1 Lecture 08.30

We began with a survey of some of the highlights of logic, mathematics, and philosophy from Pythagoras, Plato, Diophantus, and Euclid, to the Gödel Completeness Theorem, the Church-Turing Theorem (which together indicate the scope and limits of the execution of Leibniz's dream of a *charcteristica universalis* and a *calculus ratiocinator*), Tarski's proof that the theory of the real field is decidable, and the "DPRM" Theorem that the solvability of Diophantine equations in integers is algorithmically undecidable, thereby resolving Hilbert's Tenth Problem negatively and completing a mathematical arc from antiquity: geometry is decidable, while arithmetic is not. (We observed, as an aside, that Plato's awe-struck response to the Pythagoreans' demonstration that the Diophantine equation $p^2 - 2q^2 = 0$ has no solution in positive integers is the origin of Western philosophy.) In a future edition of this memoir, I may include further detail concerning these matters, but for the nonce, I want to get down to brass tacks and review the concrete problem of axiomatizability we started to engage toward the end of class.

Let $\mathbb{N} = \langle \mathbb{N}, s, 0 \rangle$, the structure consisting of the non-negative integers $\mathbb{N}$ together with the successor function s and the distinguished element 0. We started to provide an axiomatization of the first-order theory of this structure. What does this mean?

Definition 1 *Let* A *be a relational structure (for example* $\mathbb{N}$*). If* θ *is a first-order sentence, we write* $\mathsf{A} \models \theta$ *for* θ is true in A*, alternatively,* A satisfies θ*. If* Σ *is a set of first-order sentences we write* $\mathsf{A} \models \Sigma$ *if and only if* $\mathsf{A} \models \theta$*, for every* $\theta \in \Sigma$*.* $\mathsf{Th}(\mathsf{A}) = \{\theta \mid \mathsf{A} \models \theta\}$*. We say two structures* A *and* B *are* elementarily equivalent *(and write* $\mathsf{A} \equiv \mathsf{B}$*) if and only if* $\mathsf{Th}(\mathsf{A}) = \mathsf{Th}(\mathsf{B})$*. We say that a set of sentences (it could be a singleton)* axiomatizes *(the theory of) a structure* A *if and only if* $A \models \Sigma$ *and for every structure* B*, if* $\mathsf{B} \models \Sigma$*, then* $\mathsf{B} \equiv \mathsf{A}$*. The idea is that* no better description of A can be given by first-order sentences than is already given by Σ.

We wrote down a few sentences that describe basic properties of the structure $\mathbb{N}$.

1. $(\forall x)(sx \neq 0)$ (0 has no predecessor);
2. $(\forall x)(x \neq 0 \to (\exists y) sy = x)$ (everything but 0 has a predecessor);
3. $(\forall x)(\forall y)(sx = sy \to x = y)$ (the successor function is injective)

Let's call the conjunction of these sentences φ. We noted that if $A \models \varphi$ then A has a substructure (perhaps all of A) that is isomorphic to $\mathbb{N}$; in particular every such structure is infinite. Indeed, φ essentially says that the universe of any satisfying structure is *Dedekind infinite*: it supports a function which is injective but not surjective. It took a (mere?) moment to recognize that φ does not axiomatize $\mathbb{N}$. For example, the structure which consists of $\mathbb{N}$ together with an additional element a with $s(a) = a$ satisfies φ, so we realized we'd need to rule out the existence of fixed points under successor. We added the sentence $\alpha_1 : (\forall x)sx \neq x$. And it didn't take long to realize that we'd need to add infinitely many sentences α_n the n^{th} of which rules out successor cycles of length n. If we write s^n for n iterations of s we may write α_n as $(\forall x)s^n x \neq x$.

Let $\Sigma = \{\varphi\} \cup \{\alpha_n \mid n \geq 1\}$. We ended with the following questions:

- Does Σ axiomatize $\mathbb{N}$?
- Is something even stronger true, namely, does Σ categorize $\mathbb{N}$? (We say a set of sentences Σ *categorizes* a structure A just in case for every structure B, $B \models \Sigma$ if and only if B is isomorphic to A.)

We'll pursue these questions at the start of class on Thursday.

2 Lecture 09.01

We'd mentioned "isomorphism" last time, but hadn't defined it.

Definition 2 *Suppose* $\mathsf{A} = \langle A, E^A, s^A, 0^A \rangle$ *and* $\mathsf{B} = \langle B, E^B, s^B, 0^B \rangle$. *A function f is an* isomorphism *from* A *onto* B *if and only f is a bijection of A onto B and*

- $f(0^A) = 0^B$;
- *for all* $a \in A$, $f(s^A(a)) = s^B(f(a))$;
- *for all* $a_1, a_2 \in A$, $\langle a_1, a_2 \rangle \in E^A$ *if and only if* $\langle f(a_1), f(a_2) \rangle \in E^B$.

We say A *is* isomorphic to B *(and write* $\mathsf{A} \cong \mathsf{B}$*) if and only if there is an isomorphism from* A *onto* B.

We attacked the questions we'd asked at the end of the last class meeting. In particular, we began by showing that

Lemma 1 Σ does not *categorize* $\mathbb{N}$*, that is, there is a structure* A *such that* $\mathsf{A} \models \Sigma$ *but* $\mathsf{A} \ncong \mathbb{N}$.

Proof: Let $\mathbb{Z}$=$\langle \mathbb{Z}, s^{\mathbb{Z}} \rangle$ be the structure consisting of all the integers with their usual successor function. For any set I, let $\mathbb{Z}_I$ be the disjoint union of I many copies of $\mathbb{Z}$, that is, the universe of $\mathbb{Z}_I$ is the cartesian product $I \times \mathbb{Z}$, and $s^{\mathbb{Z}_I}(\langle i, p \rangle) = \langle i, p+1 \rangle$. Finally, let A_I be the union of the structures $\mathbb{N}$ and $\mathbb{Z}_I$. We observed that for every set I, $\mathsf{A}_I \models \Sigma$. Moreover, for all I and J, $\mathsf{A}_I \cong \mathsf{A}_J$ if and only if there is a bijection from I onto J (that is, the cardinality of I equals the cardinality of J). It follows at once that Σ does not categorize $\mathbb{N}$. ■

That leaves the question, "Does Σ axiomatize $\mathbb{N}$?" (Recall that a set of sentences Γ axiomatizes a structure C if and only if for every structure B, $\mathsf{B} \models \Gamma$ if and only if $\mathsf{B} \equiv \mathsf{C}$.) We made a start on answering this question by establishing:

Lemma 2 *For every structure* B, $\mathsf{B} \models \Sigma$ *if and only if for some set* I, $\mathsf{B} \cong \mathsf{A}_I$.

Proof: As just noted in the proof of Lemma 1, for every I, $\mathsf{A}_I \models \Sigma$. Thus, it suffices to show that for every B, if $\mathsf{B} \models \Sigma$, then $\mathsf{B} \cong \mathsf{A}_I$ for some I.

Suppose $\mathsf{B} \models \Sigma$. Define an equivalence relation $\approx$ on the universe of B as follows. For all $a, b \in B$

$$a \approx b \text{ if and only if for some } n,\ (s^{\mathsf{B}})^n(a) = b \text{ or } (s^{\mathsf{B}})^n(b) = a.$$

That is, $a \approx b$ just in case one of a or b is reachable from the other by finitely many applications (perhaps 0) of the successor function in B. We argued that $\approx$ factors B into equivalence classes of exactly two isomorphism types: the equivalence class of 0^{B} (which we called the standard part of B) is isomorphic to $\mathbb{N}$, and is the unique such equivalence class; all the other equivalence classes (if any) are isomorphic to $\mathbb{Z}$. ■

Next, we introduced one of the fundamental concepts of logic, the notion of logical consequence.

Definition 3 *Let Γ be a set of first-order sentences and θ a first-order sentence. θ is a* logical consequence *of Γ (written $\Gamma \models \theta$) if and only if for every structure B, if $\mathsf{B} \models \Gamma$, then $\mathsf{B} \models \theta$. $\mathsf{Cn}(\Gamma) = \{\theta \mid \Gamma \models \theta\}$.*

We gave an alternative characterization of axiomatizability in terms of consequence .

Lemma 3 *A set of sentences Γ axiomatizes a structure A if and only if $\mathsf{Cn}(\Gamma) = \mathsf{Th}(\mathsf{A})$.*

(Make sure you can prove this; ask about it next time, if you can't.)

Next time, we will establish

Proposition 1 *Σ axiomatizes $\mathbb{N}$.*

as a corollary to Lemmas 2 and 3, and the following fundamental result concerning first-order logic.

Theorem 1 (Löwenheim-Skolem) *If Γ is a countable set of first-order sentences such that $\mathsf{B} \models \Gamma$ for some infinite structure B, then for every infinite set C there is a structure C with universe C such that $\mathsf{C} \models \Gamma$. In other words, if a countable set of first-order sentences is satisfied by an infinite model, then it is satisfied by a model of cardinality κ for every infinite cardinal κ.*

It follows immediately from Lemma 1 and Proposition 1 (why?) that

Corollary 1 *$\mathsf{Th}(\mathbb{N})$ does not categorize $\mathbb{N}$ (and thus there is no set of first-order sentences that categorizes $\mathbb{N}$).*

We observed that there are sentences of other logical languages that do categorize $\mathbb{N}$. Recall the basic properties of successor we wrote down in our first class meeting:

1. $(\forall x)(sx \neq 0)$ (0 has no predecessor);

2. $(\forall x)(x \neq 0 \to (\exists y) sy = x)$ (everything but 0 has a predecessor);
3. $(\forall x)(\forall y)(sx = sy \to x = y)$ (the successor function is injective)

Let's call the conjunction of these sentences φ. The conjunction of φ with the following sentence of the infinitary language $L_{\omega_1\omega}$ categorizes $\mathbb{N}$:

$$(\forall x)(x = 0 \vee x = s0 \vee x = ss0 \vee \ldots \vee x = s^n 0 \vee \ldots).$$

Also, φ conjoined with the following second-order axiom of induction categorizes $\mathbb{N}$:

$$(\forall P)[(P(0) \wedge (\forall x)(P(x) \to P(sx))) \to (\forall x)P(x)].$$

For next time, please read Section 3 of the Barwise *Handbook of Mathematical Logic* Chapter: **The formalization of first-order logic** (pages 15-20 of the .pdf available on Canvas) and prove (by induction on formulas) that for every first-order formula φ with free variables among $v_1, \ldots, v_n$, and structures A and B, if f is an isomorphism from A onto B, then for all $a_1, \ldots, a_n \in A$,

$$\mathsf{A} \models \varphi[a_1, \ldots, a_n] \text{ if and only if } \mathsf{B} \models \varphi[f(a_1), \ldots, f(a_n)].$$

3 Lecture 09.06

We presented the syntax and semantics of first-order languages closely adhering to the treatment in the Barwise reading recommended in the last memoir. We also began to do the exercise suggested at the end of that memoir. In particular, we proved the following lemma by induction on terms.

Lemma 4 *Let A and B be structures, f an isomorphism of A onto B, s an assignment of values in the universe of A to variables, that is, $s : V \mapsto A$, and t a term. The f-image of the denotation of the term t in the structure A with respect to the assignment s is equal to the denotation of the term t in the structure B with respect to the assignment $f \circ s$, that is,*

$$f(t^{\mathsf{A}}[s]) = t^{\mathsf{B}}[f \circ s].$$

4 Lecture 09.08

We began by proving

Proposition 2 *Let A and B be structures, f an isomorphism of A onto B, s an assignment of values in the universe of A to variables, that is, $s : V \mapsto A$, and φ a formula. Then,*

$$\mathsf{A} \models \varphi[s] \textit{ if and only if } \mathsf{B} \models \varphi[f \circ s]$$

Proof: The proof proceeds by induction on formulas, with Lemma 4 used to verify the base case for atomic formulas. ■

We next proceeded with the proof of Proposition 1. The next definition introduces some of the important concepts we will use in the proof.

