The Classical Theory of Fields

Fourth Revised English Edition

Course of Theoretical Physics
Volume 2

L.D. Landau and E.M. Lifshitz

THE CLASSICAL THEORY OF FIELDS

Fourth Revised English Edition

L. D. LANDAU AND E. M. LIFSHITZ
Institute for Physical Problems, Academy of Sciences of the U.S.S.R.

Translated from the Russian
by
MORTON HAMERMESH
University of Minnesota

BUTTERWORTH HEINEMANN

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CONTENTS

EXCERPTS FROM THE PREFACES TO THE FIRST AND SECOND EDITIONS

THIS book is devoted to the presentation of the theory of the electromagnetic and gravitational fields, i.e. electrodynamics and general relativity. A complete, logically connected theory of the electromagnetic field includes the special theory of relativity, so the latter has been taken as the basis of the presentation. As the starting point of the derivation of the fundamental relations we take the variational principles, which make possible the attainment of maximum generality, unity and simplicity of presentation.

In accordance with the overall plan of our Course of Theoretical Physics (of which this book is a part), we have not considered questions concerning the electrodynamics of continuous media, but restricted the discussion to "microscopic electrodynamics"—the electrodynamics of point charges *in vacuo*.

The reader is assumed to be familiar with electromagnetic phenomena as discussed in general physics courses. A knowledge of vector analysis is also necessary. The reader is not assumed to have any previous knowledge of tensor analysis, which is presented in parallel with the development of the theory of gravitational fields.

Moscow, December 1939
Moscow, June 1947

L. LANDAU, E. LIFSHITZ

PREFACE TO THE FOURTH ENGLISH EDITION

THE first edition of this book appeared more than thirty years ago. In the course of reissues over these decades the book has been revised and expanded; its volume has almost doubled since the first edition. But at no time has there been any need to change the method proposed by Landau for developing the theory, or his style of presentation, whose main feature was a striving for clarity and simplicity. I have made every effort to preserve this style in the revisions that I have had to make on my own.

As compared with the preceding edition, the first nine chapters, devoted to electrodynamics, have remained almost without changes. The chapters concerning the theory of the gravitational field have been revised and expanded. The material in these chapters has increased from edition to edition, and it was finally necessary to redistribute and rearrange it.

I should like to express here my deep gratitude to all of my helpers in this work—too many to be enumerated—who, by their comments and advice, helped me to eliminate errors and introduce improvements. Without their advice, without the willingness to help which has met all my requests, the work to continue the editions of this course would have been much more difficult. A special debt of gratitude is due to L. P. Pitaevskii, with whom I have constantly discussed all the vexing questions.

The English translation of the book was done from the last Russian edition, which appeared in 1973. No further changes in the book have been made. The 1994 corrected reprint includes the changes made by E. M. Lifshitz in the Seventh Russian Edition published in 1987.

I should also like to use this occasion to sincerely thank Prof. Hamermesh, who has translated this book in all its editions, starting with the first English edition in 1951. The success of this book among English-speaking readers is to a large extent the result of his labour and careful attention.

E. M. LIFSHITZ

PUBLISHER'S NOTE

As with the other volumes in the *Course of Theoretical Physics,* the authors do not, as a rule, give references to original papers, but simply name their authors (with dates). Full bibliographic references are only given to works which contain matters not fully expounded in the text.

EDITOR'S PREFACE TO THE SEVENTH RUSSIAN EDITION

E. M. Lifshitz began to prepare a new edition of *Teoria Polia* in 1985 and continued his work on it even in hospital during the period of his last illness. The changes that he proposed are made in the present edition. Of these we should mention some revision of the proof of the law of conservation of angular momentum in relativistic mechanics, and also a more detailed discussion of the question of symmetry of the Christoffel symbols in the theory of gravitation. The sign has been changed in the definition of the electromagnetic field stress tensor. (In the present edition this tensor was defined differently than in the other volumes of the Course.)

June 1987 L. P. PITAEVSKII

NOTATION

Three-dimensional quantities

Three-dimensional tensor indices are denoted by Greek letters
Element of volume, area and length: dV, $d\mathbf{f}$, $d\mathbf{l}$
Momentum and energy of a particle: $\mathbf{p}$ and $\mathscr{E}$
Hamiltonian function: $\mathscr{H}$
Scalar and vector potentials of the electromagnetic field: ϕ and $\mathbf{A}$
Electric and magnetic field intensities: $\mathbf{E}$ and $\mathbf{H}$
Charge and current density: ρ and $\mathbf{j}$
Electric dipole moment: $\mathbf{d}$
Magnetic dipole moment: $\mathfrak{m}$

Four-dimensional quantities

Four-dimensional tensor indices are denoted by Latin letters $i, k, l, \ldots$ and take on the values 0, 1, 2, 3
We use the metric with signature $(+\,-\,-\,-)$
Rule for raising and lowering indices—see p. 14
Components of four-vectors are enumerated in the form $A^i = (A^0, \mathbf{A})$
Antisymmetric unit tensor of rank four is e^{iklm}, where $e^{0123} = 1$ (for the definition, see p. 17)
Element of four-volume $d\Omega = dx^0 dx^1 dx^2 dx^3$
Element of hypersurface dS^i (defined on pp. 20–21)
Radius four-vector: $x^i = (ct, \mathbf{r})$
Velocity four-vector: $u^i = dx^i/ds$
Momentum four-vector: $p = (\mathscr{E}/c, \mathbf{p})$
Current four-vector: $j^i = (c\rho, \rho\mathbf{v})$
Four-potential of the electromagnetic field: $A^i = (\phi, \mathbf{A})$
Electromagnetic field four-tensor $F_{ik} = \dfrac{\partial A_k}{\partial x^i} - \dfrac{\partial A_i}{\partial x^k}$ (for the relation of the components of F_{ik} to the components of $\mathbf{E}$ and $\mathbf{H}$, see p. 65)
Energy-momentum four-tensor T^{ik}(for the definition of its components, see p. 83)

CHAPTER 1

THE PRINCIPLE OF RELATIVITY

§ 1. Velocity of propagation of interaction

For the description of processes taking place in nature, one must have a *system of reference*. By a system of reference we understand a system of coordinates serving to indicate the position of a particle in space, as well as clocks fixed in this system serving to indicate the time.

There exist systems of reference in which a freely moving body, i.e. a moving body which is not acted upon by external forces, proceeds with constant velocity. Such reference systems are said to be *inertial*.

If two reference systems move uniformly relative to each other, and if one of them is an inertial system, then clearly the other is also inertial (in this system too every free motion will be linear and uniform). In this way one can obtain arbitrarily many inertial systems of reference, moving uniformly relative to one another.

Experiment shows that the so-called *principle of relativity* is valid. According to this principle all the laws of nature are identical in all inertial systems of reference. In other words, the equations expressing the laws of nature are invariant with respect to transformations of coordinates and time from one inertial system to another. This means that the equation describing any law of nature, when written in terms of coordinates and time in different inertial reference systems, has one and the same form.

The interaction of material particles is described in ordinary mechanics by means of a potential energy of interaction, which appears as a function of the coordinates of the interacting particles. It is easy to see that this manner of describing interactions contains the assumption of instantaneous propagation of interactions. For the forces exerted on each of the particles by the other particles at a particular instant of time depend, according to this description, only on the positions of the particles at this one instant. A change in the position of any of the interacting particles influences the other particles immediately.

However, experiment shows that instantaneous interactions do not exist in nature. Thus a mechanics based on the assumption of instantaneous propagation of interactions contains within itself a certain inaccuracy. In actuality, if any change takes place in one of the interacting bodies, it will influence the other bodies only after the lapse of a certain interval of time. It is only after this time interval that processes caused by the initial change begin to take place in the second body. Dividing the distance between the two bodies by this time interval, we obtain the *velocity of propagation of the interaction*.

We note that this velocity should, strictly speaking, be called the *maximum* velocity of propagation of interaction. It determines only that interval of time after which a change occurring in one body *begins* to manifest itself in another. It is clear that the existence of a maximum velocity of propagation of interactions implies, at the same time, that motions of bodies with greater velocity than this are in general impossible in nature. For if such a motion could occur, then by means of it one could realize an interaction with a velocity exceeding the maximum possible velocity of propagation of interactions.

Interactions propagating from one particle to another are frequently called "signals", sent out from the first particle and "informing" the second particle of changes which the first has experienced. The velocity of propagation of interaction is then referred to as the *signal velocity.*

From the principle of relativity it follows in particular that the velocity of propagation of interactions is the *same* in *all* inertial systems of reference. Thus the velocity of propagation of interactions is a universal constant. This constant velocity (as we shall show later) is also the velocity of light in empty space. The velocity of light is usually designated by the letter c, and its numerical value is

$$c = 2.998 \times 10^{10} \text{ cm/sec}. \tag{1.1}$$

The large value of this velocity explains the fact that in practice classical mechanics appears to be sufficiently accurate in most cases. The velocities with which we have occasion to deal are usually so small compared with the velocity of light that the assumption that the latter is infinite does not materially affect the accuracy of the results.

The combination of the principle of relativity with the finiteness of the velocity of propagation of interactions is called the *principle of relativity of Einstein* (it was formulated by Einstein in 1905) in contrast to the principle of relativity of Galileo, which was based on an infinite velocity of propagation of interactions.

The mechanics based on the Einsteinian principle of relativity (we shall usually refer to it simply as the principle of relativity) is called *relativistic.* In the limiting case when the velocities of the moving bodies are small compared with the velocity of light we can neglect the effect on the motion of the finiteness of the velocity of propagation. Then relativistic mechanics goes over into the usual mechanics, based on the assumption of instantaneous propagation of interactions; this mechanics is called *Newtonian* or *classical.* The limiting transition from relativistic to classical mechanics can be produced formally by the transition to the limit $c \to \infty$ in the formulas of relativistic mechanics.

In classical mechanics distance is already relative, i.e. the spatial relations between different events depend on the system of reference in which they are described. The statement that two nonsimultaneous events occur at one and the same point in space or, in general, at a definite distance from each other, acquires a meaning only when we indicate the system of reference which is used.

On the other hand, time is absolute in classical mechanics; in other words, the properties of time are assumed to be independent of the system of reference; there is one time for all reference frames. This means that if any two phenomena occur simultaneously for any one observer, then they occur simultaneously also for all others. In general, the interval of time between two given events must be identical for all systems of reference.

It is easy to show, however, that the idea of an absolute time is in complete contradiction to the Einstein principle of relativity. For this it is suffcient to recall that in classical mechanics, based on the concept of an absolute time, a general law of combination of velocities is valid, according to which the velocity of a composite motion is simply equal to the (vector) sum of the velocities which constitute this motion. This law, being universal, should also be applicable to the propagation of interactions. From this it would follow that the velocity of propagation must be different in different inertial systems of reference, in contradiction to the principle of relativity. In this matter experiment completely confirms the principle of relativity. Measurements first performed by Michelson (1881) showed complete lack of dependence of the velocity of light on its direction of propagation; whereas according to classical mechanics the velocity of light should be smaller in the direction of the earth's motion than in the opposite direction.

Thus the principle of relativity leads to the result that time is not absolute. Time elapses differently in different systems of reference. Consequently the statement that a definite time interval has elapsed between two given events acquires meaning only when the reference frame to which this statement applies is indicated. In particular, events which are simultaneous in one reference frame will not be simultaneous in other frames.

To clarify this, it is instructive to consider the following simple example.

Let us look at two inertial reference systems K and K' with coordinate axes XYZ and $X'\,Y'\,Z'$ respectively, where the system K' moves relative to K along the $X(X')$ axis (Fig. 1).

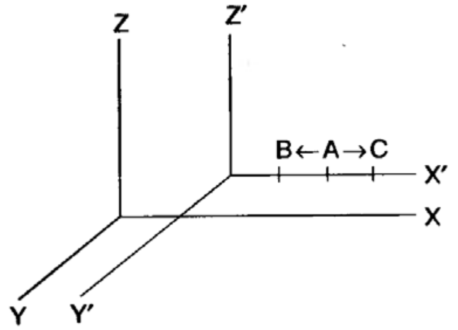

FIG. 1.

Suppose signals start out from some point A on the X' axis in two opposite directions. Since the velocity of propagation of a signal in the K' system, as in all inertial systems, is equal (for both directions) to c, the signals will reach points B and C, equidistant from A, at one and the same time (in the K' system)

But it is easy to see that the same two events (arrival of the signal at B and C) can by no means be simultaneous for an observer in the K system. In fact, the velocity of a signal relative to the K system has, according to the principle of relativity, the same value c, and since the point B moves (relative to the K system) toward the source of its signal, while the point C moves in the direction away from the signal (sent from A to C), in the K system the signal will reach point B earlier than point C.

Thus the principle of relativity of Einstein introduces very drastic and fundamental changes in basic physical concepts. The notions of space and time derived by us from our daily experiences are only approximations linked to the fact that in daily life we happen to deal only with velocities which are very small compared with the velocity of light.

§ 2. Intervals

In what follows we shall frequently use the concept of an *event*. An event is described by the place where it occurred and the time when it occurred. Thus an event occurring in a certain material particle is defined by the three coordinates of that particle and the time when the event occurs.

It is frequently useful for reasons of presentation to use a fictitious four-dimensional space, on the axes of which are marked three space coordinates and the time. In this space events are represented by points, called *world points.* In this fictitious four-dimensional space there corresponds to each particle a cetain line, called a *world line.* The points of this line determine the coordinates of the particle at all moments of time. It is easy to show that to a particle in uniform rectilinear motion there corresponds a straight world line.

We now express the principle of the invariance of the velocity of light in mathematical form. For this purpose we consider two reference systems K and K' moving relative to each other with constant velocity. We choose the coordinate axes so that the axes X and X' coincide, while the Y and Z axes are parallel to Y' and Z'; we designate the time in the systems K and K' by t and t'.

Let the first event consist of sending out a signal, propagating with light velocity, from a point having coordinates $x_1y_1z_1$ in the K system, at time t_1 in this system. We observe the propagation of this signal in the K system. Let the second event consist of the arrival of the signal at point $x_2y_2z_2$ at the moment of time t_2. The signal propagates with velocity c; the distance covered by it is therefore $c(t_1-t_2)$. On the other hand, this same distance equals $[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2]^{\frac{1}{2}}$. Thus we can write the following relation between the coordinates of the two events in the K system:

$$(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2-c^2(t_2-t_1)^2=0. \tag{2.1}$$

The same two events, i.e. the propagation of the signal, can be observed from the K' system:

Let the coordinates of the first event in the K' system be $x_1'y_1'z_1't_1'$, and of the second: $x_2'y_2'z_2't_2'$. Since the velocity of light is the same in the K and K' systems, we have, similarly to (2.1):

$$(x_2'-x_1')^2+(y_2'-y_1')^2+(z_2'-z_1')^2-c^2(t_2'-t_1')^2=0. \tag{2.2}$$

If $x_1y_1z_1t_1$ and $x_2y_2z_2t_2$ are the coordinates of *any* two events, then the quantity

$$s_{12}=[c^2(t_2-t_1)^2-(x_2-x_1)^2-(y_2-y_1)^2-(z_2-z_1)^2]^{\frac{1}{2}} \tag{2.3}$$

is called the *interval* between these two events.

Thus it follows from the principle of invariance of the velocity of light that if the interval between two events is zero in one coordinate system, then it is equal to zero in all other systems.

If two events are infinitely close to each other, then the interval ds between them is

$$ds^2=c^2dt^2-dx^2-dy^2-dz^2. \tag{2.4}$$

The form of expressions (2.3) and (2.4) permits us to regard the interval, from the formal point of view, as the distance between two points in a fictitious four-dimensional space (whose axes are labelled by x, y, z, and the product ct). But there is a basic difference between the rule for forming this quantity and the rule in ordinary geometry: in forming the square of the interval, the squares of the coordinate differences along the different axes are summed, not with the same sign, but rather with varying signs.†

As already shown, if $ds=0$ in one inertial system, then $ds'=0$ in any other system. On

† The four-dimensional geometry described by the quadratic form (2.4) was introduced by H. Minkowski, in connection with the theory of relativity. This geometry is called *pseudo-euclidean,* in contrast to ordinary euclidean geometry.

the other hand, ds and ds' are infinitesimals of the same order. From these two conditions it follows that ds^2 and ds'^2 must be proportional to each other:

$$ds^2 = a ds'^2$$

where the coefficient a can depend only on the absolute value of the relative velocity of the two inertial systems. It cannot depend on the coordinates or the time, since then different points in space and different moments in time would not be equivalent, which would be in contradiction to the homogeneity of space and time. Similarly, it cannot depend on the direction of the relative velocity, since that would contradict the isotropy of space.

