Locality for Finite Quantifier Depth Infinitary Logic with Generalized Unary Quantifiers

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1 Introduction

To say that a logic is *local* is, roughly, to say that it cannot distinguish structures which look alike on small scales. This paper gives simple statements of results by Hella, Libkin, and Nurmonen showing that $\mathcal{L}_{\infty\omega}(\mathcal{Q}_u)^\omega$ (the finite quantifier-depth fragment of infinitary logic extended with generalized unary quantifiers) is Hanf-local, and that Hanf locality implies Gaifman locality.

2 Preliminaries

We adopt a convention that Gothic letters denote structures, and the corresponding Roman letters denote the universe of that structure. For example, $\mathfrak{A}$ would be a structure with universe A.

Let $\mathfrak{A}$ be an σ-structure with universe A, and $G(A)$ its Gaifman graph with edge set $E^{G(A)}$. We say that $a, b \in A$ are adjacent if either $a = b$ or $(a, b) \in E^{G(A)}$. We use $d(a, b)$ to denote the distance of $a, b \in A$; we define $d(a, b)$ to be the length of the shortest path from a to b in $G(A)$, taking $d(a, a) := 0$. Given $\overline{a} = (a_0, ..., a_n), \overline{b} = (b_0, ..., b_n)$, we define $d(\overline{a}, b) := \min_i d(a_i, b)$ and $d(\overline{a}, \overline{b}) := \min_{i,j} d(a_i, b_j)$. We define the r-ball around $\overline{a}$ by

$$B_r(\overline{a}) := \{b \in U^{\mathcal{G}} \mid d(\overline{a}, b) \leq r\}$$

Suppose that the relations of σ are R_i, $i \in [k]$, each of arity p_i. For an n-tuple $\overline{a} = (a_0, ..., a_{n-1})$ in a structure $\mathfrak{A}$, its r-neighbourhood is the $\sigma' := \sigma \cup \{a_0, ..., a_{n-1}\}$ structure

$$N_r^{\mathfrak{A}}(\overline{a}) := \langle B_r^{\mathfrak{A}}(\overline{a}), R_1^{\mathfrak{A}} \cap B_r^{\mathfrak{A}}(\overline{a})^{p_1}, ..., R_k^{\mathfrak{A}} \cap B_r^{\mathfrak{A}}(\overline{a})^{p_k}, a_0, ..., a_{n-1}\rangle$$

ie the elements of the r-neighbourhood are the elements in the r-ball, relations are inherited from the parent structure, and the elements of $\overline{a}$ are added as extra constants. Write $N_r^{\mathfrak{A}}(\overline{a}) \cong N_r^{\mathfrak{A}}(\overline{b})$ to indicate that $N_r^{\mathfrak{A}}(\overline{a})$ and $N_r^{\mathfrak{B}}(\overline{b})$ are isomorphic.

Given an n-tuple $\overline{a} = (a_0, ..., a_{n-1})$, m-tuple $\overline{b} = (b_0, ..., b_{m-1})$ and an element c, define $\overline{a}c := (a_0, ..., a_{n-1}, c)$ and $\overline{a}\overline{b} := (a_0, ..., a_{n-1}, b_0, ..., b_{m-1})$.

A k-ary query on a structure $\mathfrak{A}$ is a function $Q : \mathfrak{A} \to A^k$ which is closed under isomorphism.

2.1 Hanf Locality

Let τ be an isomorphism type of structures with signature $\sigma_c := \sigma \cup \{c\}$, where c is constant. For a structure $\mathfrak{A}$, we say that $a \in A$ *d-realizes* τ (notation: $\tau_d(\mathfrak{A}, a) = \tau$) if $N_d^{\mathfrak{A}}(a)$ has isomorphism type τ. We denote by $|[\mathfrak{A}, \tau]_d|$ the number of elements in A which d-realize τ. We say that σ-structures $\mathfrak{A}, \mathfrak{B}$ are d-equivalent (and write $\mathfrak{A} \leftrightarrows_d \mathfrak{B}$) if $|[\mathfrak{A}, \tau]_d| = |[\mathfrak{B}, \tau]_d|$ for every isomorphism type of σ_c structures.

