^2 \right)^{\frac{1}{2}} \leqq \left(\sum_{\lambda=N}^{\infty} | q(S)_{\lambda\mu} |^2 \right)^{\frac{1}{2}}$$

can be made arbitrarily small by choosing an N, independent of R, making the last expression small. Hence, (75) can be integrated term by term and gives

$$(76) \qquad \int D^{(j)}(R)^*_{kl} q(RS)_{\kappa\mu} dR = \sum_{\lambda} \int D^{(j)}(R)^*_{kl} q(R)_{\kappa\lambda} q(S)_{\lambda\mu} dR.$$

[26] H. Weyl and F. Peter, Math. Annal. 97, 737, 1927.

Substituting $\sum_m D^{(j)}(RS)_{km} D^{(j)}(S^{-1})_{ml}$ for $D^{(j)}(R)_{kl}$ one obtains

$$\sum_m D^{(j)}(S^{-1})^*_{ml} \int D^{(j)}(RS)^*_{km} q(RS)_{\kappa\mu} dR = \sum_\lambda q(S)_{\lambda\mu} \int D^{(j)}(R)^*_{kl} q(R)_{\kappa\lambda} dR. \tag{77}$$

In the invariant integral on the left of (77), R can be substituted for RS and we obtain, for (74) and the unitary character

$$\sum_m D^{(j)}(S)_{lm} C^{\kappa\mu}_{jkm} = \sum_\lambda q(S)_{\lambda\mu} C^{\kappa\lambda}_{jkl}. \tag{78}$$

Multiplying (78) by $D^{(h)}(S)^*_{in}$, the integration on the right side can be carried out term by term again, since the sum over λ converges uniformly

$$\sum_{\lambda=N}^{\infty} | C^{\kappa\lambda}_{jkl} q(S)_{\lambda\mu} | \leqq \left(\sum_{\lambda=N}^{\infty} | C^{\kappa\lambda}_{jkl} |^2 \sum_{\lambda=N}^{\infty} | q(S)_{\lambda\mu} |^2 \right)^{\frac{1}{2}} \leqq \left(\sum_{\lambda=N}^{\infty} | C^{\kappa\lambda}_{jkl} |^2 \right)^{\frac{1}{2}}.$$

This can be made arbitrarily small, as even $\sum_\lambda \sum_{jkl} (2j+1)^{-1} | C^{\kappa\lambda}_{jkl} |^2$ converges, for (74) and (72a). The integration of (78) yields thus

$$\sum_\lambda C^{\kappa\lambda}_{jkl} C^{\lambda\mu}_{hin} = \delta_{jh} \delta_{li} C^{\kappa\mu}_{jkn}. \tag{79}$$

From $q(R)q(E) = q(R)$ follows $q(E) = 1$ and then $q(R^{-1}) = q(R)^{-1} = q(R)^\dagger$. This, with the similar equation for $D^{(j)}(R)$ gives

$$\begin{aligned} \sum_{jkl} C^{\kappa\lambda}_{jkl} D^{(j)}(R^{-1})_{kl} &= q(R^{-1})_{\kappa\lambda} = q(R)^*_{\lambda\kappa} \\ &= \sum_{jkl} C^{\lambda\kappa *}_{jlk} D^{(j)}(R)^*_{lk} = \sum_{jkl} C^{\lambda\kappa *}_{jlk} D^{(j)}(R^{-1})_{kl}, \end{aligned} \tag{80}$$

or

$$C^{\kappa\lambda}_{jkl} = C^{\lambda\kappa *}_{jlk}. \tag{81}$$

On the other hand $q(E)_{\kappa\lambda} = \delta_{\kappa\lambda}$ yields

$$\sum_{jk} C^{\kappa\lambda}_{jkk} = \delta_{\kappa\lambda}. \tag{82}$$

These formulas suffice for the reduction of $q(R)$. Let us choose for every finite irreducible representation $D^{(j)}$ an index k, say $k = 0$. We define then, in the original space of the representation $q(R)$ vectors $v^{(\kappa jl)}$ with the components

