A Reflected Skorokhod Equation Approach to Modeling AMM Arbitrage

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Problem: AMM price with a no-arbitrage band

- External market (CEX) price: P_t^{cex}, AMM marginal price: P_t^{amm}.
- Log-prices: $S_t := \log P_t^{\mathrm{cex}}$, $A_t := \log P_t^{\mathrm{amm}}$.
- Log-mispricing: $Y_t := A_t - S_t$.

No-arbitrage band induced by friction/fees:

$$Y_t \in [-\varepsilon, \varepsilon] \qquad (\varepsilon > 0). \tag{1}$$

Reflected dynamics: Skorokhod equation

Inside the band (no arb trades):

$$dY_t = dA_t - dS_t \approx -dS_t.$$

We are forced inside the band by arbitrager actions dK_t:

$$dY_t = -\,dS_t + dK_t, \qquad Y_t \in [-\varepsilon, \varepsilon].$$

Reflected dynamics: Skorokhod equation

At the boundaries: cumulative "push" processes K^+ (upward at $Y = -\varepsilon$) and K^- (downward at $Y = +\varepsilon$):

$$\boxed{dY_t = -\,dS_t + dK_t^+ - dK_t^-, \qquad Y_t \in [-\varepsilon, \varepsilon],} \tag{2}$$

here is arbitrager total push representation:

$$K_t := K_t^+ + K_t^-. \tag{3}$$

Support/minimality:

$$\int_0^t \mathbf{1}_{\{Y_u > -\varepsilon\}}\, dK_u^+ = 0, \qquad \int_0^t \mathbf{1}_{\{Y_u < \varepsilon\}}\, dK_u^- = 0. \tag{4}$$

Goal: approximate $\mathbb{E}[dK_t/dt]$

Assume S_t is a continuous semimartingale (diffusion-like).

Strategy: apply Itô to Y_t^2 and use reflection support to isolate dK_t.

Step 1: Itô for Y_t^2

For $f(y) = y^2$, Itô gives

$$d(Y_t^2) = 2Y_t\, dY_t + d\langle Y\rangle_t\,. \tag{5}$$

Intuition (discrete expansion):

$$(Y + \Delta Y)^2 - Y^2 = 2Y\,\Delta Y + (\Delta Y)^2, \quad (\Delta W)^2 \sim \Delta t \Rightarrow d\langle Y\rangle\,.$$

Step 2: quadratic variation term

From the Skorokhod SDE, $K^{\pm}$ have finite variation, so they contribute no quadratic variation:

$$d\langle Y\rangle_t = d\langle -S\rangle_t = d\langle S\rangle_t \,. \tag{6}$$

Step 3: substitute dY_t and expand

Insert (2) into (5) and use (6):

$$d(Y_t^2) = 2Y_t\big(-\ dS_t + dK_t^+ - dK_t^-\big) + d\left\langle S\right\rangle_t \tag{7}$$

$$= -2Y_t\, dS_t + 2Y_t\, dK_t^+ - 2Y_t\, dK_t^- + d\left\langle S\right\rangle_t . \tag{8}$$

Step 4: reflection support simplifies K-terms

On the support of dK_t^+ we have $Y_t = -\varepsilon$; on the support of dK_t^- we have $Y_t = +\varepsilon$:

$$Y_t \, dK_t^+ = (-\varepsilon) \, dK_t^+, \qquad Y_t \, dK_t^- = (+\varepsilon) \, dK_t^-.$$

Therefore

$$2Y_t \, dK_t^+ - 2Y_t \, dK_t^- = -2\varepsilon \, dK_t, \qquad K_t = K_t^+ + K_t^-.$$

$$\boxed{d(Y_t^2) = -2Y_t \, dS_t \; - \; 2\varepsilon \, dK_t \; + \; d\langle S \rangle_t .} \tag{9}$$

Step 5: expectation + micro-stationarity closure

Taking expectations in (9) and (informally) dividing by dt:

$$\frac{d}{dt}\mathbb{E}[Y_t^2] \approx -2\varepsilon\,\mathbb{E}\left[\frac{dK_t}{dt}\right] + \frac{d}{dt}\langle S\rangle_t\,. \tag{10}$$

Micro-stationarity / fast mixing: $\frac{d}{dt}\mathbb{E}[Y_t^2] \approx 0$.