There is also a notion of d-equivalence that we will find useful which takes into account parameters. Suppose that $\mathfrak{A}, \mathfrak{B}$ are two structures with common signature. Given $\overline{a} \in A^n$, $\overline{b} \in B^n$, if there is a bijection $f : A \to B$ such that for all $c \in A$ we have that $N_d^{\mathfrak{A}}(\overline{a}c) \cong N_d^{\mathfrak{B}}(\overline{b}f(c))$, we write

$$(\mathfrak{A}, \overline{a}) \leftrightarrows_d (\mathfrak{B}, \overline{b})$$

Note that this generalizes the original definition; take $\overline{a} = \emptyset$.

The $\leftrightarrows_d$ relation says that there is a bijection f which maps all $c \in A$ to elements $f(c) \in B$ which have the same neighbouroods. Intuitively, $\mathfrak{A} \leftrightarrows_d \mathfrak{B}$ says that the two structures $\mathfrak{A}, \mathfrak{B}$ look alike on local scales up to size d.

Definition 1. *A k-ary query Q is* Hanf-local *if there is a $d \geq 0$ such that for every two structures $\mathfrak{A}, \mathfrak{B}$ of same signature and for k-tuples $\overline{a} \in A$, $\overline{b} \in B$, we have that*

$$\big((\mathfrak{A}, \overline{a}) \leftrightarrows_d (\mathfrak{B}, \overline{b})\big) \implies \big(\overline{a} \in Q(\mathfrak{A}) \iff \overline{b} \in Q(\mathfrak{B})\big)$$

The *Hanf locality rank* of a query Q, denoted $hlr(Q)$, is taken to be the least d that witnesses that Q is Hanf-local.

We say that a logic is Hanf-local if every query definable in that logic is Hanf-local. Hanf originally showed that First-Order logic was Hanf local; a more recent result of Fagin, Stockmeyer, and Vardi modifies this result for the finite case.

Theorem 1. *[5] For $n > 0$, there exists a $d > 0$ such that if $\mathfrak{A} \leftrightarrows_d \mathfrak{B}$ then $\mathfrak{A}, \mathfrak{B}$ agree on all FO sentences of quantifier-rank no greater than n. In particular, d can be taken to be 3^{n-1}.*

2.2 Gaifman Locality

While Hanf locality deals with two structures, Gaifman locality deals with one.

Definition 2. *Let Q be a k-ary query on structures of signature $\mathcal{L}$. Q is* Gaifman local *if there is a $d \geq 0$ such that for any $\mathcal{L}$-structure $\mathfrak{A}$ and k-tuples $\overline{a_0}, \overline{a_1} \in A^k$,*

$$N_d^{\mathfrak{A}}(\overline{a_0}) \cong N_d^{\mathfrak{A}}(\overline{a_1}) \implies \big(\overline{a_0} \in Q(\mathfrak{A}) \iff \overline{a_1} \in Q(\mathfrak{A})\big)$$

Gaifman showed that all FO queries are Gaifman-local[6].

2.3 Unary Quantifiers

Suppose that $\mathcal{L}$ is some logic, and let σ_k be a signature consisting of k unary symbols. Let $\mathcal{K}$ be a class of σ_k-structures which is closed under isomorphism. $\mathcal{L}(\mathcal{Q}_\mathcal{K})$ adds to the formulae of $\mathcal{L}$ by adding the rule

$$\text{if } \varphi_1(x_1, \overline{y}_1), ..., \varphi_1(x_k, \overline{y}_k) \text{ are formulae, then so is } \mathcal{Q}_\mathcal{K} x_1...x_k(\varphi_1(x_1, \overline{y}_1), ..., \varphi_1(x_k, \overline{y}_k))$$