Definition 4 $\mathsf{Mod}(\Gamma) = \{\mathsf{B} \mid \mathsf{B} \models \Gamma\}$. Γ *is* satisfiable *if and only if* $\mathsf{Mod}(\Gamma) \neq \emptyset$. Γ *is* complete *if and only if for all structures* $\mathsf{B}, \mathsf{C} \in \mathsf{Mod}(\Gamma)$*,* $\mathsf{B} \equiv \mathsf{C}$*.* Γ *is* κ-categorical *if and only if for all structures* $\mathsf{B}, \mathsf{C} \in \mathsf{Mod}(\Gamma)$*, if* $|\mathsf{B}| = |\mathsf{C}| = \kappa$*, then* $\mathsf{B} \cong \mathsf{C}$*.*

Since $\Sigma \subseteq \mathsf{Th}(\mathbb{N})$, in order to establish Proposition 1, it suffices to show that

Lemma 5 Σ *is complete.*

Lemma 5 is a corollary to the next two results. The first of these, a corollary to Theorem 1 (the Löwenheim-Skolem Theorem), provides a useful completeness criterion.

Corollary 2 (Łos-Vaught) *If* Γ *is a countable set of first-order sentences which has no finite models and which is* κ*-categorical for some infinite cardinal* κ*, then* Γ *is complete.*

Proof: Let Γ be a countable set of first-order sentences which has no finite models and let κ be an infinite cardinal such that Γ is κ-categorical. Suppose moreover, toward a contradiction, that Γ is not complete. Then there are infinite structures $\mathsf{A}, \mathsf{B} \in \mathsf{Mod}(\Gamma)$ such that $A \not\equiv B$. It follows from Theorem 1 that there are structures C and D such that $|\mathsf{C}| = |\mathsf{D}| = \kappa$, $\mathsf{C} \equiv \mathsf{A}$, and $\mathsf{D} \equiv \mathsf{B}$. But then, $\mathsf{C} \cong \mathsf{D}$ and $\mathsf{C} \not\equiv \mathsf{D}$, which contradicts Proposition 2. ■

Lemma 6 *For every uncountable cardinal* κ*,* Σ *is* κ*-categorical.*

The proof of Lemma 6 uses the following basic result of cardinal arithmetic.

Theorem 2 *For all infinite cardinals* κ *and cardinals* λ*,*

$$\kappa + \lambda = \kappa \times \lambda = \max\{\kappa, \lambda\}.$$

Proof (of Lemma 6): Recall that the universe of A_I (call it U_I) is equal to $\{0, 1, 2, \ldots\} \cup (I \times \{\ldots - 1, 0, 1 \ldots\})$. By Theorem 2, we thus have

$$|U_I| = \aleph_0 + \max\{|I|, \aleph_0\}.$$

It follows at once that $|U_I| = \aleph_0$, if $|I| \leq \aleph_0$, and $|U_I| = |I|$, if $|I| > \aleph_0$. Recall from the proof of Lemma 1 that for all I and J, $\mathsf{A}_I \cong \mathsf{A}_J$ if and only if $|I| = |J|$, and from Lemma 2 that for every structure B, $\mathsf{B} \models \Sigma$ if and only if for some set I, $\mathsf{B} \cong \mathsf{A}_I$. It follows at once that there are countably many countable models of Σ up to isomorphism, and for every uncountable cardinal κ exactly one model of Σ of cardinality κ up to isomorphism. ■

This concludes the proof that Σ axiomatizes $\mathbb{N}$. We noted that a sea-change in the study of model theory took place with Michael Morley's 1963 result that

Theorem 3 *If Γ is a countable set of first-order sentences and Γ is κ-categorical for some $\kappa > \aleph_0$, then Γ is κ-categorical for every $\kappa > \aleph_0$.*

Another such "uncountably" categorical theory is the theory of algebraically closed fields of a fixed characteristic. This follows from the fact that an algebraically closed field is determined up to isomorphism by its characteristic and its transcendence degree, in much the way that a model of Σ is determined up to isomorphism by the multiplicity of its $\mathbb{Z}$-chains.

At the end of class we introduced theory of dense linear order without end points via the conjunction δ of the following sentences.

- $(\forall x)x \not< x$ ($<$ is irreflexive)
- $(\forall x)(\forall y)(\forall z)((x < y \wedge y < z) \rightarrow x < z)$ ($<$ is transitive)
- $(\forall x)(\forall y)(x \neq y \rightarrow (x < y \vee y < x))$ ($<$ is linear)
- $(\forall x)(\forall y)(x < y \rightarrow (\exists z)(x < z \wedge z < y))$ ($<$ is dense)
- $(\forall x)(\exists y)x < y$ ($<$ has no maxima)
- $(\forall x)(\exists y)y < x$ ($<$ has no minima)

Try to prove the following result (due to Georg Cantor).

Theorem 4 *δ is $\aleph_0$-categorical.*

(A proof is given in Lecture 6.)

5 Lecture 09.13

The following Compactness Theorem expresses a fundamental property of first-order logic.

Theorem 5 (Gödel-Mal'cev) *If every finite subset of a set of first-order sentences Γ is satisfiable, then Γ itself is satisfiable. Equivalently, if a first-order sentence θ is a logical consequence of a set of first-order sentences Γ, then θ is a logical consequence of a finite subset of Γ.*

Definition 5 *The* signature *(also called* language*) of a structure A (written $\mathbf{L}(\mathsf{A})$) is the collection of relation, function, and constant symbols interpreted by A. If $\mathbf{L}(\mathsf{A}) \subseteq \mathbf{L}(\mathsf{B})$ and $\mathsf{B}\restriction\mathbf{L}(\mathsf{A}) = \mathsf{A}$ we say that A is a* reduct *of B, alternatively, B is an* expansion *of A.*

The proof of the following proposition illustrates the use of the Compactness Theorem in conjunction with the expansion of a signature by the addition of new constant symbols.

Proposition 3 *If Γ is a set of first-order sentences and Γ has arbitrarily large finite models, then Γ has an infinite model.*

Proof: Let $\mathbf{L}$ be the set of relation, function, and constant symbols appearing in Γ. Introduce an infinite set of constant symbols $\{c_i \mid i \in N\}$ disjoint from $\mathbf{L}$ and let $\Lambda = \{c_i \neq c_j \mid i < j\}$. Suppose Γ has arbitrarily large finite models. Then every finite subset of $\Gamma \cup \Lambda$ is satisfiable. Therefore, by the Compactness Theorem, there is a structure B with $\mathsf{B} \models \Gamma \cup \Lambda$. It follows at once that the universe of B is infinite. Let A be the reduct of B to $\mathbf{L}$. Then A is an infinite model of Γ. ■

We observed that essentially the same argument can be used to establish an "upward" version of the Löwenheim-Skolem Theorem.

Theorem 6 *If Γ has an infinite model, then for every infinite cardinal κ, Γ has a model of cardinality at least κ.*

Proof: Repeat the proof of Proposition 3 with a set of new constant symbols $\{c_\beta \mid \beta < \kappa\}$ of cardinality κ. ■

Definition 6 *A structure* A *is* finitely axiomatizable *if and only if there is a sentence θ which axiomatizes* A.

We remarked that the Compactness Theorem can be applied to show that

Proposition 4 $\mathbb{N}$ *is not finitely axiomatizable.*

Proof: Forthcoming as a solution to a Bring-Back Exam problem. ■

We next introduced the notion of first-order definability.

Definition 7 *Let θ be a first-order formula with free variables among $v_1, \ldots, v_n$, and let $a_1, \ldots, a_n$ be members of the universe of a structure* A. *We write $\mathsf{A} \models \theta[a_1, \ldots, a_n]$ for $\mathsf{A} \models \theta[s]$, where $s(v_i) = a_i$ for $1 \leq i \leq n$. The set $\{\langle a_1, \ldots, a_n\rangle \mid \mathsf{A} \models \theta[a_1, \ldots, a_n]\}$ is the n-ary* relation defined by θ on A*; we denote this relation by $\theta[\mathsf{A}]$. We say that a relation $R \subseteq (U^{\mathsf{A}})^n$ is definable on A if and only if $R = \theta[\mathsf{A}]$ for some first-order formula θ.*

Definition 8 *If f is an isomorphism from A onto itself, we say f is an* automorphism *of A, and we write $\mathsf{Aut}(\mathsf{A})$ for the set of automorphisms of A. A structure A is* rigid *if and only if $|\mathsf{Aut}(\mathsf{A})| = 1$, that is, the only automorphism of A is the identity map.*

The following result, a corollary to the "isomorphism preservation property," establishes an important connection between automorphisms and definability. Recall that $f[X] = \{f(a) \mid a \in X\}$.

Corollary 3 *If A is a structure, $X \subseteq U^{\mathsf{A}}$ is definable, and $f \in \mathsf{Aut}(\mathsf{A})$, then $f[X] = X$.*

Therefore, in order to show that a set X is *not* definable on a given structure A, it suffice to exhibit $f \in \mathsf{Aut}(A)$ with $f[X] \neq X$. On finite structures, this technique is of universal applicability.

Proposition 5 *If* A *is a finite structure,* $X \subseteq U^{\mathsf{A}}$*, and for every* $f \in \mathsf{Aut}(\mathsf{A})$*,* $f[X] = X$*, then* X *is definable on* A*.*

Proof: Forthcoming as a solution to a Bring-Back Exam problem. ■

Alas, this technique is not generally applicable on infinite structures. For example, consider $\mathbb{N}$, an infinite structure of countable signature. Since the identity map fixes all subsets, the technique cannot be used to show the undefinability of any subset of $\mathbb{N}$. But there are sets of positive integers that are not definable on $\mathbb{N}$. This follows by a "cardinality argument": there are uncountably many sets of positive integers, but only countably many sets definable on $\mathbb{N}$. The following proposition characterizes those sets which are definable on $\mathbb{N}$.

Proposition 6 *If* $X \subseteq \{0, 1, 2, \ldots\}$ *is definable on* $\mathbb{N}$*, then either* X *or its complement is finite.*

Proof: Forthcoming as a solution to a Bring-Back Exam problem. ■

6 Lecture 09.15

Recall that the theory of dense linear order was discussed in class last Thursday and that an account of that discussion is included in the memoir of Lecture 4. We formulated the theory as a finite conjunction of first-order sentences δ and stated the following theorem, which we now prove.

Theorem 7 (Cantor) δ *is* $\aleph_0$*-categorical.*

Theorem 7 is an immediate corollary to the next two results, which involve the notion of partially isomorphic structures. A function f is a *partial isomorphism* from A to B if and only if f is an isomorphism from a substructure of A onto a substructure of B. The next definition introduces the key notion in the proof of Theorem 7.

