1 Inverse Matrices

An *inverse matrix* is invertible if there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.

1.1 Examples

- $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ are inverses
- Since $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we can write $B = A^{-1}$
- Not all matrices have inverses. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has no inverse.

1.2 Connection to Linear Transformations

A^{-1} is the inverse transformation with respect to matrix A (or T_A if A is viewed as the linear transformation). The inverse of T is a linear transformation whose associated matrix is A^{-1}.

2 2x2 Inverse Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then if $ad - bc \neq 0$:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This is a special non-unique formula (works only for 2x2). For larger matrices, it's more complicated to compute (we'll see this later).

2.1 Using Cross-Section to Find the Inverse

Suppose A has inverse A^{-1} Then $AA^{-1} = [A_1 A_2] \begin{bmatrix} | & | \\ x & y \\ | & | \end{bmatrix} = I_2$

So $A_1 x = e_1$ and $A_2 y = e_2$

To find A^{-1}, we must solve the system $Ax = e_i$ for each column vector of the identity matrix.

Instead of solving $[Ax], [Ay], \ldots, [Az]$ individually, we can streamline the process by solving one augmented matrix:

$$[A|e_1, e_2, \ldots]$$

If A doesn't have n pivots (one in each row), then A^{-1} does not exist.

If A has n pivots, then the algorithm produces A^{-1}.

2.2 Summary

- If A doesn't have n pivots, then A^{-1} doesn't exist
- If A has n pivots, then we can find A^{-1} using the RREF of $[A|I]$

2.3 Example

Find A^{-1}, if it exists:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ 3 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & | & -1 & 1 & 0 \\ 0 & 1 & 0 & | & 3 & -2 & -1 \\ 0 & 0 & 1 & | & 1 & -1 & 1 \end{bmatrix}$$

Therefore,

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 3 & -2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \quad \text{(Check: } AA^{-1} = I\text{)}$$

3 Using Inverses to Solve Systems

If A exists, we can solve $Ax = b$ by multiplying $A^{-1}b$:

$$\begin{aligned} A^{-1}Ax &= A^{-1}b \\ (A^{-1}A)x &= A^{-1}b \\ Ix &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

Here $A^{-1}b$ is a solution to $Ax = b$. However, even if A has n pivots (is $n \times n$ invertible), this solution is unique.