1 Inverse Matrices

An *inverse matrix* is invertible if there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.

1.1 Examples

- $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$ are inverses
- Since $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we can write $B = A^{-1}$
- Not all matrices have inverses. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has no inverse
- Similarly, $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq I$

1.2 Connection to Linear Transformations

$Ax = b$ has a unique solution if and only if A is invertible. The inverse transformation takes b back to x.

2 2x2 Inverse Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then if $ad - bc \neq 0$:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note: $\det A = ad - bc$, and $AA^{-1} = I$ and $A^{-1}A = I$

This is a special case of a more general formula (involving the determinant and adjugate matrix, which are complicated to compute but follow this idea).

2.1 Using Gauss-Jordan to Find the Inverse

Suppose A has inverse A^{-1}. Then:

$AA^{-1} = I$ $[A|I] \to [I|A^{-1}]$

So to find A^{-1}, we must solve the system $Ax = e_i$ for each i.

Instead of solving $[A|e_1], [A|e_2], \ldots, [A|e_n]$ individually, we can streamline the process and solve $[A|I]$ at once.

If A doesn't have n pivots (one in each row), then A^{-1} does not exist.

If A has n pivots, then the algorithm produces A^{-1}.

2.2 Summary for Gauss-Jordan Method

- If A doesn't have n pivots, then A^{-1} doesn't exist
- If A has n pivots, then we get A^{-1}

3 Example

Find A^{-1} if it exists:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}$$

$$[A|I] = \begin{bmatrix} 2 & 1 & 3 & | & 1 & 0 & 0 \\ 2 & 6 & 8 & | & 0 & 1 & 0 \\ 6 & 8 & 18 & | & 0 & 0 & 1 \end{bmatrix}$$

(Steps of row operations omitted for brevity)

$$[I|A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & | & 3 & -3 & 1 \\ 0 & 1 & 0 & | & -5 & 4 & -1 \\ 0 & 0 & 1 & | & 1 & -1 & 0 \end{bmatrix}$$

Therefore:

$$A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -5 & 4 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

(Check: verify $AA^{-1} = I$)

4 Using Inverses to Solve Systems

If A exists, we can solve $Ax = b$ by multiplying both sides by A^{-1}:

$$\begin{aligned} A^{-1}Ax &= A^{-1}b \\ Ix &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

Note: $A^{-1}b$ has solution x if b (whenever some A has n pivots), this solution is unique.