Key approximation:

$$\boxed{\mathbb{E}\left[\frac{dK_t}{dt}\right] \approx \frac{1}{2\varepsilon}\,\frac{d\,\langle S\rangle_t}{dt}.} \tag{11}$$

Drift + martingale

If (locally) S_t admits a drift + martingale decomposition, e.g.

$$dS_t = \mu_t \, dt + \sigma_t \, dW_t,$$

then

$$-2Y_t \, dS_t = -2Y_t \mu_t \, dt \; - \; 2Y_t \sigma_t \, dW_t.$$

Under standard integrability, $\mathbb{E}\left[\int_0^t Y_u \sigma_u \, dW_u\right] = 0$. On micro timescales one often assumes drift is small vs volatility, hence

$$\mathbb{E}[-2Y_t \, dS_t] \approx 0.$$

When is the approximation exact?

Canonical driftless case:

$$S_t = \sigma W_t, \qquad d\left\langle S \right\rangle_t = \sigma^2 \, dt.$$

If Y_t is reflected Brownian motion on $[-\varepsilon, \varepsilon]$ started in stationarity (uniform), then $\mathbb{E}[Y_t^2]$ is constant and

$$\mathbb{E}\left[\frac{dK_t}{dt}\right] = \frac{\sigma^2}{2\varepsilon}.$$

What the reflection model omits (vs. simulator)

Reflected-diffusion idealization assumes *instantaneous, unconstrained* arbitrage.

Simulator features not captured:

- Gas/fixed costs (dead-zone / impulse behavior).
- State-dependent fees (effective f_{eff} varies with imbalance).
- Oracle/repeg dynamics (extra channels moving A_t).

From reflection to volume and fee-driven growth

Reflection as arbitrage proxy: to change AMM log-price by dK_t, trade notional volume dV_t via a depth map $\Lambda(D_t)$:

$$dV_t \approx \Lambda(D_t)\, dK_t. \tag{12}$$

Fee retention grows pool scale D_t with effective fee rate f_{eff}:

$$dD_t \approx f_{\mathrm{eff}}\, dV_t \approx f_{\mathrm{eff}}\, \Lambda(D_t)\, dK_t. \tag{13}$$

Variance-driven growth (combining closures)

Combine (13) with (11):

$$dD_t \approx f_{\text{eff}}\,\Lambda(D_t)\,dK_t \approx f_{\text{eff}}\,\Lambda(D_t)\,\frac{1}{2\varepsilon}\,d\,\langle S\rangle_t\,.$$

If $\Lambda(D) = \lambda D$ (approximately linear in pool scale), then

$$d\log D_t \approx \frac{f_{\text{eff}}\,\lambda}{2\varepsilon}\,d\,\langle S\rangle_t\,.$$

Mechanism:

higher volatility $\Rightarrow$ more boundary hits $\Rightarrow$ more arb $\Rightarrow$ more fees $\Rightarrow$ growth.

Experimental pipeline and metric

For each dataset-year (e.g. `btc2021`):

1. Estimate candle drift/variance (μ, σ^2) from OHLC.
2. Choose a micro-closure $\Rightarrow$ per-candle boundary flows (dK^+, dK^-).
3. Apply a common macro-closure to update scale D and compute
$$g_{\text{model}} := \frac{D_T}{D_0}.$$
4. Compare to simulator `xcp_profit_trade` via yearly-detrended relative error (%).

Candle statistics estimator (OHLC $\to \mu, \sigma^2$)

For candle length dt with open/high/low/close (O, H, L, C):

$$\mu = \frac{\log C - \log O}{dt}, \tag{14}$$

$$\sigma^2 = \max\left\{0, \frac{(\log H - \log O)(\log H - \log C) + (\log L - \log O)(\log L - \log C)}{dt}\right\}. \tag{15}$$

Micro-closures compared (A / B)

All models output (dK^{+}, dK^{-}) per candle, then use the same macro integration.