$\mathcal{Q}_\mathcal{K}$ binds x_i in the i^{th} formula for all $i \in [k]$. Let $\varphi_i[\mathfrak{A}, \overline{a_i}] := \{a \in A \mid \mathfrak{A} \models \varphi_i(a, \overline{a_i})\}$. We then define the interpretation of $\mathcal{Q}_\mathcal{K}$ by

$$\mathfrak{A} \models \mathcal{Q}_\mathcal{K} x_1...x_k(\varphi_1(x_1, \overline{y}_1), ..., \varphi_1(x_k, \overline{y}_k)) \iff (A, \varphi_1[\mathfrak{A}, \overline{a_1}], ..., \varphi_k[\mathfrak{A}, \overline{a_k}]) \in \mathcal{K}$$

The $\mathcal{Q}_\mathcal{K}$ described above is called a *unary quantifier*. When $\mathcal{Q}$ is a set of unary quantifiers, the logic $\mathcal{L}(\mathcal{Q})$ is defined by the corresponding rule for each $Q_\mathcal{K} \in \mathcal{Q}$. Examples of unary quantifiers include the Hartig (equicardinality) and Rescher (larger cardinality) quantifiers. We write $\mathcal{Q}_u$ for the set of all unary quantifiers.

Infinitary logic $\mathcal{L}_{\infty\omega}$ extends first-order logic by allowing infinite disjunctions and conjunctions. $\mathcal{L}_{\infty\omega}$ is too expressive to be tractable though; as it can express any isomorphism-closed class of structures, we are interested in fragments of lesser expressive power. In CIS 518 we studied $\mathcal{L}^k_{\infty\omega}$, that is, the k-variable fragment of $\mathcal{L}_{\infty\omega}$. Here, we concern ourselves with the fragment of $\mathcal{L}_{\infty\omega}(\mathcal{Q}_u)$ which only contains sentences of finite quantifier rank; we denote this logic by $\mathcal{L}_{\infty\omega}(\mathcal{Q}_u)^\omega$. A result of Hella[4] shows that the bijective Ehrenfeucht-Fraisse Game[1] characterizes the expressivity of $\mathcal{L}_{\infty\omega}(\mathcal{Q}_u)^\omega$.

3 $\mathcal{L}_{\infty\omega}(\mathcal{Q}_u)^\omega$ is Hanf-Local

In what follows, we will require the next three basic claims.

Lemma 1. *Suppose $\mathfrak{A}$ is a σ-structure and $f : N^\mathfrak{A}_r(\overline{a}) \to N^\mathfrak{A}_r(\overline{b})$ is an isomorphism. For $d \leq r$, let f_d be the restriction of f to $B^\mathfrak{A}_d(\overline{a})$. Then $f_d : N^\mathfrak{A}_d(\overline{a}) \to N^\mathfrak{A}_d(\overline{b})$ is an isomorphism.*

Proof. As f is an isomorphism it suffices to show that f maps $B_d(\overline{a})$ onto $B_d(\overline{b})$. Suppose $x \in B_d(\overline{a})$ and $d(\overline{a}, x) \leq i < d$. Then there is a path of i from some $a \in \overline{a}$ to x. So we can find elements $x_1, ..., x_i$ and tuples $\overline{t}_1, ..., \overline{t}_{i+i}$ such that $i < d$, every $\overline{t}_i$ is in some relation in σ, and the following hold

- $\overline{a}, x_1 \in \overline{t}_1$
- $(\forall k \in [1, i-1])(x_k, x_{k+1} \in \overline{t}_{k+1})$

[1] This is the n-round EF game in which in each round the duplicator picks a bijection from A to B, the spoiler picks a point $a_i \in A$, and the duplicator wins if after the last round if $\{(a_i, f(a_i)) \mid i \in [n]\}$ is a partial isomorphism $\mathfrak{A} \to \mathfrak{B}$

- $x_i, x \in \bar{t}_{i+1}$.

As f is an isomorphism of $N_r^{\mathcal{A}}(\overline{a})$ onto $N_r^{\mathcal{A}}(\overline{a})$ and $d < r$ we then have

- $\bar{b}, f(x_1) \in f(\bar{t}_1)$
- $(\forall k \in [1, i-1])\Big(f(x_k), f(x_{k+1}) \in f(\bar{t}_{k+1})\Big)$
- $f(x_i), f(x) \in f(\bar{t}_{i+1})$.