Definition 9 A *is* partially isomorphic to B *(*$\mathsf{A} \cong_p \mathsf{B}$*) if and only if there is a nonempty set* P *of partial isomorphisms of* A *to* B *with the following properties:*

> *Forth Property: for every* $a \in U^{\mathsf{A}}$ *and* $f \in \mathsf{P}$*, there is a* $g \in \mathsf{P}$ *such that* $f \subseteq g$ *and* $a \in \mathsf{dom}(g)$*.*
>
> *Back Property: for every* $b \in U^{\mathsf{B}}$ *and* $f \in \mathsf{P}$*, there is a* $g \in \mathsf{P}$ *such that* $f \subseteq g$ *and* $b \in \mathsf{ran}(g)$*.*

We write $\mathsf{P} : \mathsf{A} \cong_p \mathsf{B}$*, if* P *witnesses that* $\mathsf{A} \cong_p \mathsf{B}$*.*

Theorem 7 is a corollary of the following two results.

Theorem 8 *If* A *and* B *are dense strict linear orders without endpoints, then* $A \cong_p B$*.*

Proof: Let A and B be dense strict linear orders without endpoints, and let P be the set of finite partial isomorphisms from A to B. It is easy to verify that P witnesses $\mathsf{A} \cong_p \mathsf{B}$. ■

Theorem 9 *If* A *and* B *are countable and* $\mathsf{A} \cong_p \mathsf{B}$*, then* $\mathsf{A} \cong \mathsf{B}$.

Proof: Suppose that A and B are countable and that $\mathsf{P} : \mathsf{A} \cong_p \mathsf{B}$. Let $U^{\mathsf{A}} = \{a_0, a_1, \ldots\}$ and $U^{\mathsf{A}} = \{b_0, b_1, \ldots\}$. We construct a sequence of partial isomorphisms $f_0, f_1, \ldots$ such that for all $i \in \omega$

1. $f_i \in \mathsf{P}$
2. $f_i \subseteq f_{i+1}$
3. $a_i \in f_{2i}$ and $b_i \in f_{2i+1}$.

The construction is by induction, alternately using the forth property and the back property at even and odd stages, respectively. Let $f = \bigcup_{i\in\omega} f_i$. It is easy to verify that f is an isomorphism of A onto B. ■

The following result is an immediate corollary of Theorem 7 via the Łos-Vaught Theorem.

Corollary 4 δ *is complete.*

Let $\mathbb{R}$ be the structure consisting of the collection of real numbers with its usual ordering and let $\mathbb{Q}$ be the structure consisting of the collection of rational numbers with its usual ordering. It is follows immediately from Corollary 4 that $\mathbb{R} \equiv \mathbb{Q}$, and therefore that the least upper-bound principle cannot be expressed in first-order logic.

7 Lecture 09.20

We discussed the upcoming exam. We began by considering definability over finite graphs. We used the example of a simple graph G with $U^{\mathsf{G}} = \{1, 2, 3, 4\}$ with vertex 2 of degree three and vertices 1, 3, and 4 of degree one. We noted that of the 24 permutations of U^{G}, exactly six are automorphisms, namely those which leave 2 fixed.

Definition 10 *Let* A *be a structure and* $a \in U^{\mathsf{A}}$*. The* orbit *of* a *in* A *(written* $\mathsf{orb}(a, \mathsf{A})$*) is* $\{f(a) \mid f \in \mathsf{Aut}(\mathsf{A})\}$*. The collection of orbits of the automorphism group acting on* G *(written* $\mathsf{Orbs}(\mathsf{A})$*) is the set* $\{\mathsf{orb}(a, \mathsf{A}) \mid a \in U^{\mathsf{A}}\}$.

We noted that in the example given above, the orbits of the automorphism group acting on G are $\{2\}$ and $\{1, 3, 4\}$, and that the definable sets are those, together with U^{G} and $\emptyset$ (the universe and the empty set are definable on every structure). We reminded ourselves that for all $f \in \mathsf{Aut}(A)$ and $X \subseteq U^{\mathsf{A}}$, if X is definable on A, then for every $a \in U^{\mathsf{A}}$, $a \in X$ if and only if $f(a) \in X$. It follows at once that if X is definable on A and Y is an orbit of the automorphism group acting on A, then either $Y \subseteq X$ or $Y \cap X = \emptyset$. It follows that on finite structures, the definable sets are exactly the unions of orbits (including the empty union). We suggested this as an approach to the solution of exam problem 5(b), namely, it suffices to show that every orbit is definable.

We next considered an application of the Compactness Theorem relevant to the solution of exam problems 6 and 7. We undertook to analyze the definable subsets of the structure $\mathbb{N}^< = \langle \omega, < \rangle$ consisting of the natural numbers with their usual ordering. We first observed that both zero and the successor relation are definable on this structure and hence that for every $i \in \omega$, $\{i\}$ is definable on $\mathbb{N}^<$, from which it follows immediately that every finite and co-finite subset of ω is definable as well. We began to give an argument, via an application of the Compactness Theorem and the "method of constants," that these are the only sets definable on $\mathbb{N}^<$.

Suppose for *reductio ad absurdum* that there is a formula $\theta(v)$ which defines an infinite and co-infinite subset X of ω on $\mathbb{N}^<$. We introduced two new constant symbols c, d and sets of sentences

$$\Gamma_X = \{(\exists y_1) \dots (\exists y_n)(y_1 < y_2 \wedge \dots y_{n-1} < y_n \wedge y_n < c \mid 1 < n\} \cup \{\theta(c)\},$$

and

$$\Gamma_{\overline{X}} = \{(\exists y_1) \dots (\exists y_n)(y_1 < y_2 \wedge \dots y_{n-1} < y_n \wedge y_n < d \mid 1 < n\} \cup \{\neg\theta(d)\}.$$

It follows from the fact that both X and $\overline{X}$ are infinite, that every finite subset of the set of sentences $\Sigma = \mathsf{Th}(\mathbb{N}^<) \cup \Gamma_X \cup \Gamma_{\overline{X}}$ is satisfiable. Hence, by the Compactness Theorem, there is a structure B with $\mathsf{B} \models \Sigma$. We observed that B consists of the ordered sum of $\mathbb{N}^<$ with an ordered product of (ordered) $\mathbb{Z}$-chains. Let A be the reduct of B to the signature of $\mathbb{N}^<$ and let $a = c^{\mathsf{B}}$ and $b = d^{\mathsf{B}}$. Observe that a and b lie on $\mathbb{Z}$-chains in A. We suggested that B could be chosen in such a way that there is an $f \in \mathsf{Aut}(\mathsf{A})$ with $f(a) = b$. This would contradict $\mathsf{A} \models \theta[a]$ and $\mathsf{B} \models \neg\theta[b]$.

8 Lecture 09.22

Consider the following problem! Show that for every linear ordering O the structure A_{O} consisting of the ordered sum of $\mathbb{N}^<$ and the ordered product $\mathsf{O} \times \mathbb{Z}^<$ is elementarily equivalent to $N^<$. The method developed in today's lecture will enable us to attack this problem.

Roland Fraïssé employed partial isomorphisms to provide a useful characterization of elementary equivalence (an alternative formulation of this characterization was given by Andrzej Ehrenfeucht in terms of a combinatorial game).

The Ehrenfeucht game has two players, Spoiler and Duplicator; the equipment for the game consists of "boards" corresponding to the graphs G and H and pebbles $a_1, a_2, \dots$ and $b_1, b_2, \dots$. The game is organized into rounds $r_1, r_2, \dots$. At each round r_i the Spoiler plays first and picks one of the pair of pebbles a_i or b_i to play onto a vertex of G or H, respectively; the Duplicator then plays the remaining pebble of the pair onto a vertex of the structure into which the Spoiler did not play. This completes the round. Let v_i (resp. w_i) be the vertex of G (resp. H) pebbled at round i, let G_i and H_i be the subgraphs of G and H induced by $\{v_1, \dots, v_i\}$ and $\{w_1, \dots, w_i\}$, respectively, and let $R_i = \{\langle v_j, w_j\rangle \mid 1 \leq j \leq i\}$. The Duplicator wins the game at round r_i if the relation R_i is an isomorphism from G_i onto H_i. The Duplicator has a winning strategy for the i-round game, if she has a method of play which results in a win for her no matter how the Spoiler plays. In this case, we write $G \sim^i H$.

We showed that

Proposition 7 *For every strict linear ordering* O *and for every* $i \in \omega$,

$$\mathbb{N}^< \sim^i \mathsf{A}_\mathsf{O}.$$

Definition 11 *Let* A, B *be structures for a finite relational language (no function symbols) and let* $\mathsf{P}_0 \supseteq \ldots \supseteq \mathsf{P}_n$ *be a sequence of non-empty sets of partial isomorphisms from* A *to* B*. We say that* $\mathsf{P}_0, \ldots, \mathsf{P}_n$ *is an* n-EF-*sequence (and write* $\mathsf{A} \cong^n \mathsf{B}$*) if and only if for every* $0 \leq m < n$,

> *Limited Forth Property: for every* $a \in U^\mathsf{A}$ *and* $f \in \mathsf{P}_{m+1}$*, there is a* $g \in \mathsf{P}_m$ *such that* $f \subseteq g$ *and* $a \in \mathsf{dom}(g)$.
>
> *Limited Back Property: for every* $b \in U^\mathsf{B}$ *and* $f \in \mathsf{P}_{m+1}$*, there is a* $g \in \mathsf{P}_m$ *such that* $f \subseteq g$ *and* $b \in \mathsf{ran}(g)$.

Definition 12 *The quantifier rank of a first-order formula is defined by the following induction.*

1. $\mathsf{qr}(\varphi) = 0$*, if* φ *is atomic.*
2. $\mathsf{qr}(\varphi \wedge \psi) = \mathsf{qr}(\varphi \vee \psi) = \mathsf{qr}(\varphi \to \psi) = \max(\mathsf{qr}(\varphi), \mathsf{qr}(\psi))$.
3. $\mathsf{qr}(\neg\varphi) = \mathsf{qr}(\varphi)$.
4. $\mathsf{qr}((\exists x)\varphi) = \mathsf{qr}((\forall x)\varphi) = \mathsf{qr}(\varphi) + 1$.

We say structures A *and* B *are* n-equivalent *(and write* $\mathsf{A} \equiv^n \mathsf{B}$*) if and only if for all sentences* φ*, if* $\mathsf{qr}(\varphi) \leq n$*, then* $\mathsf{A} \models \varphi$ *if and only if* $\mathsf{B} \models \varphi$.

Theorem 10 (Ehrenfeucht-Fraïssé**)** *If* A *and* B *are structures for a finite relational language, then for every* $n \in \omega$*, the following are equivalent.*

- $\mathsf{A} \equiv^n \mathsf{B}$;
- $\mathsf{A} \cong^n \mathsf{B}$;
- $\mathsf{A} \sim^n \mathsf{B}$.

Corollary 5 (Ehrenfeucht-Fraïssé**)** *If* A *and* B *are structures for a finite relational language, then the following are equivalent.*

- $\mathsf{A} \equiv \mathsf{B}$;
- $\mathsf{A} \cong^n \mathsf{B}$*, for all* $n \in \omega$;
- $\mathsf{A} \sim^n \mathsf{B}$*, for all* $n \in \omega$.

As a corollary to Proposition 7 and Corollary 5 we now have

Corollary 6 *For every strict linear ordering* O, $\mathbb{N}^< \equiv \mathsf{A}_\mathsf{O}$.

9 Lecture 09.29

We went over the Bring Back Examination. Highlights included the following two propositions. The first is a corollary to the Compactness Theorem.