Let us consider three reference systems K, K_1, K_2, and let V_1 and V_2 be the velocities of systems K_1 and K_2 relative to K. We then have:

$$ds^2 = a(V_1) ds_1^2, \qquad ds^2 = a(V_2) ds_2^2.$$

Similarly we can write

$$ds_1^2 = a(V_{12}) ds_2^2,$$

where V_{12} is the absolute value of the velocity of K_2 relative to K_1. Comparing these relations with one another, we find that we must have

$$\frac{a(V_2)}{a(V_1)} = a(V_{12}). \tag{2.5}$$

But V_{12} depends not only on the absolute values of the vectors $\mathbf{V}_1$ and $\mathbf{V}_2$, but also on the angle between them. However, this angle does not appear on the left side of formula (2.5). It is therefore clear that this formula can be correct only if the function $a(V)$ reduces to a constant, which is equal to unity according to this same formula.

Thus,

$$ds^2 = ds'^2, \tag{2.6}$$

and from the equality of the infinitesimal intervals there follows the equality of finite intervals: $s = s'$.

Thus we arrive at a very important result: the interval between two events is the same in all inertial systems of reference, i.e. it is invariant under transformation from one inertial system to any other. This invariance is the mathematical expression of the constancy of the velocity of light.

Again let $x_1 y_1 z_1 t_1$ and $x_2 y_2 z_2 t_2$ be the coordinates of two events in a certain reference system K. Does there exist a coordinate system K', in which these two events occur at one and the same point in space?

We introduce the notation

$$t_2 - t_1 = t_{12}, \qquad (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = l_{12}^2.$$

Then the interval between events in the K system is:

$$s_{12}^2 = c^2 t_{12}^2 - l_{12}^2$$

and in the K' system

$$s_{12}'^2 = c^2 t_{12}'^2 - l_{12}'^2,$$

whereupon, because of the invariance of intervals,

$$c^2t_{12}^2 - l_{12}^2 = c^2t_{12}'^2 - l_{12}'^2.$$

We want the two events to occur at the same point in the K' system, that is, we require $l_{12}' = 0$. Then

$$s_{12}^2 = c^2t_{12}^2 - l_{12}^2 = c^2t_{12}'^2 > 0.$$

Consequently a system of reference with the required property exists if $s_{12}^2 > 0$, that is, if the interval between the two events is a real number. Real intervals are said to be *timelike*.

Thus, if the interval between two events is timelike, then there exists a system of reference in which the two events occur at one and the same place. The time which elapses between the two events in this system is

$$t_{12}' = \frac{1}{c}\sqrt{c^2t_{12}^2 - l_{12}^2} = \frac{s_{12}}{c}. \tag{2.7}$$

If two events occur in one and the same body, then the interval between them is always timelike, for the distance which the body moves between the two events cannot be greater than ct_{12}, since the velocity of the body cannot exceed c. So we have always

$$l_{12} < ct_{12}.$$

Let us now ask whether or not we can find a system of reference in which the two events occur at one and the same time. As before, we have for the K and K' systems $c^2t_{12}^2 - l_{12}^2 = c^2t_{12}'^2 - l_{12}'^2$. We want to have $t_{12}' = 0$, so that

$$s_{12}^2 = -l_{12}'^2 < 0.$$

Consequently the required system can be found only for the case when the interval s_{12} between the two events is an imaginary number. Imaginary intervals are said to be *spacelike*.

Thus if the interval between two events is spacelike, there exists a reference system in which the two events occur simultaneously. The distance between the points where the events occur in this system is

$$l_{12}' = \sqrt{l_{12}^2 - c^2t_{12}^2} = is_{12}. \tag{2.8}$$

The division of intervals into space- and timelike intervals is, because of their invariance, an absolute concept. This means that the timelike or spacelike character of an interval is independent of the reference system.

Let us take some event O as our origin of time and space coordinates. In other words, in the four-dimensional system of coordinates, the axes of which are marked x, y, z, t, the world point of the event O is the origin of coordinates. Let us now consider what relation other events bear to the given event O. For visualization, we shall consider only one space dimension and the time, marking them on two axes (Fig. 2). Uniform rectilinear motion of a particle, passing through $x = 0$ at $t = 0$, is represented by a straight line going through O and inclined to the t axis at an angle whose tangent is the velocity of the particle. Since the maximum possible velocity is c, there is a maximum angle which this line can subtend with the t axis. In Fig. 2 are shown the two lines representing the propagation of two signals (with the velocity of light) in opposite directions passing through the event O (i.e. going through $x = 0$ at $t = 0$). All lines representing the motion of particles can lie only in the regions aOc and dOb. On the lines ab and cd, $x = \pm ct$. First consider events whose world points lie within the region aOc. It is easy to show that for all the points of this region $c^2t^2 - x^2 > 0$.

In other words, the interval between any event in this region and the event O is timelike. In this region $t > 0$, i.e. all the events in this region occur "after" the event O. But two events which are separated by a timelike interval cannot occur simultaneously in any reference system. Consequently it is impossible to find a reference system in which any of the events in region aOc occurred "before" the event O, i.e. at time $t < 0$. Thus all the events in region aOc are future events relative to O in *all* reference systems. Therefore this region can be called the *absolute future* relative to O.

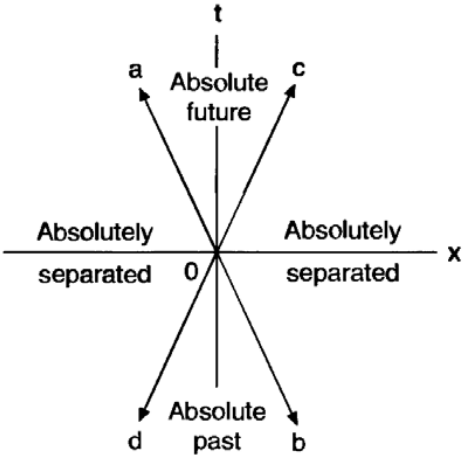

FIG. 2

In exactly the same way, all events in the region bOd are in the *absolute past* relative to O; i.e. events in this region occur before the event O in all systems of reference.

Next consider regions dOa and cOb. The interval between any event in this region and the event O is spacelike. These events occur at different points in space in every reference system. Therefore these regions can be said to be *absolutely remote* relative to O. However, the concepts "simultaneous", "earlier", and "later" are relative for these regions. For any event in these regions there exist systems of reference in which it occurs after the event O, systems in which it occurs earlier than O, and finally one reference system in which it occurs simultaneously with O.

Note that if we consider all three space coordinates instead of just one, then instead of the two intersecting lines of Fig. 2 we would have a "cone" $x^2 + y^2 + z^2 - c^2t^2 = 0$ in the four-dimensional coordinate system x, y, z, t, the axis of the cone coinciding with the t axis. (This cone is called the *light cone*.) The regions of absolute future and absolute past are then represented by the two interior portions of this cone.

Two events can be related causally to each other only if the interval between them is timelike; this follows immediately from the fact that no interaction can propagate with a velocity greater than the velocity of light. As we have just seen, it is precisely for these events that the concepts "earlier" and "later" have an absolute significance, which is a necessary condition for the concepts of cause and effect to have meaning.

§ 3. Proper time

Suppose that in a certain inertial reference system we observe clocks which are moving relative to us in an arbitrary manner. At each different moment of time this motion can be considered as uniform. Thus at each moment of time we can introduce a coordinate system rigidly linked to the moving clocks, which with the clocks constitutes an inertial reference system.

In the course of an infinitesimal time interval dt (as read by a clock in our rest frame) the moving clocks go a distance $\sqrt{dx^2 + dy^2 + dz^2}$. Let us ask what time interval dt' is indicated for this period by the moving clocks. In a system of coordinates linked to the moving clocks, the latter are at rest, i.e., $dx' = dy' = dz' = 0$. Because of the invariance of intervals

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt'^2,$$

from which

$$dt' = dt\sqrt{1 - \frac{dx^2 + dy^2 + dz^2}{c^2dt^2}}.$$

But

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} = v^2,$$

where v is the velocity of the moving clocks; therefore

$$dt' = \frac{ds}{c} = dt\sqrt{1 - \frac{v^2}{c^2}}. \tag{3.1}$$

Integrating this expression, we can obtain the time interval indicated by the moving clocks when the elapsed time according to a clock at rest is $t_2 - t_1$:

$$t_2' - t_1' = \int_{t_1}^{t_2} dt\sqrt{1 - \frac{v^2}{c^2}}. \tag{3.2}$$

The time read by a clock moving with a given object is called the *proper time* for this object. Formulas (3.1) and (3.2) express the proper time in terms of the time for a system of reference from which the motion is observed.

As we see from (3.1) or (3.2), the proper time of a moving object is always less than the corresponding interval in the rest system. In other words, moving clocks go more slowly than those at rest.

Suppose some clocks are moving in uniform rectilinear motion relative to an inertial system K. A reference frame K' linked to the latter is also inertial. Then from the point of view of an observer in the K system the clocks in the K' system fall behind. And conversely, from the point of view of the K' system, the clocks in K lag. To convince ourselves that there is no contradiction, let us note the following. In order to establish that the clocks in the K' system lag behind those in the K system, we must proceed in the following fashion. Suppose that at a certain moment the clock in K' passes by the clock in K, and at that moment the readings of the two clocks coincide. To compare the rates of the two clocks in K and K' we must once more compare the readings of the same moving clock in K' with the clocks in K. But now we compare this clock with *different* clocks in K—with those past which the clock in K' goes at ths new time. Then we find that the clock in K' lags behind the clocks in K with which it is being compared. We see that to compare the rates of clocks in two reference frames we require several clocks in one frame and one in the other, and that therefore this process is not symmetric with respect to the two systems. The clock that appears to lag is always the one which is being compared with different clocks in the other system.

If we have two clocks, one of which describes a closed path returning to the starting point (the position of the clock which remained at rest), then clearly the moving clock appears to lag relative to the one at rest. The converse reasoning, in which the moving clock would be considered to be at rest (and vice versa) is now impossible, since the clock describing a closed trajectory does not carry out a uniform rectilinear motion, so that a coordinate system linked to it will not be inertial.

Since the laws of nature are the same only for inertial reference frames, the frames linked to the clock at rest (inertial frame) and to the moving clock (non-inertial) have different properties, and the argument which leads to the result that the clock at rest must lag is not valid.

The time interval read by a clock is equal to the integral

$$\frac{1}{c}\int_a^b ds,$$

taken along the world line of the clock. If the clock is at rest then its world line is clearly a line parallel to the t axis; if the clock carries out a nonuniform motion in a closed path and returns to its starting point, then its world line will be a curve passing through the two points, on the straight world line of a clock at rest, corresponding to the beginning and end of the motion. On the other hand, we saw that the clock at rest always indicates a greater time interval than the moving one. Thus we arrive at the result that the integral

$$\int_a^b ds,$$

taken between a given pair of world points, has its maximum value if it is taken along the straight world line joining these two points.†

§ 4. The Lorentz transformation

Our purpose is now to obtain the formula of transformation from one inertial reference system to another, that is, a formula by means of which, knowing the coordinates x, y, z, t, of a certain event in the K system, we can find the coordinates x', y', z', t' of the same event in another inertial system K'.

In classical mechanics this question is resolved very simply. Because of the absolute nature of time we there have $t = t'$; if, furthermore, the coordinate axes are chosen as usual (axes X, X' coincident, Y, Z axes parallel to Y', Z', motion along X, X') then the coordinates y, z clearly are equal to y', z', while the coordinates x and x' differ by the distance traversed by one system relative to the other. If the time origin is chosen as the moment when the two coordinate systems coincide, and if the velocity of the K' system relative to K is V, then this distance is Vt. Thus

† It is assumed, of course, that the points a and b and the curves joining them are such that all elements ds along the curves are timelike.

This property of the integral is connected with the pseudo-euclidean character of the four-dimensional geometry. In euclidean space the integral would, of course, be a minimum along the straight line.

$$x = x' + Vt, \quad y = y', \quad z = z', \quad t = t'. \tag{4.1}$$

This formula is called the *Galileo transformation.* It is easy to verify that this transformation, as was to be expected, does not satisfy the requirements of the theory of relativity; it does not leave the interval between events invariant.

We shall obtain the relativistic transformation precisely as a consequence of the requirement that it leaves the interval between events invariant.

As we saw in § 2, the interval between events can be looked on as the distance between the corresponding pair of world points in a four-dimensional system of coordinates. Consequently we may say that the required transformation must leave unchanged all distances in the four-dimensional x, y, z, ct, space. But such transformations consist only of parallel displacements, and rotations of the coordinate system. Of these the displacement of the coordinate system parallel to itself is of no interest, since it leads only to a shift in the origin of the space coordinates and a change in the time reference point. Thus the required transformation must be expressible mathematically as a rotation of the four-dimensional x, y, z, ct, coordinate system.

Every rotation in the four-dimensional space can be resolved into six rotations, in the planes xy, zy, xz, tx, ty, tz (just as every rotation in ordinary space can be resolved into three rotations in the planes xy, zy and xz). The first three of these rotations transform only the space coordinates; they correspond to the usual space rotations.

Let us consider a rotation in the tx plane; under this, the y and z coordinates do not change. In particular, this transformation must leave unchanged the difference $(ct)^2 - x^2$, the square of the "distance" of the point (ct, x) from the origin. The relation between the old and the new coordinates is given in most general form by the formulas:

$$x = x' \cosh \psi + ct' \sinh \psi, \quad ct = x' \sinh \psi + ct' \cosh \psi, \tag{4.2}$$

where ψ is the "angle of rotation"; a simple check shows that in fact $c^2t^2 - x^2 = c^2t'^2 - x'^2$. Formula (4.2) differs from the usual formulas for transformation under rotation of the coordinate axes in having hyperbolic functions in place of trigonometric functions. This is the difference between pseudo-euclidean and euclidean geometry.

We try to find the formula of transformation from an inertial reference frame K to a system K' moving relative to K with velocity V along the x axis. In this case clearly only the coordinate x and the time t are subject to change. Therefore this transformation must have the form (4.2). Now it remains only to determine the angle ψ, which can depend only on the relative velocity V.†

Let us consider the motion, in the K system, of the origin of the K' system. Then $x' = 0$ and formulas (4.2) take the form:

$$x = ct' \sinh \psi, \quad ct = ct' \cosh \psi,$$

or dividing one by the other,

$$\frac{x}{ct} = \tanh \psi.$$

But x/t is clearly the velocity V of the K' system relative to K. So

† Note that to avoid confusion we shall always use V to signify the constant relative velocity of two inertial systems, and v for the velocity of a moving particle, not necessarily constant.

$$\tanh \psi = \frac{V}{c}.$$

From this

$$\sinh \psi = \frac{\dfrac{V}{c}}{\sqrt{1 - \dfrac{V^2}{c^2}}}, \qquad \cosh \psi = \frac{1}{\sqrt{1 - \dfrac{V^2}{c^2}}}.$$

Substituting in (4.2), we find:

$$x = \frac{x' + Vt'}{\sqrt{1 - \dfrac{V^2}{c^2}}}, \quad y = y', \quad z = z', \quad t = \frac{t' + \dfrac{V}{c^2}x'}{\sqrt{1 - \dfrac{V^2}{c^2}}}. \tag{4.3}$$

This is the required transformation formula. It is called the *Lorentz transformation,* and is of fundamental importance for what follows.

The inverse formulas, expressing x', y', z', t' in terms of x, y, z, t, are most easily obtained by changing V to $-V$ (since the K system moves with velocity $-V$ relative to the K' system). The same formulas can be obtained directly by solving equations (4.3) for x', y', z', t'.

It is easy to see from (4.3) that on making the transition to the limit $c \to \infty$ and classical mechanics, the formula for the Lorentz transformation actually goes over into the Galileo transformation.

For $V > c$ in formula (4.3) the coordinates x, t are imaginary; this corresponds to the fact that motion with a velocity greater than the velocity of light is impossible. Moreover, one cannot use a reference system moving with the velocity of light—in that case the denominators in (4.3) would go to zero.

