LECTURE 1

Special relativity is based on <u>2 AXIOMS</u> :

① GALILEO'S PRINCIPLE OF RELATIVITY

There exist coordinate systems, called <u>INERTIAL</u>, such that

a. All coordinate systems in UNIFORM RECTILINEAR MOTION with respect to another inertial system are inertial; and

b. the laws of physics at any time are the same in all inertial systems.

These statements are equivalent to say that:
* SPACE IS HOMOGENEOUS (no special point) and ISOTROPIC (no special direction)
* TIME IS HOMOGENEOUS (no preferred time)

Galileo's principle of relativity is the axiom of non relativistic mechanics.
We will now add Einstein's crucial insight:

② EINSTEIN'S PRINCIPLE

The speed of light, c, is <u>the same</u> in any inertial system.

In a certain inertial frame we describe physical phenomena taking place at a certain time t and point in space $\vec{x}$. We set

$$\boxed{x^\mu = (ct, \vec{x})} \qquad \mu = 0, 1, 2, 3$$

This is a FOUR-VECTOR which gives the coordinates of an EVENT taking place at time t and point $\vec{x}$

An event is thus a point in SPACETIME (also called MINKOWSKI SPACE).

Note that we inserted a factor of c in front of t so that $[x^\mu] =$ LENGTH.

In order to understand the implications of the relativity principle we need to specify how coordinates transform between TWO INERTIAL FRAMES. We assume the transformation to be linear:

$$\boxed{x'^\mu = \Lambda^\mu{}_\nu x^\nu}$$

Λ is x-independent and ν is summed over (Einstein's Convention: repeated indices are summed over)

We choose a LINEAR transformation so that a uniform rectilinear motion in one frame is seen as a uniform rectilinear motion in another frame.

Now we discuss how to CONSTRAIN THE FORM OF Λ starting from the axioms we have discussed. We will discover Lorentz transformations!

DERIVING THE INVARIANCE OF INTERVALS

Let's consider two events differing by $\Delta\vec{x}$ and Δt. We define the

$$\text{INVARIANT LENGTH} \equiv \Delta S^2 \equiv (c\Delta t)^2 - (\Delta\vec{x})^2 \equiv (\Delta x_0)^2 - (\Delta\vec{x})^2$$

We now show that $\Delta S' = \Delta S$.

First if the two events are associated to a particle travelling at the speed of light then because the speed of light is the same in any inertial frame we must have

$$c^2(t_2'-t_1')^2 - (\vec{x}_2'-\vec{x}_1')^2 = c^2(t_2-t_1)^2 - (\vec{x}_2-\vec{x}_1)^2, \text{ or } \boxed{\Delta S' = \Delta S}$$

In this case ΔS was $=0$. If the particle's velocity is less than the speed of light we could say that

$$\Delta S' = \alpha(|\vec{v}|)\,\Delta S \qquad \text{so that } \Delta S' = 0 \text{ if } \Delta S = 0.$$

The function α could depend on the modulus of the relative velocity of the two frames. Not on the positions or on the direction of $\vec{v}$ in order not to violate homogeneity and isotropy of space (i.e. no preferred point or direction).
By swapping the roles of the two frames one gets $\Delta S = \alpha(|\vec{v}|)\,\Delta S'$ $\Rightarrow$ $\alpha^2 = 1$.

Because we consider coordinate transformations that are continuously connected to the identity transformation we conclude that $\alpha = 1$ and hence

$$\boxed{\Delta S' = \Delta S.}$$

LORENTZ TRANSFORMATIONS

These are defined as those transformations leaving the invariant length ΔS ... invariant!
Remember the physical origin : c must be the same in any inertial frame.
The invariant length

$$\Delta S = (c\Delta t)^2 - (\Delta\vec{x})^2 = (\Delta x^0)^2 - (\Delta\vec{x})^2$$

Can be written as

$$\boxed{\Delta S = g_{\mu\nu}\,\Delta x^\mu \Delta x^\nu}$$

where

$$\boxed{g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}}$$

- $g_{\mu\nu}$ is called the (flat) METRIC

The condition $\Delta S' = \Delta S$ can be rewritten as

$$\Delta S' = g_{\mu\nu}\,\Delta x^{\mu\prime}\Delta x^{\nu\prime} = \Delta S = g_{\mu\nu}\,\Delta x^\mu \Delta x^\nu$$

Since $\Delta x^{\mu\prime} = \Lambda^\mu{}_\rho\,\Delta x^\rho$ we get

$$g_{\mu\nu}\,\Lambda^\mu{}_\rho\,\Lambda^\nu{}_\sigma\,\Delta x^\rho \Delta x^\sigma = g_{\rho\sigma}\,\Delta x^\rho \Delta x^\sigma$$

For this to be valid for any choice of Δx, we must have

$$\boxed{g_{\mu\nu}\,\Lambda^\mu{}_\rho\,\Lambda^\nu{}_\sigma = g_{\rho\sigma}}$$

or

$$\boxed{\Lambda^{\mu}{}_{\rho}\, g_{\mu\nu}\, \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}}$$ or

$$(\Lambda^T)_{\rho}{}^{\mu}\, g_{\mu\nu}\, \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}$$ i.e., in

matrix form,

$$\boxed{\Lambda^T g \Lambda = g}$$

This equation defines the LORENTZ GROUP

GROUP

DEF. of GROUP G :

A set of elements, Λ equipped with a COMPOSITION LAW "$\circ$" such that $\forall\, \Lambda_1, \Lambda_2 \in G$ the composition $\Lambda_1 \circ \Lambda_2 \in G$ and such that the following is true:

1) $\Lambda_1 \circ (\Lambda_2 \circ \Lambda_3) = (\Lambda_1 \circ \Lambda_2) \circ \Lambda_3$ ASSOCIATIVITY

2) $\exists$ unit element $\mathbb{1}$ (often called e) such that

$$\Lambda \circ \mathbb{1} = \mathbb{1} \circ \Lambda = \Lambda$$ EXISTENCE OF UNIT ELEMENT

3) $\forall\, \Lambda \in G$, $\exists\, \Lambda^{-1}$ such that

$$\Lambda^{-1} \circ \Lambda = \Lambda \circ \Lambda^{-1} = \mathbb{1}$$ EXISTENCE of INVERSE

<u>Note</u> : in general

$$\Lambda_1 \circ \Lambda_2 \neq \Lambda_2 \circ \Lambda_1 \quad !$$

If this happens $\forall \Lambda_1, \Lambda_2$ then the group is called ABELIAN.