Corollary 7 *If* A *is finitely axiomatizable, and* Γ *axiomatizes* A*, then for some finite* $\Delta \subseteq \Gamma$*,* Δ *axiomatizes* A*.*

Proof: Suppose A is finitely axiomatizable, and let θ be the conjunction of the sentences of such a finite axiomatization. Suppose also that Γ axiomatizes A. It follows at once that $\Gamma \models \theta$, and hence, by the Compactness Theorem, that for some finite $\Delta \subseteq \Gamma$, $\Delta \models \theta$. From which it follows that Δ axiomatizes A. ■

Definition 13 *A* basic formula *is an atomic formula or the negation of an atomic formula; a* basic sentence *is a variable-free basic formula. Let* A *be a structure and let* $C_{\mathsf{A}} = \{c_a \mid a \in U^A\}$ *be a set of constant symbols disjoint from the signature of* A*. (We will sometimes call these the* diagram constants *for* A*.)* A^d *is the expansion of* A *to the signature including* C_{A} *with* $c_a^{\mathsf{A}} = a$*, for every* $a \in U^A$*. The* diagram of A *is the set of basic sentences contained in* $\mathsf{Th}(\mathsf{A}^d)$*.*

Proposition 8 B *contains a substructure isomorphic to* A *if and only* B *has an expansion which satisfies the diagram of* A*.*

10 Lecture 10.04

We introduced a useful definition.

Definition 14 *A set of sentences* Γ *is* finitely satisfiable *if and only if every finite subset of* Γ *is satisfiable.*

We began to present a proof of the Compactness Theorem.

Theorem 11 (Compactness Theorem Restated) *Let* Γ *be a set of first-order sentences. If* Γ *is finitely satisfiable, then* Γ *is satisfiable.*

We started with a very special case of this theorem.

Theorem 12 (Compactness Theorem - Truth-Functional) *Let* Γ *be a set of quantifier-free and identity-free first-order sentences in a relational language, that is, a language with no function symbols. If every finite subset of* Γ *is satisfiable, then* Γ *itself is satisfiable.*

We will prove Theorem 12 by reducing it to the Compactness Theorem for Truth-Functional Logic. We require a further definition in order to state this theorem.

Definition 15 *A formula* θ *of a first-order language is* prime *if and only if* θ *is atomic or* θ *is of the form* $(\forall x)\psi$ *or* $(\exists x)\psi$*. We write* $\mathfrak{P}(L)$ *for the collection of prime formulas of the first-order language* L *and* $\mathfrak{P}_c(L)$ *for the collection of* *prime sentences of L, that is, prime formulas without free variables. We omit L when it is clear from the context. A* truth-assignment *ϵ to $\mathfrak{P}_c(L)$ is a map from $\mathfrak{P}_c(L)$ to $\{0,1\}$ (we think of 1 as representing truth and 0 as representing falsity). We write $\overline{\epsilon}$ for the canonical extension of ϵ to a truth-assignment to all sentences of L. We say that a sentence $\theta \in L$ is* truth-functionally satisfiable *if and only if there is a truth-assignment ϵ such that $\overline{\epsilon}(\theta) = 1$, and we say that a set $\Gamma \subseteq L$ is* truth-functionally satisfiable *if and only if each of its elements is. A set of sentences Γ is* finitely truth-functionally satisfiable *if and only if every finite subset of Γ is truth-functionally satisfiable. We abbreviate "finitely truth-functionally satisfiable" to "ftfs" and "truth-functionally satisfiable" to "tfs."*

If Σ is a set of sentences of L, then the canonical truth-assignment *determined by Σ (written ϵ_Σ) is defined as follows. For every $\theta \in \mathfrak{P}_c(L), \epsilon_\Sigma(\theta) = 1$ if and only if $\theta \in \Sigma$.*

We first prove the Compactness Theorem for Truth Functional Logic for countable languages.

Theorem 13 (Truth-functional Compactness Theorem) *Let Γ be a set of sentences of a countable first-order language. If Γ is ftfs, then Γ is tfs.*

Theorem 13 is a corollary to the next two lemmata.

Lemma 7 *If Γ is a ftfs set of sentences of a countable language L, then there is a $\Gamma \subseteq \Sigma$ such that Σ is ftfs and Σ is complete.*

Lemma 8 *Let Σ be a set of sentences of L. If Σ is ftfs and complete, then for every sentence θ of L, $\overline{\epsilon_\Sigma}(\theta) = 1$ if and only if $\theta \in \Sigma$.*

We will derive Lemma 7 from the following two results, the second of which is a fundamental theorem of infinitary combinatorics.

Sublemma 1 *If Γ is a ftfs set of sentences of L, then for every sentence θ of L, either $\Gamma \cup \{\theta\}$ or $\Gamma \cup \{\neg\theta\}$ is ftfs.*

Proof: Suppose Γ is a ftfs set of sentences of L, but that neither $\Gamma \cup \{\theta\}$ nor $\Gamma \cup \{\neg\theta\}$ is ftfs. Then there are finite $\Delta, \Delta^* \subseteq \Gamma$ such that neither $\Delta \cup \{\theta\}$ nor $\Delta^* \cup \{\neg\theta\}$ is ftfs. But then the finite set $\Delta \cup \Delta^* \subseteq \Gamma$ is not satisfiable, which contradicts our hypothesis that Γ is a ftfs. ■

In order to state the next result we require the following definition.

Definition 16 *Let $\mathbb{B}$ be the set of finite sequences of 0's and 1's. T is a* binary tree *if and only if $T \subseteq \mathbb{B}$ and T is closed under initial segments. If $s \in \mathbb{B}$ and $i \in \{0,1\}$ we write $s * i$ for the sequence which results by adjoining i at the terminus of s. If p is an infinite sequence of 0's and 1's we write $\overline{p}(n)$ for the finite initial segment of p of length n (so for every such p, $\overline{p}(0)$ is the empty sequence).*

Theorem 14 (König Infinity Lemma for Binary Trees) *Suppose T is a binary tree and for every $n \in \omega$ there is an $s \in T$ with length n. Then, there is an infinite sequence p such that for every n, $\overline{p}(n) \in T$. Briefly, every infinite binary tree has an infinite path.*

Proof: Construct by induction a sequence $<>= s_0 \preceq s_1 \preceq \ldots$ of nodes of T, each an initial segment of its successor. At stage n, suppose that the sub-tree of T determined by s_n is infinite (this condition holds at stage 0, by hypothesis). By the Pigeon Hole Principle, either $s_n * 0$ or $s_n * 1$ determines an infinite sub-tree of T, so we may choose $s_n \preceq s_{n+1}$ so that the sub-tree of T determined by s_{n+1} is infinite. Let p be the unique infinite sequence with $\overline{p}(n) = s_n$. ∎

Proof of Lemma 7: Suppose that Γ is finitely satisfiable. Since L is countable, we may enumerate all the sentences of L as $\theta_1, \theta_2, \ldots$. Let $\chi^i = \theta$, if $i = 1$ and let $\chi^i = \neg\theta$, if $i = 0$. For $s \in \mathbb{B}$ of length n, let $\Delta_s = \{\chi^{s_j} \mid 1 \leq j \leq n\}$. Let $T_\Gamma = \{s \in \mathbb{B} \mid \Delta_s \text{ is ftfs}\}$. It follows at once from Sublemma 1 that T is an infinite tree. Hence, by Theorem 14, T has an infinite path p. Let $\Sigma = \bigcup_n \Delta_{\overline{p}(n)}$. It follows at once from the construction and Sublemma 1 that Σ contains Γ, is ftfs, and is complete. ∎

11 Lecture 10.11

We concluded the proof of Theorem 13 and extended it to apply to languages of arbitrary infinite cardinality. The latter requires only a modification to the proof of Lemma 7, since that is the only point in the argument that used the hypothesis that the L is countable, via its enumeration of the sentences of L in order type ω and its invocation of the König Infinity Lemma. We now present the general result and its proof.

Lemma 9 *If Γ is a ftfs set of sentences of a language L, then there is a $\Gamma \subseteq \Sigma$ such that Σ is ftfs and Σ is complete.*

Proof: Suppose that Γ is a ftfs set of sentences of a language L, with $|L| = \kappa$. We may then enumerate the sentences of L in a κ-length sequence $\langle \theta_\beta \mid \beta < \kappa \rangle$. We define a sequence of sets of L sentences, $\langle \Sigma_\beta \mid \beta < \kappa \rangle$ by recursion as follows. $\Sigma_0 = \Gamma$. $\Sigma_{\beta+1} = \Sigma_\beta \cup \{\theta_\beta\}$ if $\Sigma_\beta \cup \{\theta_\beta\}$ is ftfs; $\Sigma_{\beta+1} = \Sigma_\beta \cup \{\neg\theta_\beta\}$, otherwise. If $\gamma < \kappa$ is a limit ordinal, $\Sigma_\gamma = \bigcup_{\beta<\gamma} \Sigma_\beta$. Finally, we let $\Sigma = \bigcup_{\beta<\kappa} \Sigma_\beta$.

It follows immediately from our construction that $\Gamma \subseteq \Sigma$ and that Σ is complete. It remains to show that Σ is ftfs. We show by induction that for every $\beta < \kappa$ that Σ_β is ftfs, from which it follows directly that Σ is ftfs. $\Sigma_0 = \Gamma$ which is ftfs by hypothesis. Suppose Σ_β is ftfs. It follows from our construction and Sublemma 1 that $\Sigma_{\beta+1}$ is ftfs. Finally, suppose that γ is a limit ordinal, and that for every $\beta < \gamma$, Σ_β is ftfs. Let $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Sigma_\gamma$. Note that by our construction, for every $\alpha < \beta < \kappa, \Sigma_\alpha \subseteq \Sigma_\beta$. It follows at once that there is a $\beta < \gamma$ such that $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Sigma_\beta$. But then, $\{\varphi_1, \ldots, \varphi_n\}$ is tfs since, by induction hypothesis, Σ_β is ftfs. ∎

We concluded the proof of Theorem 13 by establishing Lemma 8.

Proof of Lemma 8: Suppose that Σ is ftfs and complete. We show by induction that for every sentence θ of L, $\overline{\text{€}_\Sigma}(\theta) = 1$ if and only if $\theta \in \Sigma$. The basis case for truth-functionally prime sentences is immediate from the definition of €. Suppose, as induction hypothesis, that the result holds for φ and χ. That is,

$$\overline{€_\Sigma}(\varphi) = 1 \text{ if and only if } \varphi \in \Sigma \text{ and,}$$

$$\overline{€_\Sigma}(\chi) = 1 \text{ if and only if } \chi \in \Sigma.$$

We show that

$$\overline{€_\Sigma}(\varphi \wedge \chi) = 1 \text{ if and only if } \varphi \wedge \chi \in \Sigma.$$

By the definition of truth-functional satisfaction,

$$\overline{€_\Sigma}(\varphi \wedge \chi) = 1 \text{ if and only if } \overline{€_\Sigma}(\varphi) = 1 \text{ and } \overline{€_\Sigma}(\chi) = 1.$$

By the induction hypothesis, we have

$$\overline{€_\Sigma}(\varphi) = 1 \text{ and } \overline{€_\Sigma}(\chi) = 1 \text{ if and only if } (\varphi \in \Sigma \text{ and } \chi \in \Sigma).$$

It only remains to show that

$$(\varphi \in \Sigma \text{ and } \chi \in \Sigma) \text{ if and only if } \varphi \wedge \chi \in \Sigma.$$

Proceeding first from left to right, suppose that

$$(\varphi \in \Sigma \text{ and } \chi \in \Sigma)$$

but that

$$\varphi \wedge \chi \notin \Sigma.$$

By the completeness of Σ, it follows that

$$\neg(\varphi \wedge \chi) \in \Sigma.$$

But then,

$$\{\varphi, \chi, \neg(\varphi \wedge \chi)\} \subseteq \Sigma,$$

which contradicts the hypothesis that Σ is ftfs. The argument for the right to left direction is virtually identical, and the cases of the other truth-functional connectives can be handled similarly. ■

We ended with a preview of the proof of the Compactness Theorem for First-Order Logic (Theorem 11). The heart of the argument consists in a reduction of satisfiability to truth-functional satisfiability.