For velocities V small compared with the velocity of light, we can use in place of (4.3) the approximate formulas:

$$x = x' + Vt', \quad y = y', \quad z = z', \quad t = t' + \frac{V}{c^2}x'. \tag{4.4}$$

Suppose there is a rod at rest in the K system, parallel to the X axis. Let its length, measured in this system, be $\Delta x = x_2 - x_1$ (x_2 and x_1 are the coordinates of the two ends of the rod in the K system). We now determine the length of this rod as measured in the K' system. To do this we must find the coordinates of the two ends of the rod (x_2' and x_1') in this system at one and the same time t'. From (4.3) we find:

$$x_1 = \frac{x_1' + Vt'}{\sqrt{1 - \dfrac{V^2}{c^2}}}, \qquad x_2 = \frac{x_2' + Vt'}{1 - \dfrac{V^2}{c^2}}.$$

The length of the rod in the K' system is $\Delta x' = x_2' - x_1'$; subtracting x_1 from x_2, we find

$$\Delta x = \frac{\Delta x'}{\sqrt{1 - \dfrac{V^2}{c^2}}}.$$

The *proper length* of a rod is its length in a reference system in which it is at rest. Let us denote it by $l_0 = \Delta x$, and the length of the rod in any other reference frame K' by l. Then

$$l = l_0\sqrt{1 - \frac{V^2}{c^2}}. \tag{4.5}$$

Thus a rod has its greatest length in the reference system in which it is at rest. Its length in a system in which it moves with velocity V is decreased by the factor $\sqrt{1 - V^2/c^2}$. This result of the theory of relativity is called the *Lorentz contraction.*

Since the transverse dimensions do not change because of its motion, the volume $\mathscr{V}$ of a body decreases according to the similar formula

$$\mathscr{V} = \mathscr{V}_0\sqrt{1 - \frac{V^2}{c^2}}, \tag{4.6}$$

where $\mathscr{V}_0$ is the *proper volume* of the body.

From the Lorentz transformation we can obtain anew the results already known to us concerning the proper time (§ 3). Suppose a clock to be at rest in the K' system. We take two events occurring at one and the same point x', y', z' in space in the K' system. The time between these events in the K' system is $\Delta t' = t_2' - t_1'$. Now we find the time Δt which elapses between these two events in the K system. From (4.3), we have

$$t_1 = \frac{t_1' + \frac{V}{c^2}x'}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad t_2 = \frac{t_2' + \frac{V}{c^2}x'}{\sqrt{1 - \frac{V^2}{c^2}}},$$

or, subtracting one from the other,

$$t_2 - t_1 = \Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{V^2}{c^2}}},$$

in complete agreement with (3.1).

Finally we mention another general property of Lorentz transformations which distinguishes them from Galilean transformations. The latter have the general property of commutativity, i.e. the combined result of two successive Galilean transformations (with different velocities $\mathbf{V}_1$ and $\mathbf{V}_2$) does not depend on the order in which the transformations are performed. On the other hand, the result of two successive Lorentz transformations does depend, in general, on their order. This is already apparent purely mathematically from our formal description of these transformations as rotations of the four-dimensional coordinate system: we know that the result of two rotations (about different axes) depends on the order in which they are carried out. The sole exception is the case of transformations with parallel vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ (which are equivalent to two rotations of the four-dimensional coordinate system about the same axis).

§ 5. Transformation of velocities

In the preceding section we obtained formulas which enable us to find from the coordinates of an event in one reference frame, the coordinates of the same event in a second reference frame. Now we find formulas relating the velocity of a material particle in one reference system to its velocity in a second reference system.

Let us suppose once again that the K' system moves relative to the K system with velocity V along the x axis. Let $v_x = dx/dt$ be the component of the particle velocity in the K system and $v'_x = dx'/dt'$ the velocity component of the same particle in the K' system. From (4.3), we have

$$dx = \frac{dx' + Vdt'}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad dy = dy', \quad dz = dz', \quad dt = \frac{dt' + \frac{V}{c^2}dx'}{\sqrt{1 - \frac{V^2}{c^2}}}.$$

Dividing the first three equations by the fourth and introducing the velocities

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \quad \mathbf{v}' = \frac{d\mathbf{r}'}{dt'},$$

we find

$$v_x = \frac{v'_x + V}{1 + v'_x \frac{V}{c^2}}, \quad v_y = \frac{v'_y\sqrt{1 - \frac{V^2}{c^2}}}{1 + v'_x\frac{V}{c^2}}, \quad v_z = \frac{v'_z\sqrt{1 - \frac{V^2}{c^2}}}{1 + v'_x\frac{V}{c^2}}. \tag{5.1}$$

These formulas determine the transformation of velocities. They describe the law of composition of velocities in the theory of relativity. In the limiting case of $c \to \infty$, they go over into the formulas $v_x = v'_x + V$, $v_y, = v'_y$, $v_z = v'_z$ of classical mechanics.

In the special case of motion of a particle parallel to the X axis, $v_x = v$, $v_y = v_z = 0$. Then $v'_y = v'_z = 0$, $v'_x = v'$, so that

$$v = \frac{v' + V}{1 + v'\frac{V}{c^2}}. \tag{5.2}$$

It is easy to convince oneself that the sum of two velocities each smaller than the velocity of light is again not greater than the light velocity.

For a velocity V significantly smaller than the velocity of light (the velocity v can be arbitrary), we have approximately, to terms of order V/c:

$$v_x = v'_x + V\left(1 - \frac{v'^2_x}{c^2}\right), \quad v_y = v'_y - v'_x v'_y \frac{V}{c^2}, \quad v_z = v'_z - v'_x v'_z \frac{V}{c^2}.$$

These three formulas can be written as a single vector formula

$$\mathbf{v} = \mathbf{v}' + \mathbf{V} - \frac{1}{c^2}(\mathbf{V}\cdot\mathbf{v}')\mathbf{v}'. \tag{5.3}$$

We may point out that in the relativistic-law of addition of velocities (5.1) the two velocities $\mathbf{v}'$ and $\mathbf{V}$ which are combined enter unsymmetrically (provided they are not both directed along the x axis). This fact is related to the noncommutativity of Lorentz transformations which we mentioned in the preceding section.

Let us choose our coordinate axes so that the velocity of the particle at the given moment lies in the XY plane. Then the velocity of the particle in the K system has components $v_x = v\cos\theta$, $v_y = v\sin\theta$, and in the K' system $v'_x = v'\cos\theta'$, $v'_y = v'\sin\theta'$ (v, v', θ, θ' are the absolute values and the angles subtended with the X, X' axes respectively in the K, K' systems), With the help of formula (5.1), we then find

$$\tan\theta = \frac{v'\sqrt{1-\frac{V^2}{c^2}}\sin\theta'}{v'\cos\theta' + V}. \tag{5.4}$$

This formula describes the change in the direction of the velocity on transforming from one reference system to another.

Let us consider a very important special case of this formula, namely, the deviation of light in transforming to a new reference system—a phenomenon known as the *aberration of light*. In this case $v = v' = c$, so that the preceding formula goes over into

$$\tan\theta = \frac{\sqrt{1-\frac{V^2}{c^2}}}{\frac{V}{c}+\cos\theta'}\sin\theta'. \tag{5.5}$$

From the same transformation formulas (5.1) it is easy to obtain for $\sin\theta$ and $\cos\theta$:

$$\sin\theta = \frac{\sqrt{1-\frac{V^2}{c^2}}}{1+\frac{V}{c}\cos\theta'}\sin\theta', \qquad \cos\theta = \frac{\cos\theta' + \frac{V}{c}}{1+\frac{V}{c}\cos\theta'}. \tag{5.6}$$

In case $V \ll c$, we find from this formula, correct to terms of order V/c:

$$\sin\theta - \sin\theta' = -\frac{V}{c}\sin\theta'\cos\theta'.$$

Introducing the angle $\Delta\theta = \theta' - \theta$ (the aberration angle), we find to the same order of accuracy

$$\Delta\theta = \frac{V}{c}\sin\theta', \tag{5.7}$$

which is the well-known elementary formula for the aberration of light.

§ 6. Four-vectors

The coordinates of an event (ct, x, y, z) can be considered as the components of a four-dimensional radius vector (or, for short, a four-radius vector) in a four-dimensional space. We shall denote its components by x^i, where the index i takes on the values 0, 1, 2, 3, and

$$x^0 = ct, \qquad x^1 = x, \qquad x^2 = y, \qquad x^3 = z.$$

The square of the "length" of the radius four-vector is given by

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.$$

It does not change under any rotations of the four-dimensional coordinate system, in particular under Lorentz transformations.

In general a set of four quantities A^0, A^1, A^2, A^3 which transform like the components of the radius four-vector x^i under transformations of the four-dimensional coordinate system is called a *four-dimensional vector* (*four-vector*) A^i. Under Lorentz transformations,

$$A^0 = \frac{A'^0 + \frac{V}{c}A'^1}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad A^1 = \frac{A'^1 + \frac{V}{c}A'^0}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad A^2 = A'^2, \quad A^3 = A'^3. \tag{6.1}$$

The square magnitude of any four-vector is defined analogously to the square of the radius four-vector:

$$(A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2.$$

For convenience of notation, we introduce two "types" of components of four-vectors, denoting them by the symbols A^i and A_i, with superscripts and subscripts. These are related by

$$A_0 = A^0, \quad A_1 = -A^1, \quad A_2 = -A^2, \quad A_3 = -A^3. \tag{6.2}$$

The quantities A^i are called the *contravariant*, and the A_i the *covariant* components of the four-vector. The square of the four-vector then appears in the form

$$\sum_{i=0}^{3} A^i A_i = A^0 A_0 + A^1 A_1 + A^2 A_2 + A^3 A_3.$$

Such sums are customarily written simply as A^iA_i, omitting the summation sign. One agrees that one sums over any repeated index, and omits the summation sign. Of the pair of indices, one must be a superscript and the other a subscript. This convention for summation over "dummy" indices is very convenient and considerably simplifies the writing of formulas.

We shall use Latin letters $i, k, l, \ldots$, for four-dimensional indices, taking on the values 0, 1, 2, 3.

In analogy to the square of a four-vector, one forms the *scalar product* of two different four-vectors:

$$A^iB_i = A^0B_0 + A^1B_1 + A^2B_2 + A^3B_3.$$

It is clear that this can be written either as A^iB_i or A_iB^i—the result is the same. In general one can switch upper and lower indices in any pair of dummy indices.†

The product A^iB_i is a *four-scalar*—it is invariant under rotations of the four-dimensional coordinate system. This is easily verified directly,‡ but it is also apparent beforehand (from the analogy with the square A^iA_i) from the fact that all four-vectors transform according to the same rule.

† In the literature the indices are often omitted on four-vectors, and their squares and scalar products are written as A^2, AB. We shall not use this notation in the present text.

‡ One should remember that the law for transformation of a four-vector expressed in covariant components differs (in signs) from the same law expressed for contravariant components. Thus, instead of (6.1), one will have:

$$A_0 = \frac{A'_0 - \frac{V}{c}A'_1}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad A_1 = \frac{A'_1 - \frac{V}{c}A'_0}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad A_2 = A'_2, \quad A_3 = A'_3.$$

The component A^0 is called the *time component,* and A^1, A^2, A^3 the *space components* of the four-vector (in analogy to the radius four-vector). The square of a four-vector can be positive, negative, or zero; such vectors are called, *timelike, spacelike,* and *null-vectors,* respectively (again in analogy to the terminology for intervals).†

Under purely spatial rotations (i.e. transformations not affecting the time axis) the three space components of the four-vector A^i form a three-dimensional vector **A**. The time component of the four-vector is a three-dimensional scalar (with respect to these transformations). In enumerating the components of a four-vector, we shall often write them as

$$A^i = (A^0, \mathbf{A}).$$

The covariant components of the same four-vector are $A_i = (A^0, -\mathbf{A})$, and the square of the four-vector is $A^iA_i = (A^0)^2 - \mathbf{A}^2$. Thus, for the radius four-vector:

$$x^i = (ct, \mathbf{r}), \qquad x_i = (ct, -\mathbf{r}), \qquad x^ix_i = c^2t^2 - \mathbf{r}^2.$$

For three-dimensional vectors (with coordinates x, y, z) there is no need to distinguish between contra- and covariant components. Whenever this can be done without causing confusion, we shall write their components as $A_\alpha(\alpha = x, y, z)$ using Greek letters for subscripts. In particular we shall assume a summation over x, y, z for any repeated index (for example, $\mathbf{A}\cdot\mathbf{B} = A_\alpha B_\alpha$).

A *four-dimensional tensor* (*four-tensor*) of the second rank is a set of sixteen quantities A^{ik}, which under coordinate transformations transform like the products of components of two four-vectors. We similarly define four-tensors of higher rank.

The components of a second-rank tensor can be written in three forms: covariant, A_{ik}, contravariant, A^{ik}, and mixed, A^i_k (where, in the last case, one should distinguish between $A_k{}^i$ and $A^i{}_k$, i.e. one should be careful about which of the two is superscript and which a subscript). The connection between the different types of components is determined from the general rule: raising or lowering a space index (1, 2, 3) changes the sign of the component, while raising or lowering the time index (0) does not. Thus:

$$A_{00} = A^{00}, \quad A_{01} = -A^{01}, \quad A_{11} = A^{11}, \ldots,$$
$$A_0{}^0 = A^{00}, A_0{}^1 = A^{01}, A_1{}^0 = -A^{01}, A_1{}^1 = -A^{11}, \ldots.$$

Under purely spatial transformations, the nine quantities A^{11}, A^{12}, ... form a three-tensor. The three components A^{01}, A^{02}, A^{03} and the three components A^{10}, A^{20}, A^{30} constitute three-dimensional vectors, while the component A^{00} is a three-dimensional scalar.

A tensor A^{ik} is said to be *symmetric* if $A^{ik} = A^{ki}$, and *antisymmetric* if $A^{ik} = -A^{ki}$. In an antisymmetric tensor, all the diagonal components (i.e. the components A^{00}, A^{11}, . . .) are zero, since, for example, we must have $A^{00} = -A^{00}$. For a symmetric tensor A^{ik}, the mixed components $A^i{}_k$ and $A_k{}^i$ obviously coincide; in such cases we shall simply write A^i_k, putting the indices one above the other.

In every tensor equation, the two sides must contain identical and identically placed (i.e. above or below) free indices (as distinguished from dummy indices). The free indices in tensor equations can be shifted up or down, but this must be done simultaneously in all terms in the equation. Equating covariant and contravariant components of different tensors is "illegal"; such an equation, even if it happened by chance to be valid in a particular reference system, would be violated on going to another frame.

† Null vectors are also said to be *isotropic*.

From the tensor components A^{ik} one can form a scalar by taking the sum

$$A^i{}_i = A^0{}_0 + A^1{}_1 + A^2{}_2 + A^3{}_3$$

(where, of course, $A_i{}^i = A^i{}_i$). This sum is called the *trace* of the tensor, and the operation for obtaining it is called *contraction.*

The formation of the scalar product of two vectors, considered earlier, is a contraction operation: it is the formation of the scalar A^iB_i from the tensor A^iB_k. In general, contracting on any pair of indices reduces the rank of the tensor by 2. For example, $A^i{}_{kli}$ is a tensor of second rank $A^i{}_kB^k$ is a four-vector, $A^{ik}{}_{ik}$ is a scalar, etc.

The unit four-tensor δ^i_k satisfies the condition that for any four-vector A^i,

$$\delta^k_i A^i = A^k. \tag{6.3}$$

It is clear that the components of this tensor are

$$\delta^k_i = \begin{cases} 1, & \text{if} \quad i = k \\ 0, & \text{if} \quad i \neq k \end{cases} \tag{6.4}$$

Its trace is $\delta^i_i = 4$.