<u>Ex</u> : show that the Lorentz transformations, satisfying $\Lambda^T g \Lambda = g$,

form indeed a group.

From $\Lambda^T g \Lambda = g$ it follows that

$$\underbrace{\det \Lambda^T}_{= \det \Lambda} \det \Lambda \det g = \det g \quad \Rightarrow \quad (\det \Lambda)^2 = 1 \quad \Rightarrow$$

$$\boxed{\det \Lambda = \pm 1}$$

It follows that the Lorentz group splits into 2 components :

- $\det \Lambda = +1$ PROPER TRANSFORMATIONS continuously connected to $\mathbb{1}$ i.e. each Λ can be written as $\Lambda_{proper} = \mathbb{1} + \varepsilon + O(\varepsilon^2)$
- $\det \Lambda = -1$ IMPROPER TRANSFORMATIONS.

<u>Ex.</u> Show that the proper Lorentz transformations form a group

An example of IMPROPER Lorentz transformation is PARITY, which acts on $(x_0, \vec{x})$ as

$$(x_0, \vec{x}) \xrightarrow{P} (x_0, -\vec{x})$$

Within the proper transformations, there are <u>2 DISCONNECTED COMPONENTS</u>:

Those with $\Lambda^0{}_0 > 0$ ← "ORTHOCRONOUS"
Those with $\Lambda^0{}_0 < 0$

Let's see this more precisely. Firstly from

$\Lambda^\mu{}_\rho \, g_{\mu\nu} \Lambda^\nu{}_\sigma = g_{\rho\sigma}$ taking $\rho = \sigma = 0$

$\Lambda^\mu{}_0 \, g_{\mu\nu} \Lambda^\nu{}_0 = 1$ or (use $g_{ij} = -\delta_{ij}$)

$\Lambda^0{}_0 \Lambda^0{}_0 - \Lambda^i{}_0 \Lambda^i{}_0 = 1 \Rightarrow$

$(\Lambda^0{}_0)^2 = 1 + \Lambda^i{}_0 \Lambda^i{}_0$,

$\Lambda^0{}_0 = \pm \sqrt{1 + \Lambda^i{}_0 \Lambda^i{}_0}$

ANOTHER USEFUL RELATION

Lorentz transformations satisfy $\boxed{\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}}$
or $\Lambda^T \eta \Lambda = \eta$.
We now show that they also satisfy

$$\boxed{\Lambda^\mu{}_\rho \eta^{\rho\sigma} \Lambda^\nu{}_\sigma = \eta^{\mu\nu}}$$

or $\Lambda \tilde{\eta} \Lambda^T = \tilde{\eta}$ (with $\tilde{\eta} \to \eta^{\mu\nu}$)

Multiply the defining condition $\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ by $\eta^{\sigma\tau} \Lambda^\lambda{}_\tau$. We get

$$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma \eta^{\sigma\tau} \Lambda^\lambda{}_\tau = \eta_{\rho\sigma} \eta^{\sigma\tau} \Lambda^\lambda{}_\tau = \Lambda^\lambda{}_\rho$$

$=$ rewrite RHS as $\Lambda^\mu{}_\rho \delta^\lambda_\mu = \Lambda^\mu{}_\rho \eta_{\mu\nu} \eta^{\nu\lambda}$

$$\Rightarrow (\Lambda^\mu{}_\rho \eta_{\mu\nu}) \Lambda^\nu{}_\sigma \eta^{\sigma\tau} \Lambda^\lambda{}_\tau = (\Lambda^\mu{}_\rho \eta_{\mu\nu}) \eta^{\nu\lambda}$$

and hence $\boxed{\Lambda^\nu{}_\sigma \eta^{\sigma\tau} \Lambda^\lambda{}_\tau = \eta^{\nu\lambda}}$

or $\Lambda \tilde{\eta} \Lambda^T = \tilde{\eta}$.

In matrix form (faster!): start from

$\Lambda^T \eta \Lambda = \eta$ and multiply by $\tilde{\eta} \Lambda^T$: we get

$$\Lambda^T \eta \Lambda \tilde{\eta} \Lambda^T = \underbrace{\eta \tilde{\eta}}_{\delta} \Lambda^T = \Lambda^T = \Lambda^T \eta \tilde{\eta} \Rightarrow$$

$$(\Lambda^T \eta) \Lambda \tilde{\eta} \Lambda^T = (\Lambda^T \eta) \tilde{\eta} \Rightarrow \boxed{\Lambda \tilde{\eta} \Lambda^T = \tilde{\eta}}$$

One important consequence is the following.
Take the $\nu=\lambda=0$ component of the new relation we found. This gives

$$\Lambda^0{}_\sigma \eta^{\sigma\tau} \Lambda^0{}_\tau = 1 \quad \text{or} \quad \boxed{\left(\Lambda^0{}_0\right)^2 - \Lambda^0{}_i \Lambda^0{}_i = 1},$$

a relation very similar to the one we derived earlier, namely $(\Lambda^0{}_0)^2 - \Lambda^i{}_0 \Lambda^i{}_0 = 1$
except that now $\Lambda^0{}_i$ appears instead of $\Lambda^i{}_0$.

We are now going to use these facts to prove a very important property of Lorentz transformations namely the fact that

THE PRODUCT OF TWO ORTHOCRONOUS TRANSFORMATION IS AN ORTHOCRONOUS TRANSFORMATION.

