Facultatea de Automatica si Calculatoare, Universitatea Politehnica din Bucuresti

Computer Science & Engineering Department

Examen Partial MN

Student: ______________________ **Grupa:** ________

Descriere curs:	**MN, An I, Semestrul II**
Titlu curs:	**Metode Numerice**
Profesor:	**Sl.Dr.Ing. Florin POP**
Data examenului:	24 Aprilie 2010 (EC105)
Intervalul orar:	Start: 09:00→ Final: 11:00
Durata examenului:	2 ore
Tip Examen:	**"Closed Book"**
Materiale Aditionale:	**Nu! (!Fara telefoane mobile!)**
Numar pagini:	____________

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Subiecte (Numarul A1)

3 puncte **1.** Fie $A \in R^{nxn}$ o matrice simetrica si pozitiv definita. Sa se determine factorizarea $A = LDL^T$, in care L este o matrice inferior-triunghiulara cu diagonala unitara, iar D este o matrice diagonala. Scrieti o functie MATLAB care calculeaza aceasta factorizare `function [L D] = lud(A)`.

3 puncte **2.** Se considera vectorii $u, v \in R^n$ ortonormati ($||u||_2 = 1, ||v||_2 = 1, u^T v = v^T u = 0$). Se formeaza vectorul $x = u + v$.

a) Sa se dea un exemplu numeric de doi vectori ortonormati.

b) Sa se calculeze $||x||_2$.

c) Se formeaza matricea $H = I_n - xx^T$. Sa se calculeze $Hu, Hv, ||H||_2$.

d) Daca $A = uv^T$, calculati $B = H^{-n} A H^n$, $n > 1$.

2 puncte **3.** Fie functia $f(x)$ data prin $x = a, 0, 1, b$ si $f(x) = y_a, y_0, y_1, y_b$.

a) Calculati polinomul Newton de interpolare si scrieti expresia erorii.

b) Scrieti o functie MATLAB pentru calculul polinomului Newton intr-un punct a.

c) Ce devin diferentele divizate cand $a \to 0$ si $b \to 1$.

2 puncte **4.** Calculati functiile spline $s_0(x)$ si $s_1(x)$ in clasa C^1 pentru functia $f(x)$ cunoscuta prin:

x	1	2	4
f	3	4	6
f'	0	2	5

(A1)

1. Vom considera descompunerea de forma

$$A = \begin{bmatrix} 1 & 0 & 0 & & 0 \\ \ell_{21} & 1 & 0 & & 0 \\ \ell_{31} & \ell_{32} & 1 & & 0 \\ \vdots & & & & \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & & 1 \end{bmatrix} \begin{bmatrix} d_{22} & & & 0 \\ & d_{33} & & \\ & & \ddots & \\ 0 & & & d_{nn} \end{bmatrix} \begin{bmatrix} 1 & \ell_{21} & \ell_{31} & & \ell_{n1} \\ 0 & 1 & \ell_{32} & & \ell_{n2} \\ & & & & \ell_{n3} \\ & & & & \vdots \\ & 0 & 0 & & 1 \end{bmatrix}$$

cu următoarele relații

$$A_{ij} = \sum_{m=1}^{\min(i,j)} L_{im} D_{mm} L_{mj}$$

* determinăm elementele din D

$$D_{pp} = A_{pp} - \sum_{m=1}^{i-1} L_{pm}^2 D_{mm}, \quad p = 1:n$$

* $$A_{ip} = \sum_{m=1}^{p-1} L_{im} D_{mm} L_{mp} + L_{ip} D_{pp}$$

$$\Rightarrow L_{ip} = \left(A_{ip} - \sum_{m=1}^{p-1} L_{im} D_{mm} L_{mp}\right) / D_{pp} \quad = p:n$$

* $$A_{pj} = \sum_{m=1}^{p-1} L_{pm} D_{mm} L_{mj} + D_{pp} L_{pj}$$, relație care conduce la același rezultat ca precedenta

```
function [L d] = lud (A)
    [m n] = size (A)
    for p = 1 : n
        d(p) = A(p,p) (A(p, 1:p-1).^2 * d(1:p-1)')
        for i=p:n
            s = A(i, 1:p-1) * A(1:p-1, p).* d(1:p-1);
            A(i,p) = (A(i,p) - s)/ d(p)
        end
```

end

L = tril(A) - diag(diag(A)) + eye(n);

2. a) $u = [1\ 0 \dots 0]^T$

$v = [0\ 1 \dots 0]^T$

$u^T v = 0 = v^T u$

b) $\|x\|_2^2 = (u+v)^T(u+v) = (u^T+v^T)(u+v) =$
$= u^T u + \underbrace{u^T v}_{0} + \underbrace{v^T u}_{0} + v^T v = 1+1 = 2$

