LGIC 010 Textbook

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0 Prelude

0.1 Prerequisite Notation

Though the course has no specific mathematical prerequisites, a general familiarity with the set of integers and some of its basic properties will be assumed. We collect here some useful facts and notations that will appear from time to time throughout the course. We'll add more as the need arises.

1. Notations for important sets of numbers
   - $\mathbb{Z} = \{\ldots - 2, -1, 0, 1, 2, \ldots\}$ (the integers)
   - $\mathbb{N} = \{0, 1, 2, \ldots\}$ (the non-negative integers, a.k.a. the natural numbers)
   - $\mathbb{N}^+ = \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ (the positive integers)
2. Important facts about numbers
   - The Least Number Principle: If X is a nonempty subset of $\mathbb{N}$, then X has a least element.
   - Principle of Mathematical Induction: If X is a subset of $\mathbb{N}$, and $0 \in X$, and if for every i, $i \in X$ implies $i + 1 \in X$, then $X = \mathbb{N}$.
   - The Pigeonhole Principle: If you distribute m pigeons into n pigeonholes and $m > n$, then some hole contains more than one pigeon.
3. Unique Factorization into Primes: Recall that $p \in \mathbb{N}^+$ is *prime* if and only if $p \neq 1$ and p is divisible only by 1 and p. Every $n \in \mathbb{N}^+$ with $n \neq 1$ can be written uniquely (up to reordering) as $p_1^{a_1} \cdots p_n^{a_n}$ where each p_i is prime and each $a_i \geq 1$.

0.2 A Combinatorial Warmup

Combinatorics is, roughly, the part of mathematics which deals with counting things. Its techniques are general, and its results tangible. Throughout this book, we will use combinatorial problems as concrete examples of problems which can be considered and solved by means of logical techniques. To get our feet wet, let's consider the following principle and question.

Principle 1. ***The Pigeonhole Principle****: If you distribute m pigeons into n pigeonholes and $m \geq n + 1$, then some hole contains at least two pigeons.*

Example 1. *Is there a numerically diverse group of Philadelphians?*

(We call a group of people numerically diverse if no two people in the group have the same number of friends in the group - we assume groups are of size at least two and that friendship is always mutual.)

We will demonstrate that the answer is no by an application of the Pigeonhole Principle.

Proof. Suppose we have a group $G = \{1, \ldots, n\}$ of n people (we use numerals to name the people for privacy concerns). For brevity, let's write p_{ij} to signify that i is a friend of j. We assume friendship is symmetric, that is, if p_{ij}, then p_{ji}, for all $i, j \in G$, and irreflexive, that is, it is not the case that p_{ii}, for all $i \in G$. Let's write $f(i)$ for the number of friends of i, that is, the number of j such that p_{ji}. Since friendship is irreflexive, the possible values of f are the n numbers $0, 1, \ldots, n-1$. We are thinking of these values as the pigeonholes for application of the principle 1 and the members of G as being placed in these holes by f. We want to argue that the value of f must agree on at least two members of G. But so far, since we have n members of G and n pigeonholes into which they are sorted by f, we may not yet draw that conclusion via principle 1. But now we consider the question, "can f really take all the values from 0 to $n - 1$?" In particular, can it take on both the value 0 and the value $n - 1$? We argue that the answer is no. Suppose that there is some i with $f(i) = 0$, that is, for every j, it is not the case that p_{ji}. Then, by symmetry, for every j, it is not the case that p_{ij}. So, if i has no friends, then the maximum number of friends of any j is $n-2$, that is, f cannot take on the value $n-1$. Thus, the possible values of f are the $n-1$ numbers $0, \ldots, n-2$. But now, by principle 1, we can conclude that f takes on the same value for at least two members of G. This concludes our argument that there cannot be a numerically diverse group of Philadelphians. ☐

The above argument presupposes that there are finitely many Philedelphians. In fact, the theorem does not hold if we allow Philadelphia to have infinitely many people. As an exercise, try to describe a numerically diverse group of infinitely many Philadelphians.

The Pigeonhole Principle can take on a more general form, the *Mean Pigeonhole Principle*, which is as follows:

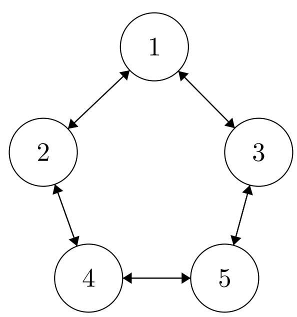
Principle 2. ***The Mean Pigeonhole Principle:*** *If you distribute m pigeons into n pigeonholes and $m \geq k \cdot n + 1$, then some hole contains at least $k+1$ pigeons.*

Note that Principle 1 is just the special case of Principle 2 for $k = 1$.

