Corporate control under common ownership*

Click to access latest version

Orestis Vravosinos†

December 11, 2025

Abstract

I study firm conduct under common ownership. I show that the current canonical *weighted average portfolio profit* model (WAPP) imposes two restrictions: (i) that the firm's strategy be Pareto efficient when the returns of a subset of "controlling shareholders" of the firm are considered, and (ii) that the distribution of power across shareholders within the firm be independent of external factors such as the stakes of the firm's shareholders in competing firms, the strategies of other firms, or market conditions. I propose the Nash bargaining (NB) model, which models the firm's behavior as the result of asymmetric Nash bargaining among shareholders. NB generalizes WAPP by allowing for external factors to influence the distribution of power across shareholders. I characterize how additional restrictions on firm conduct constrain the classes of WAPP and NB models. I discuss implications for robust testing of the common ownership hypothesis.

Keywords: common ownership, corporate governance, institutional shareholders, bargaining, Nash bargaining, Nash-in-Nash, minority shareholdings, antitrust, competition policy, oligopoly
JEL classification codes: C71, D43, G34, L11, L13, L21, L41

*Previously circulated as "A Nash-in-Nash model of corporate control and oligopolistic competition under common ownership." I am grateful to Erik Madsen for his continued guidance. I also thank Dilip Abreu, Chris Conlon, Basil Williams, and audiences at NYU for helpful comments and discussions.

†University of Chicago; e-mail: orestis.vravosinos@uchicago.edu.

1 Introduction

Perfect competition is crucial for a firm's shareholders to unanimously agree on own profit maximization (Hart, 1979), which has been the standard assumption on firm conduct since at least Fisher's (1930) separation theorem. Yet, there is evidence that market power has been widespread and increasing in the U.S. economy and around the world (De Loecker and Eeckhout, 2018; De Loecker et al., 2020). At the same time, there is concern that the rise of common ownership—where competing firms have to an extent common shareholders (Backus et al., 2021a; Antón et al., 2025a)—suppresses competition and reduces welfare by inducing firms to partially internalize their impact on competing firms' profits (Rubinstein and Yaari, 1983; Rotemberg, 1984; Matvos and Ostrovsky, 2008; Elhauge, 2016; Posner et al., 2017; He and Huang, 2017; Azar et al., 2018; Schmalz, 2018; Azar and Vives, 2021; Lu et al., 2022; Li et al., 2023; Antón et al., 2023; Newham et al., 2025; Ederer and Pellegrino, 2025; Vives and Vravosinos, 2025; DOJ & FTC, 2023).[1]

In studying such anticompetitive effects of common ownership—be it theoretically or empirically, a model of firm conduct other than own-profit maximization is often necessary (see, e.g., Backus et al., 2021b; Ederer and Pellegrino, 2025; Vives and Vravosinos, 2025). The model needs to describe how the firm's conduct is shaped by its shareholders' conflicting interests: For example, shareholders with few shares of competing firms want the firm to price more aggressively than shareholders with larger stakes in competitors. Nevertheless, multiple authors have acknowledged our limited understanding around the modeling of firm conduct under common ownership (see, e.g., Schmalz, 2018; Backus et al., 2021a; Antón et al., 2023).

Most of the literature has so far relied on what I call the *weighted average portfolio profit* model (WAPP) of O'Brien and Salop (2000).[2] Given a set $N := \{1,2,\dots,n\}$ of shareholders and a set M of firms, let s_{if} denote shareholder i's number of firm $f \in M$ shares with cash-flow rights and π_g denote firm g's profit. WAPP poses that for every shareholder i, there exists control weight $\gamma_{if}(s_{*f})$—which depends on the ownership structure $s_{*f} \equiv (s_{1f},\dots,s_{nf})$ of the firm—such that firm f's conduct can be modeled as maximizing a weighted average of its shareholders' portfolio profits, that is,

$$\sum_{i\in N} \overbrace{\gamma_{if}(s_{*f})}^{\substack{\text{shareholder } i\text{'s} \\ \text{control weight} \\ \text{over firm } f}} \times \overbrace{\sum_{g\in M} s_{ig}\pi_g}^{\substack{\text{shareholder } i\text{'s} \\ \text{portfolio profit}}} \propto \pi_f + \sum_{g\in M\setminus\{f\}} \overbrace{\frac{\sum_{i\in N}\gamma_{if}(s_{*f})s_{ig}}{\sum_{i\in N}\gamma_{if}(s_{*f})s_{if}}}^{=:\lambda_{fg}} \pi_g.$$

[1]However, some studies find little evidence that common ownership affects firm conduct (Backus et al., 2021b; Koch et al., 2021; Lewellen and Lowry, 2021). Common ownership can also have positive effects. For example, it can enhance innovation by causing firms to internalize technological spillovers (López and Vives, 2019; Antón et al., 2025b) and prevent the duplication of R&D costs across firms (Li et al., 2023). For a review of the common ownership literature, see Gerardi et al. (2024).

[2]Similar ideas can be traced back to Rotemberg (1984) and Bresnahan and Salop (1986).

Equivalently, firm f maximizes its own profit plus each other firm g's profit weighted by λ_{fg}, the Edgeworth coefficient of effective sympathy from firm f towards firm g. The literature usually makes the *proportional control* assumption that $\gamma_{if}(s_{*f}) = s_{if}$.

Although WAPP is simple and instinctively reasonable, it imposes restrictions on firm conduct that merit careful study. Much of the discussion has revolved around the "correct" mapping γ_{*f} from ownership structure to control weights. Indeed, "any formulation of γ is implicitly a model of corporate governance and one where theory offers precious little guidance" (Backus et al., 2021a). The assumption that there even *exists* a correct mapping γ_{*f} such that the firm's conduct can be modeled as maximizing a weighted average of the shareholders' portfolio profits is considered uncontroversial (Backus et al., 2020). Nonetheless, it is not clear what restrictions exactly it imposes on firm conduct. Therefore, a theoretical analysis is useful for (i) studying how properties of firm conduct translate into restrictions on the mapping γ_{*f}, (ii) evaluating the properties of firm conduct that allow it to admit a WAPP representation, and (iii) developing an alternative model of firm conduct under common ownership for when there is concern that firm conduct may not admit a WAPP representation.

I pursue each of the three objectives by studying properties of the firm's *strategic plan*. The strategic plan describes the strategy the firm will follow as a function of the strategies followed by the other firms, the stakes of each shareholder across firms, and market conditions (e.g., demand or production technology). For example, WAPP with proportional control is a strategic plan. Clearly, one cannot hope to theoretically—or, for that matter, empirically—show that a specific strategic plan is the "correct" one if for no other reason than because different firms may have different strategic plans. Indeed, there are many reasonable strategic plans. Even if we constrain attention to WAPP, empirically estimating γ's from data on market outcomes is a futile exercise given that there are usually many more shareholders than firms in an industry (Backus et al., 2020). Therefore, the focus of this paper is to characterize *classes* of strategic plans such as NB and WAPP, and subclasses thereof.

First, within the context of the WAPP model, I propose two monotonicity properties that formally characterize the intuitive notion that "more shares should lead to more control." The first property, called *rank preservation*, relates to how the firm adjusts its strategy in response to changes in its shareholders' interests. Starting from an ownership structure where firm f's shareholders' interests are aligned, consider a stock trade between two shareholders i and j of firm f in which i buys shares of another firm $g \neq f$ from j. Before the stock trade, the shareholders' interests are aligned so the firm maximizes each shareholder's portfolio profit. The stock trade causes disagreement among firm f shareholders: Shareholder i wants firm f to adjust its strategy in the direction benefiting firm g, while shareholder j wants firm f to adjust its strategy in the opposite direction. Firm f's strategic plan is rank-preserving if, in response to the stock trade, firm f adjusts its strategy in the direction preferred by the larger shareholder involved in the trade. I show that rank preservation is satisfied if and only if $s_{if} \geq s_{jf} \implies \gamma_{if}(s_{*f}) \geq \gamma_{jf}(s_{*f})$. The second monotonicity property, *stock-trade monotonicity*, requires that a shareholder i's control power over firm f—as measured through firm f's strategy adjustment in response to stock trades involving shareholder i—increases when i grows her stake in every firm, including f. I show that stock-trade monotonicity is satisfied if and only if $\gamma_{if}(s_{*f} + t(\mathbf{e}_i - \mathbf{e}_j)) \geq \gamma_{if}(s_{*f})$ for any $t > 0$, where $\mathbf{e}_i$ is the standard basis vector with 1 in its i-th entry. This implies γ_{if} (weakly) increases if shareholder i increases her stake in firm f while *no other* shareholder increases her stake in firm f. However, it does *not* imply that $s'_{if} \geq s_{if} \implies \gamma_{if}(s'_{*f}) \geq \gamma_{if}(s_{*f})$, which would be too strong (e.g., it would likely be violated for s_{*f} and s'_{*f} such that $s_{jf} > 1/2 > \max_{k \in N} s'_{kf}$ for some $j \neq i$).

I also characterize a popular generalization of WAPP with proportional control positing that there exists real function δ such that $\gamma_{if}(s_{*f}) = \delta(s_{if}) / \sum_{j \in N} \delta(s_{jf})$ (see, e.g., Backus et al., 2021a; Antón et al., 2023; Ederer and Pellegrino, 2025). Under proportional control, $\delta(s_{if}) = s_{if}$. I show that the control weights of a strategic plan that admits a WAPP representation can be expressed in this way if and only if the strategic plan satisfies three conditions: (i) anonymity, which requires that the identity of shareholders not matter for firm strategy, (ii) inclusivity, which requires that *every* shareholder of the firm—no matter how large or small—exerts some control over the firm, and (iii) independence of irrelevant shareholders (IIS), which requires that the relative control power of two shareholders over the firm be unaffected by stock trades between other shareholders of the firm. I also discuss when each of these conditions might be violated.

Second, I show that two properties are crucial for firm behavior to admit a WAPP representation: *efficiency* and *irrelevance of external factors*. Efficiency requires that for any ownership structure of the firm, there is a subset of shareholders who efficiently control the firm. Namely, the firm's strategic plan always prescribes a Pareto efficient strategy when the portfolio returns of a subset of controlling shareholders of the firm are considered. Irrelevance of external factors requires that the distribution of power across shareholders of the firm depend only on the firm's ownership structure, and not on external factors such as (i) the stakes of firm f shareholders in competing firms, (ii) the strategies of other firms, and (iii) market conditions (e.g., demand or production technology).

Nevertheless, it is plausible that the distribution of power across a firm's shareholders can depend on such external factors. The dependence may arise because the distribution of power is shaped not only by the capacity of shareholders to exercise control but also by their *incentives* to do so. Firm f shareholders with larger stakes in other firms in the same industry may pay closer attention to the industry and thus exert more control over firm f. Also, firm f shareholders who also have shares of competing firm g may have stronger incentives to exert control over firm f when firm g's strategy is such that firm

f has a lot of room to influence g's profitability. For instance, if firm g's production is large (resp. small), firm f's effect through its pricing and production strategy on firm g's profit margin will have a significant (resp. modest) impact on firm g's profits. Last, firm f shareholders who also have shares of competing firm g may have stronger incentives to try to affect firm f's strategy when firm f and g's goods are strong substitutes (than when they are weaker substitutes), in which case firm f's pricing strategy will have a more pronounced impact on firm g's profits. Indeed, as shown in section 3.2, the irrelevance of external factors condition rules out some natural benchmark strategic plans such as the one following the median shareholder's interests, which arises from majority voting as in Chiappinelli et al. (2023).

Third, I propose the Nash bargaining (NB) model of firm conduct under common ownership. NB generalizes WAPP by allowing for external factors to influence the distribution of power across a firm's shareholders while still requiring efficiency. NB represents firm conduct as the result of asymmetric Nash bargaining among the firm's shareholders. The equilibrium concept is then a Nash equilibrium in Nash bargains, or simply Nash-in-Nash equilibrium. As in WAPP, I study the constraints imposed on NB by the monotonicity properties. I show that restrictions imposed on the bargaining weights based on intuition may fail to capture the notion that "more shares should lead to more control." For example, one may instinctively think that larger shareholders having larger bargaining weights in the Nash product would capture this notion. However, this is not the case. I show that a stronger condition characterizes rank-preserving NB strategic plans.

I show that the NB model can relax the tension that arises under WAPP between (i) allowing for atomistic shareholders to collectively exert control over the firm and (ii) allowing for large shareholders to have control power. As has been noted before (see, e.g., Gramlich and Grundl, 2017; O'Brien and Waehrer, 2017; Brito et al., 2023), I show that generally, under WAPP, as ownership by a group of shareholders with aligned interests is diffused (i.e., the same number of shares is divided across more and more shareholders), the group of shareholders stops having any control over firm strategy. For example, as shareholders who do not hold shares of competing firms become dispersed, the firm tends to follow only the interests of the shareholders who are also invested in competitors. While this may be plausible, the possibility that atomistic shareholders collectively exert control over the firm is also plausible. Nonetheless, most parametrizations of the WAPP model preclude this possibility. I show that those parametrizations that do allow for atomistic shareholders to collectively exert control over the firm have an unrealistic property: They assign no control power to large shareholders when atomistic shareholders are also present. NB allows for the possibility that atomistic shareholders collectively exert control over the firm without imposing this unrealistic property.

After a discussion of related literature, section 2 presents the model. Section 3 characterizes the WAPP and NB models and studies how properties of firm conduct translate into restrictions on the parameters of the firm's objective under WAPP and NB. Section 4 discusses (i) the effects of ownership dispersion on firm conduct under WAPP and NB, (ii) possible formulations of the disagreement payoff function in NB, and (iii) the efficiency condition on the firm's strategic plan. Section 5 concludes. All proofs are gathered in the Appendix. The Online Appendix provides supplementary results.

Related literature Theoretical work on firm conduct under common ownership has primarily focused on microfounding WAPP in models of shareholder voting (see, e.g., Azar, 2017; Brito et al., 2018; Moskalev, 2019). On the other hand, Chiappinelli et al. (2023) consider a setting where shareholders elect managers through majority rule, and as a result the median shareholder's preferences dictate firm strategy. Their model allows for atomistic shareholders to collectively exert control over the firm while also allowing for large shareholders to have control power. This is possible because, as I show in section 3.2, majority voting admits a NB representation but not a WAPP one, since it induces a distribution of power across shareholders within the firm that depends on external factors.

Brito et al. (2023) propose a voting model which unlike standard formulations of WAPP, allows for atomistic shareholders to collectively exert control over the firm. However, it prevents large shareholders from having control power. Indeed, in the Online Appendix, I show that Brito et al.'s (2023) model is WAPP, which is why it cannot allow for atomistic shareholders to collectively exert control over the firm while also allowing for large shareholders to have control power.

My approach differs methodologically from previous works. Instead of making an ad-hoc assumption on the firm's objective function or microfounding a firm conduct model through shareholder voting, I discuss properties ("axioms") on firm conduct and show how these properties translate into formulations of the firm's objective function. I show that, unlike WAPP, NB allows for the distribution of power across shareholders within the firm to depend on external factors and can relax the tension that arises under WAPP between (i) allowing for atomistic shareholders to collectively exert control over the firm and (ii) allowing for large shareholders to have control power.

If every firm's strategic plan is NB, the equilibrium concept is Nash-in-Nash, which has become a standard tool since Horn and Wolinsky (1988) used it to study merger incentives when there are exclusive vertical relationships. The current paper fits into the literature that has leveraged the Nash-in-Nash solution to study equilibrium outcomes in a broad range of environments where the problem of the division of surplus between parties (e.g., upstream and downstream firms) is embedded within a larger market model. For a review of the Nash-in-Nash literature, see Collard-Wexler et al. (2019).

2 Strategic plans

A tuple $G := \langle N, M, (A_f)_{f\in M}, (\pi_f)_{f\in M}, (s_{if})_{(i,f)\in N\times M}\rangle$ characterizes an oligopoly game with common ownership, where $N := \{1,2,\ldots,n\}$ is the set of n shareholders, $M := \{1,2,\ldots,m\}$ is the set of m firms in the industry or product market, and A_f is firm f's strategy space. We will use i,j,k to denote shareholders and f,g,h, to denote firms. Let the strategy profile space be denoted by $A := \times_{f\in M} A_f$. For a strategy profile $a \equiv (a_1,\ldots,a_m) \in A$, where $a_f \in A_f$ is firm f's strategy, a_{-f} denotes the profile of strategies of all firms except f, and accordingly $A_{-f} := \times_{g\neq f} A_g$. Firm f's profit function is $\pi_f : A \to \mathbb{R}$, and $s \in S := \{s \in [0,1]^{n\times m} : \sum_{i\in N} s_{if} = 1\ \forall f \in M\}$ is the ownership matrix, where s_{if} denotes shareholder i's share of firm f.[3] This means that i has a cash-flow right over fraction s_{if} of firm f's profits. Shareholder i's total portfolio profit function is $u_i(a,s_{i*}) := \sum_{f\in M} s_{if}\pi_f(a)$.[4] A shareholder i is a shareholder of firm f if $s_{if} > 0$. $N_f(s_{*f}) := \{i \in N : s_{if} > 0\}$ is the set of shareholders of firm f.

A strategic plan $R_f : \times_{g\neq f}\Delta(A_g) \times S \times \Pi \to \Delta(A_f)$ of firm f determines the nonempty set $R_f(\alpha_{-f},s;\pi)$ of strategies deemed choosable by firm f for each strategy profile α_{-f} of the other firms, each ownership structure s, and each vector $\pi \equiv (\pi_1,\ldots,\pi_n) \in \Pi$ of profit functions, where $\Delta(A_g)$ is the set of lotteries over A_g with finite support and Π the set of possible vectors of profit functions. The strategic plan describes firm conduct for any possible profit functions π given that market conditions such as technology and demand may change. To simplify notation, I write $u_i(a,s_{i*})$ instead of $u_i(a,s_{i*};\pi)$. Abusing notation, we also write $u_i(\alpha,s_{i*})$ and $\pi_f(\alpha)$ to denote shareholder i's expected portfolio profit and firm f's expected profit, respectively.

A strategy profile is an equilibrium if every firm plays one of its choosable strategies given the other firms' strategies, the ownership structure, and the profit functions. When each firm's strategic plan results from Nash bargaining (see section 2.2), the equilibrium concept is a Nash equilibrium in Nash bargains. While the focus of the paper is a single firm's strategic plan, the Online Appendix provides sufficient conditions for the existence of a Nash-in-Nash equilibrium.

2.1 The weighted average portfolio profit (WAPP) strategic plan

Let $\Delta^n \subset \mathbb{R}^n$ denote the $(n-1)$-dimensional simplex. I first describe the strategic plan proposed by O'Brien and Salop (2000), which I call the weighted average portfolio profit strategic plan (WAPP).

[3]As in the majority of the literature, s is treated as exogenous.Denicolò and Panunzi (2025) and Piccolo and Schneemeier (2025) endogenize common ownership. The analysis goes through if we extend the model to allow for short positions (where those shorting a firm's stock do not exert control over the firm).

[4]Given a matrix x, x_{i*} and x_{*f} denote x's i-th row and f-th column, respectively. The notation for a function that maps to an $n \times m$ space is analogous.

Definition 1. Firm f's strategic plan R_f is WAPP if there exists a control power function $\gamma_{*f} : \Delta^n \to \Delta^n$ such that for every $s \in S$, $\alpha_{-f} \in \times_{g \neq f} \Delta(A_g)$, and $\pi \in \Pi$,

(i) the firm maximizes the weighted average portfolio profit of its shareholders:

$$R_f\left(\alpha_{-f}, s; \pi\right) = \arg\max_{\alpha_f \in \Delta(A_f)} \left\{ \sum_{i \in N} \gamma_{if}(s_{*f}) u_i(\alpha_f, \alpha_{-f}, s_{i*}) \right\},$$

(ii) control is exclusive to shareholders: For every $i \in N$, $s_{if} = 0 \implies \gamma_{if}(s_{*f}) = 0$.

This can be rewritten as

$$R_f\left(\alpha_{-f}, s; \pi\right) = \arg\max_{\alpha_f} \left\{ \pi_f(\alpha_f, \alpha_{-f}) + \sum_{g \in M \setminus \{f\}} \overbrace{\frac{\sum_{i \in N} \gamma_{if}(s_{*f}) s_{ig}}{\sum_{i \in N} \gamma_{if}(s_{*f}) s_{if}}}^{=:\lambda_{fg}(s) \geq 0} \pi_g(\alpha_f, \alpha_{-f}) \right\},$$

where $\lambda_{fg}(s)$ is the weight firm f places on firm g's profit with λ_{ff} normalized to 1. λ_{fg} is called the Edgeworth (1881) coefficient of effective sympathy of firm f towards firm g. The numerator of λ_{fg} is a measure of the level of cross-holdings of firm f shareholders in firm g. The denominator measures ownership concentration in firm f.

The literature often assumes $\gamma_{*f}(s_{*f}) = s_{*f}$, which it calls "proportional control." A popular generalization of proportional control specifies $\gamma_{if}(s_{*f}) = \delta(s_{if}) / \sum_{j \in N} \delta(s_{jf})$ for some non-negative and non-decreasing real function δ. Particularly, $\delta(s_{if}) = s_{if}^{\zeta}$ for some $\zeta \geq 0$ is often used (see, e.g., Backus et al., 2021a; Antón et al., 2023; Ederer and Pellegrino, 2025). $\zeta > 1$ is interpreted as large shareholders having disproportionately more power than smaller shareholders. $\zeta = 1$ corresponds to proportional control. $\zeta < 1$ is interpreted as large shareholders having less than proportionately more power than smaller shareholders. Last, Azar et al. (2018), Brito et al. (2023), and Antón et al. (2025a) consider the case where γ_{if} is shareholder i's (normalized) Banzhaf power index (Penrose, 1946; Banzhaf, 1965; Coleman, 1971). In that case, to calculate γ_{if}, one first enumerates all winning (*i.e.*, with at least 50% of the firm's shares) coalitions of shareholders where there is (at least) one swing shareholder (*i.e.*, a shareholder who is in the coalition and by leaving it would prevent the coalition from reaching majority). γ_{if} is the share of such coalitions where she is a swing shareholder, that is,

$$\gamma_{if}(s_{*f}) = \frac{\left| \left\{ T \in 2^N : \sum_{j \in T} s_{jf} \geq 1/2 > \sum_{j \in T \setminus \{i\}} s_{jf} \right\} \right|}{\sum_{k \in N} \left| \left\{ T \in 2^N : \sum_{j \in T} s_{jf} \geq 1/2 > \sum_{j \in T \setminus \{k\}} s_{jf} \right\} \right|}.$$

2.2 The Nash bargaining (NB) strategic plan

I now describe the Nash bargaining strategic plan (NB).

