

# 15-150 Spring 2016

## Lab 3

27 January 2016

The goal for the third lab is to make you more comfortable writing functions that operate on lists, and doing asymptotic analysis and proofs.

Take advantage of this opportunity to practice writing functions and proofs with the assistance of the TAs and your classmates. Today in lab we encourage you to collaborate with your classmates and to ask the TAs for help.

Remember to follow the methodology for writing functions—specifications and tests are part of your code!

## 1 Introduction

### 1.1 Getting Started

Update your clone of the `git` repository to get the files for this weeks lab as usual by running

```
git pull
```

from the top level directory (probably named 15150).

## 2 List Operations

**Task 2.1** Write a function

```
evens : int list -> int list
```

that filters out all odd elements of a list without changing the order. For example,

```
evens[0, 0, 4] = [0, 0, 4]
evens[] = []
evens[0, 0, 4, 9, 3, 2] = [0, 0, 4, 2]
```

You should use the function `evenP` that we provided at the top of your `lab03.sml` file to determine if a number is even.

**Solution 2.1** See solution in `lab03-sol.sml`.

**Task 2.2** Prof. Erdmann has decided to reward the TAs handsomely for their hard work. He wants to add a bonus \$5 to the September paycheck for each of the TAs. Write a function

```
credit : int list -> int list
```

that takes in a list representing the amounts payable to each of the TAs for the month of September. Credit \$5 to each of the TAs and make their day!

**Solution 2.2** See solution in `lab03-sol.sml`.

**Task 2.3** Define an ML function

```
reverse: int list -> int list
```

such that for all integer lists `L`, `reverse(L)` evaluates to the a list that contains all the elements of `L` in reverse order. For this task you should define a helper function that will allow you to solve the problem in  $O(n)$  work.

**Solution 2.3** See solution in `lab03-sol.sml`.

### 3 Proving Termination

Consider the function `foo : int list -> int list` given by

```
fun foo (L: int list) : int list =  
  case L of  
    [ ] => [ ]  
  | (x::R) => x :: foo(rev R)
```

where `rev` is a given function of type `int list -> int list` such that for all integer lists `L`, `rev(L)` evaluates to the reverse of list `L`.

**Task 3.1** Prove the following theorem by induction on the length of `L`

**Theorem:** For all values `L : int list`, `foo(L)` terminates.

You may use the following lemmas as facts in your proof, but be sure to cite them.

**Lemma 1:** If `L : int list` and `L` is a value, then `L = [ ]` if `length(L) = 0` and `L = x::R` for some `x : int`, `R : int list` if `length(L) > 0`

**Lemma 2:** For all values `L : int list`, `rev L` terminates and `length(rev L) = length(L)`.

You may assume `length(L)` is defined by

```
fun length([ ]) = 0  
  | length(x::R) = 1 + length(R)
```

Theorem:

Proof: By \_\_\_\_\_ (method) on \_\_\_\_\_ (variable/type)

Base Case (            ):

NTS (Need to Show):

Inductive Step (            ):

IH (Inductive Hypothesis):

NTS (Need to Show):

**Solution 3.1** We will prove the theorem by induction on the length of L

**Base Case** ( $\text{length}(L) = 0$ ):

**To Show:**  $\text{foo}(L)$  terminates for all lists of length 0

By Lemma 1, since  $\text{length}(L) = 0$ ,  $L = []$ .

```
foo([])
⇒ * case [] of [] => [] | (x::R) => x :: foo(rev R)
⇒ * []
```

Since  $\text{foo} \text{ } (L)$  evaluated to a value, this case is proven.

**Inductive Step** ( $\text{length}(L) > 0$ ):

**Inductive Hypothesis:** Assume  $\text{foo}(L')$  terminates for all lists  $L'$  of length  $n \geq 0$

**To Show:**  $\text{foo}(L)$  terminates for all lists of length  $n + 1$

Let L be a list of length  $n + 1$ .

As  $n \geq 0$ ,  $n + 1 \geq 1$ . Then by Lemma 1, since  $\text{length}(L) > 0$ ,  $L = x::R$  for some  $x : \text{int}$  and  $R : \text{int list}$ .

```
foo(x::R)
⇒ * case x::R of [] => [] | x::R => x :: foo(rev R)
⇒ * x :: foo(rev R)
⇒ * x :: foo(R')
```

for some  $R'$  where  $\text{length}(R') = \text{length}(R)$  by Lemma 2

Note  $\text{length}(x::R) = 1 + \text{length}(R)$ . We also have  $\text{length}(x::R) = n + 1$  so  $n + 1 = 1 + \text{length}(R)$ . Thus subtracting 1 from both sides, we get  $\text{length}(R) = n$ .

This means by the IH, taking  $L' = R'$ ,  $\text{foo}(R')$  terminates and

```
x :: foo(R')
⇒ * x::R'' for some R''
```

Since we showed  $\text{foo}(L)$  evaluates to a value, this case is proven.

**Have the TAs check your work in the previous section before proceeding.**

## 4 Fibonacci

### 4.1 Simple Fibonacci

Name that integer sequence:

1, 1, 2, 3, 5, 8, 13, 21, ...

That's right; it's Fibonacci!

Here's the obvious way to implement it:

```
fun fib (~1 : int) : int = 0
  | fib (0) = 1
  | fib (1) = 1
  | fib (n) = fib (n - 1) + fib (n - 2)
```

We're going to add the harmless but slightly strange base case defining the negative first element of the sequence to the definition as well; you'll see why later in lab, but just go with it for now.

We can use `fib`'s recursion pattern (that is, the different cases it covers based on its input values, as well as the ways we make recursive calls within `fib`) to think critically about its performance. One of our strongest tools to do this is *writing recurrences* (recurrences let us model the flow of data from one call to a function to another, or to a base case).

