1 Matrix Inverses

An *inverse* to a matrix A is a matrix that $AA^{-1} = A^{-1}A = I$.

1.1 Examples

- $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ are inverses
- Since $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we can write $B = A^{-1}$
- Not all matrices have inverses. $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ has no inverse
- Some $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$, but $BA \neq I$

1.2 Connection to Linear Transformations

Let T be a linear transformation with associated matrix A (in TFVS). Then the inverse transformation has associated matrix A^{-1}.

1.3 2x2 Inverse Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then if $ad - bc \neq 0$:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This is a special case of a more general formula (involving the determinant and adjugate matrix, which are complicated to compute but follow this form).

1.4 Using Gauss-Jordan to Find the Inverse

Suppose A has inverse A^{-1}. Then:

$$AA^{-1} = I \implies [A|I] \sim [I|A^{-1}]$$

So $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ as found before.

To find A^{-1}, we must solve the system $Ax = e_i$ for each standard basis vector e_i. Instead of solving $[A|e_1], [A|e_2], \ldots, [A|e_n]$ individually, we can streamline this process by solving $[A|I]$ all at once.

If A doesn't have n pivots (one in each row), then A^{-1} does not exist.

If A has n pivots, then the algorithm produces A^{-1}.

1.5 Summary: Finding Inverses for $n \times n$ Matrices

- If A doesn't have n pivots, then A^{-1} doesn't exist
- If A has n pivots, then the algorithm produces A^{-1}

1.6 Example: Find A^{-1}, if it exists

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}$$

Applying Gauss-Jordan elimination to $[A|I]$:

$$\begin{bmatrix} 2 & 1 & 3 & | & 1 & 0 & 0 \\ 2 & 6 & 8 & | & 0 & 1 & 0 \\ 6 & 8 & 18 & | & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & | & 0 & 1 & -1/3 \\ 0 & 1 & 0 & | & 1/2 & -1/4 & 1/4 \\ 0 & 0 & 1 & | & -1/6 & 1/12 & 1/12 \end{bmatrix}$$

Therefore,

$$A^{-1} = \begin{bmatrix} 0 & 1/2 & -1/6 \\ 1 & -1/4 & 1/12 \\ -1/3 & 1/4 & 1/12 \end{bmatrix}$$

1.7 Using Inverses to Solve Systems

If A exists, we can solve $Ax = b$ by multiplying both sides by A^{-1}:

$$\begin{aligned} A^{-1}Ax &= A^{-1}b \\ Ix &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

Note: $Ax = b$ has solution $x = A^{-1}b$. However, even if A has n pivots (so it's invertible), this solution is unique.