Definition 2. Firm f's strategic plan R_f is NB if there exist a bargaining power function $\beta_{*f} : \Delta^n \to \Delta^n$ and a disagreement payoff function $d_{*f} : \times_{g \neq f} \Delta(A_g) \times S \times \Pi \to \mathbb{R}^n$ such that for every $s \in S$, $\alpha_{-f} \in \times_{g \neq f} \Delta(A_g)$, and $\pi \in \Pi$,

(i) the firm maximizes the Nash product:

$$R_f\left(\alpha_{-f}, s; \pi\right) = \underset{\alpha_f \in B_f(\alpha_{-f}, s; \pi)}{\arg\max} \left\{ \prod_{i \in N_f(\beta_{*f}(s_{*f}))} \left(u_i\left(\alpha_f, \alpha_{-f}, s_{i*}\right) - d_{if}(\alpha_{-f}, s; \pi)\right)^{\beta_{if}(s_{*f})} \right\},$$

where $B_f(\alpha_{-f}, s; \pi) \coloneqq \{\alpha_f \in \Delta(A_f) : u_i(\alpha_f, \alpha_{-f}, s_{i*}) \geq d_{if}(\alpha_{-f}, s; \pi) \; \forall i \in N_f(\beta_{*f}(s_{*f}))\}$ and $N_f(\beta_{*f}(s_{*f})) \equiv \{i \in N : \beta_{if}(s_{*f}) > 0\}$,

(ii) control is exclusive to shareholders: For every $i \in N$, $s_{if} = 0 \implies \beta_{if}(s_{*f}) = 0$.

If the maximum Nash product in (i) is positive for every s, π, and α_{-f}, then we say that the NB strategic plan has strict benefits from agreement.

Remark: Given that by definition, R_f is nonempty-valued, if R_f is NB, $B_f(\alpha_{-f}, s; \pi)$ has to be nonempty.

The analysis goes through if we require that there exist function $\alpha_f^d : \times_{g \neq f} \Delta(A_g) \times S \times \pi \to \Delta(A_f)$ such that for every i, α_{-f}, s, and π, $d_{if}(\alpha_{-f}, s; \pi) = u_i(\alpha_f^d(\alpha_{-f}, s; \pi), \alpha_{-f}, s_{i*})$, where $\alpha_f^d(\alpha_{-f}, s; \pi)$ is the strategy chosen by firm f in case of disagreement when the other firms' strategies are α_{-f}, the ownership matrix is s, and the profit functions are π. It is reasonable for firm f's strategy in case of disagreement to depend on α_{-f}, s, and π. For example, if firm f's competitors produce large quantities or set low prices driving f's residual demand down, $\alpha_f^d(\alpha_{-f}, s; \pi)$ will probably reflect that f should not produce a lot in case of disagreement. Similarly, $\alpha_f^d(\alpha_{-f}, s; \pi)$ should reflect market conditions π. Section 4.2 discusses potential formulations of $\alpha_f^d(\alpha_{-f}, s; \pi)$.

3 Properties of strategic plans

This section discusses properties of strategic plans. First, it shows that assuming a strategic plan is NB is almost equivalent to requiring that the strategic plan satisfy a form of Pareto efficiency. Assuming that a strategic plan is WAPP also implies that the strategic plan must satisfy this efficiency condition but it also imposes an additional restriction: The distribution of power across shareholders within the firm must be independent of external factors such as (i) the other firms' ownership structures, (ii) the other firms' strategies, and (iii) market conditions (e.g., market demand or production technology). Second, it proposes two monotonicity properties capturing the notion that "more shares should lead to more control," and characterizes when WAPP and NB strategic plans satisfy those properties. Third, it characterizes a class of WAPP strategic plans where

$\gamma_{if}(s_{*f}) = \delta(s_{if})/\sum_{j\in N}\delta(s_{jf})$ for some real function δ, as in Backus et al. (2021a) and Antón et al. (2023).

3.1 Efficiency and internal consistency

A strategic plan is efficient if under any ownership structure, there is a subset $\widetilde{N}(s_{*f})$ of the shareholders of firm f who efficiently control the firm. Strong efficiency requires that for any strategy profile of the other firms, firm f never responds with a weakly Pareto dominated strategy—in the sense that another strategy could do at least as well for all controlling shareholders and strictly better for at least one of them. Weak efficiency requires that for any strategy profile of the other firms, firm f never responds with a strongly Pareto dominated strategy.

Definition 3. The strategic plan R_f of firm f is strongly (resp. weakly) efficient if there exists correspondence $\widetilde{N} : \Delta^n \rightrightarrows N$ such that for every $s \in S$,

(i) a nonempty set $\widetilde{N}(s_{*f})$ of shareholders control the firm,

(ii) control is exclusive to shareholders: For every $i \in N$, $s_{if} = 0 \implies i \notin \widetilde{N}(s_{*f})$, and

(iii) the firm is efficiently controlled: For every $\pi \in \Pi$ and $\alpha_{-f} \in \times_{g\neq f}\Delta(A_g)$, there do not exist $\alpha'_f \in \Delta(A_f)$ and $\alpha_f \in R_f(\alpha_{-f}, s; \pi)$ such that $u_i(\alpha'_f, \alpha_{-f}, s_{i*}) \geq u_i(\alpha_f, \alpha_{-f}, s_{i*})$ for all $i \in \widetilde{N}(s_{*f})$ with at least one (resp. every) inequality strict.

Firm f's strategic plan R_f is internally consistent if, in addition, for every $\pi \in \Pi$, $s \in S$, $\alpha_{-f} \in \times_{g\neq f}\Delta(A_g)$, $\alpha_f, \alpha'_f \in R_f(\alpha_{-f}, s; \pi)$, and $\alpha''_f \in \Delta(A_f)$

(iv) $u_i(\alpha_f, \alpha_{-f}, s_{i*}) = u_i(\alpha'_f, \alpha_{-f}, s_{i*})$ for all $i \in \widetilde{N}(s_{*f})$, and

(v) if $u_i(\alpha_f, \alpha_{-f}, s_{i*}) = u_i(\alpha''_f, \alpha_{-f}, s_{i*})$ for all $i \in \widetilde{N}(s_{*f})$, then $\alpha''_f \in R_f(\alpha_{-f}, s; \pi)$.

Internal consistency requires that the firm's strategic plan prescribes strategies that are unique up to payoff-equivalent strategies. With Pareto-dominated strategies already ruled out by efficiency, condition (iv) requires that firm f's controlling shareholders not be willing to agree to two different strategies α_f and α'_f when α_f is strictly preferred to α'_f by one controlling shareholder and α'_f is strictly preferred to α_f by another shareholder. In that case, it does not make sense that each of the two controlling shareholders is willing to agree to both policies. Also, if firm f's controlling shareholders are willing to agree to strategy α_f, condition (v) requires that they also be willing to agree to any other strategy that delivers the same portfolio profit to each one of them.

Let $\mathcal{U}_f(\alpha_{-f}, s; \pi) := \{v \in \mathbb{R}^{|N_f(s_{*f})|} : \exists \alpha_f \in \Delta(A_f) \text{ such that } u_i(\alpha_f, \alpha_{-f}, s_{i*}) = v_i$ for every $i \in N_f(s_{*f})\}$ denote the (convex) portfolio profit possibility set of the shareholders of firm f when the other firms' strategies are α_{-f}, the ownership matrix is s, and the profit functions are π. Proposition 1 studies the efficiency and internal consistency properties of WAPP and NB strategic plans.

Proposition 1. Consider a firm $f \in M$.

(i) If R_f is WAPP, then it is strongly efficient.

(ii) If R_f is NB, then it is weakly efficient.

(iii) If R_f is NB with strict benefits from agreement, then it is strongly efficient and internally consistent.

(iv) If R_f is strongly efficient and internally consistent, then it is NB.

(v) Assume $\mathcal{U}_f(\alpha_{-f}, s; \pi)$ is strictly convex for every s, α_{-f}, and π. If R_f is WAPP or NB, then it is strongly efficient and internally consistent.

Parts (ii)-(iv) show that assuming a strategic plan is NB is approximately equivalent to assuming it is efficient and internally consistent. When $\mathcal{U}_f$ is strictly convex, the class of WAPP strategic plans is a subset of the class of NB strategic plans, which coincides with the class of strongly efficient and internally consistent strategic plans. Without assuming $\mathcal{U}_f$ is strictly convex, from parts (i) and (iv), it follows that the class of WAPP and internally consistent strategic plans is a subset of the class of NB strategic plans. The only case where R_f can be WAPP but not NB is when $\mathcal{U}_f(\alpha_{-f}, s; \pi)$ is only weakly convex and R_f is *not* internally consistent. In that case, for some $(\alpha_{-f}, s; \pi)$, a WAPP R_f can choose all strategies that result in portfolio profit profiles of firm f's controlling shareholders across a linear segment of the boundary of $\mathcal{U}_f(\alpha_{-f}, s; \pi)$, which an NB R_f cannot do. Therefore, barring a small and arguably uninteresting class of internally inconsistent WAPP strategic plans when $\mathcal{U}_f$ is only weakly convex, the class of WAPP strategic plans is a subset of the class of NB strategic plans.

3.2 (Ir)relevance of external factors

Any model of firm conduct must capture the fact that the distribution of power across a firm's shareholders depends on factors *internal* to firm f. Particularly, the number of shares held by each shareholder must play an important role: Larger shareholders can be expected to have greater power in shaping firm conduct than smaller ones. Indeed, NB and WAPP can capture this given that γ_{*f} and β_{*f} depend on s_{*f}.

At the same time, the distribution of power across firm f shareholders may depend also on *external* factors such as (i) the stakes of firm f shareholders in competing firms, (ii) the strategies of other firms, and (iii) market conditions (e.g., demand or production technology). First, firm f shareholders with larger stakes in competitors may pay closer attention to the market and thus exert more control over firm f given that firm f's strategy affects the shareholders' portfolio returns not only through its effect on its own profitability but also through its impact on other competing firms' profits. Indeed, there is theoretical and empirical support that investors pay closer attention to a stock when that stock is a larger part of their portfolios (Van Nieuwerburgh and Veldkamp, 2010; Fich et al., 2015; Iliev et al., 2021). Relatedly, Gilje et al. (2020) argue that the extent to which common ownership affects firm conduct depends on whether and which shareholders are attentive to firm conduct, which is sensitive to the weight of said firm in each shareholder's portfolio. Second, firm f shareholders who also have shares of competing firm g may have stronger incentives to exert control over firm f when firm g's strategy is such that firm f has a lot of room to influence g's profitability. In other words, firm g's strategy can affect the stakes in firm f's strategy of a firm f shareholder who is also a shareholder of firm g. For instance, if firm g expands its production capacity and orders large input quantities to scale up production, firm f's effect through its pricing and production strategy on firm g's profit margin will have a large impact on firm g's profits. On the other hand, if firm g scales back production or even exits a market that firm f operates in, firm f has limited room to affect firm g's profitability. Third, firm f shareholders who also have shares of competing firm g may have stronger incentives to affect firm f's strategy when firm f and g's goods are strong substitutes, since in that case firm f's strategy will have a more pronounced impact on firm g's profits.

WAPP precludes such dependence of the distribution $\gamma_{*f}(s_{*f})$ of power across a firm's shareholders on *external* factors, while NB allows for it. To see this, let A_f be a subset of a Euclidean space with $R_f(\alpha_{-f}, s; \pi)$ pinned down by the first order condition (FOC). The FOC under WAPP is $\sum_{i \in N} \gamma_{if}(s_{*f}) \partial u_i(a_f, a_{-f}, s_{i*})/\partial a_f|_{a_f = R_f(\alpha_{-f}, s; \pi)} = \mathbf{0}$, where $\partial u_i(a_f, a_{-f}, s_{i*})/\partial a_f$ is the gradient with respect to a_f. Under NB, the FOC is $\sum_{i \in N_f(\beta_{*f}(s_{*f}))} \widetilde{\gamma}_{if}(a_{-f}, s; \pi) \partial u_i(a_f, a_{-f}, s_{i*})/\partial a_f|_{a_f = R_f(\alpha_{-f}, s; \pi)} = \mathbf{0}$, where

$$\widetilde{\gamma}_{if}(\alpha_{-f}, s; \pi) := \frac{\frac{\beta_{if}(s_{*f})}{u_i\left(R_f(\alpha_{-f}, s; \pi), \alpha_{-f}, s_{i*}\right) - d_{if}(\alpha_{-f}, s; \pi)}}{\sum_{j \in N_f(\beta_{*f}(s_{*f}))} \frac{\beta_{jf}(s)}{u_j\left(R_f(\alpha_{-f}, s; \pi), \alpha_{-f}, s_{j*}\right) - d_{jf}(\alpha_{-f}, s; \pi)}}$$

is the disagreement-adjusted control power of shareholder i over firm f at $(\alpha_{-f}, s; \pi)$. It measures shareholder control accounting for the fact that the further $u_i(\alpha_f, \alpha_{-f}, s_{i*})$ is from $d_{if}(\alpha_{-f}, s; \pi)$, the more shareholder i has to lose in case of disagreement. NB allows for $\widetilde{\gamma}_{*f}(\alpha_{-f}, s; \pi)$ to depend on the other firms' strategies or ownership structures and on market conditions, since shareholders' portfolio profits can depend on those—be it in case of agreement or disagreement. Equivalently, we can write

$$\left.\frac{\partial \pi_f(a_f, \alpha_{-f})}{\partial a_f}\right|_{a_f = R_f(\alpha_{-f}, s; \pi)} + \sum_{g \in M \setminus \{f\}} \widetilde{\lambda}_{fg}(\alpha_{-f}, s) \left.\frac{\partial \pi_g(a_f, \alpha_{-f})}{\partial a_f}\right|_{a_f = R_f(\alpha_{-f}, s; \pi)} = \mathbf{0},$$

where $\widetilde{\lambda}_{fg}(\alpha_{-f}, s; \pi) := \sum_{i \in N_f(\beta_{*f}(s_{*f}))} \widetilde{\gamma}_{if}(\alpha_{-f}, s; \pi) s_{ig} / \sum_{i \in N_f(\beta_{*f}(s_{*f}))} \widetilde{\gamma}_{if}(\alpha_{-f}, s; \pi) s_{if}$ is the weight firm f places on firm g's profit.

In fact, the irrelevance of external factors for the distribution of power across shareholders within the firm is what characterizes WAPP as a special case of NB. To see this, define a generalized weighted average portfolio profit strategic plan (GWAPP) as follows.

Definition 4. Firm f's strategic plan R_f is GWAPP if there exists a control power function $\gamma_{*f} : \times_{g \neq f} \Delta(A_g) \times S \times \Pi \to \Delta^n$ such that for every $s \in S$, $\alpha_{-f} \in \times_{g \neq f} \Delta(A_g)$, and $\pi \in \Pi$,

(i) the firm maximizes the weighted average portfolio profit of its shareholders:

$$R_f\,(\alpha_{-f}, s; \pi) = \operatorname*{arg\,max}_{\alpha_f \in \Delta(A_f)} \left\{ \sum_{i \in N} \gamma_{if}(\alpha_{-f}, s; \pi) u_i(\alpha_f, \alpha_{-f}, s_{i*}) \right\},$$

(ii) control is exclusive to shareholders: For every $i \in N$, $s_{if} = 0 \implies \gamma_{if}(\alpha_{-f}, s; \pi) = 0$,

(iii) external factors can influence a shareholder's magnitude of control power but not whether she has control power or not: For every $s' \in S$, $\alpha'_{-f} \in \times_{g \neq f} \Delta(A_g)$, and $\pi' \in \Pi$ such that $s_{*f} = s'_{*f}$, $N_f(\gamma_{*f}(\alpha_{-f}, s; \pi)) = N_f(\gamma_{*f}(\alpha'_{-f}, s'; \pi'))$

Proposition 2 shows that, barring a class of internally inconsistent GWAPP strategic plans when $\mathcal{U}_f(\alpha_{-f}, s; \pi)$ is only weakly convex, the class of GWAPP strategic plans is a subset of the class of NB strategic plans. When $\mathcal{U}_f(\alpha_{-f}, s; \pi)$ is strictly convex, the class of GWAPP strategic plans coincides with the class of NB strategic plans. An advantage of the NB formulation over GWAPP is that it can be more natural to specify firm f's strategy in case of disagreement—and thereby indirectly specify how s, α_{-f}, and π affect γ_{*f}.

Proposition 2. Consider a firm $f \in M$.

(i) If R_f is GWAPP, then it is strongly efficient.

(ii) Assume that $\mathcal{U}_f(\alpha_{-f}, s; \pi)$ is strictly convex for every s, α_{-f}, and π. Then, R_f is GWAPP if and only if it is NB.

Proposition 2 shows that assuming a strategic plan is GWAPP is approximately equivalent to assuming it is NB. When $\mathcal{U}_f$ is strictly convex, the class of GWAPP strategic plans coincides with the class of NB strategic plans. Without assuming $\mathcal{U}_f$ is strictly convex, from part (i) of Proposition 2 and part (iv) of Proposition 1, it follows that the class of GWAPP and internally consistent strategic plans is a subset of (and almost equivalent to) the class of NB and internally consistent strategic plans. The only difference is that NB and internally consistent strategic plans can be only weakly strongly efficientm while GWAPP strategic plans are strongly efficient.

Restricting attention to WAPP strategic plans essentially restricts attention to GWAPP strategic plans with $\gamma_{*f}(\alpha_{-f}, s; \pi)$ independent of α_{-f}, π, and s_{*g} for $g \neq f$. This restriction can rule out some natural benchmark strategic plans. For example, let A_f be a real interval, constrain attention to pure strategies, and assume $u_i(a_f,a_{-f},s_{i*})$ is single-peaked in a_f. Under majority voting, $R_f(a_{-f},s;\pi)$ is the median shareholder's most preferred strategy, as in Chiappinelli et al. (2023).[5] This R_f is strongly efficient and internally consistent, since it satisfies the conditions of Definition 3 for $\widetilde{N}(s_{*f}) = N_f(s_{*f})$. Therefore, by Proposition 1, it is NB. However, it is not WAPP.

Here is why. Assume by contradiction that R_f is WAPP, and fix some a_{-f} and π. First, observe that for every s and i, $\gamma_{if}(s_{*f}) = 0$ if i is *not* a median shareholder. To see this, take any s such that not all firm f shareholders have the same most preferred strategy and any $i \in N_f(s_{*f})$ that is not a median shareholder. There exists s' with $s_{j*} = s'_{j*}$ for every $j \in N_f(s_{*f}) \setminus \{i\}$, $s_{if} = s'_{if}$, and $s_{i*} \neq s'_{i*}$, such that the preferences of the median firm f shareholder under s are the same as under s' and thus $R_f(a_{-f},s;\pi) = R_f(a_{-f},s';\pi)$, but i's preferences under s_{i*} are different from her preferences under s'_{i*}.[6] $R_f(a_{-f},s;\pi) = R_f(a_{-f},s')$ implies

$$\arg\max_{a_f \in A_f} \left\{ \sum_{j \in N_f(s_{*f})} \gamma_{jf}(s_{*f}) u_j(a_f, a_{-f}, s_{j*}) \right\} = \arg\max_{a_f \in A_f} \left\{ \sum_{j \in N_f(s'_{*f})} \gamma_{jf}(s'_{*f}) u_j(a_f, a_{-f}, s'_{j*}) \right\}$$

or equivalently, given that $s_{*f} = s'_{*f}$ and $s_{j*} = s'_{j*}$ for every $j \in N_f(s_{*f}) \setminus \{i\}$,

$$\begin{aligned} &\arg\max_{a_f \in A_f} \left\{ \sum_{j \in N_f(s_{*f})} \gamma_{jf}(s_{*f}) u_j(a_f, a_{-f}, s_{j*}) \right\} \\ &= \arg\max_{a_f \in A_f} \left\{ \gamma_{if}(s_{*f}) u_i(a_f, a_{-f}, s'_{i*}) + \sum_{j \in N_f(s_{*f}) \setminus \{i\}} \gamma_{jf}(s_{*f}) u_j(a_f, a_{-f}, s_{j*}) \right\}, \end{aligned}$$

which holds if and only if $\gamma_{if}(s_{*f}) = 0$. We have thus shown that for every s and i, $\gamma_{if}(s_{*f}) = 0$ if i is *not* a median shareholder.

Now, notice that there exist s and s' such that $s_{*f} = s'_{*f}$ and $M_f(s) \cap M_f(s') = \emptyset$, where $M_f(s)$ is the set of median shareholders under s and $M_f(s')$ is the set of median shareholders under s'. For example, there can be a unique median shareholder under s and a different unique median shareholder under s' although $s_{*f} = s'_{*f}$. We have then that for every $i \notin M_f(s)$, $\gamma_{if}(s_{*f}) = 0$, and for every $i \notin M_f(s')$, $\gamma_{if}(s'_{*f}) = 0$, which given that $s_{*f} = s'_{*f}$, implies $\gamma_{if}(s_{*f}) = 0$ for every $i \in N_f(s_{*f})$, a contradiction.

We conclude that the strategic plan resulting from majority voting induces a distribution of power across shareholders within the firm that depends on external factors. Indeed,

[5]More than 90% of S&P 500 companies follow a majority voting standard for uncontested director elections (see https://www.conference-board.org/publications/corporate-board-practices-2021-edition).

[6]For this and following arguments, some minimal conditions are required that make π evaluated at a_{-f} non-trivial. For example, under minimal conditions on π, i's preferences under s_{i*} are indeed different from her preferences under s'_{i*} for some $s'_{i*} \neq s_{i*}$.

if in the argument above, γ_{*f} was allowed to depend on the whole ownership matrix s, no contradiction would arise. Similar arguments can be made when instead of varying s, we vary a_{-f} or π. The strategic plan resulting from majority voting is GWAPP with $\gamma_{*f}(a_{-f},s;\pi)$ such that $\sum_{i\in M_f(a_{-f},s;\pi)}\gamma_{if}(a_{-f},s;\pi)=1$, where $M_f(a_{-f},s;\pi)$ is the set of median shareholders.

3.3 Monotonicity

In this section, I study monotonicity properties of strategic plans capturing the notion that "more shares should lead to more control." Before proceeding, we need to develop a language to talk about stock trades. For every shareholder $i\in N_f(s_{*f})$ of firm f, $(\lambda_{i;f1},\lambda_{i;f2},\cdots,\lambda_{i;fm})\equiv\lambda_{i;f*}:=\frac{1}{s_{if}}s_{i*}\equiv(s_{i1}/s_{if},s_{i2}/s_{if},\ldots,s_{im}/s_{if})$, is the vector of weights i wants firm f to place on firms' profits with the weight on firm f's profit normalized to 1.