We begin with a a straightforward lemma that codifies the fact that first-order satisfaction and truth-functional satisfaction specify the same semantics for the truth-functional connectives.

Definition 17 *Let* A *be an* L*-structure. The canonical truth-assignment determined by* A *(written* $€_\mathsf{A}$*) is defined as follows. For every truth-functionally prime sentence* θ *of* L*,* $€_\mathsf{A}(\theta) = 1$ *if and only if* $\mathsf{A} \models \theta$*.*

Lemma 10 *If* A *is an* L*-structure, then for every sentence* θ *of* L*,*

$$\mathsf{A} \models \theta \text{ if and only if } €_\mathsf{A}(\theta) = 1.$$

It is easy to see that the converse does not hold, that is, there are truth-assignments $€$ to the prime formulas of a first-order language L such that there is no structure which satisfies every sentence of L satisfied by $€$; for example, suppose $€((\forall x)x \neq x) = 0$. For every first-order language L, we will construct a set of L^*-sentences Ω_L (in an expansion of L^* of L by constant symbols) and prove that it has the following properties.

Lemma 11 *If A is an L-structure, then there is an an expansion A^* of A such that $\mathsf{A}^* \models \Omega_L$.*

Lemma 12 *If $€$ is a truth a assignment to the truth-functionally prime sentences of L^* such that $\overline{€}(\theta) = 1$ for every $\theta \in \Omega_L$, then there is an L^*-structure B such that for every L^*-sentence θ,*

$$\mathsf{B} \models \theta \textit{ if and only if } \overline{€}(\theta) = 1.$$

To prepare for tomorrow's lecture, reconstruct for yourself our derivation of Theorem 11 from Theorem 13 and Lemmas 10-12.

12 Lecture 10.13

To set the stage for this lecture, we'll begin by summarizing our derivation of Theorem 11 from Theorem 13 and Lemmas 10-12. We begin with a useful definition.

Definition 18 *A truth assignment $€$ to the prime sentences of L satisfies a set of L-sentences Γ (written $€ \models \Gamma$) if and only if $\overline{€}(\theta) = 1$ for every $\theta \in \Gamma$.*

Proof of Theorem 11 from Theorem 13 and Lemmas 10-12: Suppose Γ is a finitely satisfiable set of L-sentences and let Δ be a finite subset of Γ. Then there is an L-structure A such that $\mathsf{A} \models \Delta$, and hence, by Lemma 11, there is an L^*-structure A^* such that $\mathsf{A}^* \models \Delta \cup \Omega_L$. By Lemma 10, it follows that $€_{\mathsf{A}^*} \models \Delta \cup \Omega_L$. Therefore, $\Gamma \cup \Omega_L$ is ftfs. Hence, by Theorem 11, $\Gamma \cup \Omega_L$ is tfs. Therefore, by Lemma 12, Γ is satisfiable. ■

We proceeded to prove Lemmas 10 and 12. Our first task was, given a language L, to define the language L^* and the set of L^*-sentences Ω_L. Recall that the difficulty we are trying to overcome is that there are truth-functionally satisfiable sentences, such as $\neg c = c$ and $\neg(\exists x)Rxx \wedge Rcc$, that are unsatisfiable. We must choose Ω_L in such a way that its satisfying truth-assignment respect the logical properties of equality and quantification. We thus divide Ω_L into two groups of axioms: the equality axioms Λ_L and the quantifier axioms Ξ_L. We first describe the language L^* which expands L with a collection of new constant symbols. We first define a sequence of expanded languages L_i and a sequence of sets of constant symbols, C_i, simultaneously, by recursion:

- $L_0 = L$.
- $L_{n+1} = L_n \cup C_n$.

- For each $\theta(x) \in L_0$, a formula with exactly one free variable x, let $c_{\theta(x)}$ be a new constant symbol not in L, and let C_0 be the set of all such constants for $\theta(x) \in L_0$.
- For each $\theta(x) \in L_{n+1} - L_n$, a formula with exactly one free variable x, let $c_{\theta(x)}$ be a new constant symbol not in L_n, and let C_{n+1} be the set of all such constants for $\theta(x) \in L_{n+1} - L_n$.

Finally, we let $C = \bigcup_n C_n$ and $L^* = L \cup C$. We call $c_{\theta(x)}$ the *witnessing constant* or *Henkin constant* for $\theta(x)$, the latter term for Leon Henkin who introduced this method of proving the Compactness Theorem and related results. The rationale for this name is apparent from the quantifier axioms Ξ_L which we now state:

1. (Existential Henkin Axioms) $(\exists x)\theta(x) \rightarrow \theta(c_{\theta(x)})$, for every $\theta(x) \in L^*$ with one free variable.
2. (Universal Henkin Axioms) $\theta(c_{\neg\theta(x)}) \rightarrow (\forall x)\theta(x)$, for every $\theta(x) \in L^*$ with one free variable.
3. (Universal Instantiation) Every L^*-sentence of the form $(\forall x)\theta(x) \rightarrow \theta(t)$, where t is a variable-free term of L^*
4. (Existential Generalization) Every L^*-sentence of the form $\theta(t) \rightarrow (\exists x)\theta(x)$, where t is a variable-free term of L^*

Next we state the Equality Axioms Λ_L. For all variable-free terms, r, s, t and $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$, all n-ary relation symbols R and n-ary function symbols f of L^* all sentences of the following forms

1. (Reflexivity) $r = r$
2. (Symmetry) $r = s \rightarrow s = r$
3. (Transitivity) $(r = s \wedge s = t) \rightarrow r = t$
4. (Leibniz's Law for Relations) $(s_1 = t_1 \wedge \ldots \wedge s_n = t_n) \rightarrow (Rs_1 \ldots s_n \rightarrow Rt_1 \ldots t_n)$
5. (Leibniz's Law for Functions) $(s_1 = t_1 \wedge \ldots \wedge s_n = t_n) \rightarrow (fs_1 \ldots s_n = ft_1 \ldots t_n)$

Proof of Lemma 11: Suppose A is an L-structure. We wish to show that A can be expanded to an L^*-structure which satisfies Ω_L. First note that all the Equality Axioms are valid sentences of L^* as are the Universal instantiation and Existential Generalization Axioms. It follows that any expansion whatsoever of A to L^* satisfies all these axioms. Hence, it suffices to construct an expansion which satisfies the Henkin Axioms. It is easy to verify that any exapnsion which satisfies the Existential Henkin Axioms will also satisfy the Universal Henkin Axioms. We complete the proof by constructing an expansion which satisfies the Existential Henkin Axioms. The construction proceeds by recursion on n, supplying suitable interpretations for the constants in C_n on the assumption that there is an expansion A_n of A to L_n which satisfies all the Existential Henkin Axioms of L_n. We let $\mathsf{A}_0 = \mathsf{A}$, since $L_0 = L$ and A is by hypothesis, an L-structure. Let $(\exists x)\theta(x)$ be an L_n sentence. If $\mathsf{A}_n \models (\exists x)\theta(x)$ let the denotation of $c_{\theta(x)}$ in A be equal to a, for some $a \in U^{\mathsf{A}_n}$ such that $\mathsf{A}_n \models \theta[a]$. If $\mathsf{A}_n \not\models (\exists x)\theta(x)$, we choose the denotation of $c_{\theta(x)}$ arbitrarily. We let A^* be the expansion of A to L^* which agrees when restricted to each L_n with expansion A_n. It is clear from the construction that A^* satisfies all the Existential Henkin Axioms. ■

We proceeded to the proof of the "crucial lemma," the heart of the reduction of first-order to truth-functional satisfaction.

Proof of Lemma 12: Suppose that the truth-assignment $\in$ satisfies Ω_L. We construct an L^*-structure B such that L^*-structure B such that for every L^*-sentence θ,

$$\mathsf{B} \models \theta \text{ if and only if } \in \models \theta.$$

Define a relation $\sim$ on the witnessing constants C by

$$c \sim d \text{ if and only if } \in \models c = d.$$

It follows at once from the fact that $\in$ satisfies the Equality Axioms, that $\sim$ is an equivalence relation on C. For $c \in C$, let $\hat{c}$ be the $\sim$-equivalence class of C and let $U^{\mathsf{B}} = \{\hat{c} \mid c \in C\}$.

Next time, we will complete the definition of the structure B and conclude the proof.

13 Lecture 10.18

We continued the **Proof of Lemma 12**. Our next task is to define the interpretations of relation, function, and constant symbols of L. We define the B interpretation of an n-ary relation symbol R as follows.

$$R^{\mathsf{B}}(\hat{c}_1, \ldots, \hat{c}_n) \text{ if and only if } \in \models Rc_1 \ldots c_n$$

In order to verify the legitimacy of this definition, we need to show that it is independent of the choice of representatives of $\sim$-equivalence classes. But this follows immediately from the fact that $\in$ satisfies Leibniz's Law for Relations.

Interpreting function symbols provides a more substantial challenge! What would be a suitable value $\hat{c}$, for $f^{\mathsf{B}}(\hat{c}_1, \ldots, \hat{c}_n)$. If there were a $c \in C$ such that

$$\in \models fc_1 \ldots c_n = c,$$

then we could set

$$f^{\mathsf{B}}(\hat{c}_1, \ldots, \hat{c}_n) = \hat{c}$$

, for some such $c \in C$, and as in the case of relation symbols, the fact that $\in$ satisfies Leibniz's Law for Functions would guarantee that our definition is independent of the choice of representatives of equivalence classes. But why chould there be such a $c \in C$. We argue as follows.

By the the reflexivity of identity we have:

$$€ \models fc_1 \ldots c_n = fc_1 \ldots c_n.$$

Hence, by Existential Generalization, it follows that

$$€ \models (\exists x) fc_1 \ldots c_n = x.$$

Therefore, by the Existential Henkin Axioms,

$$€ \models fc_1 \ldots c_n = c_{fc_1 \ldots c_n = x}.$$

The interpretations of constant symbols are handled exactly as are function symbols (indeed, we could even assimilate constant symbols to that case by regarding constants as functions of zero arguments). *Note that we use all of the strength of the hypothesis that* $€ \models \Omega_L$ *in legitimating the definition of* B (except the axioms involving the universal quantifier).

We conclude the proof by establishing, by induction on L^*-sentences, that for every L^*-sentence θ,

$$\mathsf{B} \models \theta \text{ if and only if } € \models \theta.$$

In order to prove the basis case, we must first establish that B evaluates closed terms appropriately, that is, we need to show, by induction on closed terms t that

$$t^{\mathsf{B}} = \hat{c} \text{ if and only if } € \models t = c. \tag{1}$$

We did the induction in class, and leave it as an exercise here for practice in syntactic inductions.