Example 2. *Say a group of people has three-mutuality if it contains either a group of three mutual friends or a group of three mutual strangers. How large a group of people can lack three-mutuality?*

We show that the largest such group has five members. To do this, we will give an example of a pattern of friendship among a group of five people that lacks three-mutuality, and show that every pattern of friendship among six or more people has three-mutuality. To show that every friendship pattern on six or more people lacks three-mutuality, we will use the Mean Pigeonhole Principle.

Proof. The diagram below shows a "friendship pentagon". Nodes represent people, and an edge between people represents friendship. It is easily checked that the diagram lacks 3-mutuality.



Next, we show that every group of $n \geq 6$ people must have 3-mutuality. Again, write p_{ij} to denote that i is a friend of j.

Let $G = \{1, \ldots, 6\}$ and sort the five people $2, \ldots, 6$ into two pigeonholes according to the truth value, true ($\top$) or false ($\bot$) of $p_{12}, \ldots, p_{16}$. That is, sort people $2, \ldots, 6$ into two groups, one group which are all friends of 1, and one group all of which are not friends with 1. By Principle 2, one of these holes, suppose it's the $\top$ one, contains at least three members of G.

Now, either two of these are friends, in which case they, together with 1 form a collection of three mutual friends, or none of them of friends, in which case they themselves form a collection of three mutual strangers. The argument is analogous in the case that three members of G were sorted into the $\bot$ pigeonhole. ☐

We might wonder whether every natural number n has a k such that every group of at least size k has n-mutuality. This happens to be true (try proving it!). The *Ramsey number* $R_{m,n}$ is the least k such that every set of k people must have either a group of m mutual friends or n mutual strangers. In the previous example, we showed that $R_{3,3} = 6$. Higher Ramsey numbers are much harder to compute. We know that

$R_{4,4} = 18$. $R_{5,5}$ is currently known to be between 43 and 48. $R_{6,6}$ is somewhere between 102 and 165.

As an exercise, prove that $R_{m,n} = R_{n,m}$ for all m, n.

"Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack." - Paul Erdos

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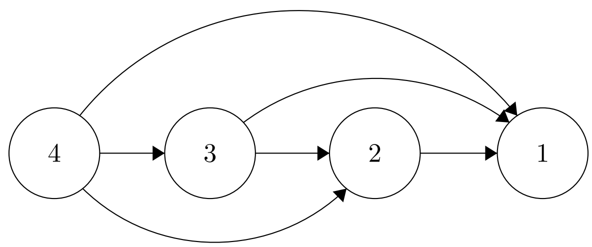
Love differs from friendship in that there are narcissists (so we can't assume the relation is irreflexive) and is not always requited (so we can't assume the relationship is symmetric). This difference between friendship and love allows the existence of numerically diverse groups of lovers, that is, groups where each person in the group loves a different number of people in the group. Consider, for example, a group of four people, $\{1, 2, 3, 4\}$. Suppose that 1 doesn't love anyone, 2 loves 1, 3 loves both 1 and 2, and 4 loves all of $1, 2$, and 3, and that this exhausts all the love among our group of four. We achieve numerical diversity at the sacrifice of requital.



How many different patterns of love might obtain among a group of four people $\{1, 2, 3, 4\}$? Let's recycle the sentence letters and use p_{ij} to signify the statement that i loves j; note that 16 sentence letters would be required to record all the relevant statements. Since each pattern of love among $1, 2, 3, 4$ is determined by assigning one of the truth values $\top$ or $\bot$ to each of these 16 sentence letters, we can conclude that the number of such patterns is 2^{16}. Why? Because there are two assignments to p_{11} and for each of these, there are two assignments to p_{12}, and thus $2 \cdot 2 = 2^2$ assignments to them jointly (this observation is given the exalted title, "The Product Rule"). Thus, by iterating application of the product rule another fourteen times, we arrive at the conclusion that there are 2^{16} possible truth assignments to the 16 sentence letters.

$2^{16} = 65536$. It's kind of amazing that there are as many as 65,536 different potential love-scenarios at a table for four!