As f is an isomorphism from $N_d(\overline{a})$ to $N_d(\bar{b})$, we have that each $f(\bar{t}_k) \in R_m \cap B_d(\bar{b})$ for some m, so $f(x) \in B_d(\bar{b})$. The same argument applied to f^{-1} shows that for all $y \in B_d(\bar{b})$, $f^{-1}(y) \in B_d(\overline{a})$. So f_d maps $B_d(\overline{a})$ to $B_d(\bar{b})$, and the claim follows. □

Lemma 2. *Suppose $\mathfrak{A}$ is a σ-structure and $f : N_r^{\mathfrak{A}}(\overline{a}) \to N_r^{\mathfrak{A}}(\bar{b})$ is an isomorphism. For $d+l \leq r$ and a tuple $\overline{x} \in B_l^{\mathfrak{A}}(\overline{a})$, we have that $f(B_d^{\mathfrak{A}}(\overline{x})) = B_d^{\mathfrak{A}}(f(\overline{x}))$, and $N_d^{\mathfrak{A}}(\overline{x}) \cong N_d^{\mathfrak{A}}(f(\overline{x}))$.*

Proof. The argument in the above proof shows one inclusion: for any x with $d(\overline{a}, x) \leq l$, f maps $B_d(x)$ onto $B_d(f(x))$ for $d \leq r - l$, from which it follows that f maps $B_d(\overline{x})$ onto $B_d(f(\overline{x}))$.

As $d + l < r$ and $\overline{x} \in B_l^{\mathfrak{A}}(\overline{a})$, $B_d(\overline{x}) \subseteq B_r(\overline{a})$. Then $B_d(f(\overline{x})) \subseteq B_r(\bar{b})$. So $N_d^{\mathfrak{A}}(\overline{x}) \cong N_d^{\mathfrak{A}}(f(\overline{x}))$. □

Lemma 3. *Let $\mathfrak{A}, \mathfrak{B}$ be σ-structures, $\overline{a_0} \in A^n, \overline{b_0} \in B^n, \overline{a_1} \in A^m, \overline{b_1} \in B^m$. Suppose that $N_r^{\mathfrak{A}}(\overline{a_0}) \cong N_r^{\mathfrak{B}}(\overline{b_0})$ and $N_r^{\mathfrak{A}}(\overline{a_1}) \cong N_r^{\mathfrak{B}}(\overline{b_1})$, and let $\overline{a} := \overline{a_0 a_1}$, $\bar{b} := \overline{b_0 b_1}$. Lastly assume that $d(\overline{a_1}, \overline{a_2}) > 2r+1$ and that $d(\overline{b_1}, \overline{b_2}) > 2r+1$ as well. Then $N_r^{\mathfrak{A}}(\overline{a}) \cong N_r^{\mathfrak{B}}(\bar{b})$.*

Proof. Let R be any relation in σ interpreted in $N_r^{\mathfrak{A}}(\overline{a})$. Then any tuple in R is composed solely of elements in $B_r^{\mathfrak{A}}(\overline{a_0})$ or solely of elements in $B_r^{\mathfrak{A}}(\overline{a_1})$, as $d(\overline{a_0}, \overline{a_1}) > 2r+1$. Similarly, any tuple in any σ-relation in $N_r^{\mathfrak{B}}(\bar{b})$ has all its components in $B_r^{\mathfrak{B}}(\overline{b_0})$ or $B_r^{\mathfrak{B}}(\overline{b_1})$. So the isomorphism from $N_r^{\mathfrak{A}}(\overline{a})$ to $N_r^{\mathfrak{B}}(\bar{b})$ can be defined componentwise on $\overline{a_0}, \overline{a_1}$. □

Lemma 4. *Suppose $\mathfrak{A} \leftrightarrows_d \mathfrak{B}$ and $N_{3d+1}^{\mathfrak{A}}(\overline{a}) \cong N_{3d+1}^{\mathfrak{B}}(\bar{b})$. Then $(\mathfrak{A}, \overline{a}) \leftrightarrows_d (\mathfrak{B}, \bar{b})$.*

Proof. To avoid notational confusion between the universe of $\mathfrak{B}$ and balls of radius r (which we would normally both denote with the letter B), we break with normal convention and instead use $\mathcal{B}$ to denote the universe of $\mathfrak{B}$, and $\mathcal{A}$ as the universe of $\mathfrak{A}$.