By raising the one index or lowering the other in δ^k_i, we can obtain the contra- or covariant tensor g^{ik} or g_{ik}, which is called the *metric tensor*. The tensors g^{ik} and g_{ik} have identical components, which can be written as a matrix:

$$(g^{ik}) = (g_{ik}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.5}$$

(the index i labels the rows, and k the columns, in the order 0, 1, 2, 3). It is clear that

$$g_{ik}A^k = A_i, \quad g^{ik}A_k = A^i. \tag{6.6}$$

The scalar product of two four-vectors can therefore be written in the form:

$$A^iA_i = g_{ik}A^iA^k = g^{ik}A_iA_k. \tag{6.7}$$

The tensors δ^i_k, g_{ik}, g^{ik} are special in the sense that their components are the same in all coordinate systems. The *completely antisymmetric unit tensor* of fourth rank, e^{iklm}, has the same property. This is the tensor whose components change sign under interchange of any pair of indices, and whose nonzero components are ± 1. From the antisymmetry it follows that all components in which two indices are the same are zero, so that the only non-vanishing components are those for which all four indices are different. We set

$$e^{0123} = +1 \tag{6.8}$$

(hence $e_{0123} = -1$). Then all the other nonvanishing components e^{iklm} are equal to $+1$ or -1, according as the numbers i, k, l, m can be brought to the arrangement 0, 1, 2, 3 by an even or an odd number of transpositions. The number of such components is $4! = 24$. Thus,

$$e^{iklm}e_{iklm} = -24. \tag{6.9}$$

With respect to rotations of the coordinate system, the quantities e^{iklm} behave like the components of a tensor; but if we change the sign of one or three of the coordinates the components e^{iklm}, being defined as the same in all coordinate systems, do not change, whereas some of the components of a tensor should change sign. Thus e^{iklm} is, strictly speaking, not a tensor, but rather a *pseudotensor*. Pseudotensors of any rank, in particular *pseudoscalars,* behave like tensors under all coordinate transformations except those that cannot be reduced to rotations, i.e. reflections, which are changes in sign of the coordinates that are not reducible to a rotation.

The products $e^{iklm}e^{prst}$ form a four-tensor of rank 8, which is a true tensor; by contracting on one or more pairs of indices, one obtains tensors of rank 6, 4, and 2. All these tensors have the same form in all coordinate systems. Thus their components must be expressed as combinations of products of components of the unit tensor δ^i_k — the only true tensor whose components are the same in all coordinate systems. These combinations can easily be found by starting from the symmetries that they must possess under permutation of indices.†

If A^{ik} is an antisymmetric tensor, the tensor A^{ik} and the pseudotensor $A^{*ik} = \frac{1}{2} e^{iklm} A_{lm}$ are said to be *dual* to one another. Similarly, $e^{iklm} A_m$ is an antisymmetric pseudotensor of rank 3, dual to the vector A^i. The product $A^{ik} A^*_{ik}$ of dual tensors is obviously a pseudoscalar.

In this connection we note some analogous properties of three-dimensional vectors and tensors. The completely antisymmetric unit pseudotensor of rank 3 is the set of quantities $e_{\alpha\beta\gamma}$ which change sign under any transposition of a pair of indices. The only nonvanishing components of $e_{\alpha\beta\gamma}$ are those with three different indices. We set $e_{xyz} = 1$; the others are 1 or -1, depending on whether the sequence α, β, γ can be brought to the order x, y, z by an even or an odd number of transpositions.‡

† For reference we give the following formulas:

$$e^{iklm}e_{prst} = -\begin{vmatrix} \delta^i_p & \delta^i_r & \delta^i_s & \delta^i_t \\ \delta^k_p & \delta^k_r & \delta^k_s & \delta^k_t \\ \delta^l_p & \delta^l_r & \delta^l_s & \delta^l_t \\ \delta^m_p & \delta^m_r & \delta^m_s & \delta^m_t \end{vmatrix}, \quad e^{iklm}e_{prsm} = -\begin{vmatrix} \delta^i_p & \delta^i_r & \delta^i_s \\ \delta^k_p & \delta^k_r & \delta^k_s \\ \delta^l_p & \delta^l_r & \delta^l_s \end{vmatrix}$$

$$e^{iklm}e_{prlm} = -2(\delta^i_p\delta^k_r - \delta^i_r\delta^k_p), \qquad e^{iklm}e_{prlm} = -6\delta^i_p.$$

The overall coefficient in these formulas can be checked using the result of a complete contraction, which should give (6.9).

As a consequence of these formulas we have:

$$e^{prst}A_{lp}A_{kr}A_{is}A_{mt} = -Ae_{iklm}.$$

$$e^{iklm}e^{prst}A_{tp}A_{kr}A_{ls}A_{mt} = 24A.$$

where A is the determinant formed from the quantities A_{ik}.

‡ The fact that the components of the four-tensor e^{iklm} are unchanged under rotations of the four-dimensional coordinate system, and that the components of the three-tensor $e_{\alpha\beta\gamma}$ are unchanged by rotations of the space axes are special cases of a general rule: any completely antisymmetric tensor of rank equal to the number of dimensions of the space in which it is defined is invariant under rotations of the coordinate system in the space.

The products $e_{\alpha\beta\gamma}e_{\lambda\mu\nu}$ form a true three-dimensional tensor of rank 6, and are therefore expressible as combinations of products of components of the unit three-tensor $\delta_{\alpha\beta}$.†

Under a reflection of the coordinate system, i.e. under a change in sign of all the coordinates, the components of an ordinary vector also change sign. Such vectors are said to be *polar*. The components of a vector that can be written as the cross product of two polar vectors do not change sign under inversion. Such vectors are said to be *axial*. The scalar product of a polar and an axial vector is not a true scalar, but rather a pseudoscalar; it changes sign under a coordinate inversion. An axial vector is a pseudovector, dual to some antisymmetric tensor. Thus, if $\mathbf{C} = \mathbf{A} \times \mathbf{B}$, then

$$C_\alpha = \tfrac{1}{2}e_{\alpha\beta\gamma}C_{\beta\gamma}, \quad \text{where} \quad C_{\beta\gamma} = A_\beta B_\gamma - A_\gamma B_\beta.$$

Now consider four-tensors. The space components (i, k, = 1, 2, 3) of the antisymmetric tensor A^{ik} form a three-dimensional antisymmetric tensor with respect to purely spatial transformations; according to our statement its components can be expressed in terms of the components of a three-dimensional axial vector. With respect to these same transformations the components A^{01}, A^{02}, A^{03} form a three-dimensional polar vector. Thus the components of an antisymmetric four-tensor can be written as a matrix:

$$(A^{ik}) = \begin{vmatrix} 0 & p_x & p_y & p_z \\ -p_x & 0 & -a_z & a_y \\ -p_y & a_z & 0 & -a_x \\ -p_z & -a_y & a_x & 0 \end{vmatrix}, \tag{6.10}$$

where, with respect to spatial transformations, $\mathbf{p}$ and $\mathbf{a}$ are polar and axial vectors, respectively. In enumerating the components of an antisymmetric four-tensor, we shall write them in the form

$$A^{ik} = (\mathbf{p}, \mathbf{a});$$

then the covariant components of the same tensor are

$$A_{ik} = (-\mathbf{p}, \mathbf{a}).$$

Finally we consider certain differential and integral operations of four-dimensional tensor analysis.

The four-gradient of a scalar ϕ is the four-vector

† For reference, we give the appropriate formulas:

$$e_{\alpha\beta\gamma}e_{\lambda\mu\nu} = \begin{vmatrix} \delta_{\alpha\lambda} & \delta_{\alpha\mu} & \delta_{\alpha\nu} \\ \delta_{\beta\lambda} & \delta_{\beta\mu} & \delta_{\beta\nu} \\ \delta_{\gamma\lambda} & \delta_{\gamma\mu} & \delta_{\gamma\nu} \end{vmatrix}.$$

Contracting this tensor on one, two and three pairs of indices, we get:

$$e_{\alpha\beta\gamma}e_{\lambda\mu\gamma} = \delta_{\alpha\lambda}\delta_{\beta\mu} - \delta_{\alpha\mu}\delta_{\beta\lambda},$$

$$e_{\alpha\beta\gamma}e_{\lambda\beta\gamma} = 2\delta_{\alpha\lambda},$$

$$e_{\alpha\beta\gamma}e_{\alpha\beta\gamma} = 6.$$

$$\frac{\partial \phi}{\partial x^i} = \left(\frac{1}{c}\frac{\partial \phi}{\partial t}, \nabla \phi\right).$$

We must remember that these derivatives are to be regarded as the covariant components of the four-vector. In fact, the differential of the scalar

$$d\phi = \frac{\partial \phi}{\partial x^i} dx^i$$

is also a scalar; from its form (scalar product of two four-vectors) our assertion is obvious.

In general, the operators of differentiation with respect to the coordinates x^i, $\partial/\partial x^i$, should be regarded as the covariant components of the operator four-vector. Thus, for example, the divergence of a four-vector, the expression $\partial A^i/\partial x^i$, in which we differentiate the contravariant components A^i, is a scalar.†

In three-dimensional space one can extend integrals over a volume, a surface or a curve. In four-dimensional space there are four types of integrations:

(1) Integral over a curve in four-space. The element of integration is the line element, i.e. the four-vector dx^i.

(2) Integral over a (two-dimensional) surface in four-space. As we know, in three-space the projections of the area of the parallelogram formed from the vectors $d\mathbf{r}$ and $d\mathbf{r}'$ on the coordinate planes $x_\alpha x_\beta$ are $dx_\alpha dx'_\beta - dx_\beta dx'_\alpha$. Analogously, in four-space the infinitesimal element of surface is given by the antisymmetric tensor of second rank $df^{ik} = dx^i dx'^k - dx^k dx'^i$; its components are the projections of the element of area on the coordinate planes. In three-dimensional space, as we know, one uses as surface element in place of the tensor $df_{\alpha\beta}$ the vector df_α dual to the tensor $df_{\alpha\beta}$: $df_\alpha = \frac{1}{2} e_{\alpha\beta\gamma} df_{\beta\gamma}$. Geometrically this is a vector normal to the surface element and equal in absolute magnitude to the area of the element. In four-space we cannot construct such a vector, but we can construct the tensor df^{*ik} dual to the tensor df^{ik},

$$df^{*ik} = \frac{1}{2} e^{iklm} df_{lm}. \tag{6.11}$$

Geometrically it describes an element of surface equal to and "normal" to the element of

† If we differentiate with respect to the "covariant coordinates" x_i, then the derivatives

$$\frac{\partial \phi}{\partial x_i} = \left(\frac{1}{c}\frac{\partial \phi}{\partial t}, -\nabla \phi\right)$$

form the contravariant components of a four-vector. We shall use this form only in exceptional cases [for example, for writing the square of the four-gradient $(\partial\phi/\partial x^i)(\partial\phi/\partial x_i)$].

We note that in the literature partial derivatives with respect to the coordinates are often abbreviated using the symbols.

$$\partial^i = \frac{\partial}{\partial x_i}, \quad \partial_i = \frac{\partial}{\partial x^i}.$$

In this form of writing of the differentiation operators, the co- or contravariant character of quantities formed with them is explicit. This same advantage exists for another abbreviated form for writing derivatives, using the index preceded by a comma:

$$\phi_{,i} = \frac{\partial \phi}{\partial x^i}, \quad \phi^{,i} = \frac{\partial \phi}{\partial x_i}.$$

surface df^{ik}; all segments lying in it are orthogonal to all segments in the element df^{ik}. It is obvious that $df^{ik}df^*_{ik} = 0$.

(3) Integral over a hypersurface, i.e. over a three-dimensional manifold. In three-dimensional space the volume of the parallelepiped spanned by three vectors is equal to the determinant of the third rank formed from the components of the vectors. One obtains analogously the projections of the volume of the parallelepiped (i.e. the "areas" of the hypersurface) spanned by three four-vectors dx^i, dx'^i, dx''^i; they are given by the determinants

$$dS^{ikl} = \begin{vmatrix} dx^i & dx'^i & dx''^i \\ dx^k & dx'^k & dx''^k \\ dx^l & dx'^l & dx''^l \end{vmatrix},$$

which form a tensor of rank 3, antisymmetric in all three indices. As element of integration over the hypersurface, it is more convenient to use the four-vector dS^i, dual to the tensor dS^{ikl}:

$$dS^i = -\frac{1}{6}e^{iklm}dS_{klm}, \quad dS_{klm} = e_{nklm}dS^n. \tag{6.12}$$

Here

$$dS^0 = dS^{123}, \quad dS^1 = dS^{023}, \ldots$$

Geometrically dS^i is a four-vector equal in magnitude to the "areas" of the hypersurface element, and normal to this element (i.e. perpendicular to all lines drawn in the hypersurface element). In particular, $dS^0 = dx\,dy\,dz$, i.e. it is the element of three-dimensional volume dV, the projection of the hypersurface element on the hyperplane x^0 = const.

(4) Integral over a four-dimensional volume; the element of integration is the scalar

$$d\Omega = dx^0dx^1dx^2\,dx^3 = c\,dt\,dV. \tag{6.13}$$

The element is a scalar: it is obvious that the volume of a portion of four-space is unchanged by a rotation of the coordinate system.†

Analogous to the theorems of Gauss and Stokes in three-dimensional vector analysis, there are theorems that enable us to transform four-dimensional integrals.

The integral over a closed hypersurface can be transformed into an integral over the four-volume contained within it by replacing the element of integration dS_i by the operator

$$dS_i \to d\Omega\frac{\partial}{\partial x^i}. \tag{6.14}$$

For example, for the integral of a vector A^i we have:

† Under a transformation from the integration variables x^0, x^1, x^2, x^3 to new variables x'^0, x'^1, x'^2, x'^3, the element of integration changes to $J\,d\Omega'$, where $d\Omega' = dx'^0\,dx'^1\,dx'^2\,dx'^3$

$$J = \frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)}$$

is the Jacobian of the transformation. For a linear transformation of the form $x'^i = a^i_k x^k$, the Jacobian J coincides with the determinant $|a^i_k|$ and is equal to unity for rotations of the coordinate system; this shows the invariance of $d\Omega$.

$$\oint A^i dS_i = \int \frac{\partial A^i}{\partial x^i}\, d\Omega. \tag{6.15}$$

This formula is the generalization of Gauss' theorem.

An integral over a two-dimensional surface is transformed into an integral over the hypersurface "spanning" it by replacing the element of integration df_{ik}^* by the operator

$$df_{ik}^* \to dS_i \frac{\partial}{\partial x^k} - dS_k \frac{\partial}{\partial x^i}. \tag{6.16}$$

For example, for the integral of an antisymmetric tensor A^{ik} we have:

$$\frac{1}{2}\int A^{ik}\, df_{ik}^* = \frac{1}{2}\int \left(dS_i \frac{\partial A^{ik}}{\partial x^k} - dS_k \frac{\partial A^{ik}}{\partial x^i}\right) = \int dS_i \frac{\partial A^{ik}}{\partial x^k}. \tag{6.17}$$

The integral over a four-dimensional closed curve is transformed into an integral over the surface spanning it by the substitution:

$$dx^i \to df^{ki} \frac{\partial}{\partial x^k}. \tag{6.18}$$

Thus for the integral of a vector, we have:

$$\oint A_i dx^i = \int df^{ki} \frac{\partial A_i}{\partial x^k} = \frac{1}{2}\int df^{ik}\left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}\right), \tag{6.19}$$

which is the generalization of Stokes' theorem.

PROBLEMS

1. Find the law of transformation of the components of a symmetric four-tensor A^{ik} under Lorentz transformations (6.1).

Solution: Considering the components of the tensor as products of components of two four-vectors, we get:

$$A^{00} = \frac{1}{1 - \frac{V^2}{c^2}}\left(A'^{00} + 2\frac{V}{c}A'^{01} + \frac{V^2}{c^2}A'^{11}\right), \quad A^{11} = \frac{1}{1 - \frac{V^2}{c^2}}\left(A'^{11} + 2\frac{V}{c}A'^{01} + \frac{V^2}{c^2}A'^{00}\right),$$

$$A^{22} = A'^{22}, \quad A^{23} = A'^{23}, \quad A^{12} = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left(A'^{12} + \frac{V}{c}A'^{02}\right),$$

$$A^{01} = \frac{1}{1 - \frac{V^2}{c^2}}\left[A'^{01}\left(1 + \frac{V^2}{c^2}\right) + \frac{V}{c}A'^{00} + \frac{V}{c} + A'^{11}\right],$$

$$A^{02} = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left(A'^{02} + \frac{V}{c}A'^{12}\right),$$

and analogous formulas for A^{33}, A^{13} and A^{03}.