$\Rightarrow \|x\|_2 = \sqrt{2}$

c) $Hu = (I_n - xx^T)u = u - xx^T u = u - (u+v)(u^T+v^T)u$
$= u - (u+v)(u^T u + \underbrace{v^T u}_{0}) = u - u - v = -v.$

$Hv = (I_n - xx^T)v = v - xx^T v = v - (u+v)(u^T+v^T)v$
$= v - (u+v)(u^T v + v^T v) = v - u - v = -u$

calculăm $H \cdot H^T = (I_n - xx^T)(I_n - xx^T)^T =$
$= I_n - 2xx^T + x\underbrace{x^T x}_{2} x^T = I_n$

$\Rightarrow$ H este ortogonală, deci $\|H\|_2 = 1$

d) $H^{-1} = H^T = H$

$B = H^{-n} A H^n = H^{-(n-1)}(H^{-1} A H) H^{n-1} =$
$= H^{-(n-1)}(H^T u v^T H) H^{n-1} = H^{-(n-1)} \underbrace{(v u^T)}_{A^T} H^{n-1}$

$= H^{-(n-2)}(H^T v u^T H) H^{n-2} = H^{-(n-2)} \underbrace{(u v^T)}_{A} H^{n-2}$

$\Rightarrow B = \begin{cases} A^T & \text{dacă } n \text{ impar} \\ A & \text{dacă } n \text{ par} \end{cases}$

3.

x	a	0	1	b
f	y_a	y_0	y_1	y_b.

a)

	F_0	F_1	F_2	F_3
a	$\boxed{y_a}$	$\boxed{\frac{y_0-y_a}{-a}}$	$\boxed{\frac{y_1-y_0+\frac{y_0-y_a}{a}}{1-a}}$	$\boxed{\frac{\frac{\frac{y_b-y_1}{b-1}-y_1+y_0}{b}-\frac{y_1-y_0+\frac{y_0-y_a}{a}}{1-a}}{b-a}}$
0	y_0	$\frac{y_1-y_0}{1}$	$\frac{\frac{y_b-y_1}{b-1}-y_1+y_0}{b}$	
1	y_1	$\frac{y_b-y_1}{b-1}$		
b	y_b			

polinomul de interpolare va fi:

$$P_3(x) = y_a + F_1(x-a) + F_2(x-a)x + F_3(x-a)x(x-1)$$

b)

```
function val = Newton(x, y, a)
    n=length(x);
    y = DifDiv(x,y);
    val = y(1);
    prod = 1;
    for i = 2:n
        prod = prod * (a-x(i-1));
        val = val + y(i) * prod;
    end;
```

c)

$$F_1[0,0] = \lim_{a\to 0} F_1[a,0] = \lim_{a\to 0} \frac{f(a)-f(0)}{a-0} = f'(0)$$

$$F_1[1,1] = \lim_{b\to 1} F_1[1,b] = \lim_{b\to 1} \frac{f(1)-f(b)}{1-b} = f'(1)$$

$$F_2[0,0,1] = \lim_{a\to 0} F_2[a,0,1] = \frac{F_1[0,0]-F_1[0,1]}{-1} = y_1-y_0-f'(0)$$

$$F_2[0,1,1] = \lim_{b\to 1} F_2[0,1,b] = \frac{F_1[0,1]-F_1[1,1]}{-1} = y_0-y_1+f'(1)$$

$$F_3[0,0,1,1] = \lim_{a\to 0,\, b\to 1} F_3[a,0,1,b] = \frac{F_2[0,0,1] - F_2[0,1,1]}{-1}$$

$$= 2(y_0 - y_1) + f'(0) + f'(1)$$

4. $S_0(x) = a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3$

$S_1(x) = a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3$

$$\begin{array}{lcl}
S_0(1) = 3 & & \underline{a_0 = 3} \\
S_0(2) = 4 & & a_0 + b_0 + c_0 + d_0 = 4 \\
S_1(2) = 4 & & \underline{a_1 = 4} \\
S_1(4) = 6 & \Rightarrow & a_1 + 2b_1 + 4c_1 + 8d_1 = 6 \\
S_0'(1) = 0 & & \underline{b_0 = 0} \\
S_0'(2) = 2 & & b_0 + 2c_0 + 3d_0 = 2 \\
S_1'(2) = 2 & & \underline{b_1 = 2} \\
S_1'(4) = 5 & & b_1 + 4c_1 + 12d_1 = 5
\end{array}$$

$$\begin{cases} c_0 + d_0 = 1 \\ 2c_0 + 3d_0 = 2 \end{cases} \Rightarrow \quad \underline{d_0 = 0}, \quad \underline{c_0 = 1}$$

$$\begin{cases} 4c_1 + 8d_1 = -2 \\ 4c_1 + 12d_1 = 3 \end{cases} \Rightarrow \quad 4d_1 = 5, \quad \underline{d_1 = \frac{5}{4}}, \quad c_1 = -3$$