Now we show that

$$\mathsf{B} \models Rt_1 \ldots t_n \text{ if and only if } € \models Rt_1 \ldots t_n.$$

Let

$$t_1^{\mathsf{B}} = \hat{c_1}, \ldots, t_n^{\mathsf{B}} = \hat{c_n}, \tag{2}$$

from which it follows at once from (1) that

$$€ \models t_1 = c_1 \ldots, € \models t_n = c_n. \tag{3}$$

Suppose first that

$$\mathsf{B} \models Rt_1 \ldots t_n.$$

It follows at once from the definition of satisfaction and (2) that

$$R^{\mathsf{B}}(\hat{c_1}, \ldots, \hat{c_n}).$$

But then, by the definition of R^{B} we have

$$€ \models Rc_1 \ldots c_n,$$

and thence by (3) and the fact that C satisfies Leibniz's Law for Relations

$$\mathsf{C} \models Rt_1 \dots t_n.$$

Note that each of the above implications can be reversed with the same justification, thus concluding our proof of this case of the basis of our induction. We leave the case of atomic sentences of the form $t_1 = t_2$, which can be handled similarly, to the reader.

We leave the induction case for truth-functional compounds, which is entirely straightforward, to the reader. this leaves the case of the quantifiers. We treat the existential case and leave the universal case, which can be handled similarly, to the reader. So suppose, as induction hypothesis, that for every variable-free term t,

$$\mathsf{B} \models \theta(t) \text{ if and only if } \mathsf{C} \models \theta(t). \tag{4}$$

We need to show that

$$\mathsf{B} \models (\exists x)\theta(x) \text{ if and only if } \mathsf{C} \models (\exists x)\theta(x). \tag{5}$$

Suppose first that

$$\mathsf{B} \models (\exists x)\theta(x).$$

Then, by the definition of satisfaction and the definition of B, for some $c \in C$,

$$\mathsf{B} \models \theta(c).$$

Hence, by (4),

$$\mathsf{C} \models \theta(c).$$

Therefore, by the fact that C satisfies the Existential Instantiation Axioms,

$$\mathsf{C} \models (\exists)\theta(x).$$

Now, proceeding in the other direction, suppose that

$$\mathsf{C} \models (\exists)\theta(x).$$

Since C satisfies the Existential Henkin Axioms, we have

$$\mathsf{C} \models \theta(c_{\theta(x)}).$$

Hence, by (4),

$$\mathsf{B} \models \theta(c_{\theta(x)}).$$

Then, by the definition of satisfaction and the definition of B,

$$\mathsf{B} \models (\exists x)\theta(x).$$

This concludes the induction. ■

As a corollary to Lemmas 10-12, we have the following result which will prove useful in establishing the Löwenheim-Skolem Theorem. Recall that L-structures A and B are elementarily equivalent ($\mathsf{A} \equiv \mathsf{B}$) if and only if they satisfy exactly the same L-sentences.

Corollary 8 *Suppose* A *is an* L*-structure. Then, there is an* L*-structure* B *such that* $\mathsf{B} \equiv \mathsf{A}$ *and* $|\mathsf{B}| \leq |L|$.

Proof: Let A be an L-structure. By Lemma 11, let the L^*-structure A^* be the "Henkin" expansion of A to the language $L^* = L \cup C$, where C is the set of witnessing. Note that $|C| = |L|$ and that $\mathsf{A}^* \models \Omega_L$. Next, by Lemma 10, pick a truth assignment $\in$ to the prime sentences of L^* such that for every L^*-sentence θ,

$$\in \models \theta \text{ if and only if } \mathsf{A}^* \models \theta. \tag{6}$$

It follows that $\in$ satisfies Ω_L, hence, by Lemma 12, we may construct a structure B with the following properties.

$$\text{For every } L^*\text{-sentence } \theta,\ \mathsf{B} \models \theta \text{ if and only if } \in \models \theta. \tag{7}$$

$$\text{For every } b \in U^{\mathsf{B}},\ \text{there is a } c \in C,\ \text{such that } b = \hat{c}. \tag{8}$$

Let B^- be the reduct of B to L. It follows from (6) and (7) that $\mathsf{B}^- \equiv \mathsf{A}$. Moreover, it follows from (8) that $|\mathsf{B}^-| \leq |L|$. ■

We proceeded to prove the

Theorem 15 (Löwenheim-Skolem) *Suppose* L *is a countable language and* Γ *is a set of* L*-sentences. If* Γ *has an infinite model, then for every infinite cardinal* κ*,* Γ *as a model of cardinality* κ*.*

Proof: Suppose Γ is a set of sentences in a countable language L and that Γ has an infinite model. Fix an infinite cardinal κ and let $C = \{c_\beta, \beta < \kappa\}$ be a set of κ new constant symbols. Let $L' = L \cup C$ and note that $|L'| = \kappa$. Let $\Sigma = \Gamma \cup \{c_\alpha \neq c_\beta \mid \alpha < \beta < \kappa\}$. It follows at once, from the hypothesis that Γ has an infinite model and the Compactness Theorem, that for some L'-structure A, $\mathsf{A} \models \Sigma$, and for every such structure $|\mathsf{A}| \geq \kappa$. On the other hand, it follows from Corollary 8 that there is a structure$\mathsf{B}$ such that $\mathsf{B} \models \Sigma$ and $|\mathsf{B}| \leq \kappa$. The reduct of B to L is a model of Γ of cardinality κ. ■

We observed a surprising consequence of Theorem 15 ("Skolem's Paradox") that Zermelo-Fraenkel set theory has a countable model, despite the fact that it proves that there are uncountably many subsets of ω. We will discuss the resolution of this apparent paradox next time.

14 Lecture 10.20

We briefly discussed the resolution of Skolem's Paradox. Zermelo-Fraenkel set theory (ZF) is a first-order theory in the language of graphs (one binary relation E called membership, so models of ZF are directed graphs). If ZF has a model, then it has a countable model, by the Löwenheim-Skolem Theorem. We may as well suppose this countable model, call it A, is a transitive, epsilon model, that is, U^{A} is a collection of sets closed under the membership relation, and E^{A} is the membership relation restricted to U^{A}. Since A is countable and transitive, every $a \in U^{\mathsf{A}}$ is a countable set. Since it is a theorem of ZF that there are uncountable sets, $\mathsf{A} \models \neg(\exists f)\theta[a]$, where a is, for example, the power set of ω in A, that is, the members of a are exactly the subsets of ω that are members of U^{A}, and $\theta(f,x)$ expresses the condition "f is a bijection between ω and x." The resolution of the paradox is now clear. Though there are bijections between a and ω, none of these are in U^{A}; this is hardly surprising, U^{A} is after all *countable*.

Sam asked why one should suppose there are set-theoretic models of ZF at all, since the universe of sets is not a set. As it happens, we are on the verge of proving two theorems germane to answering this question: the Soundness and Completeness Theorems for first-order logic. Let's state these, and then return to the question. But first we need to discuss the notion of formal proof.

Thus far, we have been engaged almost exclusively with the model-theoretic aspects of logic, that is, questions related to the expressive power of first-order logic in two senses: which classes of structures can be axiomatized by first-order theories and what sorts of sets and relations can be defined by first-order formulas in a fixed structure. Now we will turn our attention to another aspect of logic, namely, what sentences can be *derived* from a set of first-order sentences. We will define a formal derivation relation $\Pi(d,\Gamma,\theta)$ – the sequence of formulas d is a derivation of the formula θ from the set of sentences Γ. The definition will be such that there exists a mechanical procedure which, when given an oracle for membership in Γ can decide whether or not the relation $\Pi(d,\Gamma,\theta)$ holds for arbitrary inputs d and θ. We write $\Gamma \vdash \theta$ for $(\exists d)\Pi(d,\Gamma,\theta)$, that is θ is derivable from Γ. Our goal in constructing such a system of derivation is to derive formulas θ which are logical consequences of Γ, that is, we require that our system be set up so as to satisfy the *Soundness Property*: for all sets of first-order sentences Γ and all formulas θ,

$$\text{if } \Gamma \vdash \theta, \text{ then } \Gamma \models \theta. \tag{9}$$

Clearly, with sufficient care in our definition of derivation, we can guarantee that condition (11) holds. Moreover, we would also like our system to afford the opportunity to derive every logical consequence of a given set of first-order sentences Γ, that is, we wish to our system to satisfy the *Completeness Property*: for all sets of first-order sentences Γ and all formulas θ,

$$\text{if } \Gamma \models \theta, \text{ then } \Gamma \vdash \theta. \tag{10}$$

Remarkably, Kurt Gödel, in his 1930 doctoral dissertation, established that the notion of derivation can be defined in such a way that both the Soundness and Completeness Properties are satisfied. Immediately below, we will define one such notion $\Pi(d,\Gamma,\theta)$ with associated notion of derivability $\Gamma \vdash \theta$ and establish the following two results.

Theorem 16 *For all sets of first-order sentences* Γ *and all formulas* θ*,*

$$\textit{if } \Gamma \vdash \theta, \textit{ then } \Gamma \models \theta.$$

Theorem 17 *For all sets of first-order sentences* Γ *and all formulas* θ*,*

$$\textit{if } \Gamma \models \theta, \textit{ then } \Gamma \vdash \theta.$$

We present a "Hilbert-style" formal system of derivation based on a collection of axioms and rules of inference. These are adapted from the set of sentences Ω_L introduced in our proof of the Compactness Theorem. Indeed, our proof of the Completeness Theorem will rely heavily on our argument for the Compactness Theorem. As usual, the background context is a fixed first-order language L which may contain arbitrary sets of relation, function, and constant symbols.

Axioms

- Tautologies: Every truth-functionally valid formula of L is an axiom.
- Equality Axioms: Each formula of one of the following forms is an axiom.
  - (Reflexivity) $x = x$
  - (Symmetry) $x = y \to y = x$
  - (Transitivity) $(x = y \wedge y = z) \to x = z$
  - (Leibniz's Law for Relations)

$$(x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \to (Rx_1 \ldots x_n \to Ry_1 \ldots y_n)$$

  - (Leibniz's Law for Functions)

$$(x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \to (fx_1 \ldots x_n = fy_1 \ldots y_n)$$

- Quantifier Axioms: Each formula of one of the following forms is an axiom.
  - (Universal Instantiation) $(\forall x)\theta(x) \to \theta(t)$, where t is a term of L free for x in θ
  - (Existential Generalization) $\theta(t) \to (\exists x)\theta(x)$, where t is a term of L free for x in θ

Rules of Inference

- (*Modus Ponens*) Infer the formula θ from the formulas $\varphi \to \theta$ and φ.
- (Universal Generalization) Infer the formula $\theta \to (\forall x)\varphi(x)$ from the formula $\theta \to \varphi(y)$, where y is not free in θ and x is free for y in φ.
- (Existential Instantiation) Infer the formula $(\exists x)\varphi(x) \to \theta$ from the formula $\varphi(y) \to \theta$ where y is not free in θ and x is free for y in φ.