2. The same for the antisymmetric tensor A^{ik}.

Solution: Since the coordinates x^2 and x^3 do not change, the tensor component A^{23} does not change, while the components A^{12}, A^{13} and A^{02}, A^{03} transform like x^1 and x^0:

$$A^{23} = A'^{23}, \quad A^{12} = \frac{A'^{12} + \frac{V}{c}A'^{02}}{\sqrt{1 - \frac{V^2}{c^2}}}, \qquad A^{02} = \frac{A'^{02} + \frac{V}{c}A'^{12}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

and similarly for A^{13}, A^{03}.

With respect to rotations of the two-dimensional coordinate system in the plane x^0x^1 (which are the transformations we are considering) the components $A^{01} = -A^{10}$, $A^{00} = A^{11} = 0$, form an antisymmetric of tensor of rank two, equal to the number of dimensions of the space. Thus, (see the remark on p. 19) these components are not changed by the transformations:

$$A^{01} = A'^{01}.$$

§ 7. Four-dimensional velocity

From the ordinary three-dimensional velocity vector one can form a four-vector. This four-dimensional velocity (*four-velocity*) of a particle is the vector

$$u^i = \frac{dx^i}{ds}. \tag{7.1}$$

To find its components, we note that according to (3.1),

$$ds = cdt\sqrt{1 - \frac{v^2}{c^2}},$$

where v is the ordinary three-dimensional velocity of the particle. Thus

$$u^1 = \frac{dx^1}{ds} = \frac{dx}{cdt\sqrt{1 - \frac{v^2}{c^2}}} = \frac{v_x}{c\sqrt{1 - \frac{v^2}{c^2}}},$$

etc. Thus

$$u^i = \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{\mathbf{v}}{c\sqrt{1 - \frac{v^2}{c^2}}}\right). \tag{7.2}$$

Note that the four-velocity is a dimensionless quantity.

The components of the four-velocity are not independent. Noting that $dx_i dx^i = ds^2$, we have

$$u^i u_i = 1. \tag{7.3}$$

Geometrically, u^i is a unit four-vector tangent to the world line of the particle.

Similarly to the definition of the four-velocity, the second derivative

$$w^i = \frac{d^2x^i}{ds^2} = \frac{du^i}{ds}$$

may be called the four-acceleration. Differentiating formula (7.3), we find:

$$u_i w^i = 0, \tag{7.4}$$

i.e. the four-vectors of velocity and acceleration are "mutually perpendicular".

PROBLEM

Determine the relativistic uniformly accelerated motion, i.e. the rectilinear motion for which the acceleration w in the proper reference frame (at each instant of time) remains constant.

Solution: In the reference frame in which the particle velocity is $v = 0$, the components of the four-acceleration $w^i = (0, w/c^2, 0, 0)$ (where w is the ordinary three-dimensional acceleration, which is directed along the x axis). The relativistically invariant condition for uniform acceleration must be expressed by the constancy of the four-scalar which coincides with w^2 in the proper reference frame:

$$w^i w_i = \text{const} \equiv -\frac{w^2}{c^4}.$$

In the "fixed" frame, with respect to which the motion is observed, writing out the expression for $w^i w_i$ gives the equation

$$\frac{d}{dt}\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} = w, \quad \text{or} \quad \frac{v}{\sqrt{1-\frac{v^2}{c^2}}} = wt + \text{const}.$$

Setting $v = 0$ for $t = 0$, we find that const = 0, so that

$$v = \frac{wt}{\sqrt{1+\frac{w^2t^2}{c^2}}}.$$

Integrating once more and setting $x = 0$ for $t = 0$, we find:

$$x = \frac{c^2}{w}\left(\sqrt{1+\frac{w^2t^2}{c^2}} - 1\right).$$

For $wt \ll c$, these formulas go over the classical expressions $v = wt$, $x = wt^2/2$. For $wt \to \infty$, the velocity tends toward the constant value c.

The proper time of a uniformly accelerated particle is given by the integral

$$\int_0^t \sqrt{1-\frac{v^2}{c^2}}\,dt = \frac{c}{w}\sinh^{-1}\left(\frac{wt}{c}\right).$$

As $t \to \infty$, it increases much more slowly than t, according to the law $c/w \ln(2wt/c)$.

CHAPTER 2

RELATIVISTIC MECHANICS

§ 8. The principle of least action

In studying the motion of material particles, we shall start from the Principle of Least Action. The *principle of least action* is defined, as we know, by the statement that for each mechanical system there exists a certain integral S, called the *action*, which has a minimum value for the actual motion, so that its variation δS is zero.†

To determine the action integral for a free material particle (a particle not under the influence of any external force), we note that this integral must not depend on our choice of reference system, that is, it must be invariant under Lorentz transformations. Then it follows that it must depend on a scalar. Furthermore, it is clear that the integrand must be a differential of the first order. But the only scalar of this kind that one can construct for a free particle is the interval ds, or $\alpha\, ds$, where α is some constant. So for a free particle the action must have the form

$$S = -\alpha \int_a^b ds,$$

where ${}_a\int^b$ is an integral along the world line of the particle between the two particular events of the arrival of the particle at the initial position and at the final position at definite times t_1 and t_2, i.e. between two given world points; and α is some constant characterizing the particle. It is easy to see that α must be a positive quantity for all particles. In fact, as we saw in § 3, ${}_a\int^b ds$ has its maximum value along a straight world line; by integrating along a curved world line we can make the integral arbitrarily small. Thus the integral ${}_a\int^b ds$ with the positive sign cannot have a minimum; with the opposite sign it clearly has a minimum, along the straight world line.

The action integral can be represented as an integral with respect to the time

$$S = \int_{t_1}^{t_2} L dt.$$

The coefficient L of dt represents the *Lagrange function* of the mechanical system. With the aid of (3.1), we find:

† Strictly speaking, the principle of least action asserts that the integral S must be a minimum only for infinitesimal lengths of the path of integration. For paths of arbitrary length we can say only that S must be an extremum, not necessarily a minimum. (See *Mechanics*, § 2.)

$$S = -\int_{t_1}^{t_2} \alpha c \sqrt{1 - \frac{v^2}{c^2}}\, dt,$$

where v is the velocity of the material particle. Consequently the Lagrangian for the particle is

$$L = -\alpha c\sqrt{1 - v^2/c^2}\,.$$

The quantity α, as already mentioned, characterizes the particle. In classical mechanics each particle is characterized by its mass m. Let us find the relation between α and m. It can be determined from the fact that in the limit as $c \to \infty$, our expression for L must go over into the classical expression $L = mv^2/2$. To carry out this transition we expand L in powers of v/c. Then, neglecting terms of higher order, we find

$$L = -\alpha c\sqrt{1 - \frac{v^2}{c^2}} \approx -\alpha c + \frac{\alpha v^2}{2c}.$$

Constant terms in the Lagrangian do not affect the equation of motion and can be omitted. Omitting the constant αc in L and comparing with the classical expression $L = mv^2/2$, we find that $\alpha = mc$.

Thus the action for a free material point is

$$S = -mc\int_a^b ds \tag{8.1}$$

and the Lagrangian is

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}}. \tag{8.2}$$

§ 9. Energy and momentum

By the *momentum* of a particle we can mean the vector $\mathbf{p} = \partial L/\partial \mathbf{v}$ ($\partial L/\partial \mathbf{v}$ is the symbolic representation of the vector whose components are the derivatives of L with respect to the corresponding components of $\mathbf{v}$). Using (8.2), we find;

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{9.1}$$

For small velocities ($v \ll c$) or, in the limit as $c \to \infty$, this expression goes over into the classical $\mathbf{p} = m\mathbf{v}$. For $v = c$, the momentum becomes infinite.

The time derivative of the momentum is the force acting on the particle. Suppose the velocity of the particle changes only in direction, that is, suppose the force is directed perpendicular to the velocity. Then

$$\frac{d\mathbf{p}}{dt} = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}\frac{d\mathbf{v}}{dt}. \tag{9.2}$$

If the velocity changes only in magnitude, that is, if the force is parallel to the velocity, then

$$\frac{d\mathbf{p}}{dt} = \frac{m}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} + \frac{d\mathbf{v}}{dt}. \tag{9.3}$$

We see that the ratio of force to acceleration is different in the two cases.

The *energy* $\mathscr{E}$ of the particle is defined as the quantity †

$$\mathscr{E} = \mathbf{p} \cdot \mathbf{v} - L.$$

Substituting the expressions (8.2) and (9.1) for L and $\mathbf{p}$, we find

$$\mathscr{E} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{9.4}$$

This very important formula shows, in particular, that in relativistic mechanics the energy of a free particle does not go to zero for $v = 0$, but rather takes on a finite value

$$\mathscr{E} = mc^2. \tag{9.5}$$

This quantity is called the ***rest energy*** of the particle.

For small velocities ($v/c << 1$), we have, expanding (9.4) in series in powers of v/c,

$$\mathscr{E} \approx mc^2 + \frac{mv^2}{2},$$

which, except for the rest energy, is the classical expression for the kinetic energy of a particle.

We emphasize that, although we speak of a "particle", we have nowhere made use of the fact that it is "elementary". Thus the formulas are equally applicable to any composite body consisting of many particles, where by m we mean the total mass of the body and by v the velocity of its motion as a whole. In particular, formula (9.5) is valid for any body which is at rest as a whole. We call attention to the fact that in relativistic mechanics the energy of a free body (i.e. the energy of any closed system) is a completely definite quantity which is always positive and is directly related to the mass of the body. In this connection we recall that in classical mechanics the energy of a body is defined only to within an arbitrary constant, and can be either positive or negative.

The energy of a body at rest contains, in addition to the rest energies of its constituent particles, the kinetic energy of the particles and the energy of their interactions with one another. In other words, mc^2 is not equal to $\sum m_a c^2$ (where m_a are the masses of the particles), and so m is not equal to $\sum m_a$. Thus in relativistic mechanics the law of conservation of mass does not hold: the mass of a composite body is not equal to the sum of the masses of its parts. Instead only the law of conservation of energy, in which the rest energies of the particles are included, is valid.

Squaring (9.1) and (9.4) and comparing the results, we get the following relation between the energy and momentum of particle:

† *See Mechanics,* § 6.

$$\frac{\mathscr{E}^2}{c^2} = p^2 + m^2c^2. \tag{9.6}$$

The energy expressed in terms of the momentum is called the Hamiltonian function $\mathscr{H}$:

$$\mathscr{H} = c\sqrt{p^2 + m^2c^2}. \tag{9.7}$$

For low velocities, $p \ll mc$, and we have approximately

$$\mathscr{H} \approx mc^2 + \frac{p^2}{2m},$$

i.e., except for the rest energy we get the familiar classical expression for the Hamiltonian.

From (9.1) and (9.4) we get the following relation between the energy, momentum, and velocity of a free particle:

$$\mathbf{p} = \mathscr{E}\frac{\mathbf{v}}{c^2}. \tag{9.8}$$

For $v = c$, the momentum and energy of the particle become infinite. This means that a particle with mass m different from zero cannot move with the velocity of light. Nevertheless, in relativistic mechanics, particles of zero mass moving with the velocity of light can exist.† From (9.8) we have for such particles:

$$P = \frac{\mathscr{E}}{c}. \tag{9.9}$$

The same formula also holds approximately for particles with nonzero mass in the so-called *ultrarelativistic* case, when the particle energy $\mathscr{E}$ is large compared to its rest energy mc^2.

We now write all our formulas in four-dimensional form. According to the principle of least action,

$$\delta S = -mc\delta\int_a^b ds = 0.$$

To set up the expression for δS, we note that $ds = \sqrt{dx_i dx^i}$ and therefore

$$\delta S = -mc\int_a^b \frac{dx_i \delta dx^i}{ds} = -mc\int_a^b u_i d\delta x^i.$$

Integrating by parts, we obtain

$$\delta S = -mcu_i\delta x^i\Big|_a^b + mc\int_a^b \delta x^i \frac{du_i}{ds}ds. \tag{9.10}$$

As we know, to get the equations of motion we compare different trajectories between the same two points, i.e. at the limits $(\delta x^i)_a = (\delta x^i)_b = 0$. The actual trajectory is then determined

† For example, light quanta and neutrinos.

from the condition $\delta S = 0$. From (9.10) we thus obtain the equations $du_i/ds = 0$; that is, a constant velocity for the free particle in four-dimensional form.

To determine the variation of the action as a function of the coordinates, one must consider the point a as fixed, so that $(\delta x^i)_a = 0$. The second point is to be considered as variable, but only actual trajectories are admissible, i.e., those which satisfy the equations of motion. Therefore the integral in expression (9.10) for δS is zero. In place of $(\delta x^i)_b$ we may write simply δx^i, and thus obtain

$$\delta S = -\,mcu_i\delta x^i. \tag{9.11}$$

The four-vector

$$p_i = -\frac{\partial S}{\partial x^i} \tag{9.12}$$

is called the *momentum four-vector.* As we know from mechanics, the derivatives $\partial S/\partial x$, $\partial S/\partial y$, $\partial S/\partial z$ are the three components of the momentum vector $\mathbf{p}$ of the particle, while the derivative $-\partial S/\partial t$ is the particle energy $\mathscr{E}$. Thus the covariant components of the four-mementum-are $p_i = (\mathscr{E}/c, -\,\mathbf{p})$, while the contravariant components are†

$$p^i = (\mathscr{E}/c, \mathbf{p}). \tag{9.13}$$

From (9.11) we see that the components of the four-momentum of a free particle are:

$$p^i = mcu^i. \tag{9.14}$$

Substituting the components of the four-velocity from (7.2), we see that we actually get expressions (9.1) and (9.4) for $\mathbf{p}$ and $\mathscr{E}$.

Thus, in relativistic mechanics, momentum and energy are the components of a single four-vector. From this we immediately get the formulas for transformation of momentum and energy from one inertial system to another. Substituting (9.13) in the general formulas (6.1) for transformation of four-vectors, we find:

$$p_x = \frac{p_x' + \dfrac{V}{c^2}\mathscr{E}'}{\sqrt{1-\dfrac{V^2}{c^2}}}, \quad p_y = p_y', \quad p_z = p_z', \quad \mathscr{E} = \frac{\mathscr{E}' + Vp_x'}{\sqrt{1-\dfrac{V^2}{c^2}}}, \tag{9.15}$$

where p_x, p_y, p_z are the components of the three-dimensional vector $\mathbf{p}$.

From the definition (9.14) of the four-momentum, and the identity $u^iu_i = 1$, we have, for the square of the four-momentum of a free particle:

$$p_i p^i = m^2c^2. \tag{9.16}$$

Substituting the expressions (9.13), we get back (9.6).

By analogy with the usual definition of the force, the force four-vector is defined as the derivative:

$$g^i = \frac{dp^i}{ds} = mc\frac{du^i}{ds}. \tag{9.17}$$

† We call attention to a mnemonic for remembering the definition of the physical four-vectors: the *contravariant* components are related to the corresponding three-dimensional vectors ($\mathbf{r}$ for x^i, $\mathbf{p}$ for p^i) with the "right", positive sign.

Its components satisfy the identity $g_i u^i = 0$. The components of this four-vector are expressed in terms of the usual three-dimensional force vector $\mathbf{f} = d\mathbf{p}/dt$:

$$g^i = \left(\frac{\mathbf{f} \cdot \mathbf{v}}{c^2 \sqrt{1 - \frac{v^2}{c^2}}}, \frac{\mathbf{f}}{c \sqrt{1 - \frac{v^2}{c^2}}} \right). \tag{9.18}$$

The time component is related to the work done by the force.

The relativistic Hamilton–Jacobi equation is obtained by substituting the derivatives $-\partial S/\partial x^i$ for p_i in (9.16):

$$\frac{\partial S}{\partial x_i} \frac{\partial S}{\partial x^i} \equiv g^{ik} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} = m^2 c^2, \tag{9.19}$$

or, writing the sum explicitly:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{\partial S}{\partial x}\right)^2 - \left(\frac{\partial S}{\partial y}\right)^2 - \left(\frac{\partial S}{\partial z}\right)^2 = m^2 c^2. \tag{9.20}$$

The transition to the limiting case of classical mechanics in equation (9.19) is made as follows. First of all we must notice that just as in the corresponding transition with (9.7), the energy of a particle in relativistic mechanics contains the term mc^2, which it does not in classical mechanics. Inasmuch as the action S is related to the energy by $\mathscr{E} = -(\partial S/\partial t)$, in making the transition to classical mechanics we must in place of S substitute a new action S' according to the relation:

$$S = S' - mc^2 t.$$

Substituting this in (9.20), we find

$$\frac{1}{2m}\left[\left(\frac{\partial S'}{\partial x}\right)^2 + \left(\frac{\partial S'}{\partial y}\right)^2 + \left(\frac{\partial S'}{\partial z}\right)^2\right] - \frac{1}{2mc^2}\left(\frac{\partial S'}{\partial t}\right)^2 + \frac{\partial S'}{\partial t} = 0.$$

In the limit as $c \to \infty$, this equation goes over into the classical Hamilton–Jacobi equation.