$\longrightarrow$

To this end, consider $\Lambda \equiv U \circ V$

with U and V being both orthochronous Lorentz transformations. Then

$(UV)^0{}_0 > 1$ is what we have to show.

$$(UV)^0{}_0 = U^0{}_\mu V^\mu{}_0 = U^0{}_0 V^0{}_0 + U^0{}_i V^i{}_0$$

Let's call $\vec{U} = (U^0{}_1, U^0{}_2, U^0{}_3)$, $\vec{V} = (V^1{}_0, V^2{}_0, V^3{}_0)$

Then $(UV)^0{}_0 = U^0{}_0 V^0{}_0 + \vec{U}\cdot\vec{V}$

Now recall that for any Lorentz transformation

$(\Lambda^0{}_0)^2 - \Lambda^i{}_0 \Lambda^i{}_0 = 1$ and also

$(\Lambda^0{}_0)^2 - \Lambda^0{}_i \Lambda^0{}_i = 1$, Use this to write

$(V^0{}_0)^2 - 1 = \vec{V}^2$ and $(U^0{}_0)^2 - 1 = \vec{U}^2$

The triangular inequality tells us that

$$|\vec{U}\cdot\vec{V}| \leq |U||V| = \sqrt{(U^0{}_0)^2-1}\,\sqrt{(V^0{}_0)^2-1} \Rightarrow$$

$$(UV)^0{}_0 = U^0{}_0 V^0{}_0 + \vec{U}\cdot\vec{V} \geq U^0{}_0 V^0{}_0 - |\vec{U}||\vec{V}| =$$

$$= U^0{}_0 V^0{}_0 - \sqrt{(U^0{}_0)^2-1}\,\sqrt{(V^0{}_0)^2-1} \geq 1$$

The last statement can be quickly proved by noticing that, since U and V are orthochronous, one can set

$$U^0{}_0 = \cosh x \,, \quad V^0{}_0 = \cosh y$$

(recall that $\cosh(\text{anything}) \geq 1$ always!)

Since $\sqrt{\cosh^2 x - 1}\,\sqrt{\cosh^2 y - 1} = \pm \sinh x \sinh y$

we get

$$\cosh x \cosh y - \sqrt{\cosh^2 x - 1}\,\sqrt{\cosh^2 y - 1} =$$

$$= \cosh x \cosh y \mp \sinh x \sinh y =$$

$$= \cosh(x \mp y) \geq 1 \; !$$

This is what we wanted to show.

An important property of orthocronous transformations is :

> ORTHOCRONOUS TRANSFORMATIONS LEAVE THE <u>SIGN</u> OF THE TIME COMPONENT OF A TIMELIKE VECTOR UNCHANGED.

To see this let v^μ be a timelike vector, $v^\mu v_\mu \equiv v^2 > 0$. Under a Lorentz transformation

$$v^0 \to v^{0'} = \Lambda^0{}_\mu v^\mu = \Lambda^0{}_0 v^0 + \Lambda^0{}_i v^i \equiv$$

$$\equiv \Lambda^0{}_0 v^0 + \vec{\Lambda} \cdot \vec{v} \quad \text{where I set } \vec{\Lambda}_i \equiv \Lambda^0{}_i$$

By Schwartz inequality $|\vec{\Lambda} \cdot \vec{v}| \leq |\vec{\Lambda}|\,|\vec{v}|$

Let's consider first $v^0 > 0$. Then

$$v^{0'} \geq \Lambda^0{}_0 v^0 - |\vec{\Lambda}|\,|\vec{v}|$$

Now $(\Lambda^0{}_0)^2 = 1 + \Lambda^0{}_i \Lambda^0{}_i = 1 + \vec{\Lambda}^2$ and

since $\Lambda^0{}_0 > 0 \Rightarrow \Lambda^0{}_0 = +\sqrt{1+\vec{\Lambda}^2} \Rightarrow$

$$v^{0'} \geq \sqrt{1+\vec{\Lambda}^2}\, v^0 - |\vec{\Lambda}|\,|\vec{v}| \geq |\vec{\Lambda}|\left(v^0 - |\vec{v}|\right)$$

But $v^0 > |\vec{v}|$ since $v_\mu v^\mu > 0$ and $v_0 > 0 \Rightarrow$

$\boxed{v^{0'} > 0} \Rightarrow$ the sign of $v^{0'}$ is the same as v^0.

If v^0 was negative, $v^0 < 0$, then

$$v^{0'} = \Lambda^0{}_0 v^0 + \vec{\Lambda} \cdot \vec{v} \leq \Lambda^0{}_0 v^0 + |\vec{\Lambda}|\,|\vec{v}| =$$

$$= \sqrt{1+\vec{\Lambda}^2}\, v^0 + |\vec{\Lambda}|\,|\vec{v}| \leq |\vec{\Lambda}| \left(v^0 + |\vec{v}|\right)$$

But $$v^2 = \underbrace{(\underset{<0}{v^0 - |\vec{v}|})}_{<0} \underbrace{(v^0 + |\vec{v}|)} > 0$$

thus $v^0 + |\vec{v}| < 0$

$\Rightarrow$ $v^{0'} < 0$ again confirming that

an orthochronous Lorentz transformation does not change the sign of the time component of a timelike vector.

What do orthocronous transformations represent?

Consider the events of coordinates $(t, \vec{0})$. As t varies they describe the story of a clock at rest at the origin.

Now let's see the same event from a different frame related by a Lorentz transformation Λ. Then

$\boxed{t' = \Lambda^0{}_0 \, t}$ $\Rightarrow$ the hands of the

clocks in O' move in the same direction as those in O: the time arrow is not reversed by the ORTHOCRONOUS transformations.