$$C^{\kappa 1}_{jkl}, C^{\kappa 2}_{jkl}, C^{\kappa 3}_{jkl}, \cdots.$$

The vectors $v^{(\kappa jl)}$ for different j or l are orthogonal, the scalar product of those with the same j and l is independent of l. This follows from (79) and (81)

$$(v^{(\mu j'l')}, v^{(\kappa jl)}) = \sum_\lambda C^{\mu\lambda *}_{j'kl'} C^{\kappa\lambda}_{jkl} = \sum_\lambda C^{\kappa\lambda}_{jkl} C^{\lambda\mu}_{j'l'k} = \delta_{jj'} \delta_{ll'} C^{\kappa\mu}_{jkk}. \tag{83}$$

The $v^{(\kappa jl)}$ for all κ, j, l, form a complete set of vectors. In order to show this, it is sufficient to form, for every ν, a linear combination from them, the ν component of which is 1, all other components 0. This linear combination is

$$\sum_{\kappa jl} C^{\nu\kappa}_{jlk} v^{(\kappa jl)}. \tag{84}$$

In fact, the λ component of (84) is, on account of (79) and (82)

$$\sum_{\kappa j l} C^{\nu\kappa}_{jlk} C^{\kappa\lambda}_{jkl} = \sum_{jl} C^{\nu\lambda}_{jll} = \delta_{\nu\lambda}. \tag{85}$$

However, two v with the same j and l but different first indices κ are not orthogonal. We can choose for every j an l, say $l = 0$ and go through the vectors $v^{(1j0)}$, $v^{(2j0)}$, $\cdots$ and, following Schmidt's method, orthogonalize and normalize them. The vectors obtained in this way shall be denoted by

$$w^{(nj0)} = \sum_{\lambda} \alpha^{j}_{n\lambda} v^{(\lambda j0)}. \tag{86}$$

Then, since according to (83) the scalar products $(v^{(\kappa jl)}, v^{(\lambda jl)})$ do not depend on l, the vectors

$$w^{(njl)} = \sum_{\lambda} \alpha^{j}_{n\lambda} v^{(\lambda jl)} \tag{86a}$$

will be mutually orthogonal and normalized also and the vectors $w^{(njl)}$ for all n, j, l will form a complete set of orthonormal vectors. The same holds for the set of the conjugate complex vectors $w^{(njl)*}$. Using these vectors as coördinate axes for the original representation $q(R)$, we shall find that $q(R)$ is completely reduced. The ν component of the vector $q(R)v^{(\kappa jl)*}$ obtained by applying $q(R)$ on $v^{(\kappa jl)*}$ is

$$\sum_{\mu} q(R)_{\nu\mu}(v^{(\kappa jl)*})_{\mu} = \sum_{\mu} q(R)_{\nu\mu} C^{\mu\kappa}_{jlk}. \tag{87}$$

The right side is uniformly convergent. Hence, its product with $(2h + 1)$ $D^{(h)}(R)^{*}_{in}$ can be integrated term by term giving

$$\sum_{\mu} \int (2h + 1) D^{(h)}(R)^{*}_{in} q(R)_{\nu\mu} C^{\mu\kappa}_{jlk}\, dR = \sum_{\mu} C^{\nu\mu}_{hin} C^{\mu\kappa}_{jlk} = \delta_{hj}\delta_{ln} C^{\nu\kappa}_{jik}. \tag{88}$$

Thus we have for almost all R

$$\sum_{\mu} q(R)_{\nu\mu}(v^{(\kappa jl)*})_{\mu} = \sum_{i} C^{\nu\kappa}_{jik} D^{(j)}(R)_{il} = \sum_{i} D^{(j)}(R)_{il}(v^{(\kappa ji)*})_{\nu}, \tag{88a}$$

or

$$q(R)v^{(\kappa jl)*} = \sum_{i} D^{(j)}(R)_{il} v^{(\kappa ji)*}. \tag{88b}$$

Since both sides are supposed to be strongly continuous functions of R, (88b) holds for every R. In (86a), for every n, the summation must be carried out only over a finite number of λ. We can write therefore immediately

$$q(R)w^{(njl)*} = \sum_{i} D^{(j)}(R)_{il} w^{(nji)*}. \tag{89}$$

This proves that the original representation decomposes in the coördinate system of the w into well known finite irreducible representations $D^{(j)}(R)$. Since the w form a complete orthonormal set of vectors, the transition corresponds to a unitary transformation.