Definition 19 *The sequence of L-formulas $d = \langle \delta_1, \ldots, \delta_n$ is a* derivation *of the L-formula θ from the set of L-sentences Γ (written $\Pi(d, \Gamma, \theta)$) if and only if $\delta_n = \theta$ and for every $1 \leq i \leq n$, $\delta_i \in \Gamma$ or δ_i is an axiom, or δ_i may be inferred from formula(s) δ_j (and δ_k) by one of the rules of inference where $j, k < i$. The L-sentence θ is* derivable *frok Γ (written $\Gamma \vdash \theta$) if and only if $(\exists d)\Pi(d, \Gamma, \theta)$.*

Proof of Theorem 16: We show by complete induction on the length of derivations that

$$\text{if } \Pi(d, \Gamma, \theta(x_1, \ldots, x_m)), \text{ then } \Gamma \models (\forall x_1) \ldots (\forall x_m)\theta(x_1, \ldots, x_m). \tag{11}$$

Suppose for some n that $d = \langle \delta_1, \ldots, \delta_n \rangle$ is a derivation, and that for all derivations d^* of length strictly less than n, (11) holds for d^* in place of d. We proceed by cases. Suppose $\Pi(d, \Gamma, \theta(x_1, \ldots, x_m))$, and thus $\delta_n = \theta(x_1, \ldots, x_m)$. First, if $\theta \in \Gamma$ the conclusion follows immediately, since in this case θ contains no free variables, as Γ is a set of L-sentences, and every set of sentences implies each of its members. Suppose then that $\theta(x_1, \ldots, x_m)$ is an axiom. If it is a Tautology, the desired conclusion follows at once from Lemma 10. If θ is an Equality Axiom or a Quantifier Axiom, the conclusion again follows immediately, since all such axioms are evidently valid. Suppose then that θ is inferred by *Modeus Ponens* from formulas $\varphi \to \theta$ and φ where, by induction hypothesis,

$$\Gamma \models (\forall x_1) \ldots (\forall x^*_m)(\varphi(x_1, \ldots, x^*_m) \to \theta(x_1, \ldots, x_m)$$

and

$$\Gamma \models (\forall x_1) \ldots (\forall x^*_m)\varphi(x_1, \ldots, x^*_m),$$

where $m \leq m^*$ and all variables occurring free in φ are among $x_1, \ldots, x^*_m$. It follows at once, since no variables occur free in any formula in Γ, that for every structure A and for every variable assignment s with range contained in U^{A},

$$\Gamma \models (\varphi(x_1, \ldots, x^*_m) \to \theta(x_1, \ldots, x_m)[s]$$

and

$$\Gamma \models \varphi(x_1, \ldots, x^*_m)[s].$$

It now follows that for every such assignment s

$$\Gamma \models \theta(x_1, \ldots, x_m)[s],$$

and hence that

$$\Gamma \models (\forall x_1) \ldots (\forall x_m)\theta(x_1, \ldots, x^*_m).$$

Finally, suppose that θ is inferred by Existential Instantiation from $\chi(y) \to \varphi$ where y does not occur free in φ and x is free for y in χ (we will leave the case of formulas inferred by Universal Generalization as an exercise). Thus, $\theta(x_1, \ldots, x_n)$ is the formula $(\exists x)\chi \to \varphi$, and all the free variables of $\chi(y) \to \varphi$ are among $x_1, \ldots, x_n, y$. By induction hypothesis,

$$\Gamma \models (\forall y)(\forall x_1) \ldots (\forall x_n)(\chi(y) \to \varphi).$$

Therefore, for every variable assignment s with range contained in U^{A},

$$\Gamma \models (\chi(y) \to \varphi)[s].$$

Suppose, toward a contradiction, that for some s,

$$\Gamma \not\models (\exists x)\chi \to \varphi[s].$$

Then,

$$\Gamma \models (\exists x)\chi[s] \text{ and } \Gamma \not\models \varphi[s].$$

But then, for some $a \in U^{\mathsf{A}}$,

$$\Gamma \models \chi(y)[s(y|a)] \text{ and } \Gamma \not\models \varphi[s(y|a)],$$

since x is free for y in $\chi(y)$, and y does not occur free in φ, which contradicts our hypothesis. ■

15 Lecture 10.25

Proof of Theorem 17: The proof makes heavy use of the reduction of first-order satisfiability to truth-functional satisfiability achieved in the proof of the Compactness Theorem. Suppose that Γ is a set of first-order sentences and that $\Gamma \models \theta$. It follows that $\Gamma \cup \{\neg\theta\}$ is not satisfiable, and hence, from the proof of the Compactness Theorem, that $\Gamma \cup \{\neg\theta\} \cup \Omega_L$ is not truth-functionally satisfiable. Hence, by the Compactness Theorem for truth-functional logic, there is a finite set $\Delta \subseteq \Gamma \cup \Omega_L$ such that $\Delta \cup \{\neg\theta\}$ is not truth-functionally satisfiable. Let $\Delta = \{\varphi_1, \ldots, \varphi_m, \chi_1, \ldots, \chi_n\}$, where each of the φ_i is either a member of Γ or a non-Henkin axiom of Ω_L, and each of the χ_i is a Henkin axiom. Moreover, the Henkin axioms χ_i are arranged in descending order of rank, where the rank of a Henkin axiom χ is the least i such that $\chi \in L_i$ (see the bottom of page 16 for the definition of the L_i). It follows at once that

$$(\varphi_1 \to (\ldots(\varphi_n \to (\chi_1 \to (\ldots(\chi_n \to \theta)\ldots) \tag{12}$$

is a tautology. Next, we replace the Henkin constants appearing in (12) with distinct variables having no occurrences, either free or bound, in (12), yielding the formula

$$(\varphi_1^* \to (\ldots(\varphi_n^* \to (\chi_1^* \to (\ldots(\chi_n^* \to \theta)\ldots) \tag{13}$$

Note that (13) is still a tautology, and that θ is unchanged by this substitution since it is a sentence of $L_0 = L$. Moreover, each of the formulas φ_i^* is still either an element of Γ or an Equality Axiom or a Quantifier Axiom. We now construct a derivation d of θ. The first "line" of d is the tautology (13). We then perform n applications of *Modus Ponens* to arrive at the formula

$$(\chi_1^* \to (\ldots(\chi_n^* \to \theta)\ldots). \tag{14}$$

Now, (14) is of the form

$$((\exists x)\eta(x) \to \eta(y)) \to \psi, \tag{15}$$

where, by our convention on ordering the χ_i descending by rank, y does not occur free in ψ. Now both

$$\neg(\exists x)\eta(x) \to \psi \text{ and } \eta(y) \to \psi \tag{16}$$

can be derived from (15) by "tautological reasoning," that is, invoking suitable tautologies and applications of *Modus Ponens*. Finally, we may derive

$$(\exists x)\eta(x) \rightarrow \psi \tag{17}$$

from the latter formula, via Existential Instantiation, and thence, θ, via "tautological reasoning." ∎

16 Lecture 10.27

We discussed the significance of the Soundness and Completeness Theorems: the semi-decidability of validity and the connection with intuitive validity. We also began to discuss Lindstrom's Theorem. A fuller account is forthcoming.

17 Lecture 11.03

We discussed Bring Back Examination II. In that context, we proved a stronger version of the Downward Löwenheim-Skolem Theorem.

Definition 20 B *is an* elementary substructure *of* A *(written* $\mathsf{B} \preceq \mathsf{A}$ *if and only if* $\mathsf{B} \subseteq \mathsf{A}$ *and for every formula* $\theta(x_1, \ldots, x_n)$ *and* $b_1, \ldots, b_n \in U^{\mathsf{B}}$

$$\mathsf{A} \models \theta(x_1, \ldots, x_n)[b_1, \ldots, b_n] \text{ if and only if } \mathsf{B} \models \theta(x_1, \ldots, x_n)[b_1, \ldots, b_n].$$

Note that if $\mathsf{B} \preceq \mathsf{A}$, then $\mathsf{B} \equiv \mathsf{A}$, but that that there are structures A and B such that $\mathsf{B} \subseteq \mathsf{A}$ and $\mathsf{B} \equiv \mathsf{A}$, but $\mathsf{B} \not\preceq \mathsf{A}$. the next lemma provides a useful sufficient condition guaranteeing that one structure is an elementary submodel of another.

Lemma 13 (Tarski-Vaught Criterion) *Let* $\mathsf{B} \subseteq \mathsf{A}$ *and suppose the following condition is satisfied for every formula* $\theta(y, x_1, \ldots, x_n)$ *of* L*:*

$$\forall b_1, \ldots, b_n \in U^{\mathsf{B}}(\mathsf{A} \models \exists y\theta(y)[b_1, \ldots, b_n] \rightarrow \exists b \in U^{\mathsf{B}}(\mathsf{A} \models \theta[b, b_1, \ldots, b_n])).$$

Then $\mathsf{B} \preceq \mathsf{A}$.

Proof: The lemma may be proved by a straightforward induction on formulas. ∎

Theorem 18 (Löwenheim-Skolem) *Let* A *be a structure for a countable language* L *and let* $X \subseteq U^{\mathsf{A}}$ *be countable. Then there is a countable structure* B *such that* $\mathsf{B} \preceq \mathsf{A}$ *and* $X \subseteq U^{\mathsf{B}}$.

Proof: Let L be a countable language and let A be a structure for L. For each formula of L of the form $\theta(y, x_1, \ldots, x_n)$ let $f_\theta : U^{\mathsf{A}^n} \longrightarrow U^{\mathsf{A}}$ satisfy the following condition:

$$\forall b_1, \ldots, b_n \in U^{\mathsf{B}}(\mathsf{A} \models \exists y\theta[b_1, \ldots, b_n] \rightarrow \mathsf{A} \models \theta[f_\theta(b_1, \ldots, b_n), b_1, \ldots, b_n]).$$

The existence of such an f_θ is guaranteed by the axiom of choice. The set of all the f_θ for θ a formula of L is called a set of *Skolem functions* for A. Note that this set is countable, since L is countable. For $X \subseteq U^{\mathsf{A}}$ let $H(X)$ be the closure of X under a set of Skolem functions for A and let $\mathsf{H}(\mathsf{X})$ be the substructure of A with universe $H(X)$. $\mathsf{H}(\mathsf{X})$ is called the *Skolem hull* of X in A. Note that if X is countable then $H(X)$ is countable and that by the foregoing condition, for every formula $\theta(y, x_1, \ldots, x_n)$ of L

$$\forall b_1, \ldots, b_n \in H(X)(\mathsf{A} \models \exists y\theta[b_1, \ldots, b_n] \rightarrow \exists b \in H(X)(\mathsf{A} \models \theta[b, b_1, \ldots, b_n])).$$

Now let $\mathsf{B} = \mathsf{H}(X)$. It follows at once from Lemma 13 that $\mathsf{B} \preceq \mathsf{A}$. ■

18 Lecture 11.08

We discussed abstract logics and Lindstrom's Theorem. An abstract logic $\mathcal{L}$ assigns to every signature σ a pair $\langle \mathcal{S}^{\mathcal{L}}_\sigma, \models^{\mathcal{L}}_\sigma \rangle$ where $\mathcal{S}^{\mathcal{L}}_\sigma$ is the set of $\mathcal{L}$-sentences of signature σ and $\models^{\mathcal{L}}_\sigma$ is the satisfaction relation of $\mathcal{L}$ for structures and sentences of signature σ. (We will suppress superscripts and subscripts on $\mathcal{S}$ and $\models$ when they are clear from the context.) We will suppose throughout that all signatures σ are purely relational, that is, they include only relation symbols. In addition, we introduce a single logical 0-ary relation constant $\top$ whose semantics is given by, for all A, $\mathsf{A} \models \top$. If $\theta \in \mathcal{S}^{\mathcal{L}}_\sigma$, we define $\mathsf{Mod}(\theta) = \{\mathsf{A} \mid \mathsf{A} \models^{\mathcal{L}}_\sigma \theta\}$, and call this the $\mathcal{L}$-class defined by θ. We say the logic $\mathcal{L}'$ extends the logic $\mathcal{L}$ if and only if every $\mathcal{L}$-class is an $\mathcal{L}'$-class.

The following notions are needed for the statement of Lindstrom's Theorem.