An example of non-orthocronous transformation is TIME-REVERSAL:

$$T : (x_0, \vec{x}) \longrightarrow (x_0', \vec{x}') = (-x_0, \vec{x}).$$

THE RELATIVITY PRINCIPLE states that

$$\boxed{\text{THE LAWS OF PHYSICS ARE INVARIANT UNDER PROPER, ORTHOCRONOUS LORENTZ TRANSFORMATIONS}}$$

SUMMARISING

All Lorentz transformations fall into 3 classes:

1. PROPER ORTHOCRONOUS (AKA <u>RESTRICTED</u>), $L_+^\uparrow$

$\det \Lambda = +1 \qquad \Lambda^0{}_0 \geq 1$

2. PROPER NON-ORTOCHRONOUS, $L_+^\downarrow$ (e.g. $x^\mu \to -x^\mu$)

$\det \Lambda = +1, \quad \Lambda^0{}_0 \leq -1$

3. IMPROPER ORTHOCRONOUS, $L_-^\uparrow$

$\det \Lambda = -1, \quad \Lambda^0{}_0 \geq 1 \qquad$ [eg PARITY]

4. IMPROPER NON-ORTHOCRONOUS $L_-^\downarrow$

$\det \Lambda = -1 \qquad \Lambda^0{}_0 \leq -1 \qquad$ [eg TIME REVERSAL]

- The orthocronous transf. $L_+^\uparrow \cup L_-^\uparrow$ are a subgroup
- The proper transf $L_+^\uparrow \cup L_+^\downarrow$ are a ~~group~~ sub group
- The restricted transf. $L_+^\uparrow$ are a subgroup.

NOTE :

For a long time physicists thought that PARITY would be an invariance of the laws of physics. Classical physics is indeed invariant under P. However, WEAK INTERACTIONS violate P. This was suggested by Lee and Yang (Nobel prize) in relation to the decay of NEUTRAL K mesons and shown experimentally by Mme Wu in 1957.

EXAMPLES OF LORENTZ TRANSFORMATIONS

We make 2 examples of particular transformations

1. ROTATIONS. These transf do not touch time. So, separating

$\mu = (0, i)$ $\quad i = 1 2 3$ then our condition

$$\Lambda^T g \Lambda = g$$

becomes, calling $\Lambda \to R$

(recall $g = \mathrm{diag}\,(1, -1 -1 -1)$)

$$\boxed{R^T R = \mathbb{1}}$$

which is indeed the definition of a rotation. For instance, a rotation by an angle θ about the z axis has the form

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

② BOOSTS   A boost along the x-axis for instance.

This corresponds to a matrix of the form

$$\Lambda = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In order to be a boost we need $\Lambda^T g \Lambda = g$

or $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or

$a^2 - c^2 = 1$ $ab - cd = 0$

$b^2 - d^2 = -1$

Usually solved by setting $a = \cosh\theta = d$
$b = -\sinh\theta = c$

Then we have

$x^{0'} = \cosh\theta\, x^0 - \sinh\theta\, x^1$
$x^{1'} = -\sinh\theta\, x^0 + \cosh\theta\, x^1$ or

$$\begin{pmatrix} x^{0'} \\ x^{1'} \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

NOTE : Usually the transformations are written as

$\boxed{\begin{aligned} x^{0'} &= \gamma(x^0 - \beta x^1) \\ x^{1'} &= \gamma(x^1 - \beta x^0) \end{aligned}}$ with $\boxed{\begin{aligned} \beta &= \frac{v}{c} \\ \gamma &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}}$

$\gamma = \cosh\theta$, $\gamma\beta = \sinh\theta$.

v is the velocity of the 2nd inertial frame wrt the first inertial frame.

Indeed we can set $x' = 0$ to study the motion of a point at rest in O' from the perspective of O. Then we get

$$x' = 0 = \gamma(x - \beta x^0) \Rightarrow \beta = \frac{x}{x^0} = \frac{v}{c}$$

Because this is $-\sinh\theta \, x^0 + \cosh\theta \, x = 0$

we also have $\frac{x}{x^0} = \tanh\theta$ or

$\boxed{\beta = \tanh\theta}$ and also $\gamma = \cosh\theta$

so that indeed $\gamma\beta = \tanh\theta \cosh\theta = \sinh\theta$.

<u>Note</u> : θ is called <u>RAPIDITY</u>

VECTORS : COVARIANT VS CONTRAVARIANT

$x^\mu \equiv (x^0, \vec{x})$ CONTRAVARIANT VECTOR

We define $\boxed{x_\mu = g_{\mu\nu} x^\nu} \equiv (x^0, -\vec{x})$

COVARIANT VECTOR.

Of course $x \cdot y = x^\mu y^\nu g_{\mu\nu} = x^\mu y_\mu = x_\mu y^\mu$

is a Lorentz invariant.

HOW DO COVARIANT VECTORS TRANSFORM?

We start from $x_\mu \equiv g_{\mu\nu} x^\nu$ and

$x'_\mu = g_{\mu\nu} x^{\nu'} = g_{\mu\nu} \Lambda^\nu{}_\rho x^\rho = (g\Lambda) x$

Use $\Lambda^T g \Lambda = g$ or

$g\Lambda = (\Lambda^T)^{-1} g \Rightarrow$

$g_{\mu\nu} \Lambda^\nu{}_\rho = (\Lambda^T)^{-1}{}_\mu{}^\nu g_{\nu\rho} = (\Lambda^{-1})^\nu{}_\mu g_{\nu\rho} \Rightarrow$

$x'_\mu = (\Lambda^{-1})^\nu{}_\mu g_{\nu\rho} x^\rho = (\Lambda^{-1})^\nu{}_\mu x_\nu =$

$= x_\nu (\Lambda^{-1})^\nu{}_\mu$

$\boxed{x'_\mu = x_\nu (\Lambda^{-1})^\nu{}_\mu}$ COVARIANT VECTORS TRASFORM WITH Λ^{-1}

Redone faster:

We have $x'_\mu = g_{\mu\nu} \Lambda^\nu{}_\rho g^{\rho\sigma} x_\sigma$

or $X' = g \Lambda g X$ where X denotes x_μ.