Friendship, as compared to love, is relatively tame in terms of the number of scenarios that might arise. Let's return to using p_{ij} to indicate that i and j are friends. In virtue of the fact that friendship is symmetric and irreflexive, a friendship-scenario is determined by assigning one of the truth values $\top$ or $\bot$ to each of the 6 sentence letters p_{ij}, for $1 \leq i < j \leq 4$. Hence, there are only $2^6 = 64$ possible patterns of friendship among the group of four, less than 1/1000 of the number of potential love-scenarios.

In general, how many possible friendship scenarios are there among a group of n people? Well, every pair can either be friends or not friends, so there are 2^{num_pairs} possibilities. How many pairs are their, in terms of n?

0.3 Review

Concept Review

- **Pigeonhole Principle**: If you have $n + 1$ pigeons and you try to fit them all into n holes, then there has to be at least one hole with $k > 1$ pigeons.
- **The Mean Pigeonhole Principle**: If you distribute m pigeons into n pigeonholes and $m \geq k \cdot n + 1$, then some hole contains at least $k + 1$ pigeons.
- **Product Rule**: If there are n ways to do a first action and m ways to do a second action, there are $n \cdot m$ ways to do both action 1 and action 2.

Problems

1. Let X be a set, $|X| = n$ (we write $|X|$ for the size of the set X). How many subsets does X have?
2. How many subsets of even size does a set X of size $n > 0$ have?
3. Prove that the Cartesian plane cannot be colored using only two colors (Red/Blue) such that all points 1 unit away from each other are different colors.
4. Prove that for any set of $n \geq 2$ numbers, there are 2 numbers whose difference is divisible by $n - 1$.
5. Show that for any $n \in \mathbb{N}$, there is a number k whose base ten numeral contains only "5"s and "0"s such that k is divisible by n.

Solutions

1. There are 2^n many subsets of a set of size n. To see why this is the case, note that every element of the set can be either in or not in any given subset. Hence there are two choices for each of the n elements of the set, and by the product rule 2^n choices in total.

2. 2^{n-1}. We show that for every X of size at least one, the number of even-size subsets of X is equal to the number of odd-size subsets of X; it then follows from the result of the preceding problem that the answer is 2^{n-1}.

   First, suppose that the size of X is odd. Then complementation induces a one-to-one correspondence between the odd-size and even-size subsets of X. That is, we associate to each odd-size subset $Y \subseteq X$, the even-size subset $X - Y$. If, on the other hand, the size $n > 1$ of X is even, we argue as follows. Let a be an element of X and consider the set $W = X - \{a\}$. Since the size of W is odd, we already know that it has the same number of subsets of even-size as it does of odd-size; that is, there are the same number of subsets of X of odd-size that exclude a as there are subsets of X of even-size that exclude a. From this it follows at once that also X has the same number of sets of even-size that include a as it does subsets of odd-size that include a. Thus, X has the same number of subsets of odd-size as it does subsets of even-size.

3. Consider an equilateral triangle with unit-length sides. We have three points pairwise one-unit apart and only two colors. The answer follows by application of the pigeonhole principle.

4. Note that there are $n-1$ remainders when dividing by $n-1$. Hence by the pigeonhole principle two of our n numbers must have the same remainder when divided by $n-1$. Their difference is divisible by $n-1$.

5. Consider the first $n+1$ elements of the set $\{5, 55, 555...\}$. We know from above that this set has two numbers whose difference is divisible by n. Note that the difference of any two numbers in this set is written using only 5s and 0s.

1 Truth-Functional Logic

1.1 Introduction to Truth-Functional Logic

Throughout the course we will see a few different systems for formalizing statements. Each consists of a formal language to represent statements, and a way to interpret the meaning of statements in that language. Truth-functional logic is the simplest of these systems we will learn.

Components of Truth Functional Logic

1. Language (the *Syntax*)
   (a) sentence letters
   (b) connectives
2. Interpretation (the *Semantics*)
   (a) A function that assigns $\top$ or $\bot$ (true or false) to each sentence letter, called a **truth-assignment**
   (b) Fixed **truth-functional semantics** for each connective

Sentence letters such as $p, q, r, \ldots$ schematize statements (in natural language) which are true or false, and **connectives** such as $\wedge, \vee, \neg, \supset, \ldots$ are used to combine sentence letters into compound schemata.

