Stat 230 Chapter 9

Multivariate Distributions

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Abstract

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1 Basic Terminology and Techniques

Definition 1.1: Joint Probability Function

Let $X_1, \dots, X_k$ be discrete random variables. We call

$$f(x_1, \dots, x_k) = P(X_1 = x_1, \dots, X_k = x_k)$$

the **joint probability function** of $(X_1, \dots, X_k)$.

The joint probability function satisfy the following properties:

1. $\forall (x_1, \dots, x_k) \quad f(x_1, \dots, x_k) \geq 0$
2. $\sum f(x_1, \dots, x_k) = 1$

Definition 1.2: Marginal Probability Functions

If X, Y are two random variables, then

$$f_1(x) = \sum_{\forall y} f(x, y)$$
$$f_2(y) = \sum_{\forall x} f(x, y)$$

In general, we call

$$f_k(x_k) = P(X_k = x_k)$$

the **marginal probability function** of X_k.

Theorem 1.1: Independent Random Variables

Recall that, for events A and B, we have defined A and B to be independent if and only if $P(AB) = P(A)P(B)$. To extend this concept to random variables (X, Y), we define two random variables are independent if and only if **their joint probability function is the product of the marginal probability functions**, i.e.

$$\forall (x, y) \quad f(x, y) = f_1(x) f_2(y) \implies X \text{ and } Y \text{ are independent.}$$

Theorem 1.2: Conditional Probability Functions

Recall that, for events A and B, we have defined the conditional probability of event A given event B to be

$$P(A|B) = \frac{P(AB)}{P(B)}, \quad P(B) > 0.$$

Since

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}, \quad P(Y = y) > 0,$$

we define the conditional probability function of X given $Y = y$ to be

$$f_1(x|y) = \frac{f(x, y)}{f_2(y)}, \quad f_2(y) > 0.$$

Similarly, the conditional probability function of Y given $X = x$ is

$$f_2(y|x) = \frac{f(x, y)}{f_1(x)}, \quad f_1(x) > 0.$$

Theorem 1.3: Independent Poisson Random Variables

If $X \sim Poisson(\mu_1)$ and $Y \sim Poisson(\mu_2)$ independently, then

$$T = X + Y \sim Poisson(\mu_1 + \mu_2).$$

The probability function of T is

$$P(T = t) = \frac{(\mu_1 + \mu_2)^t}{t!} e^{-(\mu_1+\mu_2)}, \quad t = 0, 1, 2, \ldots$$

Theorem 1.4: Independent Binomial Random Variables

If $X \sim Binomial(n, p)$ and $Y \sim Binomial(m, p)$ independently, then

$$T = X + Y \sim Binomial(n + m, p).$$

The probability function of T is

$$P(T = t) = \binom{n+m}{t} p^t (1 - p)^{n+m-t}.$$

Remark 1

You can also prove **Theorem 1.3** and **Theorem 1.4** using the moment generating function (by showing that in each case T has the corresponding mgf for that distribution).

2 Multinomial Distribution

Definition 2.1: Multinomial Distribution

Suppose an experiment is repeated independently n times with k distinct types of outcome each time. Let the probability of these k types be $p_1, \ldots, p_k$ each time. Let X_i be the number of times the ith type occurs. Then $(X_1, \ldots, X_k)$ has a multinomial distribution:

$$f(x_1, \ldots, x_k) = \frac{n!}{x_1!x_2!\cdots x_k!}\, p_1^{x_1}p_2^{x_2}\cdots p_k^{x_k}.$$

Remark 2: Multinomial Distribution

1. $\sum_i p_i = 1, \sum_i x_i = n$, and we can drop one variable.
2. *Remark on the formula*: there are $\dfrac{n!}{x_1!x_2!\cdots x_k!}$ different outcomes of the n trials in which x_i are of ith type; each of these arrangement has probability $p_1^{x_1}p_2^{x_2}\cdots p_k^{x_k}$.

3 Expectation for Multivariate Distributions: Covariance and Correlation

Definition 3.1: Expected Value

We extend the definition of expected value to multivariate distributions:

$$E[g(X,Y)] = \sum_{(x,y)} g(x,y)f(x,y)$$
$$E[g(X_1,\ldots,X_n)] = \sum_{(x_1,\ldots,x_n)} g(x_1,\ldots,x_n)f(x_1,\ldots,x_n)$$

The most common one:

$$E[XY] = \sum_{(x,y)} xyf(x,y)$$

Theorem 3.1: Property of Multivariate Expectation

$$E[ag_1(X,Y)+bg_2(X,Y)] = aE[g_1(X,Y)]+bE[g_2(X,Y)]$$
$$X \text{ and } Y \text{ are independent} \implies E[g_1(X)g_2(Y)] = E[g_1(X)]E[g_2(Y)].$$

Definition 3.2: Covariance

The **covariance** of X and Y, denoted $Cov(X,Y)$ or σ_{XY}, is

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E(XY) - E(X)E(Y).$$

Theorem 3.2: Properties of Covariance

1. X and Y are independent $\implies Cov(X,Y)=0$.
2. $Cov(X,X) = Var(X)$.
3. $Cov(X,Y) = Cov(Y,X)$.
4. $Cov(aX+b,Y) = a\cdot Cov(X,Y)$.
5. $Cov(aX+bY,cU+dV) = acCov(X,U)+adCov(X,V)+bcCov(Y,U)+bdCov(Y,V)$.
6. $|Cov(X,Y)| \leq SD(X)\cdot SD(Y)$.

Theorem 3.3: Covariance vs. Variance

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y).$$

Definition 3.3: Covariance Coefficient

Like the covariance, the correlation coefficient measure the extent of linear relationship between X and Y. Unlike the covariance, it does not depend on the unit of measurement of X and Y.

$$\rho = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

$-1 \leq \rho \leq 1$, where ± 1 indicates perfect linear relationship; $\rho = 0$ indicates X and Y are independent (we say they are uncorrelated).

4 Mean and Variance of a Linear Combination of Random Variables

Theorem 4.1: Results for Means

1. $E(aX + bY) = aE(X) + bE(Y) = a\mu_X + b\mu_Y$. In particular, $E(X \pm Y) = \mu_X \pm \mu_Y$.
2. $E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i \mu_i$. In particular, $E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$.

Theorem 4.2: Results for Variance

1. Let X and Y be independent. Then $Var(X \pm Y) = \sigma_X^2 + \sigma_Y^2$.
2. Let $a_i \in \mathbb{R}$ and $Var(X_i) = \sigma_i^2$. Then
$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} a_i a_j Cov(X_i, X_j)$$
3. If $X_1, \ldots, X_n$ are independent, then
$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \sigma_i^2$$

5 Linear Combinations of Independent Normal Random Variables

Theorem 5.1: Linear Combinations of Independent Normal Random Variables

1. Let $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$, $a, b \in \mathbb{R}$. Then
$$Y \sim N(a\mu + b, a^2\sigma^2)$$
2. Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ independently, $a, b \in \mathbb{R}$. Then
$$aX + bY \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$$
3. Let $X_1, \ldots, X_n$ be independent $N(\mu, \sigma^2)$ random variables. Then
$$\sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2) \text{ and } \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N(\mu, \sigma^2/n)$$

6 Indicator Random Variables

Theorem 6.1: Mean and Variance of a Random Variable

Let X has some distribution. Define new random variables X_i by

$$X_i = \begin{cases} 0 & \text{if } i\text{th trial fails} \\ 1 & \text{if } i\text{th trial succeeds} \end{cases}$$

We can derive the mean and variance of X using X_i's.