$S_0(x) = 3 + (x-1)^2$

$S_1(x) = 4 + 2(x-2) - 3(x-2)^2 + \frac{5}{4}(x-2)^3$

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Subiecte (Numarul A2)

3 puncte **1.** Fie matricea $A = \begin{pmatrix} 1 & 2 \\ 3 & 13 \end{pmatrix}$. Calculati facorizarea LU-Crout ($u_{ii} = 1$) pentru maricea A. Scrieti o functie MATLAB, `function [L U] = Crout(A)`, pentru o matrice $A \in R^{nxn}$.

3 puncte **2.** Se considera un vector $x \in R^n$, $||x||_2 = 1$, si se formeaza vectorul $u = \frac{x+e_1}{\sqrt{1+x_1}}$ si matricea $H = I_n - uu^T$.

a) Sa se calculeze $||u||_2$.

b) Sa se arata ca H este ortogonala si ca $Hx = -e_1$.

2 puncte **3.** Pornind de la relatia de recurenta care defineste diferentele divizate $F_0[x_0] = f(x_0)$, $F_p[x_0, x_1, \cdots, x_p] = \frac{F_{p-1}[x_0,x_1,\cdots,x_{p-1}] - F_{p-1}[x_1,x_2,\cdots,x_p]}{x_o - x_p}$, sa se arate ca avem formula de calcul: $F_p[x_0, x_1, \cdots, x_p] = \sum_{i=0}^{p} \frac{f(x_i)}{\prod_{j=0,j\neq i}^{p}(x_i - x_j)}$. Scrieti o funcite MATLAB care calculeaza difetentele divizate: `function d = DifDiv(x,y)`.

2 puncte **4.** Pentru functia $f(x)$ cunoscuta prin tabelul urmator, calculati functiile spline $s_0(x)$ si $s_1(x)$, tensionate de clasa C^2, daca $s_0''(1) = 2$ si $s_1''(3) = -1$.

x	1	2	3
f	3	4	1

(A2)

1. $L = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \quad U = \begin{bmatrix} 1 & u_{22} \\ 0 & 1 \end{bmatrix}$

$A = L \cdot U \Rightarrow \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \cdot \begin{bmatrix} 1 & u_{22} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 13 \end{bmatrix}$

$$\begin{cases} l_{11} = 1 \\ l_{11} \cdot u_{22} = 2 \\ l_{21} = 3 \\ l_{21} \cdot u_{22} + l_{22} = 13 \end{cases} \Rightarrow \begin{cases} l_{11} = 1 \\ u_{22} = 2 \\ l_{21} = 3 \\ l_{22} = 7 \end{cases} \Rightarrow \begin{cases} L = \begin{bmatrix} 1 & 0 \\ 3 & 7 \end{bmatrix} \\ U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \end{cases}$$

```
function [L U] = Crout (A)
    [m n] = size(A);
    for p = 1 : n
        for i = p : n
            s = A(i, 1:p-1) * A(1:p-1, p);
            A(i, p) = A(i,p)-s;
        end;
        for j = p+1 : n
            s = A(p, 1:p-1) * A(1:p-1, j);
            A(p, j) = (A(p,j)-s) / A(p,p);
        end;
    end;
    *
```

```
L = triu(A);
U = A - L + eye(n);
```

Formulele folosite au fost:

$$A_{ij} = \sum_{m=1}^{\min(i,j)} L_{im} U_{mj}$$

$$A_{ip} = \sum_{m=1}^{p-1} L_{im} U_{mp} + L_{ip} \underbrace{U_{pp}}_{=1} \Rightarrow L_{ip} = A_{ip} - \sum_{m=1}^{p-1} L_{im} U_{mp} \quad i = p:n$$

$$A_{pj} = \sum_{m=1}^{p-1} L_{pm} U_{mj} + L_{pp} U_{pj} \Rightarrow U_{pj} = \left(A_{pj} - \sum_{m=1}^{p-1} L_{pm} U_{mj}\right) / L_{pp} \quad j = p+1:n$$