Statements are sentences whose truth or falsity is independent of context of utterance. For example, the sentence "I am bald" is not a statement, since the truth or falsity of a given utterance of this sentence depends not only on the speaker and the time of utterance, but also on whatever subtle contextual factors might partially restrict the the range of application of the vague term "bald." On the other hand, "eight times seven is fifty-four" is as a statement, since it's truth or falsity (in this case falsity) is independent of context of utterance. Neither of the sentences "is eight times seven fifty-four?" nor "please, let eight times seven be fifty-four," is a statement. Truth-functional logic deals with the truth or falsity of *statements* only, and we use sentence letters exclusively to schematize statements.

1.2 Basic Syntax of Truth-Functional Logic

Consider using the sentence letter p_{ij} to schematize the statement "i loves j," where $1 \leq i, j, \leq 4$. For example, p_{11} schematizes the statement "1 loves 1", or briefly, "1 is a narcissist."

Suppose we wish to schematize the following statements using those sentence letters:

1. all of 1, 2, 3, and 4 are narcissists;
2. none of 1, 2, 3, and 4 are narcissists;
3. at least one of 1, 2, 3, and 4 is a narcissist;
4. an odd number of 1, 2, 3, and 4 are narcissists.

In order to do so, we introduce the following truth-functional connectives. For each connective, we display its truth-functional interpretation via a table indicating the truth value of the compound schema as a function of the truth values of its components.

- Conjunction (and):

p	q	$(p \wedge q)$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$

- Negation (not):

p	$\neg p$
$\top$	$\bot$
$\bot$	$\top$

- Inclusive Disjunction (or)

p	q	$(p \vee q)$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\bot$

- Exclusive Disjunction (exclusive or, xor)

p	q	$(p \oplus q)$
$\top$	$\top$	$\bot$
$\top$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\bot$

- Material Conditional

p	q	$(p \supset q)$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\top$

- Material Biconditional

p	q	$(p \equiv q)$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\top$

The definitions of the truth-functional connectives suffice to determine the truth/falsity of a compound schema completely in terms of (eg as a function of) the truth/falsity of its components. Hence, the term "truth-functional logic."

We can now schematize conditions 1 – 4 in the above example as follows.

S1: $((p_{11} \wedge p_{22}) \wedge p_{33}) \wedge p_{44}$

S2: $((\neg p_{11} \wedge \neg p_{22}) \wedge \neg p_{33}) \wedge \neg p_{44}$

S3: $((p_{11} \vee p_{22}) \vee p_{33}) \vee p_{44}$

S4: $((p_{11} \oplus p_{22}) \oplus p_{33}) \oplus p_{44}$

The first three are quite straightforward to verify; the fourth we will prove later in Proposition 1.

1.3 Basic Semantics of Truth-Functional Logic

Given a truth-functional schema like $((p \wedge q) \vee r)$, we cannot determine whether the schema is true or false unless we know whether p, q, and r are true or false. That is, any schema requires a truth-assignment to its sentence letters before it can be evaluated.

Definition 1 (Truth-assignment)**.** *Let X be a set of sentence letters. A* truth-assignment *A for X is a mapping which associates with each sentence letter $q \in X$ one of the two truth values $\top$ or $\bot$; we write $A(q)$ for the value that A associates to q.*

Definition 2. *Suppose S is a truth-functional schema such that every sentence letter with an occurrence in S is a member of X. We say a truth assignment A for X* satisfies *such a schema S (and write $A \models S$) if and only if S receives the value $\top$ relative to the truth assignment A.*

Example 3. *Take the schema $S = ((p \wedge q) \vee r)$, with truth assignment A such that $A(p) = \top$, $A(q) = \bot$, and $A(r) = \bot$, we have that S receives the value $\bot$. In other words A does not satisfy S. ($A \not\models S$).*

Interpreting the Material Conditional

Let's return to our potential lovers and restrict attention to just two of them, 1 and 2. How could express the statement that all love is requited among these two sweethearts? The natural mode of expression is: if 1 loves 2, then 2 loves 1, and if 2 loves 1, then 1 loves 2. This is a perfect candidate for using the material conditional.

Using the sentence letters $p_{11}, p_{12}, p_{21}, p_{22}$ as earlier interpreted, we can express the happy state that all love among 1 and 2 is requited by the schema

$$R : (p_{12} \supset p_{21}) \wedge (p_{21} \supset p_{12})$$

or, equivalently,

$$p_{12} \equiv p_{21}$$

In how many of the possible love scenarios among 1 and 2 is all love requited? Count the number of satisfying truth-assignments to R!