Definition 21 1. *(Isomorphism Closure) A logic $\mathcal{L}$ is* isomorphism closed *if and only if for all $\mathcal{L}$-sentences θ and all structures A and B,*

$$\textit{if } \mathsf{A} \cong \mathsf{B} \textit{ and } \mathsf{A} \models \theta \textit{ then } \mathsf{B} \models \theta.$$

2. *(Negation Closure) A logic $\mathcal{L}$ is* negation closed *if and only if for every $\mathcal{L}$-sentence θ, there is an $\mathcal{L}$-sentence χ such that for every structure A*

$$\mathsf{A} \models \theta \textit{ if and only if } \mathsf{A} \not\models \chi.$$

We write $\neg\theta$ for such a sentence χ.

3. *(Finite Occurrence Property) A logic $\mathcal{L}$ has the* finite occurrence property *if and only if for every for every signature σ and every $\theta \in \mathcal{S}^{\mathcal{L}}_\sigma$, there is a finite $\tau \subseteq \sigma$ such that for all σ-structures A and B,*

$$\textit{if } \mathsf{A}{\restriction}\tau = \mathsf{B}{\restriction}\tau, \textit{ then } \mathsf{A} \models \theta \textit{ if and only if } \mathsf{B} \models \theta.$$

4. *(Admits Model Pairs) If A is a σ-structure and B is a τ-structure, the* model pair *$\langle \mathsf{A}, \mathsf{B} \rangle$ is defined as follows. Let σ' and τ' be disjoint copies of σ and τ. The signature of $\langle \mathsf{A}, \mathsf{B} \rangle$ is $\sigma' \cup \tau' \cup \{U_a, U_b\}$ where U_a and*

U_b are new unary relation symbols. The universe of $\langle \mathsf{A}, \mathsf{B}\rangle$ is $U^{\mathsf{A}} \cup U^{\mathsf{B}}$. All the relation symbols in $\sigma' \cup \tau'$ are interpreted in $\langle \mathsf{A}, \mathsf{B}\rangle$ exactly as their "progenitors" were in A and B, respectively. The interpretations of U_a and U_b in $\langle \mathsf{A}, \mathsf{B}\rangle$ are U^{A} and U^{B} respectively.

A logic $\mathcal{L}$ admits model pairs *if and only if for every pair of σ-structures A and B and every expansion C of $\langle \mathsf{A}, \mathsf{B}\rangle$ to a τ-structure, and every $\theta \in \mathcal{S}^{\mathcal{L}}_{\sigma}$, there are $\theta_a, \theta_b \in \mathcal{S}^{\mathcal{L}}_{\tau}$ such that*

$$\mathsf{A} \models \theta \textit{ if and only if } \langle \mathsf{A}, \mathsf{B}\rangle \models \theta_a \textit{ and } \mathsf{B} \models \theta \textit{ if and only if } \langle \mathsf{A}, \mathsf{B}\rangle \models \theta_b.$$

5. *(Nice) A logic $\mathcal{L}$ is* nice *if and only if $\mathcal{L}$ is isomorphism closed, negation closed, has the finite occurrence property, and admits model pairs.*

6. *((Countable) Compactness Property) A logic $\mathcal{L}$ has the* (countable) compactness property *if and only if for every (countable) collection of $\mathcal{L}$-sentences Γ, if Γ is finitely satisfiable, then Γ is satisfiable.*

7. *(Löwenheim-Skolem Property) A logic $\mathcal{L}$ has the* Löwenheim-Skolem property *if and only if for every countable collection of $\mathcal{L}$-sentences Γ, if Γ has an infinite model, then Γ has a countable model.*

We gave various examples of logics. Of course, first and foremost is first-order logic, $\mathcal{FO}$. We considered extensions of $\mathcal{FO}$ by the addition of generalized quantifiers. Let $(Q_f x)\theta(x)$ say that there are finitely many x such that $\theta(x)$. We observed that the sentence $(Q_f x)x = x$ is true in exactly those structures with finite universe. It follows at once that $\mathcal{FO} + (Q_f)$ is a proper extension of $\mathcal{FO}$ since it lacks the Compactness Property. Let $(Q_u x)\theta(x)$ say that there are uncountably many x such that $\theta(x)$. It is obvious that $\mathcal{FO} + (Q_u)$ lacks the Löwenheim-Skolem property. H. J. Keisler proved that it has the compactness property. Finally, we noted that second-order logic lacks both the compactness property and the Löwenheim-Skolem property, since it describes up to isomorphism both the natural numbers with their usual order via the second-order axiom of induction (every set containing zero and closed under successor contains all natural numbers), and the real numbers with their usual order via separability (existence of a countable dense subset) and the least upper-bound principle (every bounded non-empty set has a least upper bound).

19 Lecture 11.10

We proved Lindstrom's Theorem.

Theorem 19 *[Lindstrom] If $\mathcal{L}$ is a nice logic with the countable compactness property and the Löwenhein-Skolem property, and $\mathcal{L}$ extends $\mathcal{FO}$, then $\mathcal{L} = \mathcal{FO}$.*

Proof: Suppose that $\mathcal{L}$ satisfies the hypotheses of the theorem, but that, contrary to the conclusion, there is an $\mathcal{L}$-sentence θ of signature σ such that

$$\text{for every } \mathcal{FO}\text{-sentence } \chi \text{ of signature } \sigma,\ \mathsf{Mod}(\theta) \neq \mathsf{Mod}(\chi). \tag{18}$$

Since $\mathcal{L}$ is nice, it has the finite occurrence property, so we may, without loss of generality, assume that σ is finite. We first show that there are structures A and B such that

$$\mathsf{A} \equiv \mathsf{B}, \ \mathsf{A} \models \theta, \text{ and } \mathsf{B} \models \neg\theta. \tag{19}$$

Let $\Gamma = \{\chi \mid \chi \in \mathcal{S}_\sigma^{\mathcal{FO}} \text{ and } \theta \models \chi\}$. It follows from the compactness property and (18), that $\Gamma \not\models \theta$, and thence that there is a structure B such that $\mathsf{B} \models \Gamma$ and $\mathsf{B} \not\models \theta$. It follows via another application of the compactness property, that there is a structure A such that $\mathsf{A} \models \mathsf{Th}(B) \cup \{\theta\}$. (19) now follows immediately.

Let τ be a signature extending the signature of the model pair $\langle \mathsf{A}, \mathsf{B} \rangle$ with an infinite sequence of new relation symbols R_n, $n \geq 0$, each of arity $2n$ ($R_0 = \top$). We construct Γ, a set of $\mathcal{FO}$-sentences of signature τ, as follows. First, we guarantee that any structure satisfying Γ is a model pair, by including the following sentence π.

$$(\forall x)(U_a(x) \vee U_b(x)) \wedge (\exists x)U_a(x) \wedge (\exists x)U_b(x)$$

For each relation symbol E in σ, we write E_a and E_b for the distinct copies of E in τ. For each $n \geq 1$, let $\{\alpha_{n,i}(x_1, \ldots, x_n) \mid 1 \leq i \leq k_n\}$ be the set of all atomic formulas constructed from the variables $x_1, \ldots, x_n$ in the signature σ_a and let $\{\beta_{n,i}(y_1, \ldots, y_n) \mid 1 \leq i \leq k_n\}$ be the corresponding enumeration of atomic formulas constructed from the variables $y_1, \ldots, y_n$ in the signature σ_b. We write $\overline{x}$ for $x_1, \ldots, x_n$ and $\overline{y}$ for $y_1, \ldots, y_n$. For each $n \geq 1$, the following sentence η_n is a member of Γ.

$$(\forall \overline{x})(\forall \overline{y})((\bigwedge_{1 \leq i \leq n} U_a(x_i) \wedge \bigwedge_{1 \leq i \leq n} U_b(x_i) \wedge E_n(\overline{x}, \overline{y})) \rightarrow \bigwedge_{1 \leq i \leq k_n} (\alpha_{n,i}(\overline{x}) \leftrightarrow \beta_{n,i}(\overline{y})))$$

Finally, for each $n \geq 0$, let φ_n be the conjunction of the following two sentences.

$$\begin{aligned}&(\forall \overline{x})(\forall \overline{y})((\bigwedge_{1 \leq i \leq n} U_a(x_i) \wedge \bigwedge_{1 \leq i \leq n} U_b(x_i) \wedge E_n(\overline{x}, \overline{y})) \rightarrow \\ &\qquad\qquad (\forall x)(U_a(x) \rightarrow (\exists y)(U_b(y) \wedge E_{n+1}(\overline{x}, x, \overline{y}, y))))\end{aligned}$$

$$\begin{aligned}&(\forall \overline{x})(\forall \overline{y})((\bigwedge_{1 \leq i \leq n} U_a(x_i) \wedge \bigwedge_{1 \leq i \leq n} U_b(x_i) \wedge E_n(\overline{x}, \overline{y})) \rightarrow (\forall y)(U_b(y) \rightarrow \\ &\qquad\qquad (\exists x)(U_a(x) \wedge E_{n+1}(\overline{x}, x, \overline{y}, y))))\end{aligned}$$

We let $\Gamma = \{\pi\} \cup \{\eta_n \mid n \geq 1\} \cup \{\varphi_n \mid n \geq 0\}$ and $\Gamma_k = \{\pi\} \cup \{\eta_n \mid k \geq n \geq 1\} \cup \{\varphi_n \mid k \geq n \geq 0\}$. If $\mathsf{C} \models \Gamma$, then it consists of a model pair $\langle \mathsf{A}', \mathsf{B}' \rangle$ along with relations E_n^{C} for $n \geq 1$. Since $\mathsf{C} \models \eta_n$, if $\langle \overline{a}, \overline{b} \rangle \in E_n^{\mathsf{C}}$, then the map that sends a_i to b_i, $1 \leq i \leq n$ is a partial isomorphism from A' to B'. For each $n \geq 0$, let $\mu_n^{\mathcal{C}}$ be the set of such maps (note since $E_0 = \top$, $\mu_0^{\mathcal{C}} = \{<>\}$, the singleton of the empty map). Thus,

$$\text{if } \mathsf{A}' \cong^k \mathsf{B}', \text{ then } \mathsf{C} \models \Gamma_k, \tag{20}$$

and

$$\text{if } \mathsf{C} \models \Gamma, \text{ then } \bigcup_{n \geq 1} \mu_n \text{ witnesses } \mathsf{A}' \cong_p \mathsf{B}'. \tag{21}$$

We claim that

$$\{\theta_a\} \cup \{\neg\theta_b\} \cup \Gamma \text{ is satisfiable.} \tag{22}$$

Recall, by the Ehrenfeucht-Fraïssé Theorem, that for every k,

$$\text{if } A \equiv^k B, \text{ then } A \cong^k B. \tag{23}$$

It follows at once from (19), (20), and (22), that for every k,

$$\{\theta_a\} \cup \{\neg\theta_b\} \cup \Gamma_k \text{ is satisfiable.} \tag{24}$$

But now, it follows immediately from the countable compactness property for $\mathcal{L}$ that (22) holds, and moreover, by the Löwenheim-Skolem property, that $\{\theta_a\} \cup \{\neg\theta_b\} \cup \Gamma$, has a countable model. But then, by (21), there are countable structures A^* and B^* such that

$$\mathsf{A}^* \cong_p \mathsf{B}^*, \ \mathsf{A}^* \models \theta, \text{ and } \mathsf{B}^* \models \neg\theta. \tag{25}$$

Hence, by the Cantor Back-and-Forth Theorem, there are structures A^* and B^* such that

$$\mathsf{A}^* \cong \mathsf{B}^*, \ \mathsf{A}^* \models \theta, \text{ and } \mathsf{B}^* \models \neg\theta. \tag{26}$$

But, (26) contradicts the fact that $\mathcal{L}$ is nice, since it violates the Isomorphism Closure condition. ■