A: analytical reflected BM regulator rate

$$r = \frac{\mu}{\tanh\left(\frac{2\mu\varepsilon}{\sigma^2}\right)}, \quad r_{+} = \max\left(0, \frac{r+\mu}{2}\right), \quad r_{-} = \max\left(0, \frac{r-\mu}{2}\right), \quad dK^{\pm} = r_{\pm}\, dt. \tag{16}$$

As $\mu \to 0$,

$$r \to \frac{\sigma^2}{2\varepsilon} \quad \Rightarrow \quad \mathbb{E}\left[\frac{dK}{dt}\right] \approx \frac{\sigma^2}{2\varepsilon}.$$

B family: deterministic path clamp (discrete Skorokhod map)

$$O \to L \to H \to C \text{ or } O \to H \to L \to C \text{ (Bc)}; \qquad O \to L \to H \text{ or } O \to H \to L \text{ (Be)}.$$

Micro-closures compared (B′ / C)

B′ (Bp): event-derived scale-centered

$$d\log = \log\left(\frac{p_{\text{final}}}{p_{\text{scale,after}}}\right) - \log\left(\frac{p_{\text{before}}}{p_{\text{scale,before}}}\right); \quad dK^+ = (d\log)^+,\ dK^- = (d\log)^-.$$

C$_\Delta$ (C60): expected overshoot (analytical)

$$e_+ = \mathbb{E}[(M_\Delta - \varepsilon)^+], \qquad e_- = \mathbb{E}[(M_\Delta^- - \varepsilon)^+],$$

$$dK^+ = e_+ \frac{dt}{\Delta}, \qquad dK^- = e_- \frac{dt}{\Delta}.$$

Macro closure and centering sensitivity

Let ℓ_+, ℓ_- be depth coefficients at band edges.
Base increment:

$$\mathsf{base\_inc} = \ell_+ \, dK^+ + \ell_- \, dK^-. \tag{17}$$

With caps (optional):

$$\alpha = \min\left(1, \frac{\mathsf{cap}}{D \cdot \mathsf{base\_inc}}\right). \tag{18}$$

Update (substepped):

$$D \leftarrow D \cdot \exp\left(f_{\mathrm{eff}} \, \mathsf{base\_inc}_{\mathrm{sub}}\right). \tag{19}$$

Final profit factor:

$$g_{\mathrm{model}} = \frac{D_T}{D_0}.$$

Yearly-detrended relative error (%) – BTC

dataset	A	Bc	Be	Bp	C60
btc2019	2.6	0.0	0.0	0.1	0.2
btc2020	2.3	0.5	0.5	0.5	0.6
btc2021	5.5	0.2	0.1	0.1	0.2
btc2022	3.1	0.1	0.1	0.1	0.2
btc2023	1.0	0.1	0.1	0.1	0.0
btc2024	0.2	0.7	0.7	0.7	0.8
btc2025	0.4	0.3	0.3	0.3	0.4
btc2026	0.1	0.1	0.1	0.1	0.1
btc_avg	**1.9**	**0.3**	**0.2**	**0.3**	**0.3**

Yearly-detrended relative error (%) – ETH

dataset	A	Bc	Be	Bp	C60
eth2019	2.4	0.7	0.8	0.8	0.9
eth2020	7.6	1.0	0.8	0.9	0.6
eth2021	12.0	1.9	1.4	1.7	1.2
eth2022	2.1	0.7	0.8	0.7	0.9
eth2023	1.1	0.2	0.2	0.1	0.1
eth2024	1.3	0.4	0.4	0.4	0.5
eth2025	1.1	0.7	0.7	0.7	0.8
eth2026	0.1	0.0	0.0	0.0	0.0
eth_avg	**3.5**	**0.7**	**0.6**	**0.7**	**0.6**

Yearly-detrended relative error (%) – CRV

dataset	A	Bc	Be	Bp	C60
crv2020	6.9	1.1	1.6	1.2	1.7
crv2021	37.3	6.4	4.4	6.0	3.9
crv2022	5.3	1.3	1.7	1.1	1.8
crv2023	6.7	1.2	0.8	1.5	0.6
crv2024	3.9	0.8	1.1	0.8	1.2
crv2025	3.1	1.1	1.3	1.0	1.4
crv2026	0.2	0.0	0.0	0.0	0.0
crv_avg	**9.1**	**1.7**	**1.6**	**1.7**	**1.5**

Aggregate summary(%)

Table: Aggregate summary

Statistic	A	Bc	Be	Bp	C60
Mean (%)	4.62	0.85	0.78	0.82	0.79
Median (%)	2.40	0.70	0.70	0.70	0.60
Max (%)	37.3	6.4	4.4	6.0	3.9