Let $h : N_{3d+1}^{\mathfrak{A}}(\overline{a}) \to N_{3d+1}^{\mathfrak{B}}(\bar{b})$ be an isomorphism witnessing that $N_{3d+1}^{\mathfrak{A}}(\overline{a}) \cong N_{3d+1}^{\mathfrak{B}}(\bar{b})$. By Lemma 1, h restricted to $B_{2d+1}^{\mathfrak{A}}(\overline{a})$ is an isomorphism $h' : B_{2d+1}^{\mathfrak{A}}(\overline{a}) \to B_{2d+1}^{\mathfrak{B}}(\bar{b})$. So $|\mathcal{A} \setminus B_{2d+1}^{\mathfrak{A}}(\overline{a})| = |\mathcal{B} \setminus B_{2d+1}^{\mathfrak{B}}(\bar{b})|$.

Let $a \in B_{2d+1}^{\mathfrak{A}}(\overline{a})$ and let τ be the isomorphism type realized by a. As h maps $(3d+1)$-neighbourhoods isomorphically onto $(3d+1)$-neighbourhoods, it follows that $B_d^{\mathfrak{A}}(a) \subseteq B_{3d+1}^{\mathfrak{A}}(\overline{a})$, so $h(a) \in B_{2d+1}^{\mathfrak{B}}(\bar{b})$ realizes τ as well. So the number of elements which realize τ

is the same in $B^{\mathfrak{A}}_{2d+1}(\overline{a})$ as it is in $B^{\mathfrak{B}}_{2d+1}(\bar{b})$. Moreover, as $\mathfrak{A} \leftrightarrows_d \mathfrak{B}$, we have that $|[\mathfrak{A},\tau]_d| = |[\mathfrak{B},\tau]_d|$ for every τ. It follows that for every τ

$$|\{a \in (\mathcal{A} \setminus B^{\mathfrak{A}}_{2d+1}(\overline{a})) \mid \tau_d(\mathfrak{A}, a) = \tau\}| = |\{b \in (\mathcal{B} \setminus B^{\mathfrak{B}}_{2d+1}(\bar{b})) \mid \tau_d(\mathfrak{B}, b) = \tau\}|$$

So there is a bijection $g : \mathcal{A} \setminus B^{\mathfrak{A}}_{2d+1}(\overline{a}) \to \mathcal{B} \setminus B^{\mathfrak{B}}_{2d+1}(\bar{b})$ with the property that $N^{\mathfrak{A}}_d(a) \cong N^{\mathfrak{B}}_d(g(a))$ for all $a \in \mathcal{A} \setminus B^{\mathfrak{A}}_{2d+1}(\overline{a})$.

We now define a bijection f from $\mathcal{A}$ to $\mathcal{B}$ as

$$f(x) = \begin{cases} h(x) & \text{if } x \in B^{\mathfrak{A}}_{2d+1}(\overline{a}) \\ g(x) & \text{otherwise} \end{cases}$$

It follows that $N^{\mathfrak{A}}_d(\overline{a}x) \cong N^{\mathfrak{B}}_d(\bar{b}f(x))$ for all $x \in \mathcal{A}$. To see this, note that if $x \in B^{\mathfrak{A}}_{2d+1}(\overline{a})$ then $B^{\mathfrak{A}}_d(\overline{x}) \subseteq B^{\mathfrak{A}}_{3d+1}(\overline{a})$ and we have that $N^{\mathfrak{A}}_d(\overline{a}x) \cong N^{\mathfrak{B}}_d(\bar{b}h(x))$ as h is an isomorphism. Similarly, if $x \notin B^{\mathfrak{A}}_{2d+1}(\overline{a})$, then $f(x) = g(x)$ and so $f(x) \notin B^{\mathfrak{B}}_{2d+1}(\bar{b})$ and hence $N^{\mathfrak{A}}_d(x) \cong N^{\mathfrak{B}}_d(g(x))$. The claim then follows by Lemma 3, and the result follows. □