This completes the proof of the complete reducibility of all (finite and infinite dimensional) representations of the rotation group or unimodular unitary group. It is clear also that the same consideration applies for all closed groups, i.e., whenever the invariant integral $\int dR$ converges.

The result for the inhomogeneous Lorentz group is: For every positive numerical value of P, the representations of the little group can be, in an irreducible representation, only the $D^{(0)}, D^{(\frac{1}{2})}, D^{(1)}, \cdots$, both for P_+ and for P_-. All these representations have been found already by Majorana and by Dirac and for positive P there are none in addition to these.

B. Representations of the two dimensional Euclidean group

This group, as pointed out in Section 6, has a great similarity with the inhomogeneous Lorentz group. It is possible, again[24], to introduce "momenta", i.e. variables ξ, η and υ instead of the ζ in such a way that

$$t(x, y)\varphi(p_0, \xi, \eta, \upsilon) = e^{i(x\xi+y\eta)}\varphi(p_0, \xi, \eta, \upsilon). \tag{90}$$

Similarly, one can define again operators $R(\beta)$

$$R(\beta)\varphi(p_0, \xi, \eta, \upsilon) = \varphi(p_0, \xi', \eta', \upsilon), \tag{91}$$

where

$$\begin{aligned} \xi' &= \xi \cos \beta - \eta \sin \beta, \\ \eta' &= \xi \sin \beta + \eta \cos \beta. \end{aligned} \tag{91a}$$

Then $\delta(\beta)R(\beta)^{-1} = S(\beta)$ will commute, on account of (71c), with $t(x, y)$ and again contain ξ, η as parameter only. The equation corresponding to (57a) is

$$\delta(\beta)\varphi(p_0, \xi, \eta, \upsilon) = \sum_{\omega} S(\beta)_{\upsilon\omega}\varphi(p_0, \xi', \eta', \omega). \tag{92}$$

One can infer from (90) and (92) again that the variability domain of ξ, η can be restricted in such a way that all pairs ξ, η arise from one pair ξ_0, η_0 by a rotation, according (91a). We have, therefore two essentially different cases:

a.) $\quad \xi^2 + \eta^2 = \Xi \neq 0$

b.) $\quad \xi^2 + \eta^2 = \Xi = 0, \quad \text{i.e.} \quad \xi = \eta = 0.$

The positive definite metric in the ξ, η space excludes the other possibilities of section 6 which were made possible by the Lorentzian metric for the momenta, necessitated by (55).

Case b) can be settled very easily. The "little group" is, in this case, the group of rotations in a plane and we are interested in one and two valued irreducible representations. These are all one dimensional ($e^{is\beta}$)

$$S(\beta) = e^{is\beta} \tag{93}$$

where s is integer or half integer. These representations were also all found by Majorana and by Dirac. For $s = 0$ we have simply the equation $\Box\varphi = 0$, for $s = \pm\frac{1}{2}$ Dirac's electron equation without mass, for $s = \pm 1$ Maxwell's electromagnetic equations, etc

In case a) the little group consists only of the unit matrix and the matrix $\begin{Vmatrix} -1 & 0 \\ 0 & -1 \end{Vmatrix}$ of the two dimensional unimodular group. This group has two irreducible representations, as (1) and (−1) can correspond to the above two dimensional matrix of the little group. This gives two new representations of the whole inhomogeneous Lorentz group, corresponding to every numerical value of Ξ. Both these sets belong to class 0_+ and two similar new sets belong to class 0_-.