While the motivations for the truth-functional definitions for the other connectives normally seem evident to new logicians, the material conditional often gives people trouble. Let's consider generalized conditionals as a route to motivating the truth-functional interpretation of the conditional offered above. Of course, the statement "if an integer is divisible by six, then it is divisible by three," is true, and thence each of the following statements, which are instances of this general statement, are true.

- "If twelve is divisible by six, then twelve is divisible by three."
- "If three is divisible by six, then three is divisible by three."
- "If two is divisible by six, then two is divisible by three."

Therefore, if the conditional involved is to be understood truth-functionally, then its interpretation must satisfy the conditions imposed by the first, third, and fourth rows of the material conditional's truth-table. On the other hand, the falsity of the conditional "if twelve is divisible by six, then twelve is divisible by seven," mandates the condition imposed by the second row of the truth-table.

An Inductive Proof

Let's do a simple inductive proof about truth-functional satisfaction, as an illustration of the use of mathematical induction, especially in application to reasoning about truth-functional schemata.

Proposition 1. *For every $n \geq 2$ and every set $X = \{q_1, \ldots, q_n\}$ of n distinct sentence letters, a truth assignment A for X satisfies the schema*

$$S_n : (\ldots(q_1 \oplus q_2) \ldots \oplus q_n)$$

if and only if A assigns an odd number of the sentence letters in X the value $\top$.

Proof. We prove the proposition by induction on n.

- Basis: Examination of the truth table for $\oplus$ suffices to establish the proposition for the case $n = 2$.
- Induction Step: Suppose the proposition holds for a number $k \geq 2$, that is, for every truth assignment A for $\{q_1, \ldots, q_k\}$, $A \models S_k$ if and only if A assigns an odd number of the sentence letters in $\{q_1, \ldots, q_k\}$ the value $\top$; this is our induction hypothesis. We proceed to show that the proposition also holds for $k + 1$. Let A' be an assignment to the sentence letters $\{q_1, \ldots, q_{k+1}\}$ and let A be its restriction to $\{q_1, \ldots, q_k\}$. We consider two cases. First, suppose that $A'(q_{k+1}) = \top$. In this case, $A' \models S_{k+1}$ if and only if $A \not\models S_k$ if and only if (by our induction hypothesis) A assigns an even number of the sentence letters $\{q_1, \ldots, q_k\}$ the value $\top$. Hence, if $A'(q_{k+1}) = \top$, then $A' \models S_{k+1}$ if and only if A' assigns an odd number of the sentence letters in $\{q_1, \ldots, q_{k+1}\}$ the value $\top$. On the other hand, suppose that $A'(q_{k+1}) = \bot$. In this case, $A' \models S_{k+1}$ if and only if $A \models S_k$ if and only if (by our induction hypothesis) A assigns an odd number of the sentence letters $\{q_1, \ldots, q_k\}$ the value $\top$. Hence, if $A'(q_{k+1}) = \bot$, then $A' \models S_{k+1}$ if and only if A' assigns an odd number of the sentence letters in $\{q_1, \ldots, q_{k+1}\}$ the value $\top$. This concludes the proof, since either $A'(q_{k+1}) = \top$ or $A'(q_{k+1}) = \bot$.

□

The Centrality of Satisfaction

The satisfaction relation is the fundamental semantic relation. It is where language and the world meet; in the case to hand, language consists of truth-functional schemata and the possible worlds they describe are truth assignments to sentence letters. As the course progresses, we will encounter more textured representations of the world (relational structures) and richer languages to describe them (monadic and polyadic quantification theory). We now define some of the central notions of truth-functional logic in terms of satisfaction. These definitions will generalize directly to the more textured structures and richer languages we encounter later.

For the following definitions, we suppose that S and T are truth-functional schemata and that A ranges over truth assignments to sets of sentence letters which include all those that occur in either S or T.

Definition 3. S implies T *if and only if for every truth assignment A, if $A \models S$, then $A \models T$.*

Definition 4. S *is* equivalent *to T if and only if S implies T and T implies S*

Definition 5. S *is* satisfiable *if and only if for some A, $A \models S$.*

Definition 6. S *is* valid *if and only if every truth assignment satisfies S.*

Examples of equivalence and the material biconditional

Try to see why the following are equivalent - either by appealing to your understanding of what the connective "means" or by going back to the truth tables.

