Inverses

An $n \times n$ matrix A is invertible if there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_n$

Ex: $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ are inverses

since $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = BA$. We can write $B = A^{-1}$.

Not all $n \times n$ matrices have inverses.

Ex: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has no inverse

since $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\forall$ B.

Connection to linear transformations

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with associated matrix A (so $T(v) = Av$ $\forall v \in \mathbb{R}^n$).

The inverse of T (if it exists) is a linear transformation whose associated matrix is A^{-1}

(you'll explore this in recitation)

2x2 Inverse Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then if $ad-bc \neq 0$,

then $$A^{-1} = \frac{1}{\underbrace{ad-bc}_{\text{determinant of } A}} \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\text{Adjoint of } A}$$

We'll discuss these later in the semester.

Proof: Show $AA^{-1} = I_2$ and $A^{-1}A = I_2$ ∎

There is a general $n \times n$ inverse formula involving the determinant and adjoint. $\left(A^{-1} = \frac{1}{\det A} \operatorname{adj} A\right)$
These are complicated to compute (we'll see this later)
For now, we'll find inverses in an algorithmic way

Using Gauss-Jordan to find the inverse (if it exists)

Suppose A has inverse A^{-1}.

Let $A^{-1} = [x_1 \cdots x_n]$ (so ith column is x_i)

Then $AA^{-1} = [Ax_1 \cdots Ax_n] = I_n$

So $Ax_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← ith entry (this vector is denoted by e_i)

$Ax_i = e_i$

So to find A^{-1}, we must solve the system $Ax_i = e_i$ $\forall i$. We can do this using Gauss-Jordan Elimination applied to $[A|e_i]$ $\forall i$.

Instead of solving $[A|e_1]$, $[A|e_2]$, ..., $[A|e_n]$ individually, we can streamline the process and run Gauss-Jordan once on the matrix

$$[A \mid e_1\, e_2 \cdots e_n] = [A \mid I]$$

If A does not have n pivots (one in each row) then $Ax = e_i$ has no solution for some i.

$\Rightarrow A^{-1}$ does not exist.

If A has n pivots, then the algorithm provides:
$$[I \mid x_1\, x_2 \cdots x_n] = [I \mid A^{-1}].$$

<u>Summary</u>: Apply Gauss-Jordan to $[A|I]$ (ie. put $[A|I]$ in RREF)

- If A doesn't have n pivots, then A^{-1} doesn't exist
- If A has n pivots, A^{-1} exists and the RREF is $[I|A^{-1}]$.

Ex: Let $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 0 & 4 \\ 0 & 1 & 3 \end{bmatrix}$. Find A^{-1}, if it exists.

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 4 & 1 & 0 & 0 \\ 4 & 4 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{w/ } R2 - 2R1]{\text{Replace } R2} \left[\begin{array}{ccc|ccc} 2 & 2 & 4 & 1 & 0 & 0 \\ 0 & 0 & -8 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{swap } R2 \,\&\, R3} \left[\begin{array}{ccc|ccc} 2 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & -8 & -2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[-1/8 \cdot R3]{1/2 \cdot R1} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1/2 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/4 & -1/8 & 0 \end{array}\right]$$

$$\xrightarrow[\text{Replace } R2 \text{ w/ } R2 - 3R3]{\text{Replace } R1 \text{ w/ } R1 - 2R3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1/4 & 0 \\ 0 & 1 & 0 & -3/4 & 3/8 & 1 \\ 0 & 0 & 1 & 1/4 & -1/8 & 0 \end{array}\right] \xrightarrow[R1 - R2]{\text{Replace } R1 \text{ w/}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/8 & -1 \\ 0 & 1 & 0 & -3/4 & 3/8 & 1 \\ 0 & 0 & 1 & 1/4 & -1/8 & 0 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3/4 & -1/8 & -1 \\ -3/4 & 3/8 & 1 \\ 1/4 & -1/8 & 0 \end{bmatrix}$$

(Check: Verify $AA^{-1} = I$)

Using inverses to solve systems

If A^{-1} exists, we can solve $Ax=b$ by multiplying by A^{-1}

$$A^{-1}(Ax) = A^{-1}b$$
$$(A^{-1}A)x = A^{-1}b$$
$$Ix = A^{-1}b$$
$$x = A^{-1}b$$

Hence $Ax=b$ has solution $A^{-1}b$. Moreover, since A has n pivots (as it is invertible), this solution is unique.