Then use $\Lambda^T g \Lambda = g$ to write

$\Lambda^T g \Lambda g = g^2 \equiv \mathbb{1}$ or $g \Lambda g = (\Lambda^T)^{-1} = (\Lambda^{-1})^T$

thus

$X' = (\Lambda^{-1})^T X$ or, with indices

$$x'_\mu = \left((\Lambda^{-1})_\mu{}^\nu\right)^T x_\nu = x_\nu \Lambda^{-1\,\nu}{}_\mu$$

$$\boxed{x'_\mu = x_\nu \Lambda^{-1\,\nu}{}_\mu}$$

NUMBER OF PARAMETERS of LORENTZ GROUP

Recall our definition: $\Lambda^T g \Lambda = g$ or

$$\Lambda^\mu{}_\rho \, g_{\mu\nu} \Lambda^\nu{}_\sigma = g_{\rho\sigma}$$

Write an INFINITESIMAL Lorentz transformation as

$$\Lambda^\mu{}_\rho = g^\mu{}_\rho + \alpha^\mu{}_\rho + O(\alpha^2) = \delta^\mu{}_\rho + \alpha^\mu{}_\rho + \cdots$$

Then we get

$$(\delta^\mu{}_\rho + \alpha^\mu{}_\rho) \, g_{\mu\nu} (\delta^\nu{}_\sigma + \alpha^\nu{}_\sigma) = g_{\rho\sigma}$$ (only valid up to $O(\alpha)$)

$$\Rightarrow \delta^\mu{}_\rho \, g_{\mu\nu} \delta^\nu{}_\sigma + \alpha^\mu{}_\rho \, g_{\mu\nu} \delta^\nu{}_\sigma + \delta^\mu{}_\rho \, g_{\mu\nu} \alpha^\nu{}_\sigma = g_{\rho\sigma}$$

(discard $O(\alpha^2)$) $\Rightarrow$

$$\cancel{g_{\rho\sigma}} + \alpha_{\sigma\rho} + \alpha_{\rho\sigma} = \cancel{g_{\rho\sigma}} \Rightarrow$$

$$\boxed{\alpha_{\rho\sigma} = -\alpha_{\sigma\rho}}$$

i.e. α is a 4X4 ANTISYMMETRIC MATRIX $\Rightarrow$

$$\boxed{\frac{4 \cdot 3}{2} = 6 \text{ INDIPENDENT PARAMETERS}}$$

$\Rightarrow$ The Lorentz group has 6 parameters.

$$\boxed{\Lambda^i{}_j \text{: 3 rotations}, \quad \Lambda^i{}_0 \text{ 3 boosts}}$$

TRANSFORMATION OF DERIVATIVES

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x^{\mu'}} = \frac{\partial}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^{\mu'}}$$

$$\frac{\partial x^{\mu'}}{\partial x^\nu} = \Lambda^\mu{}_\nu \quad \text{and} \quad \frac{\partial x^\nu}{\partial x^{\mu'}} = \left(\Lambda^{-1}\right)^\nu{}_{\mu'} \quad \text{so}$$

$$\boxed{\frac{\partial}{\partial x^{\mu'}} = \frac{\partial}{\partial x^\nu} \left(\Lambda^{-1}\right)^\nu{}_{\mu'}} \qquad \text{hence}$$

$\frac{\partial}{\partial x^\mu}$ transforms as a <u>COVARIANT VECTOR</u>

$\Rightarrow$ call it ∂_μ

Note: $\partial_\mu \equiv (\partial_0, \vec{\nabla})$ (for $c=1$)

TENSORS

These are objects with many indices (of contravariant and covariant type). Each index is transformed in the way we have just found. For instance

$$F^{\mu\nu} \longrightarrow F^{\mu'\nu'} = \Lambda^{\mu'}{}_\mu \Lambda^{\nu'}{}_\nu F^{\mu\nu}, \quad \text{or}$$

↑ TWO-TENSOR

$$M^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m} \to M^{\mu_1' \cdots \mu_n'}{}_{\nu_1' \cdots \nu_m'} = \Lambda^{\mu_1'}{}_{\mu_1} \cdots \Lambda^{\mu_n'}{}_{\mu_n} \left(\Lambda^{-1}\right)^{\nu_1}{}_{\nu_1'} \cdots \left(\Lambda^{-1}\right)^{\nu_m}{}_{\nu_m'} \cdot M^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m}$$

LEVI-CIVITA TENSOR

$\varepsilon^{\mu\nu\rho\lambda}$ (totally antisymmetric)

$\downarrow$ transforms as

$$\varepsilon^{\mu\nu\rho\lambda} \to \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} \Lambda^{\rho}{}_{\gamma} \Lambda^{\lambda}{}_{\delta} \, \varepsilon^{\alpha\beta\gamma\delta} =$$

$$= \det\Lambda \; \varepsilon^{\mu\nu\rho\lambda} = \quad \text{if } \det\Lambda = +1$$

then $\varepsilon^{\mu\nu\rho\lambda} \to + \varepsilon^{\mu\nu\rho\lambda}$

Because of the appearance of $\det\Lambda$ we call this a <u>PSEUDOTENSOR</u>

MICROCAUSALITY

Consider 2 events separated by an interval

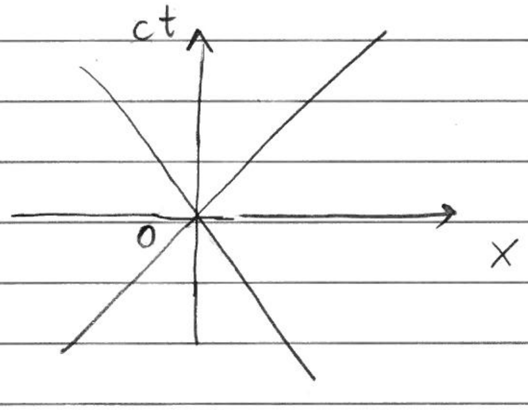
$$\Delta S = (c\Delta t)^2 - (\Delta \vec{x})^2 = (\Delta x_0)^2 - (\Delta \vec{x})^2 =$$

$$= g_{\mu\nu} \Delta x^\mu \Delta x^\nu .$$

There are several cases:

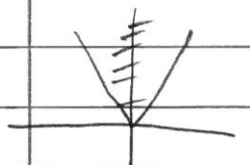
1) $\Delta S = 0$: the 2 events are connected by the propagation of a light ray.