Definition 5. Firm f shareholders unanimously agree on firm conduct under ownership matrix s if $\lambda_{i;f*}=\lambda_{j;f*}$ for every $i,j\in N_f(s_{*f})$. Then, s is called f-unanimous.

For simplicity, the remainder of section 3, let A_f be an interval and restrict attention to pure strategies. Also, fix some vector of profit functions $\pi\in\Pi$ such that for every f-unanimous s and $g\neq f$, there exists $a_{-f}\in A_{-f}$ such that $\partial\pi_g(a_f,a_{-f})/\partial a_f\neq 0$ evaluated at $a_f=R_f(a_{-f},s;\pi)$. This assumption requires a minimal level of externalities between firm f's strategy and other firms' profits. Without such externalities, firm f's shareholders' preferences over firm f's conduct would not depend on their stakes in other firms. To economize on notation, we suppress the dependence of all objects on π (e.g., by writing $R_f(a_{-f},s)$ instead of $R_f(a_{-f},s;\pi)$). Whether R_f is WAPP or NB, assume that for every a_{-f} and f-unanimous s, the firm's objective is continuously differentiable with $R_f(a_{-f},s)$ pinned down by the FOC and the second-order condition holding strictly.[7] Also, assume that under NB, there exists function $\alpha_f^d:\times\Delta(A_g)_{g\neq f}\times S\to\Delta(A_f)$ such that for every a_{-f}, s, and $i\in N$, $d_{if}(a_{-f},s)=u_i(\alpha_f^d(a_{-f},s),a_{-f},s_{i*})$, where $\alpha_f^d(a_{-f},s)$ is the strategy chosen by firm f in case of disagreement.

We will see that studying the firm's strategic plan locally, around f-unanimous matrices, is a powerful approach for deriving intuitive conditions on the firm's objective function. Starting from an f-unanimous matrix, a small stock trade where some firm f shareholders trade firm $g\neq f$ shares can cause firm f to adjust its strategy only through its effect on firm f shareholders' interests. Even if the stock trade affects the distribution of power across firm f shareholders (which is possible if firm f's strategic plan is NB), this will not play a role in how firm f adjusts its strategy in response to the stock trade: Given

[7] That is, under WAPP, $\sum_{i\in N}\gamma_{if}(s_{*f})\partial u_i(a_f,a_{-f},s_{i*})/\partial a_f|_{a_f=R_f(\alpha_{-f},s)}$ is continuously differentiable in (a_f,s). Under NB, $\sum_{i\in N_f(\beta_{*f}(s_{*f}))}\widetilde{\gamma}_{if}(a_{-f},s)\partial u_i(a_f,a_{-f},s_{i*})/\partial a_f|_{a_f=R_f(\alpha_{-f},s;\pi)}$ is continuously differentiable in (a_f,s).

that firm f shareholders unanimously agree on firm strategy to begin with, changes in the distribution of power among them are inconsequential. Simply put, by studying firm f's strategic plan around f-unanimous matrices allows us to keep all else fixed while only varying the "size" of each firm f shareholder.

Definition 6. A $(\psi, g, i, \widetilde{N})$-stock trade is an infinitesimal change in the ownership structure matrix s in direction

$$ds = \left((1-\psi)\mathbf{e}_i - \psi \sum_{j \in \widetilde{N}} \mathbf{e}_j \right) \otimes \mathbf{e}_g,$$

where $\psi \in [0,1]$, $g \in M$, $i \in N \setminus \widetilde{N}$, and $\emptyset \neq \widetilde{N} \subset N$.

In a $(\psi, g, i, \widetilde{N})$-stock trade, shareholder i buys firm g shares at rate $1-\psi$, and each shareholder in group $\widetilde{N}$ of shareholders sells firm g shares at rate ψ.

Strictly put, this is possible only when $\psi|\widetilde{N}| = 1-\psi$, or equivalently, $\psi = (|\widetilde{N}|+1)^{-1}$, so that ds points inside S. $\psi \neq (|\widetilde{N}|+1)^{-1}$ can be interpreted as follows: There are some additional firm f shareholders outside the set N without control power over firm f (e.g., no voting rights). When $\psi < (|\widetilde{N}|+1)^{-1}$, these additional shareholders sell firm g shares to shareholder i at rate $1 - \psi(|\widetilde{N}|+1)$. When $\psi > (|\widetilde{N}|+1)^{-1}$, these additional shareholders buy firm g shares from group $\widetilde{N}$ of shareholders at rate $\psi(|\widetilde{N}|+1) - 1$.[8]

3.3.1 Rank preservation

When $\widetilde{N} = \{j\}$ is a singleton, we call the stock trade a (ψ, g, i, j)-stock trade. In a $(1/2, g, i, j)$-stock trade, shareholder i buys firm g shares from shareholder j. We are now ready to study the first monotonicity property: *rank preservation*.

Definition 7. Firm f's strategic plan is rank-preserving if for any firm $g \neq f$, any strategy profile a_{-f} of the other firms, any f-unanimous ownership matrix s, and any pair of distinct shareholders $i, j \in N_f(s_{*f})$, if $s_{if} \geq s_{jf}$, then

$$\left. \frac{\partial \pi_g(a_f, a_{-f})}{\partial a_f} \right|_{a_f = R_f(a_{-f}, s)} \overset{(\text{resp. } \leq)}{\geq} 0 \implies \nabla_{ds} R_f(a_{-f}, s) \overset{(\text{resp. } \leq)}{\geq} 0.$$

Here is the intuition behind this definition. Starting from an f-unanimous ownership structure, consider a stock trade between two shareholders i and j of firm f in which i buys

[8] Alternatively, the analysis goes through if (i) we allow the number of investors n to vary and require that $R_f(a_{-f}, s) = R_f(a_{-f}, s')$ for every a_{-f}, s, s' such that $N_f(s_{*f}) = N_f(s'_{*f})$ and $s_{i*} = s'_{i*}$ for every shareholder $i \in N_f(s_{*f})$ and (ii) interpret $\psi \neq (|\widetilde{N}|+1)^{-1}$ as follows: Shareholders in $N_g(s_{*g}) \setminus N_f(s_{*f})$, who exert no control over firm f, participate in the stock trade to ensure ds points inside S. Notice that given any s_{*f}, we can construct $(s_{*g})_{g \neq f}$ such that s is f-unanimous and $N_g(s_{*g}) \setminus N_f(s_{*f}) \neq \emptyset$. However, this would complicate notation.

shares of another firm $g \neq f$ from j. Before the stock trade, firm f's strategy maximizes the portfolio profit of each of its shareholders. The stock trade causes misalignment among firm f shareholders' interests: Shareholder i wants firm f to adjust its strategy in the direction benefiting firm g, while shareholder j wants firm f to adjust its strategy in the opposite direction.[9] For example, under Bertrand competition with differentiated products, i will want firm f to price less aggressively while j will want it to price more aggressively. Firm f's strategic plan is rank-preserving if, in response to the stock trade, firm f adjusts its strategy in the direction preferred by the larger shareholder involved in the trade.

Proposition 3 characterizes rank-preserving strategic plans.

Proposition 3. Consider a firm $f \in M$.

(i) Assume that R_f is WAPP. Then, R_f is rank-preserving if and only if for every s_{*f} and every pair of firm f shareholders $i,j \in N_f(s_{*f})$, $s_{if} \geq s_{jf} \implies \gamma_{if}(s_{*f}) \geq \gamma_{jf}(s_{*f})$.

(ii) Assume that R_f is NB. Then, R_f is rank-preserving if and only if for every s_{*f} and every pair of firm f shareholders $i,j \in N_f(s_{*f})$, $s_{if} \geq s_{jf} \implies \beta_{if}(s_{*f})/s_{if} \geq \beta_{jf}(s_{*f})/s_{jf}$.

If firm f's strategic plan is WAPP, it is rank-preserving if and only if the control power function γ_{*f} preserves the ranking of shareholders in terms of the number of firm f shares they hold. One may instinctively think that under NB, $s_{if} \geq s_{jf}$ implying $\beta_{if}(s_{*f}) \geq \beta_{jf}(s_{*f})$ would be sufficient to capture the idea that larger shareholders exercise more control. However, this is not the case. For example, if $\beta_{if}(s_{*f}) = s_{if}$ for every shareholder i, then every shareholder has the same control power over firm f in terms of how firm f's strategy changes in response to stock trades around an f-unanimous ownership matrix. Under NB, the role of γ_{if} is assumed by β_{if}/s_{if}, *not* β_{if}. The division by s_{if} captures the fact that larger shareholders have more to lose in case of disagreement. Therefore, larger shareholders have more control over firm f than smaller ones if and only if their β's more than compensate for the fact that they have more to lose.

3.3.2 Stock-trade monotonicity

Before defining the second monotonicity property, *stock-trade monotonicity*, we need to define f-neutral stock trades. An f-neutral stock trade does not make firm f want to change its strategy.

[9] More generally, when R_f is neither WAPP nor NB, firm f may not maximize the portfolio profit of each of its shareholders even when their interests are perfectly aligned (i.e., s is f-unanimous). Still, the stock trade causes shareholder i (resp. j) to be more (resp. less) willing to accept an adjustment in firm f's strategy in the direction benefiting firm g.

Definition 8. Fix an f-unanimous ownership matrix s. Starting from s, a $(\psi,g,i,\widetilde{N})$-stock trade is f-neutral if for any strategy profile a_{-f} of the other firms, firm f's conduct does not change in response to the stock trade, that is, $\nabla_{ds}R_f(a_{-f},s) = 0$.

In an f-neutral $(\psi,g,i,\widetilde{N})$-stock trade, ψ captures the control power of shareholder i over firm f relative to the collective control power over firm f of group $\widetilde{N}$ of shareholders. When ψ is high, the stock trade leads to large decrease in the stakes of group $\widetilde{N}$ of shareholders in firm g and only a small increase in shareholder i's stake in firm g. Consequently, shareholder i has a *mild* desire for firm f to adjust its strategy in the direction benefiting firm g, while group $\widetilde{N}$ of shareholders have a *strong* desire for firm f to adjust its strategy in the opposite direction. The fact that firm f's conduct remains unchanged reveals that shareholder i exerts a lot of control over firm f relative to the control exerted by group $\widetilde{N}$ of shareholders. $\psi = 1$ implies that group $\widetilde{N}$ of shareholders exerts no control over firm f. $\psi = 0$ means that shareholder i exerts no control over firm f.[10]

Lemma 1 characterizes two types of f-neutral stock trades: (i) those where a firm f shareholder buys firm g shares and every other firm f shareholder sells firm g shares and (ii) those where a firm f shareholder buys firm g shares and another firm f shareholder sells firm g shares.

Lemma 1. Starting from an f-unanimous ownership matrix s,

(i) if R_f is WAPP,

  (a) a $(\psi,g,i,N_f(s_{*f}) \setminus \{i\})$-stock trade is f-neutral if and only if $\gamma_{if}(s_{*f}) = \psi$, and

  (b) a (ψ,g,i,j)-stock trade is f-neutral if and only if $\gamma_{if}(s_{*f})(1-\psi) = \gamma_{jf}(s_{*f})\psi$,

(ii) if R_f is NB,[11]

  (a) a $(\psi,g,i,N_f(s_{*f}) \setminus \{i\})$-stock trade is f-neutral if and only if

$$\frac{\beta_{if}(s_{*f})/s_{if}}{\sum_{j\in N_f(s_{*f})}\beta_{jf}(s_{*f})/s_{jf}} = \psi, \text{ and}$$

  (b) a (ψ,g,i,j)-stock trade is f-neutral if and only if

$$\frac{\beta_{if}(s_{*f})}{s_{if}}(1-\psi) = \frac{\beta_{jf}(s_{*f})}{s_{jf}}\psi.$$

Under WAPP, γ_{if} captures shareholder i's control power over firm f in the following way: If shareholder i's stake in firm g increase at rate $(1-\gamma_{if})$, while the stake in firm g of every other shareholder of firm f decreases at rate γ_{if}, firm f's conduct does not

[10]When neither i nor group $\widetilde{N}$ exert any control over firm f, a $(\psi,g,i,\widetilde{N})$-stock trade is f-neutral for any $\psi \in [0,1]$.

[11]In the statements below, for any $k \notin N_f(s_{*f})$, let $\beta_{kf}(s_{*f})/s_{kf} = 0$.

change. In this scenario, when γ_{if} is high, the other shareholders decrease their interests in firm g by a lot, which should push firm f to adjust its strategy opposite the direction that would enhance firm g's profits. However, a small increase in shareholder i's interests in firm g counteracts this effect, leaving firm f's conduct unchanged. This means that shareholder i exercises a lot of control over firm f. Given Proposition 3, one can expect that in Proposition 1, the role of γ_{if} under WAPP is assumed by β_{if}/s_{if} under NB. Indeed, this is the case with one additional caveat: β_{if}/s_{if} has to be normalized. Under NB, $\beta_{if}/s_{if}/(\sum_{j\in N_f(s_{*f})}\beta_{jf}/s_{jf})$ captures shareholder i's control power over firm f. As mentioned before, the division by s_{if} captures the fact that larger shareholders have more to lose in case of disagreement. Similarly, under WAPP, the ratio γ_{if}/γ_{jf} captures shareholder i's control power over firm f relative to shareholder j's control power over it. Under NB, the corresponding ratio is $\beta_{if}/s_{if}/(\beta_{jf}/s_{jf})$.

Having defined f-neutral stock trades, we are ready to define stock-trade monotone strategic plans.

Definition 9. Firm f's strategic plan has stock-trade monotone control if for any firm $g \neq f$, any pair of shareholders $i,j \in N$, any f-unanimous ownership matrix s, any $t \in [0, \min_{g\in M: s_{ig}>0} \min\{(1-s_{ig})/s_{ig}, s_{jg}/s_{ig}\}]$,[12] and any $\psi,\psi' \in [0,1]$, if

(i) starting from s, a $(\psi, g, i, N_f(s_{*f}) \setminus \{i\})$-stock trade is f-neutral, and

(ii) starting from s', a $(\psi', g, i, N_f(s'_{*f}) \setminus \{i\})$-stock trade is f-neutral,

then $\psi' \geq \psi$, where $s'_{k*} := s_{k*}$ for every $k \neq i,j$, $s'_{i*} := (1+t)s_{i*}$, and $s'_{j*} := s_{j*} - ts_{i*}$.

Starting from an f-unanimous ownership structure, consider a stock trade where shareholder i buys shares from shareholder j, thereby growing her stake in *every* firm by $t \times 100\%$. Firm f's strategic plan is stock-trade monotone if, in response, shareholder i's control power over firm f—as measured through an f-neutral stock trade between her and every other shareholder of firm f—increases. Stock-trade monotonicity is consistent with the idea that the more firm f shares shareholder i has, the more control she exerts over firm f. It is also consistent with the idea that the more shares i has of *other* firms in the industry, the closer attention she will pay to the industry and, thus, the more influence she will have over firm f's strategy, as discussed in section 3.2.

Proposition 4 characterizes stock-trade monotone strategic plans.

Proposition 4. Consider a firm $f \in M$.

(i) Assume that R_f is WAPP. Then, R_f has stock-trade monotone control if and only if for every s, every pair of firm f shareholders $i,j \in N_f(s_{*f})$, and every $t \in [0, \min\{s_{jf}, 1-s_{if}\}]$, $\gamma_{if}(s_{*f} + t(\mathbf{e}_i - \mathbf{e}_j)) \geq \gamma_{if}(s_{*f})$.

[12]The constraint on t guarantees that $s'_{i*} \leq (1,1,\ldots,1)$ and $s'_{j*} \geq (0,0,\ldots,0)$.

(ii) Assume that R_f is NB. Then, R_f has stock-trade monotone control if and only if for every s, every pair of firm f shareholders $i,j \in N_f(s_{*f})$, and every $t \in [0, \min\{s_{jf}, 1 - s_{if}\}]$,

$$\frac{\beta_{if}(s'_{*f})/s'_{if}}{\sum_{k\in N_f(s_{*f})} \beta_{kf}(s'_{*f})/s'_{kf}} \geq \frac{\beta_{if}(s_{*f})/s_{if}}{\sum_{k\in N_f(s_{*f})} \beta_{kf}(s_{*f})/s_{kf}},$$

where $s'_{*f} \coloneqq s_{*f} + t(\mathbf{e}_i - \mathbf{e}_j)$.

As anticipated given Lemma 1, under WAPP, shareholder i's control over firm f increases when she buys firm f shares if γ_{if} increases. Under NB, the role of γ_{if} is assumed by $\beta_{if}/s_{if}/\sum_{k\in N_f(s_{*f})} \beta_{kf}/s_{kf}$.

3.4 Independence of irrelevant shareholders

In this section, I constrain s to lie in $\{s \in S : |N_f(s_{*f})| \geq 3\}$. This constraint hardly limits the scope of the analysis given that in practice, if a firm f has some shareholders who also have stakes in competitors, then most of the time—if not always—firm f has at least three shareholders.

I characterize the generalization of WAPP with proportional control posing that there exists δ such that $\gamma_{if}(s_{*f}) = \delta(s_{if})/\sum_{j\in N} \delta(s_{jf})$. Some potentially restrictive implications of this generalization of WAPP with proportional control—and thus of WAPP with proportional control itself—are easy to see but do not meaningfully assist us in evaluating the plausibility of the formulation. For example, assuming $\gamma_{if}(s_{*f}) = \delta(s_{if})/\sum_{j\in N} \delta(s_{jf})$ does not allow for the firm's strategic plan to maximize the portfolio profit of the absolute majority shareholder when such shareholder exists. However, most publicly traded firms have no absolute majority shareholder. A characterization of this formulation of WAPP is thus useful for fully evaluating the restrictions it imposes. Proposition 5 shows that WAPP admits such a representation if and only if it satisfies three conditions.

The first condition is anonymity, which requires that the identity of shareholders not matter for firm strategy. Namely, permuting the ownership matrix s does not change the firm's strategic plan. For example, if all of shareholder i's shares across all firms are transferred to shareholder j, and all of shareholder j's shares are transferred to shareholder i, the firm will choose the same strategy as it did before the transfer.

Definition 10. A permutation matrix is an $n \times n$ matrix where (i) each row has exactly one entry of 1, (ii) each column has exactly one entry of 1, and (iii) all other entries 0.

Definition 11. Firm f's strategic plan is anonymous if for any s, a_{-f}, and permutation matrix P, $R(a_{-f},s) = R(a_{-f},Ps)$.

The second condition is inclusivity, which requires that every shareholder of a firm exert some control over the firm. Remember that in an f-neutral (ψ,g,i,j)-stock trade, ψ

captures the control power of shareholder i over firm f relative to shareholder j's control over firm f. $\psi = 0$ (resp. $\psi = 1$) means that i (resp. j) has no control over firm f.

Definition 12. Firm f's strategic plan is inclusive if for any firm $g \neq f$, any f-unanimous ownership matrix s, and any pair of firm f shareholders $i,j \in N_f(s_{*f})$, a $(0,g,i,j)$-stock trade is not f-neutral.

Remark: After relabeling of the shareholders, this also implies that a $(1,g,i,j)$-stock trade is not f-neutral either.

The third condition is independence of irrelevant shareholders (IIS), which requires that the relative control power of two shareholders over firm f be unaffected by stock trades between other shareholders of the firm.

Definition 13. Firm f's strategic plan satisfies independence of irrelevant shareholders (IIS) if for any $\psi \in [0,1]$, any firm $g \neq f$, any f-uanimous ownership matrices s and s', and any pair of shareholders $i,j \in N_f(s_{*f})$ such that $s'_{if} = s_{if}$ and $s'_{jf} = s_{jf}$, if a (ψ,g,i,j)-stock trade is f-neutral starting from s, then it is f-neutral also starting from s'.

Remark: Given any pair $i,j \in N$ of shareholders, IIS does not impose any relation between (i) the relative control power over firm f of the two shareholders under s_{*f} such that $i,j \in N_f(s_{*f})$ and $|N_f(s_{*f})| = 2$ and (ii) the control power of the two shareholders under s'_{*f} such that $i,j \in N_f(s'_{*f})$ and $|N_f(s'_{*f})| \geq 3$. This is because if $|N_f(s_{*f})| = 2$ and $|N_f(s'_{*f})| \geq 3$, $s'_{if} \neq s_{if}$ or $s'_{jf} \neq s_{jf}$. To simplify the exposition, we have constrained s to lie in $\{s \in S : |N_f(s_{*f})| \geq 3\}$.[13]

Proposition 5. Assume that R_f is WAPP. Then, R_f satisfies anonymity, inclusivity, and IIS if and only if there exists $\delta : [0,1) \to \mathbb{R}_+$ with $\delta(0) = 0$ and $\delta(x) > 0$ for every $x > 0$ such that for every s_{*f} and every $i \in N$, $\gamma_{if}(s_{*f}) = \delta(s_{if}) / \sum_{j \in N} \delta(s_{jf})$.

Remark: If R_f is also rank-preserving, δ is non-decreasing.

The three conditions are not innocuous. Anonymity may fail if different investors have different expertise or power to influence firm strategy. For example, a venture capitalist may have more resources than a retail investor or stronger incentives than an index fund to exercise control, and thus have a stronger influence on firm strategy. To be sure, while some argue that index funds exert limited control (Bebchuk and Hirst, 2019; Heath et al., 2022), there is evidence suggesting otherwise (Appel et al., 2016; Fisch et al., 2019; Shekita, 2022; Lewellen and Lewellen, 2022). Inclusivity may for example fail (i) if the firm's strategic plan maximizes the portfolio profit of the shareholder with absolute majority when such shareholder exists or (ii) if there are significant fixed costs in exerting control

[13] If we allow for any $s \in S$, in Proposition 5, we would need a separate function $\delta_2 : [0,1) \to \mathbb{R}_+$ such that for every s_{*f} with $|N_f(s_{*f})| = 2$ and every $i \in N$, $\gamma_{if}(s_{*f}) = \delta_2(s_{if}) / \sum_{j \in N} \delta_2(s_{jf})$. Clearly, both δ_2 and δ would be able to deal with s_{*f} with $|N_f(s_{*f})| = 1$.

that prevent smaller shareholders from doing so. Indeed, there is evidence pointing in this direction (Gantchev, 2013; Iliev and Lowry, 2015; Brav et al., 2022). Last, IIS may fail if a shareholder's control power over the firm depends on her probability of being pivotal in voting, which in turn depends on the number of shares held by all other shareholders. In fact, Zingales (1994, 1995) finds evidence pointing in this direction—namely, that the value of voting rights increases with the likelihood of being pivotal.

4 Discussion

In this section, I discuss (i) the effects of ownership dispersion on firm behavior under WAPP and NB, (ii) possible formulations of the disagreement payoff function in NB, and (iii) the efficiency condition on the firm's strategic plan.