2. a) $\|u\|_2^2 = u^T u = \frac{(x+e_1)^T(x+e_1)}{1+x_1} =$

$$= \frac{(x^T+e_1^T)(x+e_1)}{1+x_1} = \frac{x^Tx + x^Te_1 + e_1^Tx + e_1^Te_1}{1+x_1} =$$

$$= \frac{1+x_1+x_1+1}{1+x_1} = \frac{2(1+x_1)}{1+x_1} = 2$$

cum $\|u\|_2 \geq 0 \Rightarrow \|u\|_2 = \sqrt{2}$

b) $H^T = (I_n - uu^T)^T = I_n^T - (uu^T)^T = I_n - uu^T = H$

$$H \cdot H^T = H^2 = (I_n - uu^T)^2 = I_n - 2uu^T + u\underbrace{u^Tu}_{2}u^T$$

$$= I_n - \cancel{2uu^T} + \cancel{2uu^T} = I_n$$

$\Rightarrow H^T = H^{-1}$ și $H \cdot H^T = I_n \Rightarrow H$ ortogonală.

$$H x = (I_n - uu^T) \cdot x = x - uu^T x =$$

$$= x - \frac{(x+e_1)(x^T+e_1^T)x}{1+x_1} =$$

$$= x - \frac{xx^Tx + xe_1^Tx + e_1x^Tx + e_1e_1^Tx}{1+x_1} =$$

$$= x - \frac{x + x_1x + e_1 + e_1x_1}{1+x_1} =$$

$$= x - \frac{x(1+x_1) + e_1(1+x_1)}{1+x_1} = x - x - e_1 = -e_1$$

$\Rightarrow Hx = -e_1$

3. Vom demonstra relatia prin inducție matematică

$F_0[x_0] = f(x_0)$

$$F_1[x_0, x_1] = \frac{F_0[x_0] - F_0[x_1]}{x_0 - x_1} = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$= \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} \qquad (A)$$

Presupunem adevărat pentru p :

$$F_p[x_0, x_1, \dots, x_p] = \sum_{i=0}^{p} \frac{f(x_i)}{\prod_{\substack{j=0 \\ j\neq i}}^{p} (x_i - x_j)}$$

și vom demonstra pentru $p+1$:

$$F_{p+1}[x_0, x_1, \dots, x_{p+1}] = \frac{F_p[x_0, x_1, \dots x_p] - F_p[x_1, x_2, \dots x_{p+1}]}{x_0 - x_{p+1}}$$

$$= \frac{1}{x_0 - x_{p+1}} \sum_{i=0}^{p} \frac{f(x_i)}{\prod_{\substack{j=0 \\ j\neq i}}^{p} (x_i - x_j)} - \frac{1}{x_0 - x_{p+1}} \sum_{i=1}^{p+1} \frac{f(x_i)}{\prod_{\substack{j=1 \\ j\neq i}}^{p+1} (x_i - x_j)}$$

$$= \frac{f(x_0)}{\left[\prod_{j=1}^{p} (x_0 - x_j)\right](x_0 - x_{p+1})} + \frac{f(x_1)}{\left[\prod_{\substack{j=0 \\ j\neq 1}}^{p} (x_1 - x_j)\right](x_0 - x_{p+1})} + \dots$$

$$\dots + \frac{f(x_p)}{\left[\prod_{j=0}^{p-1} (x_p - x_j)\right](x_0 - x_{p+1})} - \frac{f(x_1)}{\left[\prod_{j=2}^{p+1} (x_1 - x_j)\right](x_0 - x_{p+1})} - \dots$$

$$\dots - \frac{f(x_{p+1})}{\left[\prod_{j=1}^{p} (x_{p+1} - x_j)\right](x_0 - x_{p+1})} =$$

$$= \frac{f(x_0)}{\prod_{j=1}^{p+1} (x_0 - x_j)} + \frac{f(x_1)\,(\cancel{x_1} - x_{p+1} - \cancel{x_1} + x_0)}{\left[\prod_{\substack{j=0 \\ j\neq 1}}^{p+1} (x_1 - x_j)\right](x_0 - x_{p+1})} + \dots$$

$$+ \dots + \frac{f(x_{p+1})}{\prod_{j=0}^{p} (x_{p+1} - x_j)} = \sum_{i=0}^{p+1} \frac{f(x_i)}{\prod_{\substack{j=0 \\ j\neq i}}^{p+1} (x_i - x_j)}$$

ceea ce demonstrează egalitate

$$\Rightarrow F_p[x_0, x_1, \dots, x_p] = \sum_{i=0}^{p} \frac{f(x_i)}{\prod_{j=0,\, j\neq i}^{p} (x_i - x_j)}$$

Diferențele divizate se vom calcula conform cu relația de recurență.

```
function d = DifDiv(x, y)
    n = length(x);
    for k = 1 : n-1
        y(k+1:n) = (y(k+1:n) - y(k))./(x(k+1:n) - x(k));
    end
    d = y(:);
```