They are on the surface of two cones (LIGHT CONES)



these are described by all trajectories of light rays from O (future cone) or to O (past cone).

2) Events with $\Delta S > 0$ $\Delta t > 0$: the INSIDE of the upper cone:



All points inside such cone can be reached by a signal emitted from O which travels at speed $\beta < 1$

Since $\Delta S = c^2 \Delta t^2 - (\Delta \vec{x})^2 > 0$

So these events can be influenced by the event at O

3) $\Delta S > 0$, $\Delta t < 0$ Likewise, these events are in the PAST wrt O : they can influence the event at O

4 Finally, $\Delta S < 0$: events separated by $\Delta S < 0$ CANNOT be connected by a signal travelling at $v < c$ hence these events ARE NOT CAUSALLY CONNECTED = they are in the <u>PRESENT of O.</u>

<u>NOTE</u> : given a four-vector x^μ, we say that if

$x^2 > 0$ then x^μ is TIMELIKE
$x^2 < 0$ " " " SPACELIKE
$x^2 = 0$ " " " LIGHTLIKE (or NULL).

x^2 is (by definition) Lorentz invariant, hence a vector is characterised by being timelike or spacelike or null.

RELATIVISTIC FORM OF MAXWELL EQUATIONS

We use natural units, where

$c = 1$
$\mu_0 = \varepsilon_0 = 1$
[$\hbar = 1$, but we won't use this now

Let's recall Maxwell's equations:

$$\vec{\nabla} \cdot \vec{E} = \rho \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t}$$

As they stand, they are not manifestly Lorentz invariant.
However, let's notice the following points:

1) From $\vec{\nabla} \cdot \vec{B} = 0$, we can write $\vec{B}$ as

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}} \quad \text{for some 3-vector } \vec{A}.$$

Then $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$ automatically.

2) From $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow$

$$\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = \vec{0} \quad \text{Swapping } \frac{\partial}{\partial t} \text{ with } \vec{\nabla}$$

$$\Rightarrow \vec{\nabla} \times \left(\vec{E} + \frac{\partial}{\partial t}\vec{A}\right) = \vec{0}.$$ Hence we

can write $\vec{E} + \frac{\partial}{\partial t}\vec{A}$ as the gradient of a function, conventionally denoted $-\varphi \Rightarrow$

$$\boxed{\vec{E} + \frac{\partial}{\partial t}\vec{A} = -\vec{\nabla}\varphi}$$

NOTE :

$\vec{A}$ IS CALLED VECTOR POTENTIAL
φ IS CALLED SCALAR POTENTIAL

3) Now look at $\vec{\nabla} \times \vec{B} = \vec{J} + \frac{\partial}{\partial t}\vec{E} \Rightarrow$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{J} + \frac{\partial}{\partial t}\left(-\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\varphi\right) \Rightarrow$$

Use $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

$\Rightarrow$

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \nabla^2\vec{A} = \vec{J} - \frac{\partial^2}{\partial t^2}\vec{A} - \frac{\partial}{\partial t}\vec{\nabla}\varphi \Rightarrow$$

$$\boxed{\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{\partial}{\partial t}\varphi\right) = \vec{J}}$$

where we swapped $\frac{\partial}{\partial t}$ with $\vec{\nabla}$.

We also recognise that $\frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box \Rightarrow$

$$\boxed{\Box\vec{A} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \partial_t\varphi\right) = \vec{J}}$$

4) $\vec{\nabla}\cdot\vec{E} = \rho$ gives

$$\vec{\nabla}\left(-\vec{\nabla}\varphi - \partial_t\vec{A}\right) = \rho \quad \text{or}$$

$$-\nabla^2\varphi - \partial_t\left(\vec{\nabla}\cdot\vec{A}\right) = \rho$$

swapped.

Now add and subtract $\partial_t^2 \varphi \Rightarrow$

$$\partial_t^2 \varphi - \nabla^2 \varphi - \partial_t (\vec{\nabla} \cdot \vec{A} + \partial_t \varphi) = \rho$$ or

$$\boxed{\Box \varphi - \partial_t (\vec{\nabla} \vec{A} + \partial_t \varphi) = \rho}$$

Remarkably, we can merge the 2 framed equations into one.

a) Note that

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \equiv \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \vec{x}} \right) = (\partial_t, +\vec{\nabla})$$ but

$$\boxed{\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial (-\vec{x})} \right) = (\partial_t, -\vec{\nabla})}$$

b) Introduce a FOUR-VECTOR

$$A^\mu \equiv (\varphi, \vec{A})$$ - Then

$$\partial_\mu A^\mu = \partial_0 A^0 + \partial_i A^i = \partial_t \varphi + \vec{\nabla} \vec{A}$$

Hence we can remarkably combine these 2 equations into

$$\boxed{\Box A^\mu - \partial^\mu (\partial_\nu A^\nu) = J^\mu}$$

which is manifestly Lorentz covariant!

COMMON REWRITING OF THIS

Look at $\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = J^\mu$

Introduce the FIELD STRENGTH

$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu}$ (whose components we will shortly study), to see that

$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu =$

$= \Box A^\nu - \partial^\nu(\partial_\mu A^\mu)$. Hence

we can rewrite our Maxwell eq. as

$\boxed{\partial_\mu F^{\mu\nu} = J^\nu}$ Beautifully compact.

WHO IS $F^{\mu\nu}$?