The final result is thus as follows: The representations P_{+j} of the first subclass P_+ can be characterized by the two numbers P and j. From these P is positive, otherwise arbitrary, while j is an integer or a half integer, positive, or zero. The same holds for the subclass P_-. There are three kinds of representations of the subclass 0_+. Those of the first kind 0_{+s} can be characterized by a number s, which can be either an integer or a half integer, positive, negative or zero. Those of the second kind $0_+(\Xi)$ are single valued and can be characterized by an arbitrary positive number Ξ, those of the third kind $0'_+(\Xi)$ are double-valued and also can be characterized by a positive Ξ. The same holds for the subclass 0_-. The representations of the other classes (0_0 and P with $P < 0$) have not been determined.

8. Representations of the Extended Lorentz Group

A.

As most wave equations are invariant under a wider group than the one investigated in the previous sections, and as it is very probable that the laws of physics are all invariant under this wider group, it seems appropriate to investigate now how the results of the previous sections will be modified if we go over from the "restricted Lorentz group" defined in section 4A, to the extended Lorentz group. This extended Lorentz group contains in addition to the translations all the homogeneous transformations X satisfying (10)

$$XFX' = F \tag{10'}$$

while the homogeneous transformations of section 4A were restricted by two more conditions. From (10′) it follows that the determinant of X can be $+1$ or -1 only. If its -1, the determinant of $X_1 = XF$ is $+1$. If the four-four element of X_1 is negative, that of $X_2 = -X_1$ is positive. It is clear, therefore, that if X is a matrix of the extended Lorentz group, one of the matrices X, XF, $-X$, $-XF$ is in the restricted Lorentz group. For $F^2 = 1$, conversely, all homogeneous transformations of the extended Lorentz group can be obtained from the homogeneous transformations of the restricted group by multiplication with one of the matrices

$$1,\ F,\ -1,\ -F. \tag{94}$$

The group elements corresponding to these transformations will be denoted by E, F, I, IF. The restricted group contains those elements of the extended group which can be reached continuously from the unity. It follows that the transformation of an element L of the restricted group by F, I, or IF gives again an element of the restricted group. This is, therefore, an invariant subgroup of the extended Lorentz group. In order to find the representations of the extended Lorentz group, we shall use again Frobenius' method.[15]

We shall denote the operators corresponding in a representation to the homogeneous transformations (94) by $d(E) = 1, d(F), d(I), d(IF)$. For deriving the equations (52) it was necessary only to assume the existence of the transformations of the restricted group, it was not necessary to assume that these are the only transformations. These equations will hold, therefore, for elements of the restricted group, in representations of the extended group also. We normalize the indeterminate factors in $d(F)$ and $d(I)$ so that their squares become unity. Then we have $d(F)d(I) = \omega d(I)d(F)$ or $d(I) = \omega d(F)d(I)d(F)$. Squaring this, one obtains $\omega^2 = \pm 1$. We can set, therefore

$$\begin{aligned} d(IF) &= d(I)d(F) = \pm d(F)d(I) \\ d(F)^2 &= d(I)^2 = 1; \qquad d(IF)^2 = \pm 1. \end{aligned} \tag{95}$$

Finally, from

$$d(F)D(L_1)d(F) = \omega(L_1)D(FL_1F) \tag{96}$$

we obtain, multiplying this with the similar equation for L_2

$$\omega(L_1)\omega(L_2) = \omega(L_1L_2)$$

which, gives $\omega(L) = 1$ as the inhomogeneous Lorentz group (or the group used in (52B)-(52D)) has the only one dimensional representation by the unity (1). In this way, we obtain

$$d(F)D(L)d(F) = D(FLF), \tag{96a}$$

$$d(I)D(L)d(I) = D(ILI), \tag{96b}$$

$$d(IF)D(L)d(IF)^{-1} = D(IFLFI). \tag{96c}$$

B.