4. Scriem relațiile de interpolare și racordare:

$$S_0(x) = a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3$$
$$S_1(x) = a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3$$

$$\left.\begin{array}{l} S_0(1) = 3 \\ S_0(2) = 4 \\ S_1(2) = 4 \\ S_1(3) = 1 \\ S_0'(2) = S_1'(2) \\ S_0''(2) = S_1''(2) \\ S_0''(1) = 2 \\ S_1''(3) = -1 \end{array}\right. \Rightarrow \begin{array}{l} \underline{a_0 = 3} \\ a_0 + b_0 + c_0 + d_0 = 4 \\ \underline{a_1 = 4} \\ a_1 + b_1 + c_1 + d_1 = 1 \\ b_0 + 2c_0 + 3d_0 = b_1 \\ 2c_0 + 6d_0 = 2c_1 \\ 2c_0 = 2 \Rightarrow \underline{c_0 = 1} \\ 2c_1 + 6d_1 = -1. \end{array}$$

$$\begin{array}{l} b_0 + d_0 = 0 \Rightarrow \underline{b_0 = -d_0} \\ b_1 + c_1 + d_1 = -3 \\ b_0 + 3d_0 = b_1 - 2 \\ 1 + 3d_0 = c_1 \\ 2c_1 + 6d_1 = -1 \end{array} \qquad \begin{array}{l} b_1 + c_1 + d_1 = -3 \\ 2d_0 = b_1 - 2 \\ 1 + 3d_0 = c_1 \\ 2c_1 + 6d_1 = -1 \end{array}$$

$$\begin{cases} 2d_0 + 2 + 1 + 3d_0 + d_1 = -3 \\ 2 + 6d_0 + 6d_1 = -1 \end{cases} \Downarrow \begin{cases} 5d_0 + d_1 = -6. \\ 6d_0 + 6d_1 = -3 \end{cases} \Downarrow$$

$$\underline{d_0 = -\frac{11}{8}} \quad \underline{d_1 = -\frac{103}{8}}$$

$$\Rightarrow \underline{c_1 = -\frac{25}{8}}, \ \underline{b_1 = -\frac{3}{4}}, \ \underline{b_0 = \frac{11}{8}}$$

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Subiecte (Numarul B1)

3 puncte **1.** Fie matricea $A = \begin{pmatrix} 1 & 2 \\ 3 & 13 \end{pmatrix}$. Calculati facorizarea LU-Doolittle ($l_{ii} = 1$) pentru maricea A. Scrieti o functie MATLAB, `function [L U] = Doolittle(A)`, $A \in R^{nxn}$.

3 puncte **2.** Un vector $u \in R^n$ poate fi adus la un vector de norma 1 prin impartirea cu norma sa. Fie reflectorii Householder $U = I_n - 2uu^T$, $||u||_2 = 1$ si $V = I_n - vv^T$, $||v||_2 = \sqrt{2}$.

a) Dati un exemplu numeric pentru vectorii u si v.

b) Calculati reflectorii U si V pentru exemplul dat la punctul a).

c) Calculati Uu si Vv.

d) Daca $A = uv^T$, calculati $B = UAU^T$ si $C = VAV^T$.

2 puncte **3.** Fie functia $f(x)$ data prin $x = a, 0, 1, b$ si $f(x) = y_a, y_0, y_1, y_b$.

a) Calculati polinomul Lagrange de interpolare.

b) Scrieti o functie MATLAB pentru calculul polinomului Lagrange intr-un punct a.

2 puncte **4.** Calculati functiile spline naturale $s_0(x)$ si $s_1(x)$ in clasa C^2 pentru functia $f(x)$:

x	1	3	4
f	-1	1	18

(B1)

1. $L = \begin{bmatrix} 1 & 0 \\ \ell_{21} & 1 \end{bmatrix}$ $\quad U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$ $\quad A = LU \Rightarrow$

$$\begin{bmatrix} 1 & 0 \\ \ell_{21} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 13 \end{bmatrix}$$

$$\begin{aligned} u_{11} &= 1 & \qquad & u_{11} = 1 \\ u_{12} &= 2 & \Rightarrow \qquad & u_{12} = 2 \\ \ell_{21} u_{11} &= 3 & & \ell_{21} = 3 \\ \ell_{21} u_{12} + u_{22} &= 13 & & u_{22} = 7 \end{aligned}$$

$$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$$

Vom folosi formulele următoare: $A_{ij} = \sum_{m=1}^{\min(i,j)} L_{im} U_{mj}$

$$A_{pj} = \sum_{m=1}^{p-1} L_{pm} U_{mj} + L_{pp} U_{pj} \Rightarrow U_{pj} = A_{pj} - \sum_{m=1}^{p-1} L_{pm} U_{mj}$$

$j = p:n$

$$A_{ip} = \sum_{m=1}^{p-1} L_{im} U_{mp} + L_{ip} U_{pp} \Rightarrow L_{ip} = \left(A_{ip} - \sum_{m=1}^{p-1} L_{im} U_{mp}\right) / U(p,p)$$