4.1 Powerlessness of diffuse ownership

In this section, I discuss the effects of ownership dispersion under WAPP and NB. Before doing so, I define an f-bianimous ownership matrix. An ownership matrix is f-bianimous if firm f shareholders can be divided into two group such that within each group, all shareholders have aligned interests.

Definition 14. Firm f shareholders are divided in their preferences on firm conduct under ownership matrix s if there exists a partition $\{N_1,N_2\}$ of $N_f(s_{*f})$ such that for every $i,j \in N_f(s_{*f})$, if (i) $i,j \in N_1$ or $i,j \in N_2$, then $\lambda_{i;f*} \equiv s_{i*}/s_{if} = s_{j*}/s_{jf} \equiv \lambda_{j;f*}$ for every $i,j \in N_f(s_{*f})$, while (ii) if $i \in N_1$ and $j \in N_2$, then $\lambda_{i;f*} \neq \lambda_{j;f*}$. Then, s is called f-bianimous.

We will study how the firm adjusts its strategy as ownership within group N_2 of its shareholders becomes dispersed.

Ownership diffusion under WAPP. Fix some $i_1 \in N_1$ and $i_2 \in N_2$. It is easy to see that for any firm $g \neq f$,

$$\lambda_{fg}(s) = \frac{\lambda_{i_1;fg} \sum_{i \in N_1} \gamma_{if}(s_{*f}) s_{if} + \lambda_{i_2;fg} \sum_{i \in N_2} \gamma_{if}(s_{*f}) s_{if}}{\sum_{i \in N_1} \gamma_{if}(s_{*f}) s_{if} + \sum_{i \in N_2} \gamma_{if}(s_{*f}) s_{if}}.$$

Consider a sequence $s(\nu)_{\nu \in \mathbb{N}}$ of f-bianimous ownership matrices such that $\lambda_{i_1;fg}$, $\lambda_{i_2;fg}$, N_1, and s_{i*} are fixed along the sequence for every $i \in N_1$, but the holdings of shareholders in N_2 are divided across more and more shareholders, so that $|N_2| \to \infty$ and $\max_{i \in N_2} s_{if} \to 0$. Then, $\sum_{i \in N_2} \gamma_{if}(s_{*f}) s_{if} \to 0$, so $\lambda_{fg}(s) \to \lambda_{i_1;fg}$ (i.e., the weight firm f assigns on firm g's profit converges to the weight group N_1 of shareholders assigns to it) unless $\sum_{i \in N_1} \gamma_{if}(s_{*f}) s_{if} \to 0$ at the same or faster rate than $\sum_{i \in N_2} \gamma_{if}(s_{*f}) s_{if} \to 0$. If $\lambda_{fg}(s) \to \lambda_{i_1;fg}$ for every firm $g \neq f$, then, under standard assumptions, firm f's strategy will converge to the strategy most preferred by the group N_1 of shareholders.[14] For $\sum_{i\in N_2}\gamma_{if}(s_{*f})s_{if}\to 0$, it is necessary that $\max_{i\in N_1}\gamma_{if}(s_{*f})\to 0$. In that case, the shareholders' ranking can be preserved only weakly in the limit (see Proposition 3). This means that under WAPP, at least one of the following must hold: (i) diffuse ownership is powerless in the sense that atomistic shareholders exert no control over the firm (given that in the limit, firm f only pursues group N_1's interests) or (ii) when some of firm f shareholders are atomistic, every individual shareholder—whether large or atomistic—should have zero control power. This suggests that under WAPP, there is a tension between (i) allowing for atomistic shareholders to collectively exert control over the firm and (ii) allowing for large shareholders to have control power.

With additional structure imposed on WAPP, the tension becomes more stark. Let $\gamma_{if}(s_{*f})=\delta(s_{if})/\sum_{j\in N}\delta(s_{jf})$ with

$$\delta(s_{if})=\begin{cases} s_{if}^{\zeta} & \text{if } s_{if}>0\\ 0 & \text{if } s_{if}=0\end{cases}$$

for $\zeta\geq 0$, as in Backus et al. (2021a), Antón et al. (2023), and Ederer and Pellegrino (2025). If $\zeta>0$, $\lambda_{fg}(s)\to\lambda_{i_1;fg}$ as $|N_2|\to\infty$ and $\max_{i\in N_2}s_{if}\to 0$. If $\zeta=0$, firm f maximizes the *unweighted* average of its shareholders' portfolio profits, and thus, it assigns weight

$$\lambda_{fg}(s)=\frac{\sum_{i\in N_f(\gamma_{*f})}\gamma_{if}(s_{*f})s_{ig}}{\sum_{i\in N_f(\gamma_{*f})}\gamma_{if}(s_{*f})s_{if}}=\frac{\sum_{i\in N_f(\gamma_{*f})}s_{ig}/|N_f(\gamma_{*f})|}{\sum_{i\in N_f(\gamma_{*f})}s_{if}/|N_f(\gamma_{*f})|}=\sum_{i\in N_f(\gamma_{*f})}s_{ig}$$

to firm g's profit.[15] Therefore, if $\zeta=0$, atomistic shareholders collectively exert control over firm f, since λ_{fg} represents all shareholders' interests and remains fixed as ownership by N_2 is diffused. However, at the same time, λ_{fg} is unreasonably high. It is equal to 1 if firm g's shareholders are a subset of firm f shareholders. To see why this can be particularly unrealistic, start from $s=I_n$, where I_n the identity matrix (*i.e.*, each firm is owned by a unique shareholder). If we slightly perturb s, so that each shareholder has some shares of every firm, the firms will switch from own-profit maximization to collectively acting as a monopolist, a stark discontinuity. The *unweighted* average (or, equivalently, sum) of the firm's shareholders' portfolio profits can arise as the firm's objective absent agency and trading frictions. Absent such frictions, shareholders could shape firm strategy to maximize their aggregate portfolio profits and then potentially make monetary transfers among themselves.

[14] For example, if A_f is compact and π_g is continuous for every firm g, then by Berge's Maximum Theorem, $R_f(a_{-f},s)\equiv\arg\max_{a_f}\{\pi_f(a_f,a_{-f})+\sum_{g\in M\setminus\{f\}}\lambda_{fg}(s)\pi_g(a_f,a_{-f})\}$ is upper-hemicontinuous in λ_{f*}, so the limit of $R_f(a_{-f},s)$ as $\lambda_{f*}(s)\to\lambda_{i_1;f*}$ (given that it exists) is a subset of $\arg\max_{a_f}\{\pi_f(a_f,a_{-f})+\sum_{g\in M\setminus\{f\}}\lambda_{i_1;fg}\pi_g(a_f,a_{-f})\}$.

[15] Dubey and Shapley (1979) study the limit behavior of Banzhaf γ_{*f}.

Ownership diffusion under NB. Fix some s, a_{-f}, $i_1 \in N_1$, and $i_2 \in N_2$, and assume that for every a_{-f}, s, and $i \in N$, $d_{if}(a_{-f},s) = u_i(\alpha_f^d(a_{-f},s),a_{-f},s_{i*})$, where $\alpha_f^d(a_{-f},s)$ the strategy chosen by firm f in case of disagreement.[16] Then, firm f's objective is

$$\prod_{i\in N_f(\beta_{*f}(s_{*f}))} (u_i(a_f,a_{-f},s_{i*}) - d_{if}(a_{-f},s))^{\beta_{if}(s_{*f})}$$
$$\propto (u_{i_1}(a_f,a_{-f},s_{i*}) - d_{i_1 f}(a_{-f},s))^{\sum_{i\in N_1}\beta_{if}(s_{*f})} (u_{i_2}(a_f,a_{-f},s_{i*}) - d_{i_2 f}(a_{-f},s))^{\sum_{i\in N_2}\beta_{if}(s_{*f})}.$$

Therefore, if we consider a sequence $s(\nu)_{\nu\in\mathbb{N}}$ of f-bianimous ownership matrices such that $|N_2| \to \infty$ and $\max_{i\in N_2} s_{if} \to 0$, the following two can hold at the same time: (i) firm f's strategy does *not* converge to the strategy most preferred by the group N_1 of shareholders and (ii) $\sum_{i\in N_1} \beta_{if}(s(\nu)_{*f})$ is bounded away from 0.[17] Thus, NB can relax the tension that arises under WAPP between (i) allowing for atomistic shareholders to (collectively) exert control over the firm and (ii) allowing for large shareholders to have control power.

4.2 Disagreement payoffs

Choosing a disagreement payoff function greatly restricts the class of strategic plans that admit a NB representation under that specific disagreement function. Such a restriction is bound to rule out many plausible strategic plans and should therefore be guided by empirical evidence. For example, in testing alternative conduct models as in Backus et al. (2021b), one can test which of several plausible disagreement payoff functions is most consistent with the data. Here, I discuss some potential formulations of the disagreement payoff function. A convex combination of those formulations is also wroth considering, where shareholder i's disagreement payoff is her expected portfolio profit under a lottery between the different formulations. For example, in case of disagreement, there is a chance that the management manages to implement their most preferred strategy; otherwise, the status quo is followed.

Management-preferred strategy. In case of disagreement, firm f's management may be able to implement their own most preferred strategy, unconstrained by the shareholders' interests. This can, for example, be the strategy that maximizes firm f's profit or the one that maximizes f's revenue.[18] Mangers may engage in "empire-building."

[16] This implies that for $i \in N_1$, $u_i(\alpha_f,\alpha_{-f},s_{i*}) - d_{if}(\alpha_{-f},s) = s_{if}/s_{i_1 f}(u_{i_1}(\alpha_f,\alpha_{-f},s_{i*}) - d_{i_1 f}(\alpha_{-f},s))$, while for $i \in N_2$, $u_i(\alpha_f,\alpha_{-f},s_{i*}) - d_{if}(\alpha_{-f},s) = s_{if}/s_{i_2 f}(u_{i_2}(\alpha_f,\alpha_{-f},s_{i*}) - d_{i_2 f}(\alpha_{-f},s))$.

[17] For example, $R_f(a_{-f},s(\nu))$ can be bounded away from the strategy most preferred by the group N_1 of shareholders if there exists $\kappa > 0$ such that $d_{i_2 f}(a_{-f},s(\nu)) > \max_{a_f} u_{i_1}(a_f,a_{-f},s_{i*}) + \kappa$ and $\sum_{i\in N_2} \beta_{if}(s(\nu)_{*f}) > 0$ for every ν large enough.

[18] Clearly, if in case of disagreement the strategy that maximizes firm f's profit is implemented, and there is some shareholder i of firm f with shares in no other firms and $\beta_{if} > 0$, then NB will prescribe the strategy that maximizes firm f's profit, even if s_{if} is small. Therefore, one needs to be careful when specifying β_{*f} and the disagreement payoff function.

To enhance their power and reputation, entrenched managers may engage in over-invest and over-produce (compared to the profit-maximizing levels) and make value-destroying acquisitions, expanding the firm beyond its optimal size (Jensen, 1986; Masulis et al., 2007; Harford et al., 2012). Therefore, the management's preferred strategy may be inefficient; namely, it can be too aggressive even for the firm's shareholders who have no stakes in competing firms. This would create incentives for shareholders to reach an agreement.

Status-quo. In case of disagreement, the firm implements some status-quo strategy. The status-quo could, for example, entail no substantial new projects or increases in production capacity and little R&D. In a dynamic setting, this could mean that the firm repeats last period's strategy. A more extreme formulation would have the firm not produce at all or even get liquidated in case of disagreement.

Random dictatorship. The Online Appendix describes formally the random dictatorship formulation of disagreement payoffs. Under this formulation, there exists a lottery weight function $\mu_{*f} : \Delta^n \to \Delta^n$ such that, in case of disagreement, for every $i \in N_f(s_{*f})$, shareholder i gets to implement her most preferred strategy with probability $\mu_{if}(s_{*f})$. While like WAPP, NB with disagreement payoffs derived from random dictatorship can be best interpreted as an as-if assumption, this formulation of disagreement payoffs has certain desirable properties. First, through the lottery weights μ_{*f}, it can account for the relative power of shareholders. Second, when A_f is convex subset and the portfolio profit $u_i(a_f,a_{-f},s_{i*})$ of every shareholder $i \in N_f(s_{*f})$ is strictly concave in a_f,[19] by Jensen's inequality, the shareholders will have strict incentives to reach an agreement. Last, disagreement payoffs derived from random dictatorship have connections to strategic foundations of Nash bargaining. Howard (1992) shows that symmetric NB with random dictatorship disagreement payoffs can be implemented as the unique perfect equilibrium outcome of a game.

4.3 Efficiency

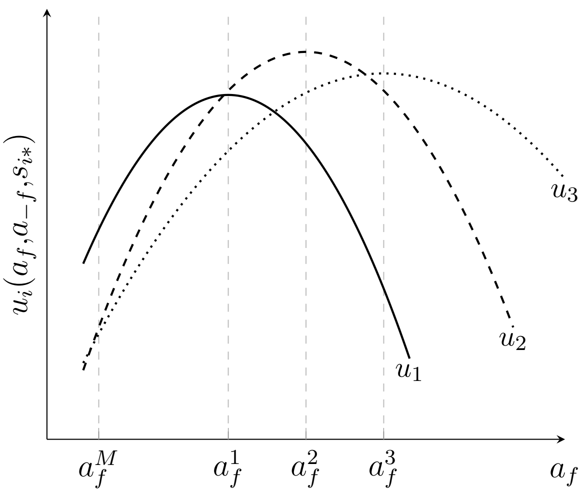
We have seen that efficiency is the crucial condition for a firm's strategic plan to be NB. WAPP imposes the additional condition that the distribution of power across shareholders within the firm be independent of external factors. We have discussed ways in which this assumption might be violated. Nevertheless, efficiency is not an innocuous assumption either. Agency costs may prevent the firm from being efficiently controlled. For example, as discussed in section 4.2, the firm's management's preferred strategy may be too aggressive even for the firm's shareholders who have no stakes in competing firms. In that case, if the management manages to bring firm strategy close enough to its preferred strategy, firm

[19] Lemma 3 in the Online Appendix provides sufficient conditions for strict concavity in a homogeneous product Cournot market.

strategy will be inefficient. This is shown in Figure 1, where firm f has three shareholders, a_f^i is shareholder i's preferred strategy, and a_f^M is the management's preferred strategy. If $R_f(a_{-f},s)$ is close to a_f^M so that $R_f(a_{-f},s) < a_f^1$, R_f is inefficient. Namely, there exists no nonempty subset of controlling shareholders for whom $R_f(a_{-f},s)$ is Pareto efficient (see Definition 3).

WAPP and NB can account for some inefficiencies of this kind if the firm's management is treated as a shareholder. For example, it can be shown thatAzar and Ribeiro's (2022) model is equivalent to allowing for the manager of firm f to be treated as a "virtual" shareholder of the firm with control power γ_f^m and "cash-flow right" s_f^m normalized to $s_f^m = 1$ (so that $s_f^m + \sum_{i \in N} s_{if} = 2$). Azar and Ribeiro (2022) employ a modification of WAPP that allows for managerial entrenchment. They assume that the manager prefers to maximize the firm's own profit, bringing the firm's objective closer to own-profit maximization. Ownership diffusion has similar effects in their model as in standard WAPP: As ownership becomes dispersed, the manager has more power and thus internalizes the shareholders' interests to a lesser degree. When all shareholders are atomistic, the manager maximizes the firm's own profit—even if all shareholders are completely diversified across the industry.

Figure 1: Inefficient strategic plan due to management entrenchment



4.4 Multidimensional strategy spaces

For some of the results, I have restricted attention to the case where A_f is a real interval and firm f's conduct around f-unanimous matrices is characterized by the FOC. This assumption need not be understood as limiting the scope of the results. Given that the firm's strategic plan needs to specify firm conduct across a range of environments, it is informative to study the strategic plan in this simple setting. The restriction to a one-dimensional strategy space makes it clear that after a $(\psi,g,i,\widetilde{N})$-stock trade, shareholder i wants firm f to adjust its strategy in the direction benefiting firm g, which is the exact opposite direction from the one in which group $\widetilde{N}$ of shareholders wants firm f to adjust its strategy. The same analysis can be performed when A_f is multi-dimensional. The restriction to the case where firm f's conduct around f-unanimous matrices is characterized by the FOC is needed exactly because, as discussed in section 3.3, local changes in firm conduct around f-unanimous matrices can affect firm f's conduct only through their effect on shareholders' interests. Monotonicity conditions on a wider range of the space S of ownership matrices would impose restrictions on how the distribution of power across shareholders might depend on external factors.

5 Conclusion

Both theoretical and empirical work has so far followed the weighted average portfolio profit (WAPP) model put forward by O'Brien and Salop (2000) to model firm conduct under common ownership despite our limited understanding of the restrictions it imposes on firm behavior. In this paper, I show that WAPP imposes two restrictions: (i) that the firm is efficiently controlled, and (ii) that the distribution of power across shareholders within the firm depends only on the firm's ownership structure, and not on external factors such as the stakes of the firm's shareholders in competing firms, the strategic choices of other firms, or market conditions (e.g., demand or production costs).

I pursue each of the three objectives by studying properties of the firm's *strategic plan*. The strategic plan describes the strategy the firm will follow as a function of the strategies followed by the other firms, the stakes of each shareholder across firms, and market conditions (e.g., demand or production technology). For example, WAPP with proportional control is a strategic plan.

I propose the Nash bargaining (NB) model of firm conduct under common ownership, a generalization of WAPP which models the firm's behavior as the result of asymmetric Nash bargaining among the firm's shareholders. NB also requires efficient control but allows for external factors to influence the distribution of power across the firm's shareholders. Indeed, I argue that external factors may play a role in the extent to which each shareholder controls the firm. In addition, I show that the NB model can relax the tension that arises under WAPP between (i) allowing for atomistic shareholders to collectively exert control over the firm while at the same time (ii) allowing for large shareholders to have control power. Last, I study the constraints imposed on the parameters of the firm's objective under WAPP and NB by additional restrictions on the firm's behavior, thereby characterizing a popular subclass of WAPP models.

This paper provides a unified axiomatic framework for studying firm conduct under common ownership. It shows the general class of NB models of firm behavior is essentially characterized by an efficiency condition on firm behavior. WAPP models are the subclass of NB models that imposes an additional condition on firm behavior: the irrelevance of external factors for the distribution of control power across the firm's shareholders. The results guide researchers and practitioners to think in two steps when deciding on a model of firm conduct under common ownership: (i) Choose between WAPP and NB depending on whether the assumption that the distribution of control power across the firm's shareholders is independent of external factors is likely to be satisfied or not, and (ii) decide on a specific parametrization of the model chosen in the first step depending on what additional conditions are likely to be satisfied.

Future theoretical and empirical work can evaluate the robustness of results obtained under WAPP by also considering NB models of firm conduct. For example, Backus et al. (2021b) find that own-firm profit maximization is more consistent with firm behavior in the ready-to-eat cereal market than WAPP with proportional control, which may serve as evidence against the "common ownership hypothesis" (i.e., that common ownership induces firms to internalize the effects of their strategic decisions on other firms' profits). An obvious robustness check would be to consider different parametrization of WAPP. However, if one is concerned that the distribution of power across shareholders may depend on external factors, this would not be enough. One would have to test NB models against own-profit maximization to more robustly evaluate the "common ownership hypothesis."

In this paper, I have studied tractable and flexible representations of firm conduct that can be embedded in theoretical and empirical studies of oligopolistic competition under common ownership. In doing so, I focused on the problem of aggregating shareholders' preferences than on whether and through what mechanisms the firm's management is induced to pursue the shareholders' aggregated preferences despite agency frictions. Still, the representations of firm conduct I derive can well apply in the presence of agency frictions. For example, as Backus et al. (2020) observe, the measure of common ownership proposed by Gilje et al. (2020) that accounts for investor attention corresponds to a specific formulation of γ_{*f} in WAPP. Another promising avenue for research examines in more detail channels through which common ownership can affect firm conduct despite agency frictions (see, e.g., Hemphill and Kahan, 2020; Antón et al., 2023).

References

Antón, M., Ederer, F., Giné, M., and Ramirez-Chiang, G. (2025a). Common ownership around the world.

Antón, M., Ederer, F., Giné, M., and Schmalz, M. (2023). Common ownership, competition, and top management incentives. *Journal of Political Economy*, 131(5):1294–1355.

Antón, M., Ederer, F., Giné, M., and Schmalz, M. (2025b). Innovation: the bright side of common ownership? *Management Science*, 71(5):3713–3733.

Appel, I. R., Gormley, T. A., and Keim, D. B. (2016). Passive investors, not passive owners. *Journal of Financial Economics*, 121(1):111–141.

Aubin, J.-P. and Ekeland, I. (1984). *Applied nonlinear analysis*. John Wiley & Sons.

Azar, J. (2017). Portfolio diversification, market power, and the theory of the firm. *SSRN Electronic Journal*.

Azar, J. and Ribeiro, R. M. (2022). Estimating oligopoly with shareholder voting models.

Azar, J., Schmalz, M. C., and Tecu, I. (2018). Anticompetitive effects of common ownership. *Journal of Finance*, 73:1513–1565.

Azar, J. and Vives, X. (2021). General equilibrium oligopoly and ownership structure. *Econometrica*, 89:999–1048.

Backus, M., Conlon, C., and Sinkinson, M. (2020). Theory and measurement of common ownership. *AEA Papers and Proceedings*, 110:557–560.

Backus, M., Conlon, C., and Sinkinson, M. (2021a). Common Ownership in America: 1980–2017. *American Economic Journal: Microeconomics*, 13:273–308.

Backus, M., Conlon, C. T., and Sinkinson, M. (2021b). Common ownership and competition in the ready-to-eat cereal industry. *SSRN Electronic Journal*.

Banzhaf, J. F. I. (1965). Weighted voting doesn't work: A mathematical analysis. *Rutgers L. Rev.*, 19:317–343.

Bebchuk, L. and Hirst, S. (2019). Index funds and the future of corporate governance: Theory, evidence, and policy. *Columbia Law Review*, 119(8):2029–2145.

Brav, A., Cain, M., and Zytnick, J. (2022). Retail shareholder participation in the proxy process: Monitoring, engagement, and voting. *Journal of Financial Economics*, 144(2):492–522.

Bresnahan, T. F. and Salop, S. C. (1986). Quantifying the competitive effects of production joint ventures. *International Journal of Industrial Organization*, 4:155–175.

Brito, D., Elhauge, E., Ribeiro, R., and Vasconcelos, H. (2023). Modelling the objective function of managers in the presence of overlapping shareholding. *International Journal of Industrial Organization*, 87:102905.