Given a representation of the extended Lorentz group, one can perform the transformations described in section 6A, by considering the elements of the restricted group only. We shall consider here only such representations of the extended group, for which, after having introduced the momenta, all representations of the restricted group are either in class 1 or 2, i.e. $P \geqq 0$ but not 0_0. Following then the procedure of section 6, one can find a set of wave functions for which the operators $D(L)$ of the restricted group have one of the forms, given in section 6 as irreducible representations. We shall proceed, next to find the operator $d(F)$. For the wave functions belonging to an irreducible $D(L)$ of the restricted group, we can introduce a complete set of orthonormal functions $\psi_1(p, \zeta), \psi_2(p, \zeta), \cdots$. We then have

$$D(L)\psi_\kappa(p, \zeta) = \sum_\mu D(L)_{\mu\kappa}\psi_\mu(p, \zeta). \tag{97}$$

The infinite matrices $D(L)_{\mu\kappa}$ defined in (97) are unitary and form a representation which is equivalent to the representation by the operators $D(L)$. The $D(L)$, $d(F)$ are, of course, operators, but the $D(L)_{\mu\kappa}$ are components of a matrix, i.e. numbers. We can now form the wave functions $d(F)\psi_1, d(F)\psi_2, d(F)\psi_3, \cdots$ and apply $D(L)$ to these. For (96a) and (97) we have

$$\begin{aligned} D(L)d(F)\psi_\kappa &= d(F)D(FLF)\psi_\kappa \\ &= d(F)\sum_\mu D(FLF)_{\mu\kappa}\psi_\mu \\ &= \sum_\mu D(FLF)_{\mu\kappa}d(F)\psi_\mu . \end{aligned} \tag{97a}$$

The matrices $D°(L)_{\mu\kappa} = D(FLF)_{\mu\kappa}$ give a representation of the restricted group (FLF is an element of the restricted group, we have a new representation by an automorphism, as discussed in section 6B). We shall find out whether $D°(L)$ is equivalent $D(L)$ or not. The translation operation in $D°$ is

$$T°(a) = d(F)T(a)d(F) = T(Fa) \tag{98}$$

which, together with (53) shows that $D°$ has the same P as $D(L)$ itself. In fact, writing

$$U_1\varphi(p, \zeta) = \varphi(Fp, \zeta) \tag{99}$$

one has $U_1^{-1} = U_1$ and one easily calculates $U_1T°(a)U_1 = T(a)$. Similarly for $U_1d°(\Lambda)U_1$ one has

$$\begin{aligned} U_1d°(\Lambda)U_1\varphi(p, \zeta) &= U_1d(F\Lambda F)U_1\varphi(p, \zeta) \\ &= d(F\Lambda F)U_1\varphi(Fp, \zeta) = \sum_\eta Q(Fp, F\Lambda F)_{\zeta\eta}U_1\varphi(F\Lambda^{-1}p, \eta) \\ &= \sum_\eta Q(Fp, F\Lambda F)_{\zeta\eta}\varphi(\Lambda^{-1}p, \eta). \end{aligned} \tag{99a}$$

This means that the similarity transformation with U_1 brings $T°(a)$ into $T(a)$ and $d°(\Lambda)$ into $Q(Fp, F\Lambda F)P(\Lambda)$. Thus the representation of the "little group" in $U_1d°(\Lambda)U_1$ is

$$q°(\lambda) = Q(Fp_0, F\lambda F).$$

For this latter matrix, one obtains from (67a)

$$\begin{aligned} q°(\lambda) &= Q(Fp_0, F\lambda F) = q(\alpha(Fp_0)^{-1}F\lambda F\alpha(Fp_0)) \\ &= q(\lambda°) \end{aligned} \tag{100}$$

where $\lambda°$ is obtained from λ by transforming it with $F\alpha(Fp_0)$.