- $p \oplus q$ is equivalent to $q \oplus p$ (commutativity of exclusive disjunction)
- $(p \oplus q) \oplus r$ is equivalent to $p \oplus (q \oplus r)$ (associativity of exclusive disjunction).
- $p \equiv q$ ius equivalent to $(p \supset q) \wedge (q \supset p)$

Note that both conjunction and inclusive disjunction are also commutative and associative, whereas the material conditional is neither.

Try to to think of examples of (binary) truth-functional connectives which are commutative but not associative, and associative but not commutative.

1.4 Review

Concept Review

Definitions

- A truth-assignment A for X is a function which maps every sentence letter $q \in X$ to either $\top$ or $\bot$. $A(q)$ is the notation for the value A associates with q.
- A schema S *implies* a schema T iff for all truth-assignments A, if $A \models S$ then $A \models T$.
- A schema S is *equivalent* to a schema T iff S and T are satisfied by exactly the same truth assignments (for all A, $A \models S$ iff $A \models T$).
- S is *satisfiable* iff there is a truth assignment that satisfies it (there exists an A such that $A \models S$)
- S is *valid* iff all truth assignments satisfy it (for all A, $A \models S$)

Syntax, Semantics The *syntax* of TF-logic is given by the rules for forming truth-functional schemata from sentence letters and connectives. The *semantics* of TF-logic are given by a *truth-assignment*, which associates with each letter a *truth-value*.

Satisfying Sentences The *truth-values* of the individual sentence letters in a schema are propagated to the whole schema by means of *truth-tables* which give fixed semantic interpretations to each of the *connectives*. We say that a truth-assignment A *satisfies* a sentence S (written $A \models S$) iff the sentence S evaluates to $\top$ under the truth-assignment A. Otherwise, we write $A \not\models S$ and say that A does not satisfy S.

Problems

1. Is "the University of Pennsylvania has a Logic major" a statement? Why or why not?
2. Is "should I major in Logic?" a statement? Why or why not?
3. Using the sentence letters p_{ij}, $q \leq i, j \leq 4$ to stand for "person i loves person j". Schematize the following statements:
   (a) Person 1 loves everyone else.
   (b) There is a Shakespearean love triangle (*i.e.*, no one has their love requited) between people 1, 2, 3, and person 4 is a Scrooge (he does not love anyone, even himself).
   (c) Everyone loves, exclusively, people with numbers lower than themselves.
4. How many truth-assignments to the given letters satisfy the following schema?
$$(p_1 \supset q_1) \wedge ... \wedge (p_5 \supset q_5)$$
5. How many truth-assignments to the set of sentence letters $X_n = \{p_1, q_1, \ldots, p_n, q_n\}$ satisfy the following schema S_n? Express your answer as a function of n and prove that it is correct by mathematical induction.
$$(p_1 \supset q_1) \wedge ... \wedge (p_n \supset q_n)$$
6. How many truth-assignments over the given letters satisfy the following schema?
$$p_1 \oplus p_2 \oplus p_3 \oplus p_4 \oplus p_5$$
7. Is the following sentence valid, satisfiable but not valid, or unsatisfiable?
$$(a \equiv b) \supset (a \vee \neg b)$$
8. Valid, satisfiable, or unsatisfiable?
$$(b \vee (b \supset a)) \wedge (\neg b \vee (a \supset b))$$
9. Valid, satisfiable, or unsatisfiable?
$$(a \equiv b) \wedge (b \equiv c) \wedge (a \oplus b)$$
10. How many truth-assignments for the given letters satisfy
$$(a \equiv b) \wedge (b \equiv c) \wedge (c \equiv d)$$
11. How many truth-assignments to the given letters satisfy
$$(a \oplus b) \vee (b \oplus c) \vee (c \oplus d)$$
12. I claim that if n people all shake hands with each other (once per pair), the total number of handshakes is $\frac{n(n-1)}{2}$. Prove this by induction.

Solutions

1. Yes, it is.[a]

2. No, it is not. It is not a statement because it expresses a question, which is not determinitely true or false.

   With that being said, you should - of course - major in logic[b].