$i = p+1:n$

```
function [L U] = Doolittle (A)
    [m n] = size (A);
    for p = 1 : n
        for j = p : n
            s = A(p, 1:p-1) * A(1:p-1, j);
            A(p, j) = A(p, j) - s;
        end;
        for i = p+1 : n
            s = A(i, 1:p-1) * A(1:p-1, p);
            A(i, p) = (A(i, p) - s) / A(p, p);
        end
end
```

$U = \mathrm{triu}(A)$;

$L = A - U + \mathrm{eye}(n)$;

2. a) $u = [1\ 0\ \dots\ 0]^T \qquad \|u\|_2 = 1.$

$v = [1\ 1\ 0 \dots 0]^T \qquad \|v\|_2 = \sqrt{2}$

b) $U = I_n - 2uu^T = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$

$V = I_n - vv^T = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & & & \\ -1 & 0 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$

c) $Uu = (I_n - 2uu^T)u = u - 2u\underbrace{u^Tu}_{1} = u - 2u = -u$

$Vv = (I_n - vv^T)v = v - v\underbrace{v^Tv}_{2} = v - 2v = -v.$

d) $A = uv^T$

$B = UAV^T = Uuv^TV^T = (Uu)\cdot(Vv)^T = -u(-v)^T = A$

sau $B = UAU^T = Uuv^TU^T = -uv^TU^T = -AU^T$

$C = VAV^T = Vuv^TV^T = Vu(-v)^T = -VA$

3.

x	a	0	1	b
f	y_a	y_0	y_1	y_b

a) $P_3(x) = y_a \dfrac{(x-0)(x-1)(x-b)}{(a-0)(a-1)(a-b)} + y_0 \dfrac{(x-a)(x-1)(x-b)}{(0-a)(0-1)(0-b)} +$

$+ y_1 \dfrac{(x-a)(x-0)(x-b)}{(1-a)(1-0)(1-b)} + y_b \dfrac{(x-a)(x-0)(x-1)}{(b-a)(b-0)(b-1)}$

b) Pentru functia MATLAB vom considera forma generală $P_n(x) = \sum_{i=0}^{n} \left(f(x_i) \prod_{\substack{j=0 \\ j \neq i}}^{n} \dfrac{x - x_j}{x_i - x_j} \right)$

```
function val = Lagrange(x, y, a)
    n = length(x);   val = 0;
    for i = 1 : n
        prod = 1;
        if i ~= j
            prod = prod * (a - x(j)) / (x(i) - x(j));
        end;
        val = val + y(i) * prod;
    end;
```

4. $S_0(x) = a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3$

$S_1(x) = a_1 + b_1(x-3) + c_1(x-3)^2 + d_1(x-3)^3$

$$\begin{array}{ll}
S_0(1) = -1 & \underline{a_0 = -1} \\
S_0(3) = 1 & a_0 + 2b_0 + 4c_0 + 8d_0 = 1 \\
S_1(3) = 1 & \underline{a_1 = 1} \\
S_1(4) = 18 & a_1 + b_1 + c_1 + d_1 = 18 \\
S_0'(3) = S_1'(3) \quad \Rightarrow & b_0 + 4c_0 + 12d_0 = b_1 \\
S_0''(3) = S_1''(3) & 2c_0 + 12d_0 = c_1 \\
S_0''(1) = 0 & \underline{c_0 = 0} \\
S_1''(4) = 0 & 2c_1 + 6d_1 = 0
\end{array}$$

$$\begin{array}{lll}
2b_0 + 8d_0 = 2 & & -b_0 + 4d_0 = 1 \\
b_1 + c_1 + d_1 = 17 & & b_0 + 24d_0 + d_1 = 17 \\
b_0 + 12d_0 = b_1 & \Rightarrow & 12d_0 + 3d_1 = 0 \quad \Rightarrow d_1 = -4d_0 \\
12d_0 = c_1 & & \Downarrow \\
c_1 + 3d_1 = 0 & &
\end{array}$$

$$\begin{cases} b_0 + 4d_0 = 1 \\ b_0 + 20d_0 = 17 \end{cases} \Rightarrow 16d_0 = 16 \Rightarrow \underline{d_0 = 1}$$

$\underline{d_1 = -4}$

$\underline{c_1 = 12}$, $\underline{b_0 = -3}$, $\underline{b_1 = 9}$

$$\Rightarrow \begin{cases} S_0(x) = -1 - 3(x-1) + (x-1)^3 \\ S_1(x) = 1 + 9(x-3) + 12(x-3)^2 - 4(x-3)^3 \end{cases}$$

Facultatea de Automatica si Calculatoare, Universitatea Politehnica din Bucuresti