$F^{\mu\nu}$ is antisymmetric $\Rightarrow \frac{4\times 3}{2} = 6$ components

(same as the # of comp. of $\vec{E}$ and $\vec{B}$)

Look at F^{0i}, F^{ij} separately.

a) $F^{0i} = \partial^0 A^i - \partial^i A^0 = \partial_t A^i + \nabla^i \varphi = -E^i$

$\Rightarrow E^i = -F^{0i}$ or $\boxed{E^i = +F^{i0}}$

b) $F^{ij} = \partial^i A^j - \partial^j A^i = -\nabla^i A^j + \nabla^j A^i$

It is easy to see that

$$\boxed{\nabla^i A^j - \nabla^j A^i = \varepsilon^{ijk} B^k}$$

Indeed (*)

$$\underbrace{\varepsilon^{ijk} \varepsilon^{ijl}}_{2\delta^{kl}} B^k = \varepsilon^{ijl} (\nabla^i A^j - \nabla^j A^i)$$

$$2 B^l = \varepsilon^{ijl} (\nabla^i A^j - \nabla^j A^i) \quad \text{or}$$

$$B^l = \frac{1}{2} \varepsilon^{lij} (\nabla^i A^j - \nabla^j A^i) = \varepsilon^{lij} \nabla^i A^j =$$

$$= (\vec{\nabla} \times \vec{A})^l \quad \underline{\underline{\text{OK}}}$$

Hence $\boxed{F^{ij} = -\varepsilon^{ijk} B^k}$

(*) or even simpler: $\nabla^1 A^2 - \nabla^2 A^1 \equiv B^3 \equiv \varepsilon^{123} B^3$ and so on.

GAUGE INVARIANCE

We wrote earlier the EOM for A_μ, in the form

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu}$$

which is an equation for A^μ, where

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Now, notice that if we vary A^μ as

$$\boxed{A^\mu \to A^\mu + \partial^\mu \Lambda(x)}$$

with $\Lambda(x)$ a(ny) function of the coordinates, then

$$F^{\mu\nu} \to \partial^\mu(A^\nu + \partial^\nu \Lambda) - \partial^\nu(A^\mu + \partial^\mu \Lambda) =$$

$$= F^{\mu\nu} + \cancel{(\partial^\mu \partial^\nu - \partial^\nu \partial^\mu)} \Lambda = F^{\mu\nu}$$

$\Rightarrow$ **the EOM are invariant in form**

This is called <u>gauge invariance</u>

It is a REDUNDANCY. It's there because we are trying to describe the photon (the quantum of the EM field), which has only 2 INDEPENDENT DEGREES OF FREEDOM (labelled by the HELICITY) using a 4-COMPONENT VECTOR ∇

A_μ

SOLUTION:

FIX THE GAUGE to remove the arbitrariness i.e. put some EXTRA CONDITIONS on A_μ in order to reduce the # of independent components.

If time permits, we will come back to this fascinating point later on.

GENERATORS AND REPRESENTATIONS

It easy to see that all Lorentz transformations can be written as the product of a RESTRICTED (= proper, orthochronous) transf. times an inversion of some type, more precisely:

$L \times$ space inversion $\longrightarrow L_-^\uparrow$

$L \times$ time inversion $\longrightarrow L_-^\downarrow$

$L \times$ space inv. $\times$ time inv $\longrightarrow L_+^\downarrow$

Furthermore, we already said we really care about $L_+^\uparrow$ mostly. Hence now we focus on them.

We wrote earlier $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \underset{\text{antisymm}}{\alpha^\mu{}_\nu}$.

We can write $\delta x^\mu = \alpha^\mu{}_\nu x^\nu$

with $\alpha_{\mu\nu} = -\alpha_{\nu\mu}$ and

$$\delta x^\mu \equiv \frac{i}{2}\alpha^{\rho\sigma}\underbrace{L_{\rho\sigma}}_{\text{generator}} x^\mu = \underbrace{\frac{i}{2}\alpha^{\rho\sigma}(2i\, x_\rho \partial_\sigma)}_{-\alpha^{\rho\mu} x_\rho}\underbrace{x^\mu}_{\delta_\sigma^\mu}$$

$$= \text{antisymmetrysing} = \frac{i}{2}\alpha^{\rho\sigma}\left[i(x_\rho\partial_\sigma - x_\sigma\partial_\rho)\right]x^\mu$$

whence

$$\boxed{L_{\rho\sigma} = i(x_\rho \partial_\sigma - x_\sigma \partial_\rho)} \qquad L_{\rho\sigma} = -L_{\sigma\rho}$$

It is easy to check that the L generators satisfy a Lie algebra commutation relations

$$[L_{\mu\nu}, L_{\rho\sigma}] = i g_{\nu\rho} L_{\mu\sigma} - i g_{\mu\rho} L_{\nu\sigma} +$$
$$- i g_{\nu\sigma} L_{\mu\rho} + i g_{\mu\sigma} L_{\nu\rho}$$

which is the Lie algebra of $SO(1,3)$.

The most general rep of generators of $SO(1,3)$ has the form

$$\boxed{M_{\mu\nu} \equiv L_{\mu\nu} + S_{\mu\nu}}$$ where the $S_{\mu\nu}$

satisfy the same Lie algebra relations as the $M_{\mu\nu}$ (and $L_{\mu\nu}$), and also $[L_{\mu\nu}, S_{\rho\sigma}] = 0$

The spatial part of the M's satisfy a Lie algebra themselves:

$$[M_{ij}, M_{kl}] = -i\delta_{jk} M_{il} + i\delta_{ik} M_{jl} + i\delta_{jl} M_{ik} + \\ -i\delta_{il} M_{jk}$$

Upon introducing $\boxed{J_i \equiv \frac{1}{2}\varepsilon_{ijk} M_{jk}}$

one can recognise that, from the M_{ij} commutation relations, it follows that

$$\boxed{[J_i, J_j] = i\varepsilon_{ijk} J_k}$$

thus the M_{ij} satisfy an $SU(2)$ algebra relation.