§ 10. Transformation of distribution functions

In various physical problems we have to deal with distribution functions for the momenta of particles: $f(\mathbf{p})dp_x dp_y dp_z$ is the number of particles having momenta with components in given intervals dp_x, dp_y, dp_z (or, as we say for brevity, the number of particles in a given volume element $d^3p \equiv dp_x dp_y dp_z$ in "momentum space"). We are then faced with the problem of finding the law of transformation of the distribution function $f(\mathbf{p})$ when we transform from one reference system to another.

To solve this problem, we first determine the properties of the "volume element" $dp_x dp_y dp_z$ with respect to Lorentz transformations. If we introduce a four-dimensional coordinate system, on whose axes are marked the components of the four-momentum of a particle, then $dp_x dp_y dp_z$ can be considered as the zeroth component of an element of the hypersurface defined by the equation $p^i p_i = m^2 c^2$. The element of hypersurface is a four-vector directed along the normal to the hypersurface; in our case the direction of the normal obviously coincides with the direction of the four-vector p_i. From this it follows that the ratio

$$\frac{dp_x dp_y dp_z}{\mathscr{E}} \tag{10.1}$$

is an invariant quantity, since it is the ratio of corresponding components of two parallel four-vectors.†

The number of particles, $f\,dp_x dp_y dp_z$, is also obviously an invariant, since it does not depend on the choice of reference frame. Writing it in the form

$$f(\mathbf{p})\mathscr{E}\frac{dp_x dp_y dp_z}{\mathscr{E}}$$

and using the invariance of the ratio (10.1), we conclude that the product $f(\mathbf{p})\mathscr{E}$ is invariant. Thus the distribution function in the K' system is related to the distribution function in the K system by the formula

$$f'(\mathbf{p}') = \frac{f(\mathbf{p})\mathscr{E}}{\mathscr{E}'}, \tag{10.2}$$

where $\mathbf{p}$ and $\mathscr{E}$ must be expressed in terms of $\mathbf{p}'$ and $\mathscr{E}'$ by using the transformation formulas (9.15).

Let us now return to the invariant expression (10.1). If we introduce "spherical coordinates" in momentum space, the volume element $dp_x dp_y dp_z$ becomes $p^2 dp\,do$, where do is the element of solid angle around the direction of the vector $\mathbf{p}$. Noting that $pdp = \mathscr{E}d\mathscr{E}/c^2$ [from (9.6)], we have:

$$\frac{p^2 dp do}{\mathscr{E}} = \frac{pd\mathscr{E}\, do}{c^2}.$$

Thus we find that the expression

$$pd\mathscr{E}\, do \tag{10.3}$$

is also invariant.

The notion of a distribution function appears in a different aspect in the kinetic theory of gases: the product $f(\mathbf{r}, \mathbf{p})\, dp_x dp_y dp_z\, dV$ is the number of particles lying in a given volume element dV and having momenta in definite intervals dp_x, dp_y, dp_z. The function $f(\mathbf{r}, \mathbf{p})$ is

† The integration with respect to the element (10.1) can be expressed in four-dimensional form by means of the δ-function (cf. the footnote on p. 74) as an integration with respect to

$$\frac{2}{c}\delta(p_i p^i - m^2c^2)d^4p, \qquad d^4p = dp^0 dp^1 dp^2 dp^3. \tag{10.1a}$$

The four components p^i are treated as independent variables (with p^0 taking on only positive values). Formula (10.1*a*) is obvious from the following representation of the delta function appearing in it:

$$\delta(p^i p_i - m^2c^2) = \delta\left((p_0)^2 - \frac{\mathscr{E}^2}{c^2}\right) = \frac{c}{2\mathscr{E}}\left[\delta\left(p_0 + \frac{\mathscr{E}}{c}\right) + \delta\left(p_0 - \frac{\mathscr{E}}{c}\right)\right], \tag{10.1b}$$

where $\mathscr{E} = c\sqrt{p^2 + m^2c^2}$. This formula in turn follows from formula (V) of the footnote on p. 74.

called the distribution function in *phase space* (the space of the coordinates and momenta of the particle), and the product of differentials $d\tau = d^3p\, dV$ is the element of volume of this space. We shall find the law of transformation of this function.

In addition to the two reference systems K and K', we also introduce the frame K_0 in which the particles with the given momentum are at rest; the proper volume dV_0 of the element occupied by the particles is defined relative to this system. The velocities of the systems K and K' relative to the system K_0 coincide, by definition, with the velocities v and v' which these particles have in the systems K and K'. Thus, according to (4.6), we have

$$dV = dV_0\sqrt{1-\frac{v^2}{c^2}}, \quad dV' = dV_0\sqrt{1-\frac{v'^2}{c^2}},$$

from which

$$\frac{dV}{dV'} = \frac{\mathscr{E}'}{\mathscr{E}}.$$

Multiplying this equation by the equation $d^3p/d^3p' = \mathscr{E}/\mathscr{E}'$, we find that

$$d\tau = d\tau', \tag{10.4}$$

i.e. the element of phase volume is invariant. Since the number of particles $f\, d\tau$ is also invariant, by definition, we conclude that the distribution function in phase space is an invariant:

$$f'(\mathbf{r}', \mathbf{p}') = f(\mathbf{r}, \mathbf{p}), \tag{10.5}$$

where $\mathbf{r}'$, $\mathbf{p}'$ are related to $\mathbf{r}$, $\mathbf{p}$ by the formulas for the Lorentz transformation.

§ 11. Decay of particles

Let us consider the spontaneous decay of a body of mass M into two parts with masses m_1 and m_2. The law of conservation of energy in the decay, applied in the system of reference in which the body is at rest, gives†

$$M = \mathscr{E}_{10} + \mathscr{E}_{20}. \tag{11.1}$$

where $\mathscr{E}_{10}$ and $\mathscr{E}_{20}$ are the energies of the emerging particles. Since $\mathscr{E}_{10} > m_1$ and $\mathscr{E}_{20} > m_2$, the equality (11.1) can be satisfied only if $M > m_1 + m_2$, i.e. a body can disintegrate spontaneously into parts the sum of whose masses is less than the mass of the body. On the other hand, if $M < m_1 + m_2$ the body is stable (with respect to the particular decay) and does not decay spontaneously. To cause the decay in this case, we would have to supply to the body from outside an amount of energy at least equal to its "binding energy" $(m_1 + m_2 - M)$.

Momentum as well as energy must be conserved in the decay process. Since the initial momentum of the body was zero, the sum of the momenta of the emerging particles must be zero: $\mathbf{p}_{10} + \mathbf{p}_{20} = 0$. Consequently $p_{10}^2 = p_{20}^2$, or

† In §§ 11–13 we set $c = 1$. In other words the velocity of light is taken as the unit of velocity (so that the dimensions of length and time become the same). This choice is a natural one in relativistic mechanics and greatly simplifies the writing of formulas. However, in this book (which also contains a considerable amount of nonrelativistic theory) we shall not usually use this system of units, and will explicitly indicate when we do.

If c has been set equal to unity in formulas, it is easy to convert back to ordinary units: the velocity is introduced to assure correct dimensions.

$$\mathscr{E}_{10}^2 - m_1^2 = \mathscr{E}_{20}^2 - m_2^2. \tag{11.2}$$

The two equations (11.1) and (11.2) uniquely determine the energies of the emerging particles:

$$\mathscr{E}_{10} = \frac{M^2 + m_1^2 - m_2^2}{2M}, \qquad \mathscr{E}_{20} = \frac{M^2 - m_1^2 + m_2^2}{2M}. \tag{11.3}$$

In a certain sense the inverse of this problem is the calculation of the total energy M of two colliding particles in the system of reference in which their total momentum is zero. (This is abbreviated as the "system of the centre of inertia" or the "C-system".) The computation of this quantity gives a criterion for the possible occurrence of various inelastic collision processes, accompanied by a change in state of the colliding particles, or the "creation" of new particles. A process of this type can occur only if the sum of the masses of the "reaction products" does not exceed M.

Suppose that in the initial reference system (the "laboratory" system) a particle with mass m_1 and energy $\mathscr{E}_1$ collides with a particle of mass m_2 which is at rest. The total energy of the two particles is

$$\mathscr{E} = \mathscr{E}_1 + \mathscr{E}_2 = \mathscr{E}_1 + m_2,$$

and their total momentum is $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1$. Considering the two particles together as a single composite system, we find the velocity of its motion as a whole from (9.8):

$$\mathbf{V} = \frac{\mathbf{p}}{\mathscr{E}} = \frac{\mathbf{p}_1}{\mathscr{E}_1 + m_2}. \tag{11.4}$$

This quantity is the velocity of the C-system with respect to the laboratory system (the L-system).

However, in determining the mass M, there is no need to transform from one reference frame to the other. Instead we can make direct use of formula (9.6), which is applicable to the composite system just as it is to each particle individually. We thus have

$$M^2 = \mathscr{E}^2 - p^2 = (\mathscr{E}_1 + m_2)^2 - (\mathscr{E}_1^2 - m_1^2),$$

from which

$$M^2 = m_1^2 + m_2^2 + 2m_2\mathscr{E}_1. \tag{11.5}$$

PROBLEMS

1. A particle moving with velocity V dissociates "in flight" into two particles. Determine the relation between the angles of emergence of these particles and their energies.

Solution: Let $\mathscr{E}_0$ be the energy of one of the decay particles in the C-system [i.e. $\mathscr{E}_{10}$ or $\mathscr{E}_{20}$ in (11.3)], $\mathscr{E}$ the energy of this same particle in the L-system, and θ its angle of emergence in the L-system (with respect to the direction of $\mathbf{V}$). By using the transformation formulas we find:

$$\mathscr{E}_0 = \frac{\mathscr{E} - Vp\cos\theta}{\sqrt{1 - V^2}},$$

so that

$$\cos\theta = \frac{\mathscr{E} - \mathscr{E}_0\sqrt{1 - V^2}}{V\sqrt{\mathscr{E}^2 - m^2}}. \tag{1}$$

For the determination of $\mathscr{E}$ from $\cos\theta$ we then get the quadratic equation

$$\mathscr{E}^2(1-V^2\cos^2\theta)-2\mathscr{E}\mathscr{E}_0\sqrt{1-V^2}+\mathscr{E}_0^2(1-V^2)+V^2m^2\cos^2\theta=0, \qquad (2)$$

which has one positive root (if the velocity v_0 of the decay particle in the C-system satisfies $v_0 > V$) or two positive roots (if $v_0 < V$).

The source of this ambiguity is clear from the following graphical construction. According to (9.15), the momentum components in the L-system are expressed in terms of quantities referring to the C-system by the formulas

$$p_x = \frac{p_0\cos\theta_0+\mathscr{E}_0V}{\sqrt{1-V^2}}, \quad p_y = p_0\sin\theta_0.$$

Eliminating θ_0, we get

$$p_y^2+(p_x\sqrt{1-V^2}-\mathscr{E}_0V)^2 = p_0^2.$$

With respect to the variables p_x, p_y, this is the equation of an ellipse with semiaxes $p_0/\sqrt{1-V^2}, p_0$, whose centre (the point O in Fig. 3) has been shifted a distance $\mathscr{E}_0V/\sqrt{1-V^2}$ from the point $\mathbf{p}=0$ (point A in Fig. 3).†

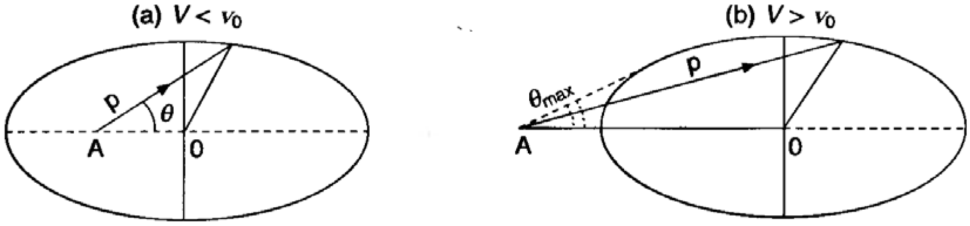

FIG. 3.

If $V > p_0/\mathscr{E}_0 = v_0$, the point A lies outside the ellipse (Fig. 3b), so that for a fixed angle θ the vector $\mathbf{p}$ (and consequently the energy $\mathscr{E}$) can have two different values. It is also clear from the construction that in this case the angle θ cannot exceed a definite value θ_{max} (corresponding to the position of the vector $\mathbf{p}$ in which it is tangent to the ellipse). The value of θ_{max} is most easily determined analytically from the condition that the discriminant of the quadratic equation (2) go to zero:

$$\sin\theta_{max} = \frac{p_0\sqrt{1-V^2}}{mV}.$$

2. Find the energy distribution of the decay particles in the L-system.

Solution: In the C-system the decay particles are distributed isotropically in direction, i.e. the number of particles within the element of solid angle $do_0 = 2\pi\ \sin\theta_0\, d\theta_0$ is

$$dN = \frac{1}{4\pi}do_0 = \frac{1}{2}\,|\,d\cos\theta_0\,|. \qquad (1)$$

The energy in the L-system is given in terms of quantities referring to the C-system by

$$\mathscr{E} = \frac{\mathscr{E}_0+p_0V\cos\theta_0}{\sqrt{1-V^2}}$$

and runs through the range of values from

$$\frac{\mathscr{E}_0-Vp_0}{\sqrt{1-V^2}} \quad\text{to}\quad \frac{\mathscr{E}_0+Vp_0}{\sqrt{1-V^2}}.$$

† In the classical limit, the ellipse reduces to a circle. (See *Mechanics*, § 16.)

Expressing $d\,|\cos\theta_0|$ in terms of $d\mathscr{E}$, we obtain the normalized energy distribution (for each of the two types of decay particles):

$$dN = \frac{1}{2Vp_0}\sqrt{1-V^2}\,d\mathscr{E}.$$

3. Determine the range of values in the L-system for the angle between the two decay particles (their separation angle) for the case of decay into two identical particles.

Solution: In the C-system, the particles fly off in opposite directions, so that $\theta_{10} = \pi - \theta_{20} \equiv \theta_0$. According to (5.4), the connection between angles in the C- and L-systems is given by the formulas:

$$\cot\theta_1 = \frac{v_0\cos\theta_0 + V}{v_0\sin\theta_0\sqrt{1-V^2}}, \qquad \cot\theta_2 = \frac{-v_0\cos\theta_0 + V}{v_0\sin\theta_0\sqrt{1-V^2}}$$

(since $v_{10} = v_{20} = v_0$ in the present case). The required separation angle is $\Theta = \theta_1 + \theta_2$, and a simple calculation gives:

$$\cot\Theta = \frac{V^2 - v_0^2 + V^2 v_0^2\sin^2\theta_0}{2Vv_0\sqrt{1-V^2}\sin\theta_0}.$$

An examination of the extreme for this expression gives the following ranges of possible values of Θ:

$$\text{for } V < v_0\text{: } 2\tan^{-1}\left(\frac{v_0}{V}\sqrt{1-V^2}\right) < \Theta < \pi;$$

$$\text{for } v_0 < V < \frac{v_0}{\sqrt{1-v_0^2}}\text{: } 0 < \Theta < \sin^{-1}\sqrt{\frac{1-V^2}{1-v_0^2}} < \frac{\pi}{2};$$

$$\text{for } V > \frac{v_0}{\sqrt{1-v_0^2}}\text{: } 0 < \Theta < 2\tan^{-1}\left(\frac{v_0}{V}\sqrt{1-V^2}\right) < \frac{\pi}{2}.$$

4. Find the angular distribution in the L-system for decay particles of zero mass.

Solution: According to (5.6) the connection between the angles of emergence in the C- and L-systems for particles with $m = 0$ is

$$\cos\theta_0 = \frac{\cos\theta - V}{1 - V\cos\theta}.$$

Substituting this expression in formula (1) of Problem 2, we find:

$$dN = \frac{(1-V^2)do}{4\pi(1-V\cos\theta)^2}.$$

5. Find the distribution of separation angles in the L-system for a decay into two particles of zero mass.

Solution: The relation between the angles of emergence, θ_1, θ_2 in the L-system and the angles $\theta_{10} \equiv \theta_0$, $\theta_{20} = \pi - \theta_0$ in the C-system is given by (5.6), so that we have for the separation angle $\Theta = \theta_1 + \theta_2$:

$$\cos\Theta = \frac{2V^2 - 1 - V^2\cos^2\theta_0}{1 - V^2\cos^2\theta_0}$$

and conversely,

$$\cos\theta_0 = \sqrt{1 - \frac{1-V^2}{V^2}\cot^2\frac{\Theta}{2}}.$$

Substituting this expression in formula (1) of problem 2, we find:

$$dN = \frac{1 - V^2}{16\,\pi V\,\sin^3\frac{\Theta}{2}} \frac{do}{\sqrt{V^2 - \cos^2\frac{\Theta}{2}}}.$$

The angle Θ takes on values from π to $\Theta_{min} = 2\cos^{-1} V$.