Brito, D., Osório, A., Ribeiro, R., and Vasconcelos, H. (2018). Unilateral effects screens for partial horizontal acquisitions: The generalized HHI and GUPPI. *International Journal of Industrial Organization*, 59:127–189.

Chiappinelli, O., Papadopoulos, K. G., and Xefteris, D. (2023). Common Ownership Unpacked. UB School of Economics Working Papers 2023/448, University of Barcelona School of Economics.

Coleman, J. S. (1971). Control of collectivities and the power of a collectivity to act.

Collard-Wexler, A., Gowrisankaran, G., and Lee, R. S. (2019). "Nash-in-Nash" Bargaining: A Microfoundation for Applied Work. *Journal of Political Economy*, 127(1):163–195.

De Loecker, J. and Eeckhout, J. (2018). Global market power. Technical report, National Bureau of Economic Research.

De Loecker, J., Eeckhout, J., and Unger, G. (2020). The rise of market power and the macroeconomic implications. *The Quarterly journal of economics*, 135(2):561–644.

Denicolò, V. and Panunzi, F. (2025). Common ownership, competition, and corporate governance. *Management Science*, 71(5):3924–3943.

Dubey, P. and Shapley, L. S. (1979). Mathematical Properties of the Banzhaf Power Index. *Mathematics of Operations Research*, 4:99–131.

Ederer, F. and Pellegrino, B. (2025). A tale of two networks: Common ownership and product market rivalry. *The Review of Economic Studies*, page rdaf057.

Edgeworth, F. Y. (1881). *Mathematical Psychics. An Essay on the Application of Mathematics to the Moral Sciences.* C. Kegan Paul and Co.

Elhauge, E. (2016). Horizontal shareholding. *Harvard Law Review*, 129:1267–1317.

Farrell, J. and Shapiro, C. (1990). Horizontal mergers: An equilibrium analysis. *American Economic Review*, 80:107–126.

Fich, E. M., Harford, J., and Tran, A. L. (2015). Motivated monitors: The importance of institutional investors' portfolio weights. *Journal of Financial Economics*, 118(1):21–48.

Fisch, J., Hamdani, A., and Solomon, S. D. (2019). The new titans of Wall Street: A theoretical framework for passive investors. *University of Pennsylvania Law Review*, pages 17–72.

Fisher, I. (1930). *The Theory of Interest: As Determined by Impatience to Spend Income and Opportunity to Invest It.* Macmillan.

Gantchev, N. (2013). The costs of shareholder activism: Evidence from a sequential decision model. *Journal of Financial Economics*, 107(3):610–631.

Gerardi, K., Lowry, M., and Schenone, C. (2024). A critical review of the common ownership literature. *Annual Review of Financial Economics*, 16.

Gilje, E. P., Gormley, T. A., and Levit, D. (2020). Who's paying attention? measuring common ownership and its impact on managerial incentives. *Journal of Financial Economics*, 137:152–178.

Gramlich, J. and Grundl, S. (2017). Estimating the competitive effects of common ownership.

Harford, J., Humphery-Jenner, M., and Powell, R. (2012). The sources of value destruction in acquisitions by entrenched managers. *Journal of financial economics*, 106(2):247–261.

Hart, O. D. (1979). On shareholder unanimity in large stock market economies. *Econometrica*, 47.

He, J. and Huang, J. (2017). Product market competition in a world of cross-ownership: Evidence from institutional blockholdings. *Review of Financial Studies*, 30:2674–2718.

Heath, D., Macciocchi, D., Michaely, R., and Ringgenberg, M. C. (2022). Do index funds monitor? *The Review of Financial Studies*, 35(1):91–131.

Hemphill, C. S. and Kahan, M. (2020). The strategies of anticompetitive common ownership. *Yale Law Journal*, 129:1392–1459.

Horn, H. and Wolinsky, A. (1988). Bilateral monopolies and incentives for merger. *The RAND Journal of Economics*, 19(3):408–419.

Howard, J. V. (1992). A social choice rule and its implementation in perfect equilibrium. *Journal of Economic Theory*, 56:142–159.

Iliev, P., Kalodimos, J., and Lowry, M. (2021). Investors' attention to corporate governance. *The Review of Financial Studies*, 34(12):5581–5628.

Iliev, P. and Lowry, M. (2015). Are mutual funds active voters? *The Review of Financial Studies*, 28(2):446–485.

Jensen, M. C. (1986). Agency costs of free cash flow, corporate finance, and takeovers. *The American economic review*, 76(2):323–329.

Koch, A., Panayides, M., and Thomas, S. (2021). Common ownership and competition in product markets. *Journal of Financial Economics*, 139:109–137.

Lewellen, J. and Lewellen, K. (2022). Institutional investors and corporate governance: The incentive to be engaged. *The Journal of Finance*, 77(1):213–264.

Lewellen, K. and Lowry, M. (2021). Does common ownership really increase firm coordination? *Journal of Financial Economics*, 141:322–344.

Li, X., Liu, T., and Taylor, L. A. (2023). Common ownership and innovation efficiency. *Journal of financial economics*, 147(3):475–497.

López, Á. L. and Vives, X. (2019). Overlapping ownership, r&d spillovers, and antitrust policy. *Journal of Political Economy*, 127(5):2394–2437.

Lu, R., Shen, Q., Wang, T., and Zhang, X. (2022). Frenemies: Corporate advertising under common ownership. *Management Science*, 68(6):4645–4669.

Masulis, R. W., Wang, C., and Xie, F. (2007). Corporate governance and acquirer returns. *the Journal of Finance*, 62(4):1851–1889.

Matvos, G. and Ostrovsky, M. (2008). Cross-ownership, returns, and voting in mergers. *Journal of Financial Economics*, 89(3):391–403.

Moskalev, A. (2019). Objective function of a non-price-taking firm with heterogeneous shareholders. *SSRN Electronic Journal*.

Newham, M., Seldeslachts, J., and Banal-Estanol, A. (2025). Common ownership and market entry: Evidence from the pharmaceutical industry. *American Economic Journal: Microeconomics*, 17(4):260–327.

O'Brien, D. P. and Salop, S. C. (2000). Competitive effects of partial ownership: Financial interest and corporate control. *Antitrust Law Journal*, 67:559–603.

O'Brien, D. P. and Waehrer, K. (2017). The competitive effects of common ownership: We know less than we think. *Antitrust Law Journal*, 81:729–776.

Penrose, L. S. (1946). The elementary statistics of majority voting. *Journal of the Royal Statistical Society*, 109:53–57.

Piccolo, A. and Schneemeier, J. (2025). Ownership and competition. *Available at SSRN 3733795*.

Posner, E. A., Morton, F. M. S., and Weyl, E. G. (2017). A proposal to limit the anti-competitive power of institutional investors. *Antitrust Law Journal*, 81:669–728.

Rotemberg, J. (1984). Financial transaction costs and industrial performance.

Rubinstein, A. and Yaari, M. E. (1983). The competitive stock market as cartel maker: Some examples.

Schmalz, M. C. (2018). Common-ownership concentration and corporate conduct. *Annual Review of Financial Economics*, 10:413–448.

Shekita, N. (2022). Interventions by common owners. *Journal of Competition Law & Economics*, 18(1):99–134.

U.S. Department of Justice and Federal Trade Commission (2023). Merger Guidelines.

Van Nieuwerburgh, S. and Veldkamp, L. (2010). Information acquisition and under-diversification. *The Review of Economic Studies*, 77(2):779–805.

Vives, X. (1999). *Oligopoly pricing: old ideas and new tools*. MIT Press.

Vives, X. and Vravosinos, O. (2025). Free Entry in a Cournot Market with Overlapping Ownership. *American Economic Journal: Microeconomics*, 17(2):292–320.

Zingales, L. (1994). The value of the voting right: A study of the Milan stock exchange experience. *The Review of Financial Studies*, 7(1):125–148.

Zingales, L. (1995). What determines the value of corporate votes? *The Quarterly Journal of Economics*, 110(4):1047–1073.

Appendix

A Proofs

Proof of Proposition 1. Part (i). Let R_f be WAPP with control power function γ_{*f}. For every $s \in S$ define $\widetilde{N}(s_{*f}) \coloneqq \{i \in N : \gamma_{if}(s_{*f}) > 0\}$, and use $\widetilde{N}(s_{*f})$ to verify that R_f satisfies the strong efficiency conditions.

Part (ii). Now, assume R_f is NB with bargaining power function β_{*f}. For every $s \in S$ define $\widetilde{N}(s_{*f}) \coloneqq \{i \in N : \beta_{if}(s_{*f}) > 0\}$, and use $\widetilde{N}(s_{*f})$ to verify that R_j satisfies the weak efficiency conditions.

Part (iii). Let R_f be NB with strict benefits form agreement and bargaining power function β_{*f}. For every $s \in S$ define $\widetilde{N}(s_{*f}) \coloneqq \{i \in N : \beta_{if}(s_{*f}) > 0\}$, and use $\widetilde{N}(s_{*f})$ to verify that R_f satisfies the strong efficiency conditions. Also, since for every s, α_{-f}, and π, there exists $u \in \mathcal{U}_f(\alpha_{-f}, s; \pi)$ such that $u_i > d_{if}(\alpha_{-f}, s; \pi)$ for every $i \in N_f(\beta_{*f}(s_{*f}))$, the Nash product $\prod_{i \in N_f(\beta_{*f}(s_{*f}))}(u_i - d_{if})^{\beta_{if}}$ is strictly quasiconcave in u where that inequality holds. Thus, since $\{u \in \mathcal{U}_f(\alpha_{-f}, s; \pi) : u_i > d_{if}(\alpha_{-f}, s; \pi)$ for every $i \in N_f(\beta_{*f}(s_{*f}))\}$ is convex for every $s \in S$, $\pi \in \Pi$, and $\alpha_{-f} \in \times_{g \neq f} \Delta(A_g)$, there exists at most one u that maximizes the Nash product, so R_f is internally consistent.

Part (iv). Let R_f be weakly efficient and internally consistent, so that there exists function $\widetilde{N}(s_{*f})$ satisfying the conditions of Definition 3. For every $s \in S$, let the bargaining power function be

$$\beta_{*f}(s_{*f}) \coloneqq \frac{1}{\left|\widetilde{N}(s_{*f})\right|} \left(\mathbb{I}\left(1 \in \widetilde{N}(s_{*f})\right) \quad \dots \quad \mathbb{I}\left(n \in \widetilde{N}(s_{*f})\right)\right),$$

where $\mathbb{I}$ is the indicator function, and for every $i \in \widetilde{N}(s_{*f})$, let the disagreement payoff function be $d_{if}(\alpha_{-f}, s; \pi) \coloneqq u_i(\widetilde{\alpha}_f(\alpha_{-f}, s; \pi), \alpha_{-f}, s)$ for some function $\widetilde{\alpha}_f(\alpha_{-f}, s; \pi)$ that is a selection from $R_f(\alpha_{-f}, s; \pi)$ (i.e., $\widetilde{\alpha}_f(\alpha_{-f}, s; \pi) \in R_f(\alpha_{-f}, s; \pi)$), and for every $i \notin \widetilde{N}(s_{*f})$, let $d_{if}(\alpha_{-f}, s; \pi) \coloneqq 0$. d_{*f} is well-defined since R_f is internally consistent. Notice that by the way β_{*f} is defined, $N_f(\beta_{*f}(s_{*f})) = \widetilde{N}(s_{*f})$. Fix arbitrary $s \in S$, $\pi \in \Pi$, and $\alpha_{-f} \in \times_{g \neq f} \Delta(A_g)$. Observe that any $\alpha_f \in R_f(\alpha_{-f}, s; \pi)$ achieves the maximum value of zero for the Nash product, so

$$R_f\left(\alpha_{-f}, s; \pi\right) \subseteq \underset{\alpha'_f \in B_f(\alpha_{-f}, s; \pi)}{\arg\max} \left\{ \prod_{i \in N_f(\beta_{*f}(s_{*f}))} \left(u_f\left(\alpha'_f, \alpha_{-f}, s_{i*}\right) - d_{if}(\alpha_{-f}, s; \pi)\right)^{\beta_{if}(s_{*f})} \right\}.$$

Now, take an arbitrary

$$\alpha_f \in \underset{\alpha'_f \in B_f(\alpha_{-f}, s; \pi)}{\arg\max} \left\{ \prod_{i \in N_f(\beta_{*f}(s_{*f}))} \left(u_i\left(\alpha'_f, \alpha_{-f}, s_{i*}\right) - d_{if}(\alpha_{-f}, s; \pi)\right)^{\beta_{if}(s_{*f})} \right\}.$$

We will show by contradiction that $\alpha_f \in R_f(\alpha_{-f}, s; \pi)$. Assume that $\alpha_f \notin R_f(\alpha_{-f}, s; \pi)$. Then, since R_f is internally consistent, there exists $j \in N_f(\beta_{*f}(s_{*f}))$ such that $u_j(\alpha_f, \alpha_{-f}, s_{j*}) \neq u_j(\widetilde{\alpha}_f(\alpha_{-f}, s; \pi), \alpha_{-f}, s_{j*}) = d_{jf}(\alpha_{-f}, s; \pi)$. Also, given that α_f maximizes the Nash product above, $\alpha_f \in B_f(\alpha_{-f}, s; \pi)$, and thus, $u_i(\alpha_f, \alpha_{-f}, s_{f*}) \geq d_{if}(\alpha_{-f}, s; \pi)$ for every $i \in N_f(\beta_{*f}(s_{*f}))$. Particularly, the inequality must hold strictly for j, that is, $u_j(\alpha_f, \alpha_{-f}, s_{j*}) > d_{jf}(\alpha_{-f}, s; \pi) \equiv u_j(\widetilde{\alpha}_f(\alpha_{-f}, s; \pi), \alpha_{-f}, s_{j*})$. But then, α_f weakly dominates $\widetilde{\alpha}_f(\alpha_{-f}, s; \pi) \in R_f(\alpha_{-f}, s; \pi)$, a contradiction to the strong efficiency of R_f. Therefore,

$$R_f\left(\alpha_{-f}, s; \pi\right) \supseteq \underset{\alpha_f \in B_f(\alpha_{-f}, s; \pi)}{\arg\max} \left\{ \prod_{i \in N_f(\beta_{*f}(s_{*f}))} \left(u_f\left(\alpha_f, \alpha_{-f}, s_{i*}\right) - d_{if}(\alpha_{-f}, s; \pi)\right)^{\beta_{if}(s_{*f})} \right\}.$$

Part (v). Let R_f be WAPP with control power function γ_{*f}. For every $s \in S$, define $\widetilde{N}(s_{*f}) = \{i \in N : \gamma_{if}(s_{*f}) > 0\}$. Given part (i) of this Proposition, it remains to show parts (iv) and (v) are satisfied. Part (v) of Definition 3 is clearly satisfied. To see that part (iv) also holds, assume by contradiction that there exist $s \in S$, $\pi \in \Pi$, $\alpha_{-f} \in \times_{h \neq f} \Delta(A_{-h})$ and $\alpha_f, \alpha'_f \in R_f(\alpha_{-f}, s; \pi)$, such that $u_j(\alpha_f, \alpha_{-f}, s) \neq u_j(\alpha'_f, \alpha_{-f}, s)$ for some shareholder j of firm f. Then, by strict convexity of $\mathcal{U}_f(\alpha_{-f}, s; \pi)$, any strict convex combination v of the two portfolio profit profiles under α_f and α'_f lies in the interior of $\mathcal{U}_f(\alpha_{-f}, s; \pi)$, and thus there exists $v' \in \mathcal{U}_f(\alpha_{-f}, s; \pi)$ such that $v' \gg v$, or equivalently α^*_f such that $u_i(\alpha^*_f, \alpha_{-f}, s_{i*}) > v_i$ for every shareholder i of firm f. But then $\sum_{i \in N} \gamma_{if}(s_{*f}) u_i(\alpha^*_f, \alpha_{-f}, s_{i*})$ is higher than the strict convex combination of $\sum_{i \in N} \gamma_{if}(s_{*f}) u_i(\alpha_f, \alpha_{-f}, s_{i*})$ and $\sum_{i \in N} \gamma_{if}(s_{*f}) u_i(\alpha'_f, \alpha_{-f}, s_{i*})$, and thus higher than each of the two (since $\alpha_f, \alpha'_f \in R_f(\alpha_{-f}, s; \pi)$, so they both maximize the WAPP objective), which contradicts that $\alpha_f, \alpha'_f \in R_f(\alpha_{-f}, s; \pi)$.

The proof for the case where R_f is NB is analogous. **Q.E.D.**

Proof of Proposition 2. Part (i). Let R_f be GWAPP with control power function γ_{*f}. Fix some α_{-f}, $(s_{*g})_{g \neq f}$, and π. For every s_{*f}, define $\widetilde{N}(s_{*f}) \coloneqq \{i \in N : \gamma_{if}(\alpha_{-f}, s; \pi) > 0\}$. By condition (iii) of Definition 4, $\widetilde{N}(s_{*f})$ is independent of α_{-f}, $(s_{*g})_{g \neq f}$, and π. Use $\widetilde{N}(s_{*f})$ to verify that R_f satisfies the strong efficiency conditions.

Part (ii). Assume R_f is GWAPP. Then, by part (i) of this Proposition, it is streongly efficient, and as in the proof of part (v) of Proposition 1, it is easy to see that R_f is also internally consistent. Part (iv) of Proposition 1 then implies that R_f is NB.

Now, assume that R_f is NB. Part (v) of Proposition 1 implies that it is strongly efficient and internally consistent. Take some arbitrary α_{-f}, s, and π. Since R_f is strongly efficient and internally consistent, there exists $\widetilde{N}(s_{*f})$ satisfying the conditions of Definition 3. Define $v^* \in \mathbb{R}^{|\widetilde{N}(s_{*f})|}$ to be such that $v^*_i = u_i(\alpha_f, \alpha_{-f}, s_{i*})$ for every $i \in \widetilde{N}(s_{*f})\}$ and every $\alpha_f \in R_f(\alpha_{-f}, s; \pi)$, which is possible because R_f is internally consistent.

Define the convex sets $V := \{v \in \mathbb{R}^{|\widetilde{N}(s_{*f})|} : v_i \geq v_i^* \text{ for every } i \in \widetilde{N}(s_{*f}) \text{ with at least one inequality strict}\}$, and $V' := \{v \in \mathbb{R}^{|\widetilde{N}(s_{*f})|} : \text{there exists } \alpha_f \in \Delta(A_f) \text{ such that } v_i = u_i(\alpha,\alpha_{-f},s_{i*}) \text{ for every } i \in \widetilde{N}(s_{*f})\}$. By strong efficiency of R_f, $V \cap V' = \emptyset$. Therefore, by the separating hyperplane theorem, there exist non-zero $\gamma_{*f}(\alpha_{-f},s;\pi) \in \mathbb{R}^{|\widetilde{N}(s_{*f})|}$ and $x^* \in \mathbb{R}$ such that $\sum_{i\in\widetilde{N}(s_{*f})} \gamma_{if}(\alpha_{-f},s;\pi)v_i \geq x^*$ for every $v \in V$ and $\sum_{i\in\widetilde{N}(s_{*f})} \gamma_{if}(\alpha_{-f},s;\pi)v_i \leq x^*$ for every $v \in V'$. Particularly, $x^* = \sum_{i\in\widetilde{N}(s_{*f})} \gamma_{if}(\alpha_{-f},s;\pi)v_i^*$, and

$$\max_{v\in V}\left\{\sum_{i\in\widetilde{N}(s_{*f})} \gamma_{if}(\alpha_{-f},s;\pi)v_i\right\} = \sum_{i\in\widetilde{N}(s_{*f})} \gamma_{if}(\alpha_{-f},s;\pi)v_i^*.$$

Therefore,

$$R_f(\alpha_{-f},s;\pi) \subseteq \underset{\alpha_f\in\Delta(A_f)}{\arg\max}\left\{\sum_{i\in\widetilde{N}(s_{*f})} \gamma_{if}(\alpha_{-f},s;\pi)u_i(\alpha_f,\alpha_{-f},s_{i*})\right\}. \quad (1)$$

It must hold that $\gamma_{*f}(\alpha_{-f},s;\pi) \in \mathbb{R}_+^{|\widetilde{N}(s_{*f})|}$. To see this, notice that if $\gamma_{jf}(\alpha_{-f},s;\pi) < 0$ for some $j \in \widetilde{N}(s_{*f})$, then $\sum_{i\in\widetilde{N}(s_{*f})} \gamma_{if}(\alpha_{-f},s;\pi)v_i \geq x^*$ will be violated for v such that v_j is large enough. Also, since $\gamma_{*f}(\alpha_{-f},s;\pi) \in \mathbb{R}_+^{|\widetilde{N}(s_{*f})|}$ is non-zero, its entries can be normalized to sum up to 1.

Also, given (1) and because $\mathcal{U}_f(\alpha_{-f},s;\pi)$ is strictly convex, for every $\alpha_f \in R_f(\alpha_{-f},s;\pi)$ and every $\alpha'_f \in \arg\max_{\alpha''_f\in\Delta(A_f)}\{\sum_{i\in\widetilde{N}(s_{*f})} \gamma_{if}(\alpha_{-f},s;\pi)u_i(\alpha''_f,\alpha_{-f},s_{i*})\}$, it must hold that $u_i(\alpha_f,\alpha_{-f},s_{i*}) = u_i(\alpha'_f,\alpha_{-f},s_{i*})$ for every $i \in \widetilde{N}(s_{*f})$.[20] Thus, given also that R_f is internally consistent (part (v) of Definition 3),

$$R_f(\alpha_{-f},s;\pi) = \underset{\alpha_f\in\Delta(A_f)}{\arg\max}\left\{\sum_{i\in\widetilde{N}(s_{*f})} \gamma_{if}(\alpha_{-f},s;\pi)u_i(\alpha_f,\alpha_{-f},s_{i*})\right\}. \quad (2)$$

We conclude that R_f is WAPP with $\gamma_{if}(\alpha_{-f},s;\pi)$ as above for every $i \in \widetilde{N}(s_{*f})$ and $\gamma_{if}(\alpha_{-f},s;\pi) = 0$ for every $i \notin \widetilde{N}(s_{*f})$. **Q.E.D.**

Proof of Proposition 3. (i) Firm f's objective function is $\sum_{k\in N} \gamma_{kf}(s_{*f})u_k(a_f,a_{-f},s_{k*})$, and the Implicit Function Theorem gives that around any f-unanimous s

$$\nabla_{ds}R_f(a_{-f},s) = -\frac{(\gamma_{if}(s_{*f}) - \gamma_{jf}(s_{*f}))\left.\frac{\partial\pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)}}{\sum_{k\in N}\gamma_{kf}(s_{*f})\left.\frac{\partial^2 u_k(a_f,a_{-f},s_{k*})}{\partial a_f^2}\right|_{a_f=R_f(a_{-f},s)}} \implies$$

$$\operatorname{sgn}\{\nabla_{ds}R_f(a_{-f},s)\} = \operatorname{sgn}\left\{(\gamma_{if}(s_{*f}) - \gamma_{jf}(s_{*f}))\left.\frac{\partial\pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)}\right\}. \quad (3)$$

[20]The argument is analogous to the one in the proof of part (v) of Proposition 1.