Recurrences for functions of two cases on natural numbers are usually of the form:

$$\begin{aligned} W_{\text{fn}}(0) &= k_0 \\ W_{\text{fn}}(n) &= k_1 + W_{\text{fn}}(< \text{new size} >) \text{ for non-zero } n \end{aligned}$$

That is, we write a case in our recurrence relation for each case of the function (the base cases and the recursive cases). Each of the base cases does some trivial *constant* amount of work, so we denote this using a constant (usually, something like  $k_i$  for some previously-unused value  $i$ ). In our recursive case, we might do some amount of constant work (which we again denote by the term  $k_1$ ), or even some amount of work dependent on the value of  $n$  (which we denote using the variable  $n$  multiplied by some amount of work), but we additionally do more work in the form of a *recursive call*. We show this by re-referencing the recurrence relation  $W_{\text{fn}}$ , but with an argument representative of the relative size of the new argument to `fn`. For example, if every recursive call decreases `n` by 1, our value for `<new size>` might be  $n - 1$ .

**Task 4.1** Write a recurrence relation for the work of `fib` from above. We've written the cases for you; all you need to do is fill in the right-hand-side of each equation. Note: we left out the case for `~1`; don't worry about it!

$$W_{\text{fib}}(0) =$$

$$W_{\text{fib}}(1) =$$

$$W_{\text{fib}}(n) =$$

<b>Solution 4.1</b>
---------------------

$$W_{fib}(0) = k_0$$

$$W_{fib}(1) = k_1$$

$$W_{fib}(n) = k_2 + W_{fib}(n-1) + W_{fib}(n-2)$$

This is not so helpful, since it says that the time to compute the  $n^{th}$  Fibonacci  $n$  is proportional to the  $n^{th}$  Fibonacci number!

However, if we can get an *upper bound* for this recurrence as follows:

$$\begin{aligned} W_{\text{fib}}(0) &= k_0 \\ W_{\text{fib}}(1) &= k_1 \\ W_{\text{fib}}(n) &\leq k_2 + 2W_{\text{fib}}(n-1) \text{ for non-zero } n \end{aligned}$$

Because  $W_{\text{fib}}(n)$  is *monotonically increasing* (it's never smaller on bigger inputs), we can pretend that it's two recursive calls on  $n-1$ .

If you write it out, you can see that the closed form of this recurrence is

$$W_{\text{fib}}(n) = k_0 + k_1 + k_2 * (2^{n+1} - 1)$$

To see this, you can write the recursion out as a tree. `fib` does `k2` work at each recursive call, so we can label each node with `k2`. Each node has two children, because each call makes two recursive calls.

```

      k2
     /  \
    k2   k2
   /  \ /  \
  k2 k2 k2 k2
...

```

The  $k_2$  is uniform, so factor it out

```

      1
     /  \
    1    1
   /  \ /  \
  1  1 1  1
...

```

We want to count the number of nodes in this tree. The total has the form  $1+2+4+8+16+\dots$ . The reason is that the tree has  $n$  levels, because the recurrence recurs on  $n-1$ , and the  $i^{th}$  level has  $2^i$  work. Thus, the total amount of work is

$$\sum_{i=0}^n 2^i$$

If you look it up, the closed form of this sum is  $2^{n+1} - 1$  (cf. how many binary numbers are there with  $n$  bits).

Once you've written out the closed form, it's clear that this recurrence is  $O(2^n)$ , just by forgetting the constants. The " $O$ " means "order". Roughly speaking, it measures the most significant (as in largest) term of a function as inputs to the function get large. We will talk more about it in lecture on Thursday.



## 4.2 Fast Fibonacci

In this problem, you will show that you can compute Fibonacci more efficiently. The key insight is that one of the recursive calls can be reused each time:

To compute	We need
<code>fib n</code>	<code>fib (n-1)</code> and <code>fib (n-2)</code>
<code>fib (n-1)</code>	<code>fib (n-2)</code> and <code>fib (n-3)</code>
<code>fib (n-2)</code>	<code>fib (n-3)</code> and <code>fib (n-4)</code>

So we really don't need two recursive calls, if we reuse the same computation of `fib n` the two times we use it. To implement this, you must *generalize* the problem so that we compute both `fib n` and `fib (n-1)`.

### 4.2.1 Programming

**Task 4.2** Implement a function

```
fastfib : int -> int * int
```

such that for all natural numbers  $n$ , `fastfib n = (fib(n - 1), fib n)`

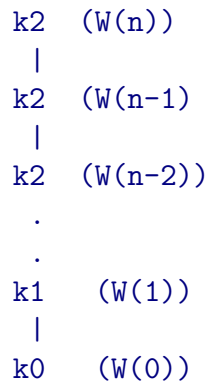
### 4.2.2 Analysis

**Task 4.3** Write a recurrence for the work of `fastfib`,  $W_{\text{fastfib}}$ .

**Solution 4.3**

$$\begin{aligned}W_{\text{fastfib}}(0) &= k_0 \\W_{\text{fastfib}}(n) &= k_2 + W_{\text{fastfib}}(n-1)\end{aligned}$$

**Task 4.4** Compute and informally justify the closed form for your recurrence.

**Solution 4.4**

There are  $n$  levels in the tree. At each level a constant amount of work is done.

$$\begin{aligned}
 W_{fastfib}(n) &= \sum_{i=0}^n k' \\
 &= k' \sum_{i=0}^n 1 \\
 &= k'(n)
 \end{aligned}$$

**Task 4.5** Give a tight big- $O$  bound for this closed form.

**Solution 4.5**

$$W_{fastfib}(n) \in O(n)$$

**Have a TA check your code and analysis for fastfib.**