6. Determine the maximum energy which can be carried off by one of the decay particles, when a particle of mass M at rest decays into three particles with masses m_1, m_2, and m_3.

Solution: The particle m_1 has its maximum energy if the system of the other two particles m_2 and m_3 has the least possible mass; the latter is equal to the sum $m_2 + m_3$ (and corresponds to the case where the two particles move together with the same velocity). Having thus reduced the problem to the decay of a body into two parts, we obtain from (11.3):

$$\mathscr{E}_{1\,max} = \frac{M^2 + m_1^2 - (m_2 + m_3)^2}{2M}.$$

§ 12. Invariant cross-section

Collision processes are characterized by their *invariant cross-sections,* which determine the number of collisions (of the particular type) occurring between beams of colliding particles.

Suppose that we have two colliding beams; we denote by n_1 and n_2 the particle densities in them (i.e. the numbers of particles per unit volume) and by $\mathbf{v}_1$ and $\mathbf{v}_2$ the velocities of the particles. In the reference system in which particle 2 is at rest (or, as one says, in the *rest frame* of particle 2), we are dealing with the collision of the beam of particles 1 with a stationary target. Then according to the usual definition of the cross-section σ, the number of collisions occurring in volume dV in time dt is

$$d\nu = \sigma v_{rel}\, n_1 n_2 dV dt.$$

where v_{rel} is the velocity of particle 1 in the rest system of particle 2 (which is just the definition of the relative velocity of two particles in relativistic mechanics).

The number $d\nu$ is by its very nature an invariant quantity. Let us try to express it in a form which is applicable in any reference system:

$$d\nu = A n_1 n_2 dV dt, \tag{12.1}$$

where A is a number to be determined, for which we know that its value in the rest frame of one of the particles is $v_{rel}\,\sigma$. We shall always mean by σ precisely the cross-section in the rest frame of one of the particles, i.e. by definition, an invariant quantity. From its definition, the relative velocity v_{rel} is also invariant.

In the expression (12.1) the product $dVdt$ is an invariant. Therefore the product An_1n_2 must also be an invariant.

The law of transformation of the particle density n is easily found by noting that the number of particles in a given volume element dV, ndV, is invariant. Writing $ndV = n_0 dV_0$ (the index 0 refers to the rest frame of the particles) and using formula (4.6) for the transformation of the volume, we find:

$$n = \frac{n_0}{\sqrt{1 - v^2}} \tag{12.2}$$

or $n = n_0 \mathscr{E}/m$, where $\mathscr{E}$ is the energy and m the mass of the particles.

Thus the statement that An_1n_2 is invariant is equivalent to the invariance of the expression $A\,\mathscr{E}_1\,\mathscr{E}_2$. This condition is more conveniently represented in the form

$$A\frac{\mathscr{E}_1\mathscr{E}_2}{p_{1i}p_2^i}=A\frac{\mathscr{E}_1\mathscr{E}_2}{\mathscr{E}_1\mathscr{E}_2-\mathbf{p}_1\cdot\mathbf{p}_2}=\text{inv}, \tag{12.3}$$

where the denominator is an invariant—the product of the four-momenta of the two particles.

In the rest frame of particle 2, we have $\mathscr{E}_2=m_2$, $\mathbf{p}_2=0$, so that the invariant quantity (12.3) reduces to A. On the other hand, in this frame $A=\sigma v_{\text{rel}}$. Thus in an arbitrary reference system,

$$A=\sigma v_{\text{rel}}\frac{p_{1i}p_2^i}{\mathscr{E}_1\mathscr{E}_2}. \tag{12.4}$$

To give this expression its final form, we express v_{rel} in terms of the momenta or velocities of the particles in an arbitrary reference frame. To do this we note that in the rest frame of particle 2,

$$p_{1i}p_2^i=\frac{m_1}{\sqrt{1-v_{\text{rel}}^2}}m_2\,.$$

Then

$$v_{\text{rel}}=\sqrt{1-\frac{m_1^2m_2^2}{(p_{1i}p_2^i)^2}}\,. \tag{12.5}$$

Expressing the quantity $p_{1i}p_2^i=\mathscr{E}_1\mathscr{E}_2-\mathbf{p}_1\cdot\mathbf{p}_2$ in terms of the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ by using formulas (9.1) and (9.4):

$$p_{1i}p_2^i=m_1m_2\frac{1-\mathbf{v}_1\cdot\mathbf{v}_2}{\sqrt{(1-v_1^2)(1-v_2^2)}},$$

and substituting in (12.5), after some simple transformations we get the following expression for the relative velocity:

$$v_{\text{rel}}=\frac{\sqrt{(\mathbf{v}_1-\mathbf{v}_2)^2-(\mathbf{v}_1\times\mathbf{v}_2)^2}}{1-\mathbf{v}_1\cdot\mathbf{v}_2} \tag{12.6}$$

(we note that this expression is symmetric in $\mathbf{v}_1$ and $\mathbf{v}_2$, i.e. the magnitude of the relative velocity is independent of the choice of particle used in defining it).

Substituting (12.5) or (12.6) in (12.4) and then in (12.1), we get the final formulas for solving our problem:

$$d\nu=\sigma\frac{\sqrt{(p_{1i}p_2^i)^2-m_1^2m_2^2}}{\mathscr{E}_1\mathscr{E}_2}n_1n_2dVdt \tag{12.7}$$

or

$$d\nu=\sigma\sqrt{(\mathbf{v}_1-\mathbf{v}_2)^2-(\mathbf{v}_1\times\mathbf{v}_2)^2}\,n_1n_2dVdt \tag{12.8}$$

(W. Pauli, 1933).

If the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are collinear, then $\mathbf{v}_1\times\mathbf{v}_2=0$, so that formula (12.8) takes the form:

$$d\nu=\sigma\,|\,\mathbf{v}_1-\mathbf{v}_2\,|\,n_1n_2dVdt. \tag{12.9}$$

PROBLEM

Find the "element of length" in relativistic "velocity space".

Solution: The required line element dl_v is the relative velocity of two points with velocities $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$. We therefore find from (12.6)

$$dl_v^2 = \frac{(d\mathbf{v})^2 - (\mathbf{v} \times d\mathbf{v})^2}{(1 - v^2)^2} = \frac{dv^2}{(1 - v^2)^2} + \frac{v^2}{1 - v^2}(d\theta^2 + \sin^2 \theta \cdot d\phi^2),$$

where θ, ϕ are the polar angle and azimuth of the direction of $\mathbf{v}$. If in place of v we introduce the new variable χ through the equation $v = \tanh \chi$, the line element is expressed as:

$$dl_v^2 = d\chi^2 + \sinh^2 \chi (d\theta^2 + \sin^2 \theta \cdot d\phi^2).$$

From the geometrical point of view this is the line element in three-dimensional Lobachevskii space—the space of constant negative curvature (see (111.12)).

§ 13. Elastic collisions of particles

Let us consider, from the point of view of relativistic mechanics, the *elastic collision* of particles. We denote the momenta and energies of the two colliding particles (with masses m_1 and m_2) by $\mathbf{p}_1$, $\mathscr{E}_1$ and $\mathbf{p}_2$, $\mathscr{E}_2$; we use primes for the corresponding quantities after collision. The laws of conservation of momentum and energy in the collision can be written together as the equation for conservation of the four-momentum:

$$p_1^i + p_2^i = p_1'^i + p_2'^i. \tag{13.1}$$

From this four-vector equation we construct invariant relations which will be helpful in further computations. To do this we rewrite (13.1) in the form:

$$p_1^i + p_2^i - p_1'^i + p_2'^i,$$

and square both sides (i.e. we write the scalar product of each side with itself). Noting that the squares of the four-momenta p_1^i and $p_1'^i$ are equal to m_1^2, and the squares of p_2^i and $p_2'^i$ are equal to m_2^2, we get:

$$m_1^2 + p_{1i} p_2^i - p_{1i} p_1'^i - p_{2i} p_1'^i = 0. \tag{13.2}$$

Similarly, squaring the equation $p_1^i + p_2^i - p_2'^i = p_1'^i$, we get:

$$m_2^2 + p_{1i} p_2^i - p_2^i p_2'^i - p_{1i} p_2'^i = 0. \tag{13.3}$$

Let us consider the collision in a reference system (the L-system) in which one of the particles (m_2) was at rest before the collision. Then $\mathbf{p}_2 = 0$, $\mathscr{E}_2 = m_2$, and the scalar products appearing in (13.2) are:

$$\begin{aligned} p_{1i} p_2^i &= \mathscr{E}_1 m_2, \\ p_{2i} p_1'^i &= m_2 \mathscr{E}_1', \\ p_{1i} p_1'^i &= \mathscr{E}_1 \mathscr{E}_1' - \mathbf{p}_1 \cdot \mathbf{p}_1' = \mathscr{E}_1 \mathscr{E}_1' - p_1 p_1' \cos \theta_1, \end{aligned} \tag{13.4}$$

where θ_1 is the angle of scattering of the incident particle m_1. Substituting these expressions in (13.2) we get:

$$\cos \theta_1 = \frac{\mathscr{E}_1'(\mathscr{E}_1 + m_2) - \mathscr{E}_1 m_2 - m_1^2}{p_1 p_1'}. \tag{13.5}$$

Similarly, we find from (13.3):

$$\cos \theta_2 = \frac{(\mathscr{E}_1 + m_2)(\mathscr{E}_2' - m_2)}{p_1 p_2'}, \tag{13.6}$$

where θ_2 is the angle between the transferred momentum $\mathbf{p}_2'$ and the momentum of the incident particle $\mathbf{p}_1$.

The formulas (13.5)–(13.6) relate the angles of scattering of the two particles in the L-system to the changes in their energy in the collision. Inverting these formulas, we can express the energies $\mathscr{E}_1', \mathscr{E}_2'$ in terms of the angles θ_1 or θ_2. Thus, substituting in (13.6) $p_1 = \sqrt{\mathscr{E}_1^2 - m_1^2}$, $p_2' = \sqrt{(\mathscr{E}_2')^2 - m_2^2}$ and squaring both sides, we find after a simple computation:

$$\mathscr{E}_2' = m_2 \frac{(\mathscr{E}_1 + m_2)^2 + (\mathscr{E}_1^2 - m_1^2)\cos^2\theta_2}{(\mathscr{E}_1 + m_2)^2 - (\mathscr{E}_1^2 - m_1^2)\cos^2\theta_2}. \tag{13.7}$$

Inversion of formula (13.5) leads in the general case to a very complicated formula for $\mathscr{E}_1'$ in terms of θ_1.

We note that if $m_1 > m_2$, i.e. if the incident particle is heavier than the target particle, the scattering angle θ_1 cannot exceed a certain maximum value. It is easy to find by elementary computations that this value is given by the equation

$$\sin \theta_{1\,\max} = \frac{m_2}{m_1}, \tag{13.8}$$

which coincides with the familiar classical result.

Formulas (13.5)–(13.6) simplify in the case when the incident particle has zero mass: $m_1 = 0$, and correspondingly $p_1 = \mathscr{E}_1$, $p_1' = \mathscr{E}_1'$. For this case let us write the formula for the energy of the incident particle after the collision, expressed in terms of its angle of deflection:

$$\mathscr{E}_1' = \frac{m_2}{1 - \cos\theta_1 + \dfrac{m_2}{\mathscr{E}_1}}. \tag{13.9}$$

Let us now turn once again to the general case of collision of particles of arbitrary mass. The collision is most simply treated in the C-system. Designating quantities in this system by the additional subscript 0, we have $\mathbf{p}_{10} = -\mathbf{p}_{20} \equiv \mathbf{p}_0$. From the conservation of momentum, during the collision the momenta of the two particles merely rotate, remaining equal in magnitude and opposite in direction. From the conservation of energy, the value of each of the momenta remains unchanged.

Let χ be the angle of scattering in the C-system—the angle through which the momenta $\mathbf{p}_{10}$ and $\mathbf{p}_{20}$ are rotated by the collision. This quantity completely determines the scattering process in the C-system, and therefore also in any other reference system. It is also convenient in describing the collision in the L-system and serves as the single parameter which remains undetermined after the conservation of momentum and energy are applied.

We express the final energies of the two particles in the L-system in terms of this parameter. To do this we return to (13.2), but this time write out the product $p_{1i}p_1'^{i}$ in the C-system:

$$p_{1i}p_1'^i = \mathscr{E}_{10}\mathscr{E}_{10}' - \mathbf{p}_{10}\cdot\mathbf{p}_{10}' = \mathscr{E}_{10}^2 - p_0^2\cos\chi = p_0^2(1-\cos\chi) + m_1^2$$

(in the C-system the energies of the particles do not change in the collision: $\mathscr{E}_{10}' = \mathscr{E}_{10}$). We write out the other two products in the L-system, i.e we use (13.4). As a result we get: $\mathscr{E}_1' - \mathscr{E}_1 = -(p_0^2/m_2)(1-\cos\chi)$. We must still express p_0^2 in terms of quantities referring to the L-system. This is easily done by equating the values of the invariant $p_{1i}p_2^i$ in the L- and C-systems:

$$\mathscr{E}_{10}\mathscr{E}_{20} - \mathbf{p}_{10}\cdot\mathbf{p}_{20} = \mathscr{E}_1 m_2,$$

or

$$\sqrt{(p_0^2+m_1^2)(p_0^2+m_2^2)} = \mathscr{E}_1 m_2 - p_0^2.$$

Solving the equation for p_0^2, we get:

$$p_0^2 = \frac{m_2^2(\mathscr{E}_1^2 - m_1^2)}{m_1^2+m_2^2+2m_2\mathscr{E}_1}. \tag{13.10}$$

Thus, we finally have:

$$\mathscr{E}_1' = \mathscr{E}_1 - \frac{m_2(\mathscr{E}_1^2-m_1^2)}{m_1^2+m_2^2+2m_2\mathscr{E}_1}(1-\cos\chi). \tag{13.11}$$

The energy of the second particle is obtained from the conservation law: $\mathscr{E}_1 + m_2 = \mathscr{E}_1' + \mathscr{E}_2'$. Therefore

$$\mathscr{E}_2' = m_2 + \frac{m_2(\mathscr{E}_1^2-m_1^2)}{m_1^2+m_2^2+2m_2\mathscr{E}_1}(1-\cos\chi). \tag{13.12}$$

The second terms in these formulas represent the energy lost by the first particle and transferred to the second particle. The maximum energy transfer occurs for $\chi = \pi$, and is equal to

$$\mathscr{E}_{2\,\max}' - m_2 = \mathscr{E}_1 - \mathscr{E}_{1\,\min}' = \frac{2m_2(\mathscr{E}_1^2-m_1^2)}{m_1^2+m_2^2+2m_2\mathscr{E}_1}. \tag{13.13}$$

The ratio of the minimum kinetic energy of the incident particle after collision to its initial energy is:

$$\frac{\mathscr{E}_{1\,\min}' - m_1}{\mathscr{E}_1 - m_1} = \frac{(m_1-m_2)^2}{m_1^2+m_2^2+2m_2\mathscr{E}_1}. \tag{13.14}$$

In the limiting case of low velocities (when $\mathscr{E} \approx m + mv^2/2$), this relation tends to a constant limit, equal to

$$\left(\frac{m_1-m_2}{m_1+m_2}\right)^2.$$

In the opposite limit of large energies $\mathscr{E}_1$, relation (13.14) tends to zero; the quantity $\mathscr{E}_{1\,\min}'$ tends to a constant limit. This limit is

$$\mathscr{E}_{1\,\min}' = \frac{m_1^2+m_2^2}{2m_2}.$$

Let us assume that $m_2 >> m_1$, i.e. the mass of the incident particle is small compared to the mass of the particle at rest. According to classical mechanics the light particle could transfer only a negligible part of its energy (see *Mechanics,* § 17). This is not the case in relativistic mechanics. From formula (13.14) we see that for sufficiently large energies $\mathscr{E}_1$ the fraction of the energy transferred can reach the order of unity. For this it is not sufficient that the velocity of m_1 be of order 1, but one must have $\mathscr{E}_1 \sim m_2$, i.e. the light particle must have an energy of the order of the rest energy of the heavy particle.