We will prove each direction separately.

$\Longleftarrow$: Assume that for every s_{*f} and every pair of firm f shareholders $i,j \in N_f(s_{*f})$, $s_{if} \geq s_{jf} \implies \gamma_{if}(s_{*f}) \geq \gamma_{jf}(s_{*f})$. We need to show that R_f is rank-preserving. Take arbitrary $a_{-f} \in A_{-f}$, f-unanimous $s \in S$, and pair of distinct shareholders $i,j \in N_f(s_{*f})$ with $s_{if} \geq s_{jf}$. It follows that $\gamma_{if}(s_{*f}) \geq \gamma_{jf}(s_{*f})$. Consider a stock trade where i buys firm $g \neq f$ shares from j. (3) gives

$$\left.\frac{\partial \pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)} \overset{\text{(resp. }\leq)}{\geq} 0 \implies \nabla_{ds} R_f(a_{-f},s) \overset{\text{(resp. }\leq)}{\geq} 0.$$

$\Longrightarrow$: Let R_f be rank-preserving. We need to show that for every s_{*f} and every pair of firm f shareholders $i,j \in N_f(s_{*f})$, $s_{if} \geq s_{jf} \implies \gamma_{if}(s_{*f}) \geq \gamma_{jf}(s_{*f})$. Take arbitrary s_{*f} and pair of firm f shareholders $i,j \in N_f(s_{*f})$ with $s_{if} \geq s_{jf}$. By assumption, there exist $(s_{*g})_{g\neq f}$, firm $g \neq f$, and $a_{-f} \in A_{-f}$ such that s is f-unanimous and $\partial \pi_g(a_f,a_{-f})/\partial a_f|_{a_f=R_f(a_{-f},s)} \neq 0$.[21] Let $ds := (\mathbf{e}_i - \mathbf{e}_j) \otimes \mathbf{e}_g$. If $\partial \pi_g(a_f,a_{-f})/\partial a_f|_{a_f=R_f(a_{-f},s)} > 0$, then given that R_f is rank-preserving, $\nabla_{ds} R_f(a_{-f},s) \geq 0$, and thus (3) implies $\gamma_{if}(s_{*f}) \geq \gamma_{jf}(s_{*f})$. Similarly, if $\partial \pi_g(a_f,a_{-f})/\partial a_f|_{a_f=R_f(a_{-f},s)} < 0$, then given that R_f is rank-preserving, $\nabla_{ds} R_f(a_{-f},s) \leq 0$, and thus (3) implies $\gamma_{if}(s_{*f}) \geq \gamma_{jf}(s_{*f})$,

(ii) Now, notice that under NB, for any f-unanimous s, $\partial u_k(a_f,a_{-f},s_{k*})/\partial a_f|_{a_f=R_f(a_{-f},s)} = 0$ for every shareholder k of firm f. For a stock trade ds, the Implicit Function Theorem then gives that around any f-unanimous s[22]

$$\begin{aligned}
\operatorname{sgn}\{\nabla_{ds} R_f(a_{-f},s)\} &= \operatorname{sgn}\left\{ \left(\widetilde{\gamma}_{if}(a_{-f},s) - \widetilde{\gamma}_{jf}(a_{-f},s)\right) \left.\frac{\partial \pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)} \right\} \\
&= \operatorname{sgn}\left\{ \frac{\left(\beta_{if}(s_{*f}) - \beta_{jf}(s_{*f}) s_{if}/s_{jf}\right) \left.\frac{\partial \pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)}}{u_i\left(R_f(a_{-f},s),a_{-f},s_{i*}\right) - u_i\left(\alpha_d(a_{-f},s),a_{-f},s_{i*}\right)} \right\} \\
&= \operatorname{sgn}\left\{ \left(\frac{\beta_{if}(s_{*f})}{s_{if}} - \frac{\beta_{jf}(s_{*f})}{s_{jf}}\right) \left.\frac{\partial \pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)} \right\}, \qquad (4)
\end{aligned}$$

where $\alpha_d(a_{-f},s)$ the strategy followed in case of disagreement. In the second line, we have used the fact that $\lambda_{i;f*} = \lambda_{j;f*}$, and in the third the fact that $u_i(R_f(a_{-f},s),a_{-f},s_{i*}) > u_i(\alpha_d(a_{-f},s),a_{-f},s_{i*})$. We will prove each direction separately.

$\Longleftarrow$: Assume that for every s_{*f} and every pair of firm f shareholders $i,j \in N_f(s_{*f})$, $s_{if} \geq s_{jf} \implies \beta_{if}(s_{*f})/s_{if} \geq \beta_{jf}(s_{*f})/s_{jf}$. We need to show that R_f is rank-preserving. Take arbitrary $a_{-f} \in A_{-f}$, f-unanimous $s \in S$, and pair of distinct shareholders $i,j \in N_f(s_{*f})$ with $s_{if} \geq s_{jf}$. It follows that $\beta_{if}(s_{*f})/s_{if} \geq \beta_{jf}(s_{*f})/s_{jf}$. Consider a stock trade

[21] For example, $(s_{*g})_{g\neq f}$ such that $s_{*g} = s_{*f}$ for every $g \neq f$ makes s f-unanimous.

[22] Notice that because $\partial u_k(a_f,a_{-f},s_{k*})/\partial a_f|_{a_f=R_f(a_{-f},s)} = 0$ for every shareholder k of firm f, the changes in $\widetilde{\gamma}_{*f}$ caused by the stock trade vanish.

where i buys firm $g \neq f$ shares from j. (4) gives

$$\left.\frac{\partial \pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)} \overset{\text{(resp. } \leq\text{)}}{\geq} 0 \implies \nabla_{ds} R_f(a_{-f},s) \overset{\text{(resp. } \leq\text{)}}{\geq} 0.$$

$\Longrightarrow$: Let R_f be NB and rank-preserving. We need to show that for every s_{*f} and every pair of firm f shareholders $i,j \in N_f(s_{*f})$, $s_{if} \geq s_{jf} \implies \beta_{if}(s_{*f})/s_{if} \geq \beta_{jf}(s_{*f})/s_{jf}$. Take arbitrary s_{*f} and pair of firm f shareholders $i,j \in N_f(s_{*f})$ with $s_{if} \geq s_{jf}$. By assumption, there exist $(s_{*g})_{g \neq f}$, firm $g \neq f$, and $a_{-f} \in A_{-f}$ such that s is f-unanimous and $\partial \pi_g(a_f,\widehat{a}_{-f})/\partial a_f|_{a_f=R_f(a_{-f},s)} \neq 0$. Let $ds := (\mathbf{e}_i - \mathbf{e}_j) \otimes \mathbf{e}_g$. If $\partial \pi_g(a_f,a_{-f})/\partial a_f|_{a_f=R_f(a_{-f},s)} > 0$, then given that R_f is rank-preserving, $\nabla_{ds} R_f(a_{-f},s) \geq 0$, and thus (4) implies $\beta_{if}(s_{*f})/s_{if} \geq \beta_{jf}(s_{*f})/s_{jf}$. Similarly, if $\partial \pi_g(a_f,a_{-f})/\partial a_f|_{a_f=R_f(a_{-f},s)} < 0$, then given that R_f is rank-preserving, $\nabla_{ds} R_f(a_{-f},s) \leq 0$, and thus (4) implies $\beta_{if}(\widehat{s}_{*f})/\widehat{s}_{if} \geq \beta_{jf}(\widehat{s}_{*f})/\widehat{s}_{jf}$. **Q.E.D.**

Proof of Lemma 1. (i-a) Firm f's objective function is $\sum_{k \in N_f(\gamma_{*f}(s'_{*f}))} \gamma_{kf}(s_{*f}) u_k(a_f,a_{-f},s_{k*})$, and the Implicit Function Theorem gives that around any f-unanimous s,

$$\begin{aligned} &\operatorname{sgn}\{\nabla_{ds} R_f(a_{-f},s)\} \\ &= \operatorname{sgn}\left\{\left((1-\psi)\gamma_{if}(s_{*f}) - \psi \sum_{j \in N_f(s_{*f}) \setminus \{i\}} \gamma_{jf}(s_{*f})\right) \left.\frac{\partial \pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)}\right\}. \end{aligned} \tag{5}$$

We will prove each direction separately.

$\Longleftarrow$: Let R_f be WAPP, and assume that $\gamma_{if}(s_{*f}) = \psi$. We need to show that the stock trade is neutral. Take arbitrary $a_{-f} \in A_{-f}$. (5) then gives

$$\operatorname{sgn}\{\nabla_{ds} R_f(a_{-f},s)\} = \operatorname{sgn}\left\{((1-\psi)\psi - \psi(1-\psi))) \left.\frac{\partial \pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)}\right\} = 0.$$

$\Longrightarrow$: Let R_f be WAPP, and assume that the stock trade is neutral. We need to show that $\gamma_{if}(s_{*f}) = \psi$. By assumption, there exists $a_{-f} \in A_{-f}$ such that $\partial \pi_g(a_f,a_{-f})/\partial a_f|_{a_f=R_f(a_{-f},s)} \neq 0$. Then, (5) implies $\gamma_{if}(s_{*f}) = \psi$.

(ii-a) Under NB, the Implicit Function Theorem gives that around any f-unanimous s

$$\begin{aligned} &\operatorname{sgn}\{\nabla_{ds} R_f(a_{-f},s)\} \\ &= \operatorname{sgn}\left\{\left((1-\psi)\widetilde{\gamma}_{if}(a_{-f},s) - \psi \sum_{j \in N_f(s_{*f}) \setminus \{i\}} \widetilde{\gamma}_{jf}(a_{-f},s)\right) \left.\frac{\partial \pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)}\right\} \\ &= \operatorname{sgn}\left\{\left((1-\psi)\frac{\beta_{if}(s_{*f})}{s_{if}} - \psi \sum_{j \in N_f(s_{*f}) \setminus \{i\}} \frac{\beta_{jf}(s_{*f})}{s_{jf}}\right) \left.\frac{\partial \pi_g(a_f,a_{-f})}{\partial a_f}\right|_{a_f=R_f(a_{-f},s)}\right\}, \end{aligned}$$

where the second line follows as in the proof of Proposition 3. Then, the result follows as in part (i).

Similar arguments prove (i-b) and (ii-b). **Q.E.D.**

Proof of Proposition 4. (i) $\Rightarrow$: Assume that R_f has stock-trade monotone control, and take arbitrary s_{*f}, pair of firm f shareholders $i,j \in N_f(s_{*f})$, and $s'_{*f} = s_{*f} + t(\mathbf{e}_i - \mathbf{e}_j)$ for some $t \in [0, \min\{s_{jf}, 1 - s_{if}\}]$. Clearly, there exist $(s_{*g})_{g\neq f}$ and $(s'_{*g})_{g\neq f}$ such that s and s' are f-unanimous ownership matrices with $s'_{i*} = (1 + t/s_{if})s_{i*}$ and $s'_{j*} = s_{j*} - (t/s_{if})s_{i*}$. Given Lemma 1, starting from s, a $(\gamma_{if}(s_{*f}),g,i,N_f(s_{*f}) \setminus \{i\})$-stock trade is f-neutral. Also, starting from s', a $(\gamma_{if}(s'_{*f}),g,i,N_f(s'_{*f}) \setminus \{i\})$-stock trade is f-neutral. Thus, given that R_f has monotone control power, $\gamma_{if}(s'_{*f}) \geq \gamma_{if}(s_{*f})$.

$\Leftarrow$: Assume that for every s_{*f}, every pair of firm f shareholders $i,j \in N_f(s_{*f})$, and $t \in [0, \min\{s_{jf}, 1 - s_{if}\}]$, $\gamma_{if}(s_{*f} + t(\mathbf{e}_i - \mathbf{e}_j)) \geq \gamma_{if}(s_{*f})$. Now, take arbitrary $g \neq f$, pair of shareholders $i,j \in N$, and f-unanimous ownership matrix s. Assume that starting from s, a $(\psi,g,i,N_f(s_{*f}) \setminus \{i\})$-stock trade is f-neutral, and that starting from s' such that $s'_{k*} = s_{k*}$ for every $k \neq i,j$, $s'_{i*} = (1 + t/s_{if})s_{i*}$, and $s'_{j*} = s_{j*} - (t/s_{if})s_{i*}$ for some $t \in [0, \min_{g\in M:s_{ig}>0} \min\{1 - s_{ig}/s_{ig}, s_{jg}/s_{ig}\}]$, a $(\psi',g,i,N_f(s'_{*f}) \setminus \{i\})$-stock trade is f-neutral. s' is f-unanimous given that s is. Then, given Lemma 1, $\psi = \gamma_{if}(s_{*f})$ and $\psi' = \gamma_{if}(s'_{*f})$, where $s'_{*f} = s_{*f} + t(\mathbf{e}_i - \mathbf{e}_j)$. Therefore, $\psi' \geq \psi$.

Part (ii) follows similarly. **Q.E.D.**

Proof of Proposition 5. (i) $\Leftarrow$: Assume that there exists $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\delta(0) = 0$ and $\delta(x) > 0$ for every $x > 0$ such that for every s and $i \in N$, $\gamma_{if}(s_{*f}) = \delta(s_{if})/\sum_{j\in N} \delta(s_{jf})$. Clearly, R_f is anonymous. To see why it is inclusive, take arbitrary firm $g \neq f$, f-unanimous ownership matrix s, and pair of firm f shareholders $i,j \in N_f(s_{*f})$.[23] Lemma 1 then implies that a $(0,g,i,j)$-stock trade is not f-neutral since $\gamma_{if}(s_{*f})(1 - 0) = \delta(s_{if})/\sum_{k\in N} \delta(s_{kf}) > 0 = \gamma_{jf}(s_{*f}) \times 0$.

It remains to show that R_f satisfies IIS. Take arbitrary $g \neq f$, f-unanimous ownership matrices s and s', pair of shareholders $i,j \in N_f(s_{*f})$, and $\psi \in [0,1]$ such that $s'_{if} = s_{if}$ and $s'_{jf} = s_{jf}$. Take any (ψ,g,i,j)-stock trade that is f-neutral starting from s. Lemma 1 implies $\gamma_{if}(s_{*f})(1-\psi) = \gamma_{jf}(s_{*f})\psi$. Multiplying both sides by $\gamma_{jf}(s'_{*f})$, we get $\gamma_{if}(s_{*f})\gamma_{jf}(s'_{*f})(1-\psi) = \gamma_{jf}(s_{*f})\gamma_{jf}(s'_{*f})\psi$. Substituting $\gamma_{if}(s_{*f})\gamma_{jf}(s'_{*f}) = \gamma_{if}(s'_{*f})\gamma_{jf}(s_{*f})$ (which i implied by IIS) in the left-hand side, we get $\gamma_{if}(s'_{*f})\gamma_{jf}(s_{*f})(1 - \psi) = \gamma_{jf}(s_{*f})\gamma_{jf}(s'_{*f})\psi$. Given that $\gamma_{jf}(s_{*f}) = \delta(s_{jf})/\sum_{k\in N} \delta(s_{kf}) > 0$, we can divide both sides by $\gamma_{jf}(s_{*f})$, which gives $\gamma_{if}(s'_{*f})(1 - \psi) = \gamma_{jf}(s'_{*f})\psi$. Therefore, given Lemma 1, the (ψ,g,i,j)-stock trade is f-neutral also starting from s'.

$\Rightarrow$: Assume that R_f satisfies anonymity, inclusivity, and IIS. That R_f is WAPP implies that there exists $\gamma_{*f} : \Delta^n \to \Delta^n$ satisfying the conditions of definition 1. By anonymity,

[23] If f has only one shareholder, then R_f is automatically inclusive.

for every a_{-f}, s, and permutation matrix P,

$$\arg\max_{a_f \in A_f} \left\{ \sum_{k \in N} \gamma_{kf}(s_{*f}) u_k(a_f, a_{-f}, s_{k*}) \right\} = \arg\max_{a_f \in A_f} \left\{ \sum_{k \in N} \gamma_{kf}(Ps_{*f}) u_k(a_f, a_{-f}, (Ps)_{k*}) \right\}. \tag{6}$$

Also, pre-multiplying $\gamma_{*f}(s_{*f})$ and s by P simply relabels firm f shareholders, so that for every a_{-f}, s, and permutation matrix P, $R_f(a_{-f},s) = R_f(a_{-f},Ps)$, or equivalently,

$$\sum_{k \in N} \gamma_{kf}(s_{*f}) u_k(a_f, a_{-f}, s_{k*}) = \sum_{k \in N} (P\gamma_{*f}(s_{*f}))_k u_k(a_f, a_{-f}, (Ps)_{k*}). \tag{7}$$

(6) combined with (7) implies that without loss, we can let $\gamma_{*f} : \Delta^n \to \Delta^n$ be such that for every s_{*f} and permutation matrix P, $\gamma_{*f}(Ps_{*f}) = P\gamma_{*f}(s_{*f})$.

To see this, for every s_{*f}, define $\gamma'_{*f}(s_{*f}) \coloneqq P^{-1}(s_{*f})\gamma(P(s_{*f})s_{*f})$, where $P(s_{*f})$ is the permutation matrix that—by pre-multiplying s_{*f}—orders the entries of s_{*f} from largest to smallest, and $P^{-1}(s_{*f})$ is its inverse (which is the permutation matrix that reorders the entries back into their original order). We have then that for every a_{-f} and s,

$$\begin{aligned}
R_f(a_{-f},s) &= \arg\max_{a_f \in A_f} \left\{ \sum_{k \in N} \gamma_{kf}(s_{*f}) u_k(a_f, a_{-f}, s_{k*}) \right\} \\
&= \arg\max_{a_f \in A_f} \left\{ \sum_{k \in N} \gamma_{kf}(P(s_{*f})s_{*f}) u_k(a_f, a_{-f}, (P(s_{*f})s)_{k*}) \right\} \\
&= \arg\max_{a_f \in A_f} \left\{ \sum_{k \in N} P^{-1}(s_{*f})\gamma_{kf}(P(s_{*f})s_{*f}) u_k(a_f, a_{-f}, (P^{-1}(s_{*f})P(s_{*f})s)_{k*}) \right\} \\
&= \arg\max_{a_f \in A_f} \left\{ \sum_{k \in N} \gamma'_{kf}(s_{*f}) u_k(a_f, a_{-f}, s_{k*}) \right\},
\end{aligned}$$

where the second line follows from (6), the third from (7), and the fourth by definition of γ'_{*f}. Thus, R_f is WAPP with control power function γ'_{*f}, which for every s_{*f} and permutation matrix X, satisfies $\gamma'_{*f}(Xs_{*f}) \equiv P^{-1}(Xs_{*f})\gamma(P(Xs_{*f})Xs_{*f}) = XP^{-1}(s_{*f})\gamma(P(s_{*f})s_{*f}) \equiv X\gamma'_{*f}(s_{*f})$, where the equivalence relations follow by definition of γ'_{*f}, and the equality because $P^{-1}(Xs_{*f}) = XP^{-1}(s_{*f})$ and $P(Xs_{*f})Xs_{*f} = P(s_{*f})s_{*f}$ by definition of the function P.

Now, notice that for any s_{*f} and $i \in N_f(s_{*f})$, $\gamma_{if}(s_{*f}) > 0$. To see this, take arbitrary s_{*f} and $i \in N_f(s_{*f})$. Clearly, there exist $(s_{*g})_{g \neq f}$ such that s is f-unanimous. By inclusivity, for any firm $g \neq f$ and shareholder $j \in N_f(s_{*f}) \setminus \{i\}$, a $(0,g,i,j)$-stock trade is not f-neutral, which by Lemma 1 means that $\gamma_{if}(s_{*f}) \neq 0$, so $\gamma_{if}(s_{*f}) > 0$.

Now, take arbitrary s_{*f}, s'_{*f}, and pair of shareholders $i,j \in N_f(s_{*f})$ such that $s'_{if} = s_{if}$ and $s'_{jf} = s_{jf}$. Clearly, there exist $(s_{*g})_{g \neq f}$ and $(s'_{*g})_{g \neq f}$ such that s and

s' are f-unanimous ownership matrices. Since R_f satisfies IIS, for any $\psi \in [0,1]$, if starting from s, a (ψ,g,i,j)-stock trade is f-neutral, then starting from s', a (ψ,g,i,j)-stock trade is again f-neutral. Given Lemma 1, (i) starting from s, a (ψ,g,i,j)-stock trade is f-neutral if and only if $\psi = \gamma_{if}(s_{*f})/(\gamma_{if}(s_{*f}) + \gamma_{jf}(s_{*f}))$, and (ii) starting from s', a (ψ',g,i,j)-stock trade is f-neutral if and only if $\psi' = \gamma_{if}(s'_{*f})/(\gamma_{if}(s'_{*f}) + \gamma_{jf}(s'_{*f}))$. Thus, $\gamma_{if}(s_{*f})/(\gamma_{if}(s_{*f}) + \gamma_{jf}(s_{*f})) = \gamma_{if}(s'_{*f})/(\gamma_{if}(s'_{*f}) + \gamma_{jf}(s'_{*f}))$, which implies $\gamma_{if}(s_{*f})/\gamma_{jf}(s_{*f}) = \gamma_{if}(s'_{*f})/\gamma_{jf}(s'_{*f})$. This means that $\gamma_{if}(s_{*f})/\gamma_{jf}(s_{*f})$ depends on s_{*f} only through s_{if} and s_{jf}. Therefore, there exists function $h_{ij} : \{(x,y) \in \mathbb{R}^2_{++} : x + y \leq 1\} \to \mathbb{R}_{++}$ such that $\gamma_{if}(s_{*f}) = h_{ij}(s_{if},s_{jf})\gamma_{jf}(s_{*f})$ for every s_{*f}. Given that for every s and P, $\gamma_{*f}(Ps_{*f}) = P\gamma_{*f}(s_{*f})$, we can drop the subscript ij from h; namely, there exists $h : \{(x,y) \in \mathbb{R}^2_{++} : x + y \leq 1\} \to \mathbb{R}_{++}$ such that $h(s_{if},s_{jf}) = \gamma_{if}(s_{*f})/\gamma_{jf}(s_{*f})$ for every s_{*f} and every $i,j \in N_f(s_{*f})$. Notice that for every s_{*f} and every $i,j,k \in N_f(s_{*f})$

$$h(s_{if},s_{jf}) = \frac{\gamma_{if}(s_{*f})}{\gamma_{jf}(s_{*f})} = \frac{\gamma_{if}(s_{*f})/\gamma_{kf}(s_{*f})}{\gamma_{jf}(s_{*f})/\gamma_{kf}(s_{*f})} = \frac{h(s_{if},s_{kf})}{h(s_{jf},s_{kf})}.$$

This means that for every $x,y,z > 0$ such that $x + y + z \leq 1$, $h(x,y) = h(x,z)/h(y,z)$. In fact, this equation must hold more generally. To see this, take arbitrary $x,y,z > 0$ such that $x + y < 1$, $x + z < 1$ and $y + z < 1$. It then holds that

$$\begin{aligned} h(x,y) &= \frac{h(x,1-\max\{x+y,x+z,y+z\})}{h(y,1-\max\{x+y,x+z,y+z\})} \\ &= \frac{h(x,1-\max\{x+y,x+z,y+z\})/h(z,1-\max\{x+y,x+z,y+z\})}{h(y,1-\max\{x+y,x+z,y+z\})/h(z,1-\max\{x+y,x+z,y+z\})} \\ &= \frac{h(x,z)}{h(y,z)}, \end{aligned} \tag{8}$$

where the first and third lines follow from $h(x,y) = h(x,z)/h(y,z)$ holding for every $x,y,z > 0$ such that $x + y + z \leq 1$.