3. (a) $p_{11} \wedge p_{12} \wedge p_{13} \wedge p_{14}$

   (b) $((p_{12} \wedge p_{23} \wedge p_{31}) \vee (p_{13} \wedge p_{21} \wedge p_{32})) \wedge \neg(p_{41} \vee p_{42} \vee p_{43} \vee p_{44})$

   (c) $p_{21} \wedge p_{31} \wedge p_{32} \wedge p_{41} \wedge p_{42} \wedge p_{43}$

4. 3^5. Note that each of the terms of the form $p_i \supset q_i$ is satisfied in three cases (check the truth table) and apply the product rule.

5. 3^n for $n \geq 1$.

   BASE CASE: Verify, via the truth table for the material conditional, that three of the four truth assignments to $X_1 = \{p_1, q_1\}$ satisfy the schema S_1.

   INDUCTION STEP: Suppose that 3^n of the 4^n truth-assignments to X_n satisfy S_n:

   $$(p_1 \supset q_1) \wedge ... \wedge (p_n \supset q_n).$$

   Let A be one such truth-assignment. Verify, using the truth-table for the material conditional, that A may be extended to exactly three distinct truth-assignments to the sentence letters X_{n+1} each of which satisfies S_{n+1}. It follows that there are $3 \cdot 3^n = 3^{n+1}$ truth-assignments to the sentences letters X_{n+1} that satisfy S_{n+1}:

   $$(p_1 \supset q_1) \wedge ... \wedge (p_n \supset q_n) \wedge (p_{n+1} \supset q_{n+1}).$$

6. $2^4 = 16$. Remember that there are 2^{n-1} ways to pick an odd-sized subset from n elements and that a sentence of the given form is satisfied iff an odd number of sentence letters are set to true.

7. This is valid. Suppose A is a truth-assignment to the sentence letters a and b. Note that if $A(a \equiv b) = \bot$, then A satisfies the given schema. So suppose $A(a \equiv b) = \top$. Then $A(a) = A(b)$, hence either $A(a) = \top$ or $A(b) = \bot$. Thus $A(a \vee \neg b) = \top$.

8. Valid. Suppose A is a truth-assignment to the sentence letters a and b. If $A(b) = \top$, then A clearly satisfies the left conjunct. If $A(b) = \bot$, then $A(b \supset a) = \top$, hence A satisfies the left conjunct as well. Similarly, if $A(b) = \top$, then A satisfies the right conjunct, and if $A(b) = \bot$, then $A(\neg b) = \top$, hence again A satisfies the right conjunct.

9. Unsatisfiable. Suppose A is a truth-assignment to the sentence letters a, b and c and A satisfies both $a \equiv b$ and $b \equiv c$. It follows that A satisfies $a \equiv c$ (in other words, $\equiv$ is *transitive*). But then A does not satisfy $a \oplus c$, since this is truth-functionally equivalent to $\neg(a \equiv c)$. So the schema is unsatisfiable.

10. 2. Picking true/false for a fixes the truth-values of the remaining letters.

11. 14. To get this answer, we note that there are 16 (2^4) truth-assignments in total; count the number which do not satisfy our sentence, and subtract that number from 16. The sentence is only not satisfied when each of a, b, c, d have the same truth-value, so there are 2 non-satisfying truth-assignments. This means there are $16 - 2 = 14$ satisfying truth assignments.

12. BASE CASE: $n = 2$. Two people shaking hands results in one handshake, and the formula gives us $\frac{2(2-1)}{2} = 1$ which is correct. Note that we pick $n = 2$ as the base case (not $n = 0$ or $n = 1$) because it doesn't really make sense to talk about those cases (since you need two people for a handshake).

    INDUCTIVE CASE: Assume that for n people, the number of handshakes (let's denote it H_n) is $H_n = \frac{n(n-1)}{2}$. We want to show (henceforth "wts") that for $n+1$ people the number of handshakes is $H_{n+1} = \frac{(n+1)n}{2}$. The number of handshakes between $n+1$ people is clearly the number of handshakes for n people (H_n) plus n, since our new person must shake hands with the n others. So we have $H_{n+1} = H_n + n = \frac{n(n-1)}{2} + n = \frac{n^2-n+2n}{2} = \frac{n^2+n}{2} = \frac{(n+1)n}{2}$, which is what we wanted to show.

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[a]Although, one might insist that there remains an element of context dependence owing to an ambiguity in the proper name "University of Pennsylvania" - those in Indiana County, Pennsylvania might well use it to refer to a different institution. This observation invites reflection upon the intriguing question whether (virtually) all sentences of ordinary language are to some extent context dependent (at least without non-ordinary supplementation).

[b]Provided you like it and want to.