A similar situation occurs for $m_2 << m_1$, i.e. when a heavy particle is incident on a light one. Here too, according to classical mechanics, the energy transfer would be insignificant. The fraction of the energy transferred begins to be significant only for energies $\mathscr{E}_1 \sim m_1^2/m_2$. We note that we are not taking simply of velocities of the order of the light velocity, but of energies large compared to m_1, i.e. we are dealing with the ultrarelativistic case.

PROBLEMS

1. The triangle *ABC* in Fig. 4 is formed by the momentum vector $\mathbf{p}$ of the impinging particle and the momenta $\mathbf{p}_1'$, $\mathbf{p}_2'$ of the two particles after the collision. Find the locus of the points *C* corresponding to all possible values of $\mathbf{p}_1'$, $\mathbf{p}_2'$.

Solution: The required curve is an ellipse whose semiaxes can be found by using the formulas obtained in problem 1 of § 11. In fact, the construction given there determined the locus of the vectors $\mathbf{p}$ in the *L*-system which are obtained from arbitrarily directed vectors $\mathbf{p}_0$ with given length p_0 in the *C*-system.

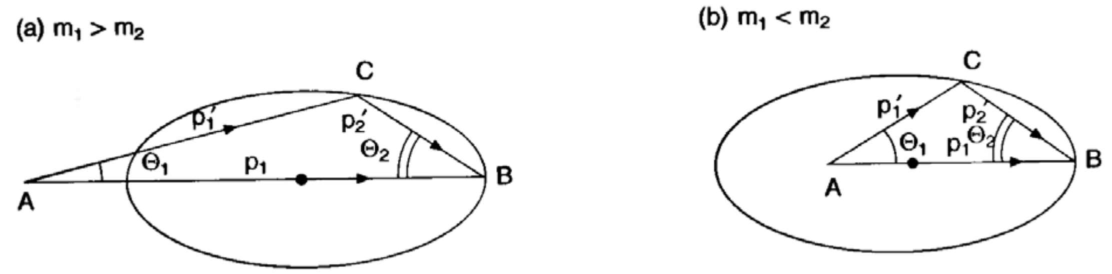

FIG. 4.

Since the absolute values of the momenta of the colliding particles are identical in the *C*-system, and do not change in the collision, we are dealing with a similar construction for the vector $\mathbf{p}_1'$, for which

$$p_0 \equiv p_{10} = p_{20} = \frac{m_2 V}{\sqrt{1 - V^2}}$$

in the *C*-system where *V* is the velocity of particle m_2 in the *C*-system, coincides in magnitude with the velocity of the centre of inertia, and is equal to $V = p_1/(\mathscr{E}_1 + m_2)$ (see (11.4)). As a result we find that the minor and major semiaxes of the ellipse are

$$p_0 = \frac{m_2 p_1}{\sqrt{m_1^2 + m_2^2 + 2m_2\mathscr{E}_1}}, \quad \frac{p_0}{\sqrt{1 - V^2}} = \frac{m_2 p_1(\mathscr{E}_1 + m_2)}{m_1^2 + m_2^2 + 2m_2\mathscr{E}_1}$$

(the first of these is, of course, the same as (13.10)).

For $\theta_1 = 0$, the vector $\mathbf{p}_1'$ coincides with $\mathbf{p}_1$, so that the distance *AB* is equal to p_1. Comparing p_1 with the length of the major axis of the ellipse, it is easily shown that the point *A* lies outside the ellipse if $m_1 > m_2$ (Fig. 4a), and inside it if $m_1 < m_2$ (Fig. 4b).

2. Determine the minimum separation angle Θ_{min} of two particles after collision of the masses of the two particles are the same ($m_1 = m_2 \equiv m$).

Solution: If $m_1 = m_2$, the point A of the diagram lies on the ellipse, while the minimum separation angle corresponds to the situation where point C is at the end of the minor axis (Fig. 5). From the construction it is clear that tan $(\Theta_{min}/2)$ is the ratio of the lengths of the semiaxes, and we find:

$$\tan\frac{\Theta_{min}}{2} = \sqrt{\frac{2m}{\mathscr{E}_1 + m}},$$

or

$$\cos\Theta_{min} = \frac{\mathscr{E}_1 - m}{\mathscr{E}_1 + 3m}.$$

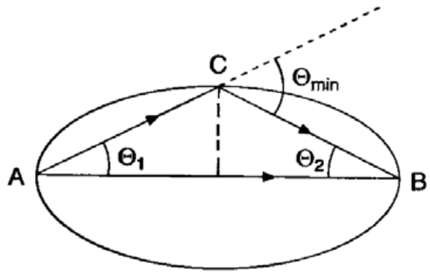

FIG. 5.

3. For the collision of two particles of equal mass m, express $\mathscr{E}_1', \mathscr{E}_2', \chi$ in terms of the angle θ_1 of scattering in the L-system.

Solution: Inversion of formula (13.5) in this case gives:

$$\mathscr{E}_1' = m\frac{(\mathscr{E}_1 + m) + (\mathscr{E}_1 - m)\cos^2\theta_1}{(\mathscr{E}_1 + m) - (\mathscr{E}_1 - m)\cos^2\theta_1}, \qquad \mathscr{E}_2' = m + \frac{(\mathscr{E}_1^2 - m^2)\sin^2\theta_1}{2m + (\mathscr{E}_1 - m)\sin^2\theta_1}.$$

Comparing with the expression for $\mathscr{E}_1'$ in terms of χ:

$$\mathscr{E}_1' = \mathscr{E}_1 - \frac{\mathscr{E}_1 - m}{2}(1 - \cos\chi),$$

we find the angle of scattering in the C-system:

$$\cos\chi = \frac{2m - (\mathscr{E}_1 + 3m)\sin^2\theta_1}{2m + (\mathscr{E}_1 - m)\sin^2\theta_1}.$$

§ 14. Angular momentum

As is well known from classical mechanics, for a closed system, in addition to conservation of energy and momentum, there is conservation of angular momentum, that is, of the vector

$$\mathbf{M} = \sum \mathbf{r} \times \mathbf{p}$$

where $\mathbf{r}$ and $\mathbf{p}$ are the radius vector and momentum of the particle; the summation runs over all the particles making up the system. The conservation of angular momentum is a consequence of the fact that because of the isotropy of space, the Lagrangian of a closed system does not change under a rotation of the system as a whole.

By carrying through a similar derivation in four-dimensional form, we obtain the relativistic expression for the angular momentum. Let x^i be the coordinates of one of the particles of the system. We make an infinitesimal rotation in the four-dimensional space. Under such a transformation, the coordinates x^i take on new values x'^i such that the differences $x'^i - x^i$ are linear functions

$$x'^i - x^i = x_k \delta\Omega^{ik} \tag{14.1}$$

with infinitesimal coefficients $\delta\Omega_{ik}$. The components of the four-tensor $\delta\Omega_{ik}$ are connected to one another by the relations resulting from the requirement that, under a rotation, the length of the radius vector must remain unchanged, that is, $x'_i x'^i = x_i x^i$. Substituting for x'^i from (14.1) and dropping terms quadratic in $\delta\Omega_{ik}$, as infinitesimals of higher order, we find

$$x^i x^k \delta\Omega_{ik} = 0.$$

This equation must be fulfilled for arbitrary x^i. Since $x^i x^k$ is a symmetric tensor, $\delta\Omega_{ik}$ must be an antisymmetric tensor (the product of a symmetrical and an antisymmetrical tensor is clearly identically zero). Thus we find that

$$\delta\Omega_{ki} = -\,\delta\Omega_{ik}. \tag{14.2}$$

The change in the action for an infinitesimal change of coordinates of the initial point a and the final point b of the trajectory has the form (see 9.11):

$$\delta S = -\,\Sigma p^i \delta x_i \Big|_a^b$$

(the summation extends over all the particles of the system). In the case of rotation which we are now considering, $\delta x_i = \delta\Omega_{ik} x^k$, and so

$$\delta S = -\,\delta\Omega_{ik}\, \Sigma\, p^i x^k \Big|_a^b.$$

If we resolve the tensor $\Sigma p^i x^k$ into symmetric and antisymmetric parts, then the first of these when multiplied by an antisymmetric tensor gives identically zero. Therefore, taking the antisymmetric part of $\Sigma p^i x^k$, we can write the preceding equality in the form

$$\delta S = -\,\delta\Omega_{ik}\, \tfrac{1}{2} \Sigma\, (p^i x^k - p^k x^i) \Big|_a^b . \tag{14.3}$$

For a closed system the action, being an invariant, is not changed by a rotation in 4-space. This means that the coefficients of $\delta\Omega_{ik}$ in (14.3) must vanish:

$$\Sigma(p^i x^k - p^k x^i)_b = \Sigma(p^i x^k - p^k x^i)_a .$$

Consequently we see that for a closed system the tensor

$$M^{ik} = \Sigma(x^i p^k - x^k p^i)\,. \tag{14.4}$$

This antisymmetric tensor is called the *four-tensor of angular momentum.* The space components of this tensor are the components of the three-dimensional angular momentum vector $\mathbf{M} = \Sigma \mathbf{r} \times \mathbf{p}$:

$$M^{23} = M_x, \quad -M^{13} = M_y, \quad M^{12} = M_z.$$

The components M^{01}, M^{02}, M^{03} form a vector $\Sigma(t\mathbf{p} - \mathscr{E}\mathbf{r}/c^2)$. Thus, we can write the components of the tensor M^{ik} in the form:

$$M^{ik} = \left[c\,\Sigma \left(t\mathbf{p} - \frac{\mathscr{E}\mathbf{r}}{c^2} \right), -\mathbf{M} \right]. \tag{14.5}$$

(Compare (6.10).)

Because of the conservation of M^{ik} for a closed system, we have, in particular,

$$\sum\left(t\mathbf{p}-\frac{\mathscr{E}\mathbf{r}}{c^2}\right)=\text{const.}$$

Since, on the other hand, the total energy $\sum\mathscr{E}$ is also conserved, this equality can be written in the form

$$\frac{\sum\mathscr{E}\mathbf{r}}{\sum\mathscr{E}}-\frac{c^2\sum\mathbf{p}}{\sum\mathscr{E}}t=\text{const.}$$

(Quantities referring to different particles are taken at the same time t).

From this we see that the point with the radius vector

$$\mathbf{R}=\frac{\sum\mathscr{E}\mathbf{r}}{\sum\mathscr{E}} \tag{14.6}$$

moves uniformly with the velocity

$$\mathbf{V}=\frac{c^2\sum\mathbf{p}}{\sum\mathscr{E}}, \tag{14.7}$$

which is none other than the velocity of motion of the system as a whole. [It relates the total energy and momentum, according to formula (9.8).] Formula (14.6) gives the relativistic definition of the coordinates of the *centre of inertia* of the system. If the velocities of all the particles are small compared to c, we can approximately set $\mathscr{E}\approx mc^2$ so that (14.6) goes over into the usual classical expression

$$\mathbf{R}=\frac{\sum m\mathbf{r}}{\sum m}.\dagger$$

We note that the components of the vector (14.6) do not constitute the space components of any four-vector, and therefore under a transformation of reference frame they do not transform like the coordinates of a point. Thus we get different points for the centre of inertia of a given system with respect to different reference frames.

PROBLEM

Find the connection between the angular momentum $\mathbf{M}$ of a body (system of particles) in the reference frame K in which the body moves with velocity $\mathbf{V}$, and its angular momentum $\mathbf{M}^{(0)}$ in the frame K_0 in which the body is at rest as a whole; in both cases the angular momentum is defined with respect to the same point—the centre of inertia of the body in the system K_0.‡

† We note that whereas the classical formula for the centre of inertia applies equally well to interacting and non-interacting particles, formula (14.6) is valid only if we neglect interaction. In relativistic mechanics, the definition of the centre of inertia of a system of interacting particles requires us to include explicitly the momentum and energy of the field produced by the particles.

‡ We remind the reader that although in the system K_0 (in which $\sum\mathbf{p}=0$) the angular momentum is independent of the choice of the point with respect to which it is defined, in the K system (in which $\sum\mathbf{p}\neq 0$) the angular momentum does depend on this choice (see *Mechanics*, § 9).

Solution: The K_0 system moves relative to the K system with velocity $\mathbf{V}$; we choose its direction for the x axis. The components of M^{ik} that we want transform according to the formulas (see problem 2 in § 6):

$$M^{12} = \frac{M^{(0)12} + \frac{V}{c} M^{(0)02}}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad M^{13} = \frac{M^{(0)13} + \frac{V}{c} M^{(0)03}}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad M^{23} = M^{(0)23}.$$

Since the origin of coordinates was chosen at the centre of inertia of the body (in the K_0 system), in that system $\sum \mathcal{E}\mathbf{r} = 0$, and since in that system $\sum \mathbf{p} = 0$, $M^{(0)02} = M^{(0)03} = 0$. Using the connection between the components of M^{ik} and the vector $\mathbf{M}$, we find for the latter:

$$M_z = M_x^{(0)}, \quad M_y = \frac{M_y^{(0)}}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad M_z = \frac{M_z^{(0)}}{\sqrt{1 - \frac{V^2}{c^2}}}.$$

CHAPTER 3

CHARGES IN ELECTROMAGNETIC FIELDS

§ 15. Elementary particles in the theory of relativity

The interaction of particles can be described with the help of the concept of a *field* of force. Namely, instead of saying that one particle acts on another, we may say that the particle creates a field around itself; a certain force then acts on every other particle located in this field. In classical mechanics, the field is merely a mode of description of the physical phenomenon—the interaction of particles. In the theory of relativity, because of the finite velocity of propagation of interactions, the situation is changed fundamentally. The forces acting on a particle at a given moment are not determined by the positions at that same moment. A change in the position of one of the particles influences other particles only after the lapse of a certain time interval. This means that the field itself acquires physical reality. We cannot speak of a direct interaction of particles located at a distance from one another. Interactions can occur at any one moment only between neighbouring points in space (contact interaction). Therefore we must speak of the interaction of the one particle with the field, and of the subsequent interaction of the field with the second particle.

We shall consider two types of fields, gravitational and electromagnetic. The study of gravitational fields is left to Chapters 10 to 14 and in the other chapters we consider only electromagnetic fields.

Before considering the interactions of particles with the electromagnetic field, we shall make some remarks concerning the concept of a "particle" in relativistic mechanics.

In classical mechanics one can introduce the concept of a rigid body, i.e., a body which is not deformable under any conditions. In the theory of relativity it should follow similarly that we would consider as rigid those bodies whose dimensions all remain unchanged in the reference system in which they are at rest. However, it is easy to see that the theory of relativity makes the existence of rigid bodies impossible in general.

Consider, for example, a circular disk rotating around its axis, and let us assume that it is rigid. A reference frame fixed in the disk is clearly not inertial. It is possible, however, to introduce for each of the infinitesimal elements of the disk an inertial system in which this element would be at rest at the moment; for different elements of the disk, having different velocities, these systems will, of course, also be different. Let us consider a series of line elements, lying along a particular radius vector. Because of the rigidity of the disk, the length of each of these segments (in the corresponding inertial system of reference) will be the same as it was when the disk was at rest. This same length would be measured by an observer at rest, past whom this radius swings at the given moment, since each of its segments is perpendicular to its velocity and consequently a Lorentz contraction does not occur. Therefore the total length of the radius as measured by the observer at rest, being the