Now, define $\delta : [0,1] \to \mathbb{R}_+$ given by

$$\delta(x) := \begin{cases} 0 & \text{if } x = 0 \\ h(x, 1/5) & \text{if } x \in (0, 3/4] \\ \frac{h(x,(1-x)/5)}{h(1/5,(1-x)/5)} & \text{if } x \in (3/4, 1) \end{cases}$$

and satisfying $\delta(x)/\delta(y) = h(x, y)$ for all $x, y \in (0,1)$ such that $x + y < 1$. To see this, notice that:

1. If $x, y \in (0, 3/4]$, then from (8) it follows that

$$\frac{\delta(x)}{\delta(y)} = \frac{h(x, 1/5)}{h(y, 1/5)} = h(x, y).$$

2. If $x \in (3/4, 1)$ (and thus $y \in (0, 1/4)$), then

$$\frac{\delta(x)}{\delta(y)} = \frac{\frac{h(x,(1-x)/5)}{h(1/5,(1-x)/5)}}{h(y,1/5)} = \frac{h(x,(1-x)/5)}{h(1/5,(1-x)/5)h(y,1/5)},$$

where $h(1/5,(1-x)/5) = 1/h((1-x)/5,1/5)$ and, given (8), $h(x,(1-x)/5) = h(x,y)/h((1-x)/5,y)$, so

$$\frac{\delta(x)}{\delta(y)} = \frac{h(x,y)}{h((1-x)/5,y)} \frac{h((1-x)/5,1/5)}{h(y,1/5)} = \frac{h(x,y)}{h((1-x)/5,y)} h((1-x)/5,y) = h(x,y),$$

where the second equality also follows from (8).

3. If $y \in (3/4, 1)$ (and thus $x \in (0, 1/4)$), then $\delta(x)/\delta(y) = (\delta(y)/\delta(x))^{-1} = (h(y,x))^{-1} = h(x,y)$, where the second equality follows from the previous case.

We have then that for every s_{*f} and distinct $i,j \in N_f(s_{*f})$ such that $s_{if} + s_{jf} < 1$, $\gamma_{if}(s_{*f})/\gamma_{jf}(s_{*f}) = \delta(s_{if})/\delta(s_{jf})$. This equality also automatically holds when $j \in N_f(s_{*f})$ but $i \notin N_f(s_{*f})$. Therefore, for every s_{*f} such that $|N_f(s_{*f})| \geq 3$ and every $j \in N_f(s_{*f})$

$$1 = \sum_{i \in N} \gamma_{if}(s_{*f}) = \sum_{i \in N} \frac{\delta(s_{if})}{\delta(s_{jf})} \gamma_{jf}(s_{*f}) = \frac{\gamma_{jf}(s_{*f})}{\delta(s_{jf})} \sum_{i \in N} \delta(s_{if}) \implies \gamma_{jf}(s_{*f}) = \frac{\delta(s_{jf})}{\sum_{i \in N} \delta(s_{if})}.$$

Also, for $j \notin N_f(s_{*f})$, it automatically holds that $\gamma_{jf}(s_{*f}) = \delta(s_{jf})/\sum_{i \in N} \delta(s_{if}) = \delta(0)/\sum_{i \in N} \delta(s_{if}) = 0$. **Q.E.D.**

Online Appendix

Corporate control under common ownership

Orestis Vravosinos

B The weighted average profit weight strategic plan (WAPW)

In Brito et al.'s (2023) voting model, when the profit relevance of shareholder bias parameter is equal to 1, the authors frame the strategic plan as—what I call—a weighted average profit weight strategic plan (WAPW).

Definition 15. Firm f's strategic plan R_f is a weighted average profit weight (WAPW) if there exists a control power function $\widehat{\gamma}_{*f} : \Delta^n \to \Delta^n$ such that for every $s \in S$ and $\alpha_{-f} \in \times_{g \neq f} \Delta(A_{-g})$

(i) (*weighted sum of firm profit maximization with weighted average profit weights*)

$$R_f\left(\alpha_{-f}, s\right) = \underset{\alpha_f \in \Delta(A_f)}{\arg\max} \left\{ \pi_f(\alpha_f, \alpha_{-f}) + \sum_{g \in M \setminus \{f\}} \left(\sum_{i \in N_f(\gamma_{*f})} \widehat{\gamma}_{if}(s_{*j}) \lambda_{i;fg} \right) \pi_g(\alpha_f, \alpha_{-f}) \right\},$$

where $N_f(\gamma_{*f}(s_{*f})) \equiv \{i \in N : \widehat{\gamma}_{if}(s_{*f}) > 0\}$,

(ii) (*control exclusive to shareholders*) for every $i \in N$, $s_{if} = 0 \implies \widehat{\gamma}_{if}(s_{*f}) = 0$.

In WAPW, the weight that the manager of firm f places on firm g's profit is a weighted average of the weights $\{\lambda_{i;fg}\}_{i \in N_f(s_{*f})}$ that the shareholders of firm f would want firm f to use. This still is WAPP, since it can be written as

$$R_f\left(\alpha_{-f}, s\right) = \underset{\alpha_f \in \Delta(A_f)}{\arg\max} \left\{ \sum_{i \in N_f(\widehat{\gamma}_{*f})} \gamma_{if}(s_{*f}) u_i(\alpha_f, \alpha_{-f}, s_{i*}) \right\},$$

where for every shareholder i of firm f

$$\gamma_{if}(s_{*f}) \coloneqq \frac{\widehat{\gamma}_{if}(s_{*f})/s_{if}}{\sum_{i \in N_f(\widehat{\gamma}_{*f})} \widehat{\gamma}_{if}(s_{*f})/s_{if}}.$$

Thus, a strategic plan is WAPP if and only if it is WAPW. Therefore, under WAPW, there is tension between (i) allowing for atomistic shareholders to collectively exert control over the firm and (ii) allowing for large shareholders to have control power. Also, the distribution of power across shareholders within the firm is independent of external factors.

The novelty is that the WAPW parametrizations considered in Brito et al. (2023) give rise to γ's that are not standard in the literature. If all shares have voting rights, proportional $\widehat{\gamma}$'s give rise to

$$\gamma_{if}(s_{*j}) = \begin{cases} 1/|N_f(s_{*f})| & \text{if } s_{if} > 0 \\ 0 & \text{if } s_{if} = 0. \end{cases}$$

That is, firm f maximizes the *unweighted* average of its shareholders' portfolio profits. Section 4.1 discusses the implications of this WAPP formulation. Banzhaf $\widehat{\gamma}$'s give rise to

$$\gamma_{if}(s_{*f}) := \begin{cases} \frac{\gamma^B_{if}(s_{*f})/s_{if}}{\sum_{j\in N_f(s_{*f})} \gamma^B_{jf}(s_{*f})/s_{jf}} & \text{if } s_{if}(s_{*f}) > 0 \\ 0 & \text{if } s_{if}(s_{*f}) = 0. \end{cases}$$

where

$$\gamma^B_{if}(s_{*f}) = \frac{\left|\left\{T \in 2^N : \sum_{k\in T} s_{kf} \geq 1/2 > \sum_{k\in T\setminus\{i\}} s_{kf}\right\}\right|}{\sum_{t\in N}\left|\left\{T \in 2^N : \sum_{k\in T} s_{kf} \geq 1/2 > \sum_{k\in T\setminus\{t\}} s_{kf}\right\}\right|}.$$

C The random dictatorship disagreement payoff function

The random dictatorship specification of the disagreement payoff function poses that in case of disagreement in a firm, the shareholders' payoffs are derived from random dictatorship: With some exogenous probability, each shareholder of the firm is chosen to implement her most preferred strategy.

Definition 16. The disagreement payoff function d_{*f} is a random dictatorship (RD) disagreement payoff function if there exist a lottery weight function $\delta_{*f} : \Delta^n \to \Delta^n$ and a choice function (in case of disagreement) $\alpha^d_f : \times_{g\neq f}\Delta(A_k) \times \{v \in \mathbb{R}^m_+ : v_f = 1\} \to \Delta(A_f)$ such that

(i) (*the choice function α^d_f for firm f is a selection from the correspondence that takes as arguments the other firms' strategies α_{-f} and a vector v of relative weights on firms' profits (with the weight on firm f's profit normalized to* 1*) and returns the firm f strategies that maximize the payoff of a shareholder with relative holdings v in the firms*) for every $v \in \{v' \in \mathbb{R}^m_+ : v'_f = 1\}$[24]

$$\alpha^d_f\,(\alpha_{-f},v) \in \arg\max_{\alpha_f\in\Delta(A_f)} \sum_{g\in M} v_g\pi_g(\alpha_f,\alpha_{-f}),$$

[24]Notice that the choice function $\alpha^d_f(\alpha_{-f}, v)$ does not depend on the absolute size of a shareholder's stakes in the firms but only on her relative holdings v. This makes sense because $\arg\max_{\alpha_f\in\Delta(A_f)} \sum_{g\in M} v_g\pi_g(\alpha_f,\alpha_{-f})$ does not change if the objective function is multiplied by a positive constant. Also, notice that the choice function is not shareholder-specific. That is, all shareholders with the same relative holdings v choose the same strategy to be implemented by firm f in case of disagreement (if they are chosen by the lottery to make a decision). Of course, both of these properties are automatically satisfied when $\arg\max_{\alpha_f\in\Delta(A_f)} \sum_{g\in M} v_g\pi_g(\alpha_f,\alpha_{-f})$ is a singleton.

and for every $s \in S$ and $\alpha_{-f} \in \times_{g \neq f} \Delta(A_g)$

(ii) (*disagreement payoffs derived from random dictatorship*)

$$d_{*f}(\alpha_{-f},s) = \sum_{i \in N_f(\delta_{*f})} \delta_{if}(s_{*f}) u\left(\alpha_f^d\left(\alpha_{-f},\lambda_{i;f*}\right),\alpha_{-f},s\right),$$

where $N_f(\delta_{*f}) \equiv \{i \in N : \delta_{if}(s_{*f}) > 0\}$ and $\lambda_{i;f*} \equiv s_{i*}/s_{if}$,

(iii) (*control exclusive to shareholders*) for every $i \in N$, $s_{if} = 0 \implies \delta_{if}(s_{*f}) = 0$.

D Equilibrium existence

Under NB, the equilibrium is a Nash equilibrium in Nash bargains. Particularly, the oligopoly game can be seen as a generalized game where a firm's strategy set depends on the other firms' strategies. Namely, when the other firms play α_{-f}, firm f can choose a strategy in $B_f(\alpha_{-f},s)$, because it needs to make sure that each controlling shareholder achieves at least her disagreement payoff. Proposition 6 provides sufficient conditions for existence of a pure equilibrium of this generalized game.

Proposition 6. Fix an $s \in S$. If for every firm $f \in M$

(i) A_f is a nonempty, compact and convex subset of a Euclidean space,

(ii) $\pi_f(a)$ is continuous in a,

(iii) for each $i \in N$, $d_{if}(a_{-f},s)$ is continuous in a_{-f},

(iv) $B_f^P(a_{-f})$ is lower hemicontinuous in a_{-f} over $a_{-f} \in \widetilde{A}_{-f}$,[25]

(v) $\pi_f(a_f,a_{-f})$ is concave in a_f for every $a_{-f} \in A_{-f}$,[26]

where $B_f^P(a_{-f}) := \{a_f \in A_f : u_i(a_f,a_{-f},s_{i*}) \geq d_{if}(a_{-f},s)\ \forall i \in N_f(\beta_{*f}(s_{*f}))\}$ and $\widetilde{A} := \{a \in A : a_f \in B_f^P(a_{-f})\ \forall f \in M\}$. Then, a pure Nash equilibrium in Nash bargains exists.

Lemma 2 provides conditions for assumption (iv) of Proposition 6 to hold.

Lemma 2. Fix an $s \in S$ and let condition (i) of Proposition 6 hold. For each firm $j \in M$ let the strategic plan R_j be $\mathrm{NB}_{\beta_{*j},d_{*j}}$. $B_j^P(a_{-j})$ is lower hemicontinuous in $a_{-j} \in \widetilde{A}_{-j}$ if any of the following three conditions hold.

(i) For every $j \in M$, conditions (ii) and (v) of Proposition 6 hold, and for every $a_{-j} \in \widetilde{A}_{-j}$ there exists $a_j \in A_j$ such that $u(a_j,a_{-j},s) \gg d_{*j}(a_{-j},s)$.

[25] Lemma 2 in the Appendix provides sufficient conditions for condition (iv) to hold.
[26] Assumption (v) guarantees that the Nash product is quasi-concave in a_f.

(ii) For every $j \in M$, conditions (ii) and (iii) of Proposition 6 hold and for every $a_{-j} \in \widetilde{A}_{-j}$, $B_j^P(a_{-j},s) \subseteq \mathrm{cl}(\{a_j \in A_j : u(a_j,a_{-j},s) \gg d_{*j}(a_{-j},s)\})$.

(iii) For every $j \in M$, $\widetilde{A}_j \subset \mathbb{R}^{r_j}$ is an r_j-dimensional compact and convex polytope.

E Competitive effects of common ownership and policy implications

This section shows that WAPP and NB can give rise to significantly different theoretical predictions and policy implications. Specifically, I look at how market outcomes change as a shareholder varies the degree of diversification of a fixed number of shares across the industry.

Consider a homogeneous product Cournot duopoly ($m = 2$) with 3 shareholders ($n = 3$), linear inverse demand $P(Q) = \max\{10 - Q,0\}$ and symmetric linear cost functions $C_1(q_1) = q_1$, $C_2(q_2) = q_2$. Under both NBRD and WAPP, let control be proportional $\beta(s) = \gamma(s) = \delta(s) = s$, and the ownership structure be

$$s = \begin{bmatrix} s_{11} & 0.45 - s_{11} \\ 1 - s_{11} & 0 \\ 0 & 0.55 + s_{11} \end{bmatrix},$$

which is indexed by the shares s_{11} of shareholder 1 in firm 1.

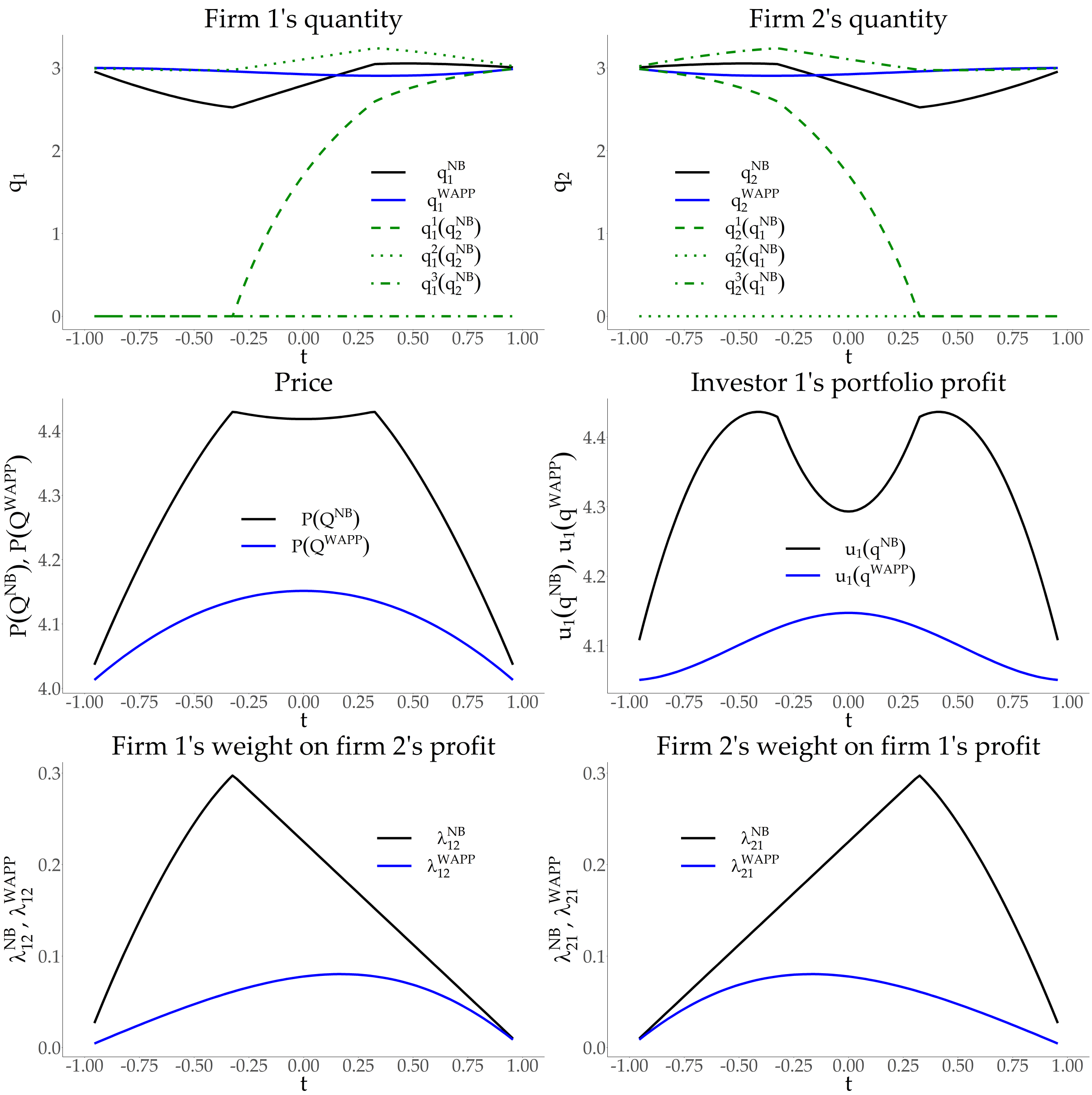
The two firms are equally efficient and shareholder 1 (e.g., a large fund) can choose how to distribute her total holdings of 0.45 in the industry between the two firms. Shareholders 2 and 3 are passive in that they are indifferent towards the capital they invest in the firms. The fund can buy shares of either firm at the same price and the rest of the capital is provided by shareholders 2 and 3. Define the normalized value $t := (s_{11} - 0.225)/0.225 \in [-1,1]$ measuring what firm the fund's holdings are concentrated in. The closer t is to 0, the higher is the fund's diversification; for $t = 0$ the equilibrium is symmetric. As t increases shareholder 1's holdings become more concentrated in firm 1.

Think of a policy that limits the degree of common ownership a shareholder can have within the industry; it specifies some $\tau \in [0,1]$ and requires that $t \in [-1, -\tau] \cup [\tau,1]$. Figure 2 shows equilibrium results under NBRD and WAPP.

If the fund only cares to maximize its portfolio profit, then under WAPP it will choose t as close to 0 as possible. Thus, the price is decreasing in the restrictiveness τ of the policy. However, under NBRD the fund picks t as close as possible to either of the two peaks (in its portfolio profit) as possible, so that the price is first constant and then decreasing in τ. Therefore, a policy that is effective in increasing consumer welfare under WAPP may be ineffective under NBRD.[27]

[27]Remember that consumer surplus is increasing in the total quantity (and thus decreasing in the price)

Figure 2: Equilibrium with a large fund and two undiversified passive shareholders for varying levels of diversification by the fund



Note: black lines represent equilibrium values under NBRD; blue ones under WAPP. Green lines show the most preferred quantity of each shareholder for each firm with the competitor's quantity taken as given (fixed at its equilibrium value). The bottom two panels plot λ_{12},λ_{21} (under WAPP) and $\widetilde{\lambda}_{12}$,$\widetilde{\lambda}_{21}$ (under NBRD).

Consider now an alternate scenario where the fund only cares to maximize its portfolio diversification, that is $\min |t|$, in order for example to mitigate risk or track an industry index. Then, under WAPP, the price is decreasing in τ. However, under NBRD, the price is first increasing and then decreasing in τ. Thus, a policy that is effective under WAPP may in fact harm consumer welfare under NBRD.

The differences in predictions between WAPP and NBRD are due to the differences (between the two strategic plans) in magnitudes of the various channels through which a change in t affects equilibrium outcomes. As t changes, both the fund's preferences and the division of power within each firm change.

Under WAPP, as t (*i.e.*, s_{11}) increases, the degree to which the fund wants firm 1 (resp. 2) to internalize firm 2's (resp. 1's) profits decreases (resp. increases), which tends to shift production towards firm 1. On the other hand, as t increases shareholder 2's control of firm 1 decreases, and shareholder 3's control of firm 2 increases, which tend to shift production towards firm 2. Under WAPP, around $t = 0$, the latter effects dominate, so that firm 2's quantity increases with t, while the quantity of firm 1 decreases making it unprofitable for the fund to pick $t \neq 0$. Also, firm 1's quantity increases faster than firm 2's quantity decreases with t (around $t = 0$), and the price has a global maximum at $t = 0$ under.

However, under NBRD, as t increases (around $t = 0$), production shifts towards firm 1, which is in the interest of the fund when $t > 0$. This makes it profitable for the fund to pick $t \neq 0$. Also, firm 1's quantity increases more slowly than firm 2's quantity decreases with t (around $t = 0$), so that the price has a local minimum at $t = 0$ under NBRD.

Similarly, based on WAPP a consumer-welfare-maximizing regulator would want to block a trade that brings t from -0.25 to 0, even though this trade would increase consumer welfare under NBRD.