The representations $D°(L)$ and $D(L)$ are equivalent if the representation

$q(\lambda)$ is equivalent to the representation which coördinates $q(\lambda^\circ)$ to λ. The $\alpha(Fp_0)$ is a transformation of the restricted group which brings p_0 into $\alpha(Fp_0)p_0 = Fp_0$. (Cf. (65).) This transformation is, of course, not uniquely determined but if $\alpha(Fp_0)$ is one, the most general can be written as $\alpha(Fp_0)\iota$, where $\iota p_0 = p_0$ is in the little group. For $q(\iota^{-1}\alpha(Fp_0)^{-1}\,\Lambda\,\alpha(Fp_0)\iota) = q(\iota)^{-1}q(\alpha(Fp_0)^{-1}\,\Lambda\,\alpha(Fp_0))\,q(\iota)$, the freedom in the choice of $\alpha(Fp_0)$ only amounts to a similarity transformation of $q^\circ(\lambda)$ and naturally does not change the equivalence or non equivalence of $q^\circ(\lambda)$ with $q(\lambda)$.

For the case P_+, we can choose p_0 in the direction of the fourth axis, with components 0, 0, 0, 1. Then $Fp_0 = p_0$ and $\alpha(Fp_0) = 1$. The little group is the group of rotations in ordinary space and $F\lambda F = \lambda$. Hence $q^\circ(\lambda) = q(\lambda)$ and $D^\circ(\Lambda)$ is equivalent to $D(\Lambda)$ in this case. The same holds for the representations of class P_-.

For 0_+ we can assume that p_0 has the components 0, 0, 1, 1. Then the components of Fp_0 are 0, 0, -1, 1. For $\alpha(Fp_0)$ we can take a rotation by π about the second axis and $F\alpha(Fp_0)$ will be a diagonal matrix with diagonal elements 1, -1, 1, 1, i.e., a reflection of the second axis. Thus if λ is the transformation in (70), $\lambda^0 = \alpha(Fp_0)^{-1}F\lambda F\alpha(Fp_0)$ is the transformation for which

$$\lambda^0\begin{pmatrix} x_4 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_4 - x_3 \end{pmatrix}\lambda^{0\dagger} = \begin{pmatrix} x_4' + x_3' & x_1' - ix_2' \\ x_1' + ix_2' & x_4' - x_3' \end{pmatrix}. \tag{101}$$

This is, however, clearly $\lambda^0 = \lambda^*$. Thus the operators of $q^0(\lambda)$ are obtained from the operators $q(\lambda)$ by (cf. (71a))

$$\begin{aligned} t^0(x, y) &= t(x, -y) \\ \delta^0(\beta) &= \delta(-\beta). \end{aligned} \tag{101a}$$

For the representations 0_{+s} with discrete s, the $q^0(\lambda)$ and $q(\lambda)$ are clearly inequivalent as $\delta^0(\beta) = (e^{-is\beta})$ and $\delta(\beta) = (e^{is\beta})$, except for $s = 0$, when they are equivalent. For the representations $0_+(\Xi)$, $0_+'(\Xi)$, the $q^0(\lambda)$ and $q(\lambda)$ are equivalent, both in the single valued and the double valued case, as the substitution $\eta \rightarrow -\eta$ transforms them into each other. The same holds for representations of the class 0_-. If $D^0(L)$ and $D(L)$ are equivalent

$$U^{-1}D^0(L)U = D(L), \tag{102}$$

the square of U commutes with all $D(L)$. As a consequence of this, U^2 must be a constant matrix. Otherwise, one could form, in well known manner,[27] an idempotent which is a function of U^2 and thus commutes with $D(L)$ also. Such an idempotent would lead to a reduction of the representation $D(L)$ of the restricted group. As a constant is free in U, we can set

$$U^2 = 1 \tag{102a}$$

[27] J. von Neumann, Ann. of Math. 32. 191. 1931; ref. 2. p. 89.

C.