Defining the boost generators $\boxed{K_i \equiv M_{0i}}$

one further finds that

$$\boxed{\begin{aligned} [K_i, K_j] &= -i\varepsilon_{ijk} J_k \\ [J_i, K_j] &= +i\varepsilon_{ijk} K_k \end{aligned}}$$

One then proceeds to introduce

$$\boxed{N_i \equiv \frac{1}{2}(J_i + iK_i)}$$

which nicely repackage the J's and the K's and satisfy

$$\boxed{\begin{aligned} &[N_i, N_j] = i\,\varepsilon_{ijk} N_k \\ &[N_i^+, N_j^+] = i\,\varepsilon_{ijk} N_k^+ \\ &\text{and } [N_i, N_j^+] = 0 \end{aligned}}$$

We have thus shown that

$$\boxed{SO(1,3) \sim SU(2) \times SU(2)}$$

where in our REAL Minkowski space, the second $SU(2)$ is the complex conjugate of the first.

NOT SO IN COMPLEX MINKOWSKI SPACE !

Also note that parity $J \to J$ $K \to -K$ here swaps the 2 $SU(2)$s as well.

Thus Lorentz group reps are not

parity invariant.

The eigenvalues:

first $SU(2)$, $N_i N_i$ } are the Casimir
second $SU(2)$ $N_i^+ N_i^+$ }

(operators that commute with all generators)

$$\boxed{\begin{array}{lll} N_i N_i & \longrightarrow \text{eigenvalue} & n(n+1) \\ N_i^+ N_i^+ & \longrightarrow \quad " & m(m+1) \end{array}}$$

where $n, m = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

Example of reps:

1) $(0,0) \rightarrow$ scalar field spin 0

2) $(\frac{1}{2}, 0) \rightarrow$ spin $\frac{1}{2}$, left-handed spinor

3) $(0, \frac{1}{2}) \rightarrow$ " ", right-handed spinor.

4) $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \rightarrow$ Dirac spinor.

POINCARÉ GROUP

Physical laws are also invariant under 4-translations $x^\mu \to x^\mu + a^\mu$

hence combining Lorentz + translations we get a 10-dim group called Poincaré group

$$\boxed{x^\mu \to x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu + a^\mu}$$

The generators:

$$\delta x^\mu = a^\mu \equiv i a_\rho P^\rho x_\mu \Rightarrow$$

$$\boxed{P^\rho = -i\partial^\rho}$$

Obviously $[P^\mu, P^\nu] = 0$ while one can find

$$\boxed{[M_{\mu\nu}, P_\rho] = -i g_{\mu\rho} P_\nu + i g_{\nu\rho} P_\mu}$$

CASIMIRS OF THE POINCARÉ GROUP

There are 2 Casimirs which define the reps of the group

1. the first one is easy to find, it's

$P_\rho P^\rho$

This is obviously invariant under Lorentz transformations (obvious since it's $P \cdot P$, can also be checked directly:

$$[M_{\mu\nu}, P_\rho P^\rho] = 2[M_{\mu\nu}, P_\rho]P^\rho =$$

$$= (-i g_{\mu\rho} P_\nu + i g_{\nu\rho} P_\mu) P^\rho = -i P_\mu P_\nu + i P_\mu P_\nu = 0)$$

and furthermore $[P^\mu, P \cdot P] = 0 \Rightarrow$ it's a Casimir.

2. To get the second Casimir we introduce the PAULI-LUBANSKI VECTOR:

$$W^\mu = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \quad , \text{ where}$$

$$[W^\mu, P^\nu] = 0$$

(Proof:

$$[W^\mu, P_\alpha] = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}[P_\nu M_{\rho\sigma}, P_\alpha] =$$

$$= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} P_\nu [M_{\rho\sigma}, P_\alpha] \quad (\text{since } [P_\nu, P_\alpha]=0)$$

$$= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} P_\nu \left(-i g_{\rho\alpha} P_\sigma + i g_{\sigma\alpha} P_\rho\right) =$$

$$= -\frac{i}{2}\left(\varepsilon^{\mu\nu\alpha\sigma} P_\nu P_\sigma - \varepsilon^{\mu\nu\rho\alpha} P_\nu P_\rho\right) = 0 - 0 = 0$$

(symm x antisymm)

Thus $W_\mu W^\mu$ is a Casimir

(commutes with P_α and is Lorentz invariant)

Also note that using $M_{\rho\sigma} = i(x_\rho \partial_\sigma - x_\sigma \partial_\rho) + S_{\rho\sigma}$

we see that $= -(x_\rho P_\sigma - x_\sigma P_\rho) + S_{\rho\sigma}$

$$W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\Big[\underbrace{-P_\nu(x_\rho P_\sigma - x_\sigma P_\rho)}_{=0 \text{ since}} + P_\nu S_{\rho\sigma}\Big]$$

$$\varepsilon^{\mu\nu\rho\sigma} P_\nu x_\rho = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}[P_\nu, x_\rho] =$$

$$= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(-i g_{\nu\rho}) = 0 \quad \text{and}$$

similarly for the second term.

REPRESENTATIONS OF THE POINCARÉ GROUP

Representation theory for the Poincaré group was worked out by Wigner — the results are:

1. The eigenvalue of $P_\rho P^\rho$ is $m^2 > 0$.

Then the eigenvalue of $W_\rho W^\rho$ is $-m^2 s(s+1)$

$s = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots$ = spin.

States are labeled by the mass (continuous) and the 3rd component S_3 of the spin

2. Eigenvalue of $P_\rho P^\rho$ is $m^2 = 0$.

If also $W_\rho W^\rho = 0$ then, since $W_\rho P^\rho = 0$, it turns out that $W_\mu \parallel P_\mu$

The constant of proportionality $W_\mu = h P_\mu$ is called the HELICITY and it is equal to

$0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2} \ldots$

So for each |value| → 2 polarisation states ±

E.g. photon, gluons ±1
graviton ±2.

3. $P_\mu P^\mu = 0$ but Continuous Spin

Not realised in Nature .....