Last, notice that the graphs of control weights γ and $\tilde{\gamma}$ differ between WAPP and NBRD. These weights capture the extent to which changes in shareholder preferences (e.g., due to a stock trade) will be accommodated by each firm. Thus, the WAPP and NBRD models will give different predictions regarding stock trade effects.

F Application: homogeneous product Cournot oligopoly

This section characterizes the Nash-in-Nash equilibrium of a homogeneous product Cournot oligopoly and studies how changes in corporate control affect equilibrium outcomes.[28]

in a homogeneous product market.

[28]As seen in section 3, the analysis is also valid under WAPP.

F.1 A Nash-in-Nash model of Cournot oligopoly with common ownership

There is a set N of n firms producing a homogeneous good. Each firm f chooses its production quantity q_f simultaneously with the other firms. Denote by $w_f \equiv q_f/Q$ firm f's market share of the total quantity $Q \coloneqq \sum_{g=1}^{n} q_g$. q_{-f} denotes the production profile of the firms other than f, and $Q_{-f} \coloneqq \sum_{g\neq f}^{n} q_g$. Firm f's production cost is given by the twice-differentiable function $C_f : \mathbb{R}_+ \to \mathbb{R}_+$ with $C_f'(q_f) > 0$ globally.

The twice-differentiable inverse demand function $P(Q)$ satisfies $P'(Q) < 0 \ \forall Q \in [0,\overline{Q})$, where $\overline{Q} \in (0, +\infty]$ is such that $P(Q) > 0 \iff Q \in [0,\overline{Q})$. $\eta(Q) \coloneqq -P/(QP')$ denotes the elasticity of demand. Firm f's profit is given by $\pi_f(q) \coloneqq q_f P(Q) - C_f(q_f)$.

Define the following index of the weight firm f places on other firms' profits

$$\overline{\lambda}_j(q,s) \coloneqq \sum_{g\in M\setminus\{f\}} w_g \widetilde{\lambda}_{fg}(q_{-f},s) \equiv \sum_{g\in M\setminus\{f\}} w_g \frac{\sum_{i\in N_f(\beta_{*f}(s_{*f}))} \widetilde{\gamma}_{if}(q_{-f},s) s_{ig}}{\sum_{i\in N_f(\beta_{*f})} \widetilde{\gamma}_{if}(q_{-f},s) s_{if}}.$$

Similarly, for each firm f and each shareholder i of firm f define $\overline{\lambda}_{i;f}(q,s_{i*}) \coloneqq \sum_{g\in M\setminus\{f\}} w_g \lambda_{i;jg}$, an index of the weight shareholder i wants firm f to place "on average" on other firms' profits.

Define also the bargaining-adjusted (i) Herfindahl-Hirschman Index (HHI) of market shares, (ii) MHHIΔ, and (iii) modified HHI, (iv) weighted average Lerner index LI, respectively given by

$$\mathrm{HHI}(q) \coloneqq \sum_{g\in M} w_g^2, \quad \mathrm{MHHI\Delta}(q,s) \coloneqq \sum_{g\in M} w_g \overline{\lambda}_g(q,s),$$
$$\mathrm{MHHI}(q,s) \coloneqq \mathrm{HHI}(q) + \mathrm{MHHI\Delta}(q,s), \quad \overline{\mathrm{LI}}(q) \coloneqq \sum_{g=1}^{m} w_g \frac{P(Q) - C_g'(q_g)}{P(Q)}.$$

F.2 The firm's problem in a homogeneous-product Cournot market

Lemma 3 provides conditions under which in a Cournot oligopoly a shareholder's portfolio profit is strictly concave in a firm's quantity.

Lemma 3. Fix a shareholder $i \in N$ and a firm $j \in M$. If for every quantity profile q such that $Q < \overline{Q}$ it holds that

$$E(Q) \sum_{k\in M} s_{ik} w_k < 1 + s_{ij} \left(1 - \frac{C''(q_j)}{P'(Q)}\right),$$

where $E(Q) \coloneqq -P''(Q)Q/P'(Q)$ the (absolute value of the) elasticity of the slope of inverse demand, then for any q_{-j}, $u_i(q,s_{i*})$ is strictly concave in q_j for every q_j such that $Q < \overline{Q}$. A sufficient condition is

$$E(Q) < \frac{1 + s_{ij}}{\max_{k\in M} s_{ik}} \ \forall Q \in \left[0,\overline{Q}\right) \quad \text{and} \quad C''(q_j) \geq 0 \ \forall q_j.$$

Lemma 4 characterizes a firm's problem in a Cournot oligopoly.

Lemma 4. Assume that assumed there exists $\overline{q} > 0$ such that $P(q) < C_j(q)/q$ for every $q > \overline{q}$ and every firm $j \in M$. Fix a firm $j \in M$ and q_{-j} and let the strategic plan R_j be $\mathrm{NB}_{\beta_{*j},d_{*j}}$. Assume that for every shareholder $i \in N$, $u_i(q,s_{i*})$ is strictly concave in q_j. Then, the following statements are true:

(i) $B^P_j(q_{-j},s) := \{q_j \in A_j : u(q_j,q_{-j},s) \geq d_{*j}(q_{-j},s)\}$ is a closed interval,

(ii) $R_j(q_{-j},s)$ is a singleton,

(iii) the Nash product is increasing (resp. decreasing) in q_j for $q_j \overset{\text{(resp.>)}}{<} R_j(q_{-j},s)$, and

(iv) if $\exists q_j$ such that $d_i(q_{-j},s) < u_i(q_j,q_{-j},s_{i*})$ for every $i \in N_j(\beta_{*j})$, then $R_j(q-j,s)$ solves the FOC.

F.3 Nash-in-Nash equilibrium characterization

Let $\widetilde{S} \subseteq S$ be an open subset of S such that for every $s \in \widetilde{S}$, there is a unique and interior equilibrium q^* where $u(\mathrm{NB}_{\beta_{*f},d_{*f}}(q^*_{-f},s),q^*_{-f},s) \gg d_{*f}(q^*_{-f},s)$ for every firm $j \in M$. $q^* : \widetilde{S} \to \mathbb{R}^m_{++}$ returns this equilibrium as a function of s.[29] Similarly, write $Q^* \equiv \sum_{g\in M} q^*_g$, $w^*_f := q^*_f/Q^*$. To simplify notation, define also $\gamma^*_{if}(s) := \widetilde{\gamma}_{ij}(q^*_{-f}(s),s)$, $\lambda^*_{fg}(s) := \widetilde{\lambda}_{fg}(q^*_{-f}(s),s)$, $\overline{\lambda}^*_f(s) := \overline{\lambda}_f(q^*(s),s)$, $\overline{\lambda}^*_{i;f}(s) := \overline{\lambda}_{i;f}(q^*(s),s_{i*})$ for every shareholder $i \in N$ and pair of distinct firms $f,g \in M$. These functions give the equilibrium values of the corresponding objects as functions of the ownership structure. $q^*(s)$ is then pinned down by the following FOCs:

$$f(q,s) := \left(\sum_{i\in N_1(\beta_{*1})} \gamma^*_{i1}(s)\frac{\partial u_i(q,s_{i*})}{\partial q_1} \quad \dots \quad \sum_{i\in N_m(\beta_{*m})} \gamma^*_{im}(s)\frac{\partial u_i(q,s_{i*})}{\partial q_m}\right)\bigg|_{q=q^*(s)} = \mathbf{0}.$$

Denote the Jacobian of $f(q,s)$ (with respect to q) by $J(q,s)$. An interior, regular equilibrium is then defined as follows.

Definition 17. An equilibrium q^* is called interior and regular if (i) $q^* \gg \mathbf{0}$, (ii) for every firm $j \in M$, $d_{N_f(\beta_{*f})f}(q^*_{-f},s) \ll u_{N_f(\beta_{*f})}(q^*_f,q^*_{-f},s)$, and (iii) $J(q^*,s)$ is negative definite.

It is a maintained assumption that the equilibrium is interior and regular. Proposition 7 derives the equilibrium markup of each firm and the relationship between the weighted average Lerner index and the MHHI.

Proposition 7. In equilibrium for every firm $j \in M$ it holds that

$$\frac{P(Q^*) - C'_f(q^*_f)}{P(Q^*)} = \frac{w^*_f + \overline{\lambda}^*_f(s)}{\eta(Q^*)}.$$

The weighted average Lerner Index is $\overline{\mathrm{LI}}(q^*) = \mathrm{MHHI}(q^*,s)/\eta(Q^*)$.

[29] I will sometimes simply write q^* instead of $q^*(s)$.

F.4 Competitive effects of changes in corporate control

Consider an exogenous change in a shareholder's control power over a firm.

Definition 18. An exogenous increase (resp. decrease) in shareholder i's control over firm f at $s \in S \times \mathbb{R}^m_+$ is a change in the strategic plan of firm f so that $\beta_{if}(s_{*f})$ changes infinitesimally by $d\beta_{if} >$ (resp. $<$) 0 with all else kept constant.[30]

Proposition 8 then studies the effects of a change in a shareholder's control over a firm.

Proposition 8. An exogenous increase (resp. decrease) in shareholder i's control over firm f causes firm f's quantity to change in the direction (resp. direction opposite to the one) preferred by shareholder i, that is

$$\operatorname{sgn}\left\{\frac{dq_f^*}{d\beta_{if}}\right\} = \operatorname{sgn}\left\{\left.\frac{\partial u_i\left(q,s_{i*}\right)}{\partial q_f}\right|_{q=q^*}\right\} = \operatorname{sgn}\left\{\overline{\lambda}_f^*(s) - \overline{\lambda}_{i;f}^*(s)\right\}.$$

Proposition 8 shows that if a firm is underproducing (resp. overproducing) relative to a shareholder's preferences and that shareholder's control over that firm increases, then the firms quantity will increase (resp. decrease). The proposition also provides an intuitive measure of whether the firm is under- or overproducing relative to the shareholder's preferences. It underproduces (resp. overproduces) if its (local) weighted average Edgeworth coefficient $\overline{\lambda}_f^*(s)$ is higher (resp. lower) than the shareholder's weighted Edgeworth coefficient.

A policy proposal by Posner et al. (2017) is to require institutional investors to be passive if they accumulate large amounts of stock in multiple competing firms. Such a policy can be understood as setting $\beta_{if} = 0$ for an investment fund i and every firm f. Provided that total quantity changes in the same direction as firm f's quantity, this policy will indeed increase consumer welfare if $\overline{\lambda}_{i;f}^*(s) > \overline{\lambda}_f^*(s)$ along a path where β_{if}'s go to 0 for every firm f.[31]

G Proofs of supplementary results

Proof of Proposition 6. The game can be seen as a generalized game where the strategy constraint correspondence is $B_j^P(a_{-j},s) := \{a_j \in A_j : u(a_j,a_{-j},s) \geq d_{*j}(a_{-j},s)\}$. The proof is composed of three steps.

Step 1: $B_j^P(a_{-j},s)$ is

[30] For the entries of β_{*f} to still sum up to 1, the other entries clearly need to decrease. However, this is just a normalization that does not affect the analysis, so it is ignored. Also, notice that an exogenous increase (resp. decrease) in d_{if} will have the same qualitative effect as an increase (resp. decrease) in β_{if}.

[31] Under WAPP, the total quantity changes in the same direction as firm f's quantity if the game is aggregative and the slope of each firm's best response function is higher than -1 (see, e.g., Farrell and Shapiro, 1990; Vives, 1999). The game is aggregative if s is such that for every firm $j \in M$, $\lambda_{fg}(s) = \lambda_{fh}(s)$ for every pair of firms $g,h \in M \setminus \{f\}$.

(i) nonempty by property (i) of disagreement payoffs of NB strategic plans,

(ii) compact as a closed subset of a compact set (since u is continuous in a_j),

(iii) upper hemicontinuous in a_{-j}, as a closed-valued correspondence to a compact space (see, e.g., Corollary 9 in p.111, Aubin and Ekeland, 1984),

(iv) lower hemicontinuous in a_{-j} by assumption.

Also, the Nash product is continuous in a_j and a_{-j} given that u and d_{*j} are. It follows then by Berge's maximum theorem that $R_j(a_{-j},s)$ is an upper hemicontinuous, nonempty-valued and compact-valued correspondence.

Step 2: For any $i \in N$ and any $a_{-j} \in A_j$ we have that $u_i(\delta a_j + (1-\delta)a_j', a_{-j}, s_{i*}) - d_{ij}(a_{-j},s)$ is concave over $B_j^P(a_{-j},s)$. It follows that for any $i \in N_j(\beta_{*j})$ and any a_{-j}

$$(u_i(\delta a_j + (1-\delta)a_j', a_{-j}, s_{i*}) - d_{ij}(a_{-j},s))^{\beta_{ij}(s_{*j})}$$

is concave (and thus log-concave) over $B_j^P(\alpha_{-j},s)$, since $a_j \mapsto u_i(\delta a_j + (1-\delta)a_j', a_{-j}, s_{i*}) - d_{ij}(a_{-j},s)$ is concave and $x \mapsto x_{ij}^{\beta}(s_{*j})$ is concave and increasing. Thus,

$$\prod_{i \in N_j(s)} (u_i(a_j, a_{-j}, s_{i*}) - d_{ij}(a_{-j},s))^{\beta_{ij}(s_{*j})}$$

is log-concave over $B_j^P(a_{-j},s)$ as a product of log-concave functions (and thus also quasi-concave in a_j for every a_{-j}). The product is also continuous in a_j and a_{-j}, and given also that $B_j^P(a_{-j},s)$ is convex for any $a_{-j} \in A_{-j}$, it follows that $R_j(a_{-j},s)$ is convex-valued.

Step 3: $G(a) := \times_{j \in M} R_j(a_{-j},s)$ is an upper hemicontinuous, nonempty-, compact- and convex-valued correspondence since R_j is for each $j \in M$. By Kakutani's fixed point theorem, G admits a fixed point, which is an equilibrium. **Q.E.D.**

Proof of Lemma 2. Part (i) follows from Proposition 4.2 in Dutang (2013), which is an application of Theorem 5.9 in Rockafellar and Wets (1997). Part (ii) follows from Proposition 4.3 in Dutang (2013); see also Theorem 13 of Hogan (1973). Part (iii) follows from Corollary 2 in Maćkowiak (2006). A similar result is also given in Claim 2 of Banks and Duggan (2004). **Q.E.D.**

Proof of Lemma 3. The derivative of $u_i(q_j, q_{-j}, s_{i*})$ with respect to q_j is given by

$$\frac{\partial u_i\left(q_j, q_{-j}, s_{i*}\right)}{\partial q_j} = s_{ij}\left(P(Q) - C'(q_j)\right) + P'(Q) \sum_{k \in M} s_{ik} q_k,$$

and the second derivative by

$$\begin{aligned}\frac{\partial^2 u_i\left(q_j,q_{-j},s_{i*}\right)}{\partial q_j^2} &= (1+s_{ij})P'(Q) - s_{ij}C''(q_j) + P''(Q)\sum_{k\in M} s_{ik}q_k \\ &= P'(Q)\left[1+s_{ij}\left(1-\frac{C''(q_j)}{P'(Q)}\right) - E(Q)\sum_{k\in M} s_{ik}w_k\right],\end{aligned}$$

and the result follows. **Q.E.D.**

Proof of Lemma 4. Since for $q_j > \overline{q}$ profit becomes negative, we can constrain each firm to choose quantity $q_j \in [0,\overline{q}]$. From continuity of u_i in q_j and the definition of B_j^P it follows then that B_j^P is compact. Especially, given strict concavity of u_i in q_j for every i, it follows that B_j^P is convex, thus a closed interval. We distinguish the following two cases:

Case 1: Given that u_i is strictly concave in q_j for every i (so u_i can be equal to d_{ij} for at most 2 values of q_j in B_j^P), the only way that $\forall q_j \in B_j^P$ there exists $i \in N$ such that $d_{ij}(q_{-j},s) = u_i(q_j,q_{-j},s_{i*})$ is for B_j^P to be a singleton. By continuity of u_i in q_j, this means that $d_{ij}(q_{-j},s)$ is equal to $\max_{q_j} u_i(q_j,q_{-j},s_{i*})$ for some $i \in N$, and the relevant results follow.

Case 2: If $\exists q_j \in B_j^P$ such that $d_{*j}(q_{-j},s) \ll u(q_j,q_{-j},s)$, we have that for every $i \in N$ and every $q_j \in B_j^P(q_{-j},s)$

$$\begin{aligned}&\frac{\partial^2\left(u_i\left(q_j,q_{-j},s_{i*}\right)-d_{ij}(q_{-j},s)\right)^{\beta_{ij}(s_{*j})}}{\partial q_j^2}\\ &= -\frac{\beta_{ij}(s_{*j})\left(1-\beta_{ij}(s_{*j})\right)}{\left(u_i\left(q_j,q_{-j},s_{i*}\right)-d_{ij}(q_{-j},s)\right)^{2-\beta_{ij}(s_{*j})}}\left(\frac{\partial u_i\left(q_j,q_{-j},s_{i*}\right)}{\partial q_j}\right)^2\\ &\quad+\frac{\beta_{ij}(s_{*j})}{\left(u_i\left(q_j,q_{-j},s_{i*}\right)-d_{ij}(q_{-j},s)\right)^{1-\beta_{ij}(s_{*j})}}\frac{\partial^2 u_i\left(q_j,q_{-j},s_{i*}\right)}{\partial q_j^2} < 0,\end{aligned}$$

by strict concavity of u_i in q_j. Also, for every i, $(u_i(q_j,q_{-j},s_{i*}) - d_{ij}(q_{-j},s))^{\beta_{ij}(s_{*j})}$ is non-negative and not identically equal to zero over B_j^P. The results then follow from Theorem 4 in Kantrowitz and Neumann (2005). **Q.E.D.**

Proof of Proposition 7. The FOCs in equilibrium give:

$$P(Q^*) - C_j'(q_j^*) + P'(Q^*)\left[q_j^* + \sum_{k\in M\setminus\{j\}} \lambda_{jk}^*(s)q_k^*\right] = 0,$$

and the result follows. **Q.E.D.**

Proof of Proposition 8. The partial derivative of $f(q,s)$ with respect to β_{ij} is

$$\begin{aligned}\frac{\partial f(q,s)}{\partial \beta_{ij}} &= \left[\frac{\gamma_{ij}^*}{\beta_{ij}}\frac{\partial u_i\left(q,s_{i*}\right)}{\partial q_j} - \frac{1}{u_i - d_{ij}}\frac{1}{\sum_{h\in N_j(\beta_{*j})}\frac{\beta_{hj}}{u_h-d_{hj}}}\sum_{t\in N_j(\beta_{*j})}\gamma_{tj}^*\frac{\partial u_t\left(q,s_{t*}\right)}{\partial q_j}\right]\cdot \mathbf{e}_j\\ &= \frac{\gamma_{ij}^*}{\beta_{ij}}\frac{\partial u_i\left(q,s_{i*}\right)}{\partial q_j}\cdot \mathbf{e}_j,\end{aligned}$$

where $\mathbf{e}_j$ the m-dimensional standard unit vector with 1 in its j-th dimension. It follows by the Implicit Function Theorem that

$$\begin{aligned}\begin{pmatrix}\frac{dq_1^*}{d\beta_{ij}}\\ \frac{dq_2^*}{d\beta_{ij}}\\ \vdots\\ \frac{dq_m^*}{d\beta_{ij}}\end{pmatrix} &= -J^{-1}(q^*,s)\left.\frac{\partial u_i\left(q,s_{i*}\right)}{\partial q_j}\right|_{q=q^*}\cdot \mathbf{e}_j = -\left(\det\left(J\right)\right)^{-1}\frac{\partial u_i}{\partial q_j}\cdot \operatorname{adj}\left(J\right)\mathbf{e}_j\\ &= -\left(\det\left(J\right)\right)^{-1}\frac{\partial u_i}{\partial q_j}\cdot\begin{pmatrix}(-1)^{1+j}\det\left(J_{-j-1}\right)\\ (-1)^{2+j}\det\left(J_{-j-2}\right)\\ \vdots\\ (-1)^{m+j}\det\left(J_{-j-m}\right)\end{pmatrix},\end{aligned}$$

where the second equality follows from the Laplace expansion, adj(J) is the adjugate or classical adjoint of J, and J_{-j-k} is the J matrix with the j-th row and k-th column removed. Since J is negative definite

$$\begin{aligned}&\operatorname{sgn}\left\{\det\left(J\right)\right\} = -\operatorname{sgn}\left\{\det\left(J_{-j-j}\right)\right\} = \operatorname{sgn}\{(-1)^m\},\\ \text{so that}\quad &\operatorname{sgn}\left\{\frac{dq_j^*}{d\beta_{ij}}\right\} = \operatorname{sgn}\left\{(-1)^{2j}\frac{\partial u_i}{\partial q_j}\right\} = \operatorname{sgn}\left\{\left.\frac{\partial u_i\left(q,s_{i*}\right)}{\partial q_j}\right|_{q=q^*}\right\},\end{aligned}$$

where

$$\begin{aligned}\left.\frac{\partial u_i\left(q,s_{i*}\right)}{\partial q_j}\right|_{q=q^*} &= \sum_{h=1}^m s_{ih}\left.\frac{\partial \pi_h(q,s_{i*})}{\partial q_j}\right|_{q=q^*} = P(Q^*)\left[s_{ij}\frac{P(Q^*) - C_j'(q_j^*)}{P(Q^*)} - \frac{\sum_{h=1}^m s_{ih}w_h^*}{\eta(Q^*)}\right]\\ &= -Q^*P'(Q^*)\left[s_{ij}\left(w_j^* + \overline{\lambda}_j\right) - \sum_{h=1}^m s_{ih}w_h^*\right] = -Q^*P'(Q^*)s_{ij}\left(\overline{\lambda}_j^* - \overline{\lambda}_{i;j}^*\right),\end{aligned}$$

and the result follows. **Q.E.D.**

References

Banks, J. and Duggan, J. (2004). Existence of Nash equilibria on convex sets.

Dutang, C. (2013). Existence theorems for generalized Nash equilibrium problems. *Journal* *of Nonlinear Analysis and Optimization: Theory & Applications*, 4:115–126.

Hosomatsu, Y. (1969). Point-to-set maps in mathematical programming. *SIAM Review*, 15:591–603.

Kantrowitz, R. and Neumann, M. M. (2005). Optimization for products of concave functions. *Rendiconti del Circolo Matematico di Palermo*, 54:291–302.

Maćkowiak, P. (2006). Some remarks on lower hemicontinuity of convex multivalued mappings. *Economic Theory*, 28:227–233.

Rockafellar, R. T. and Wets, R. J.-B. (1997). Variational analysis, volume 317. SpringerVerlag.