Returning now to equation (97a), if $D^0(L) = D(FLF)$ and $D(L)$ are equivalent ($P > 0$ or 0_+, 0_- with continuous Ξ or $s = 0$) there is a unitary matrix $U_{\mu\nu}$, corresponding to U, such that

$$\sum_\mu D(FLF)_{\kappa\mu} U_{\mu\nu} = \sum_\mu U_{\kappa\mu} D(L)_{\mu\nu} \tag{102b}$$
$$\sum_\mu U_{\kappa\mu} U_{\mu\nu} = \delta_{\kappa\nu} .$$

Let us now consider the functions

$$\varphi_\nu = \psi_\nu + \sum_\mu U_{\mu\nu} d(F)\psi_\mu . \tag{103}$$

Applying $D(L)$ to these

$$\begin{aligned} D(L)\varphi_\nu &= D(L)\psi_\nu + \sum_\mu U_{\mu\nu} D(L) d(F)\psi_\mu \\ &= D(L)\psi_\nu + \sum_\mu U_{\mu\nu} d(F) D(FLF)\psi_\mu \\ &= \sum_\mu D(L)_{\mu\nu}\psi_\mu + \sum_{\mu\kappa} U_{\mu\nu} d(F) D(FLF)_{\kappa\mu}\psi_\kappa \\ &= \sum_\mu D(L)_{\mu\nu}\left(\psi_\mu + \sum_\kappa U_{\kappa\mu} d(F)\psi_\kappa\right) = \sum_\mu D(L)_{\mu\nu}\varphi_\mu . \end{aligned} \tag{103a}$$

Similarly

$$\begin{aligned} d(F)\varphi_\nu = d(F)\psi_\nu + \sum_\mu U_{\mu\nu}\psi_\mu \\ = \sum_\mu U_{\mu\nu}\left(\psi_\mu + \sum_\kappa U_{\kappa\mu} d(F)\psi_\kappa\right) = \sum_\mu U_{\mu\nu}\varphi_\mu . \end{aligned} \tag{103b}$$

Thus the wave functions φ transform according to the representation in which $D(L)_{\mu\nu}$ corresponds to L and $U_{\mu\nu}$ to $d(F)$. The same holds for the wave functions

$$\varphi_\nu' = \psi_\nu - \sum_\mu U_{\mu\nu} d(F)\psi_\mu , \tag{104}$$

except that in this case $(-U_{\mu\nu})$ corresponds to $d(F)$. The ψ_ν and $d(F)\psi_\nu$ can be expressed by the φ and φ'. If the ψ and $d(F)\psi$ were linearly independent, the φ and φ' will be linearly independent also. If the $d(F)\psi$ were linear combinations of the ψ, either the φ or the φ' will vanish.

If we imagine a unitary representation of the group formed by the L and FL in the form in which it is completely reduced out as a representation of the group of restricted transformations L, the above procedure will lead to a reduction of that part of the representation of the group of the L and FL, for which $D(L)$ and $D(FLF)$ are equivalent.

If $D(L)_{\mu\nu}$ and $D^\circ(L)_{\mu\nu}$ are inequivalent, the ψ_κ and $d(F)\psi_\nu = \psi_\nu'$ are orthogonal. This is again a generalization of the similar rule for finite unitary representations.[28] One can see this in the following way: Denoting $M_{\kappa\nu} = (\psi_\kappa, \psi'_\nu)$ one has

$$\begin{aligned} M_{\kappa\nu} &= (\psi_\kappa, \psi'_\nu) = (D(L)\psi_\kappa, D(L)\psi'_\nu) \\ &= \sum_{\mu\lambda} D(L)^*_{\mu\kappa} D^\circ(L)_{\lambda\nu} M_{\mu\lambda}; \\ M &= D(L)^\dagger M D^\circ(L). \end{aligned}$$

Hence

$$D(L)M = MD^\circ(L); \qquad M^\dagger D(L) = D^\circ(L)M^\dagger. \tag{105}$$

From these, one easily infers that $MM^\dagger$ commutes with $D(L)$, and $M^\dagger M$ commutes with $D^\circ(L)$. Hence both are constant matrices, and if neither of them is zero, M and $M^\dagger$ are, apart from a constant, unitary. Thus $D(L)$ would be equivalent $D^\circ(L)$ which is contrary to supposition. Hence $MM^\dagger = 0$, $M = 0$ and the ψ are orthogonal to the $d(F)\psi = \psi'$. Together, they give a representation of the group formed by the restricted Lorentz group and F. If they do not form a complete set, the reduction can be continued as before.