Computer Science & Engineering Department

Examen Partial MN

Student:	______________________	**Grupa:**	________

Descriere curs:	**MN, An I, Semestrul II**
Titlu curs:	**Metode Numerice**
Profesor:	**Sl.Dr.Ing. Florin POP**
Data examenului:	24 Aprilie 2010 (EC105)
Intervalul orar:	Start: 09:00→ Final: 11:00
Durata examenului:	2 ore
Tip Examen:	**"Closed Book"**
Materiale Aditionale:	**Nu! (!Fara telefoane mobile!)**
Numar pagini:	______________

Rezultate Examen	
Subiect	Punctaj
1	/3
2	/3
3	/2
4	/2
$\sum$	/10

Subiecte (Numarul B2)

3 puncte **1.** Fie matricea $A = \begin{pmatrix} 1 & 2 \\ 3 & 13 \end{pmatrix}$. Calculati facorizarea LU-Cholesky ($A = LL^T$) pentru maricea A. Scrieti o functie MATLAB, `function [L U] = Choleschy(A)`, $A \in R^{nxn}$.

3 puncte **2.** Fie reflectorul Hauseholder $H = I - 2uu^T$. Demonstrati ca daca se alege vectorul u de forma $u = \frac{v+||v||e_1}{||v+||v||e_1||}$ atunci H este ortogonala si $Hv = -||v||e_1$.

2 puncte **3.** In expresia polinomului de interpolare Lagrange $P_n(x) = \sum_{i=0}^{n} f(x_i)L_i(x)$, multiplicatorii Lagrange sunt $L_i(x) = \prod_{j=0,j\neq i}^{n} \frac{x-x_j}{x_i-x_j}$.

a) Aratati ca $L_i(x_j) = \delta_{ij}$ si $\sum_{i=0}^{n} L_i(x) = 1$. Aratati ca multiplicatorii Lagrange sunt invarianti la o schimbare liniara de variabile.

b) Exprimati polinomul de interpolare Lagrange in cazul absciselor echidistante date prin $x = x_0 + uh$, $x_i = x_0 + ih$. Scrieti o functie MATLAB pentru calculul lui $P(u)$.

2 puncte **4.** Pentru functia $f(x)$ cunoscuta prin tabelul urmator, calculati functiile spline $s_0(x)$ si $s_1(x)$, *tensionate* de clasa C^1, daca $s_0'(1) = 2$ si $s_1'(4) = -1$.

x	1	2	4
f	3	4	6

B2)

1. $L = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix}$ $\quad A = LL^T \Rightarrow \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \cdot \begin{bmatrix} l_{11} & l_{21} \\ 0 & l_{22} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 13 \end{bmatrix}$

Observație: matricea nu este simetrică, deci nu se poate factoriza Cholesky. Dacă vom considera $a_{21} = 2$

$$\begin{aligned} l_{11}^2 &= 1 & l_{11} &= \pm 1 \\ l_{11} l_{21} &= 2 & l_{21} &= \pm 2 \\ l_{21} l_{11} &= 2 \Rightarrow & l_{22}^2 &= 13 - 4 = 9 \Rightarrow l_{22} = \pm 3 \\ l_{21}^2 + l_{22}^2 &= 13 \end{aligned}$$

deci factorizarea admite mai multe soluții.

Alegem $L = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

Pentru funcția MATLAB vom folosi formulele:

$$L_{ii} = \sqrt{A_{ii} - \sum_{k=1}^{i-1} L_{ik}^2}, \quad i = 1:n$$

$$L_{ij} = \left(A_{ij} - \sum_{k=1}^{j-1} L_{ik} L_{jk}\right) / L_{jj} \quad j = 1:i-1$$

```
function L = Cholesky (A)
    [m n] = size(A);
    for i = 1:n
        for j = 1:i-1
            s = L(i, 1:j-1) * L(j, 1:j-1)';
            L(i,j) = (A(i,j) - s) / L(j,j)
        end
        s = A(i, 1:i-1) * A(i, 1:i-1)';
        L(i,i) = sqrt(A(i,i) - s);
        L(i, i+1:n) = 0;
    end
```

2. H - ortogonală $\Rightarrow H \cdot H^T = I_n$

Calculăm pentru început $\|v + \|v\| e_1\|$ considerând norma 2 și folosind notația $k = \|v\|_2$, $v^T v = k^2$

$$\|v+ke_1\|^2 = (v+ke_1)^T(v+ke_1) = (v^T+ke_1^T)(v+ke_1) =$$
$$= v^Tv + kv^Te_1 + ke_1^Tv + k^2e_1^Te_1 =$$
$$= k^2 + 2kv_1 + k^2 = 2k(k+v_1)$$

unde v_1 este prima componentă a vectorului v.