We have the following immediate corollaries to the preceding lemma:

Corollary 1. *If a formula is Hanf-local, it is Gaifman-local.*

Proof. Suppose $\varphi(x_0, ..., x_{n-1})$ is hanf-local with Hanf locality rank d, and let $\overline{a}, \bar{b}$ be n-tuples from a structure $\mathfrak{A}$ such that $N^{\mathfrak{A}}_{3d+1}(\overline{a}) \cong N^{\mathfrak{A}}_{3d+1}(\bar{b})$. As $\mathfrak{A} \leftrightarrows_d \mathfrak{A}$, it follows by Lemma 4 that $(\mathfrak{A}, \overline{a}) \leftrightarrows_d (\mathfrak{A}, \bar{b})$. As φ is Hanf-local, $\mathfrak{A} \models \varphi(\overline{a}) \iff \mathfrak{A} \models \varphi(\bar{b})$ and so φ is Gaifman local with locality rank bounded above by $3d + 1$. □

Corollary 2. *Suppose $(\mathfrak{A}, \overline{a}) \leftrightarrows_{3d+1} (\mathfrak{B}, \bar{b})$. Then there is a bijection $f : A \to B$ which is such that $(\mathfrak{A}, \overline{a}x) \leftrightarrows_d (\mathfrak{B}, \bar{b}f(x))$ for all $x \in A$.*

Proof. Suppose $(\mathfrak{A}, \overline{a}) \leftrightarrows_{3d+1} (\mathfrak{B}, \bar{b})$. Then evidently $\mathfrak{A} \leftrightarrows_d \mathfrak{B}$. Moreover, by definition there is a bijection $f : A \to B$ such that $N^{\mathfrak{A}}_{3d+1}(\overline{a}x) \cong N^{\mathfrak{B}}_{3d+1}(\bar{b}f(x))$ for all $x \in A$. The claim then follows by Lemma 4. □

This second corollary leads to a winning strategy for the n-round bijective Ehrenfeucht-Fraisse Game, which implies the locality of $\mathcal{L}_{\infty\omega}(\mathcal{Q}_u)^\omega$.

Theorem 2. *Let $n \in \mathbb{N}^+$ and $\mathfrak{A} \leftrightarrows \mathfrak{B}$ where $d = (3^{n-1} - 1)/2$. Then the duplicator has a winning strategy in the n-round bijective EF game on $\mathfrak{A}, \mathfrak{B}$.*

Proof. Suppose that $\mathfrak{A} \leftrightarrows_d \mathfrak{B}$. Let $d_0 = 0$, $d_i = 3d_{i-1}+1$ for $i \in [1, n]$, and note that $d = d_{n-1}$. Evidently the duplicator has a winning strategy in the 0-round game; the empty map suffices. Suppose then that after round $i < n$ the spoiler has picked points $a_0, ..., a_{i-1}$, the duplicator has picked bijections $f_0, ..., f_{i-1}$, and that $(\mathfrak{A}, (a_0, ..., a_{i-1})) \leftrightarrows_{d_{n-i}} (\mathfrak{B}, (f_0(a_0), ..., f_{i-1}(a_{i-1})))$. Then Corollary 2 ensures that the duplicator can pick a bijection f_i such that for any a_i we have that

$$(\mathfrak{A}, (a_0, ..., a_{i-1}, a_i)) \leftrightarrows_{d_{n-i-1}} (\mathfrak{B}, (f_0(a_0), ..., f_{i-1}(a_{i-1}), f_i(a_i)))$$

and hence there are a_i, f_i ($i \in [0, n]$) such that $\{(a_i, f(a_i)) \mid 0 \leq i \leq n\}$ is a partial isomorphism. $\square$

Corollary 3. *$\mathcal{L}_{\infty\omega}(\mathcal{Q}_u)^\omega$ is Hanf-local.*

References

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