One sees, thus, that introducing the operation F "doubles" the number of dimensions of the irreducible representations in which the little group was the two dimensional rotation group, while it does not increase the underlying linear manifold in the other cases. This is analogous to what happens, if one adjoins the reflection operation to the rotation groups themselves.[29]

D.

The operations $d(I)$ can be determined in the same manner as the $d(F)$ were found. A complete set of orthonormal functions corresponding to an irreducible representation of the group formed by the L and FL shall be denoted by ψ_1, $\psi_2, \cdots$. For this, we shall assume (97) again, although the $D(L)$ contained therein is now not necessarily irreducible for the restricted group alone but contains, in case of 0_{+s} or 0_{-s} and finite s, both s and $-s$. We shall set, furthermore

$$d(F)\psi_\kappa = \sum_\mu d(F)_{\mu\kappa}\psi_\mu . \tag{106}$$

We can form then the functions $d(I)\psi_1$, $d(I)\psi_2, \cdots$. The consideration, contained in (97a) shows that these transform according to $D(ILI)_{\mu\kappa}$ for the transformation L of the restricted group:

$$D(L)d(I)\psi_\kappa = \sum_\mu D(ILI)_{\mu\kappa} d(I)\psi_\mu . \tag{106a}$$

Choosing for L a pure translation, a consideration analogous to that performed in (98) shows that the set of momenta in the representation $L \to D\ (ILI)$ has the opposite sign to the set of momenta in the representation $D(L)$. If the latter

[28] Cf. e.g. E. Wigner, ref. 4, Chapter XII.

[29] I. Schur, Sitz. d. kön. Preuss. Akad. pages 189, 297, 1924.

belongs to a positive subclass, the former belongs to the corresponding negative subclass and conversely. Thus the adjunction of the transformation I always leads to a "doubling" of the number of states, the states of "negative energy" are attached to the system of possible states. One can describe all states ψ_1, $\psi_2, \cdots, d(I)\psi_1, d(I)\psi_2, \cdots$ by introducing momenta p_1, p_2, p_3, p_4 and restricting the variability domain of p by the condition $\{p, p\} = P$ alone without stipulating a definite sign for p_4.

As we saw before, the $d(I)\psi_1$, $d(I)\psi_2$, are orthogonal to the original set of wave functions $\psi_1, \psi_2, \cdots$. The result of the application of the operations $D(L)$ and $d(F)$ to the $\psi_1, \psi_2, \cdots$ (i.e., the representation of the group formed by the L, FL) was given in part C. The $D(L)d(I)\psi_\kappa$ are given in (106a). On account of the normalization of $d(I)$ we can set

$$d(I)d(I)\psi_\kappa = \psi_\kappa . \tag{106b}$$

For $d(F)d(I)\psi_\kappa$ we have two possibilities, according to the two possibilities in (95). We can either set

$$d(F)d(I)\psi_\kappa = d(I)d(F)\psi_\kappa = \sum_\mu d(F)_{\mu\kappa}d(I)\psi_\mu , \tag{107}$$

or

$$d(F)\cdot d(I)\psi_\kappa = -d(I)d(F)\psi_\kappa = -\sum_\mu d(F)_{\mu\kappa}d(I)\psi_\mu . \tag{107a}$$

Strictly speaking, we thus obtain two different representations. The system of states satisfying (107) could be distinguished from the system of states for which (107a) is valid, however, only if we could really perform the transition to a new coördinate system by the transformation I. As this is, in reality, impossible, the representations distinguished by (107) and (107a) are not different in the same sense as the previously described representations are different.

I am much indebted to the Wisconsin Alumni Research Foundation for their aid enabling me to complete this research.

MADISON, WIS.