Calculăm $\|u\|_2 = \left\| \frac{v+ke_1}{\|v+ke_1\|_2} \right\|_2 = \frac{1}{\|v+ke_1\|_2} \cdot \|v+ke_1\| = 1.$

$$H \cdot H^T = (I_n - 2uu^T)(I_n - 2uu^T)^T = (I_n - 2uu^T)(I_n^T - 2uu^T) =$$
$$= I_n - 4uu^T + 4u\underbrace{u^Tu}_{1}u^T = I_n$$

$\Rightarrow H \cdot H^T = I_n$ deci H este ortogonală.

$$Hv = (I_n - 2uu^T)v = v - 2uu^Tv = v - 2 \cdot \frac{(v+ke_1)(v+ke_1)^Tv}{\|v+ke_1\|_2^2}$$
$$= v - \cancel{2}\frac{(v+ke_1) \cdot (v^T+ke_1^T)v}{\cancel{2}k(k+v_1)} = v - \frac{(v+ke_1)(v^Tv+ke_1^Tv)}{k(k+v_1)} =$$
$$= v - \frac{(v+ke_1)\cancel{(k^2+kv_1)}}{\cancel{k^2+kv_1}} = v - v - ke_1 = -ke_1 = -\|v\|e_1$$

$\Rightarrow Hv = -\|v\|e_1.$

3. a) schimbăm indicele de produs: $L_i(x) = \prod_{\substack{k=0 \\ k\neq i}}^{n} \frac{x - x_k}{x_i - x_k}$

$$L_i(x_j) = \prod_{\substack{k=0 \\ k\neq i}}^{n} \frac{x_j - x_k}{x_i - x_k} =$$

1° dacă $i=j$ atunci numitorul și numărătorul sunt identice, deci $L_i(x_i) = 1$.

2° dacă $i \neq j$ atunci la numărător va apărea un termen de forma $x_k - x_k = 0$ care va anula multiplicatorul Lagrange deci $L_i(x_j) = 0$

$\Rightarrow L_i(x_j) = \delta_{ij}$

Pentru a demonstra că $\sum_{i=0}^{n} L_i(x) = 1$ vom considera funcția unitate $f(x) = 1$ pentru care $P_n(x) = 1 \Rightarrow 1 = \sum_{i=0}^{n} L_i(x)$

O schimbare liniară de variabilă este $x = \alpha t + \beta$

$$L_i(x) = L_i(\alpha t + \beta) = \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{\alpha \cancel{t} + \beta - \alpha \cancel{t_j} - \beta}{\alpha t_i + \beta - \alpha t_j - \beta} = \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{t - t_j}{t_i - t_j}$$

$$= L_i(t)$$

b) Dacă abscisele sunt echidistante, atunci

$$L_i(x) = L_i(x_0 + uh) = L_i(u) = \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{u-j}{i-j}$$

Polinomul de interpolare va fi $P_n(x) = \sum_{i=0}^{n} \left(f(x_i) \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{u-j}{i-j} \right)$

Funcția primește ca parametru pe u și vectorul de valori pentru f, y

```
function p = echiLagrange(u, y)
    n = length(y); p = 0;
    for i = 1:n
        prod = 1;
        for j = 1:n
            if i ~= j
                prod = prod * (u-j) / (i-j);
            end;
        end;
        p = p + y(i) * prod;
    end;
```

4. Funcțiile tensionate în clasă C^1 cu condițiile dat sunt funcții parabolice:

$$S_0(x) = a_0 + b_0(x-1) + c_0(x-1)^2$$
$$S_1(x) = a_1 + b_1(x-2) + c_1(x-2)^2$$

Condițiile de interpolare, racordare și limită sunt:

$$s_0(1) = 3 \qquad \underline{a_0 = 3}$$
$$s_0(2) = 4 \qquad a_0 + b_0 + c_0 = 4 \qquad \underline{c_0 = -1}.$$
$$s_1(2) = 4 \qquad \underline{a_1 = 4} \qquad 2b_1 + 4c_1 = 2$$
$$s_1(4) = 6 \Rightarrow a_1 + 2b_1 + 4c_1 = 6 \Rightarrow b_1 + 4c_1 = -1.$$
$$s_0'(1) = 2 \qquad \underline{b_0 = 2}$$
$$s_1'(4) = -1 \qquad b_1 + 4c_1 = -1$$

$$\begin{aligned} b_1 + 2c_1 &= 1 \\ b_1 + 4c_1 &= -1 \end{aligned} \Rightarrow -2c_1 = 2 \Rightarrow \underline{c_1 = -1}, \ \underline{b_1 = 3}$$

$$\Rightarrow \begin{cases} s_0(x) = 3 + 2(x-1) + (x-1)^2 \\ s_1(x) = 4 + 3(x-2) - (x-2)^2 \end{cases}$$