PRINCIPLE OF MATHEMATICAL INDUCTION

Let P(n) be a **propositional function** defined for all positive integers **n**. P(n) is **true** for every positive integer **n** if

1. Basis Step:
   The proposition P(1) is **true**.

PRINCIPLE OF MATHEMATICAL INDUCTION

2. Inductive Step:

If P(k) is true then P(k + 1) is **true** for all integers $k \geq 1$.

$$p(k) \rightarrow P(k+1) \quad \forall k$$

EXAMPLE

Use Mathematical Induction to prove that

$$1+2+3+\dots+n=\frac{n(n+1)}{2}$$

for all integers $n \geq 1$.

SOLUTION

$$P(n): 1+2+3+\dots+n=\frac{n(n+1)}{2}$$

1. **Basis Step:**

P(1) is **true**. For $n = 1$, left hand side of P(1) is the sum of all the successive integers starting at 1 and ending at 1, so

$$L.H.S = 1$$

SOLUTION

$$L.H.S = 1$$

and

$$R.H.S = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

$$L.H.S = R.H.S$$

so the proposition is true for $n = 1$.

SOLUTION

2. **Inductive Step:** Suppose $P(k)$ is **true** for, some integers $k \geq 1$.

$$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2} \qquad \dots.(1)$$

We will use this **proposition** to show that $P(k+1)$ is **true**.

SOLUTION

If we take $n = k+1$ our proposition becomes

$$1+2+3+\ldots+(k+1)=\frac{(k+1)(k+2)}{2} \quad \ldots (2)$$

SOLUTION

Consider L.H.S. of (2)

$$1+2+3+\dots+(k+1)$$

$$=1+2+3+\dots+k+(k+1)$$

$$=\frac{k(k+1)}{2}+(k+1) \quad \text{using R.H.S of(1)}$$

$$=(k+1)\left[\frac{k}{2}+1\right]$$

SOLUTION

$$= (k+1)\left[\frac{k+2}{2}\right]$$

$$= \frac{(k+1)(k+2)}{2}$$

= R.H.S of proposition (2)

EXERCISE

Use mathematical induction to prove that

$$1+3+5+\ldots+(2n-1) = n^2$$

for all integers $n \geq 1$.

SOLUTION

Let $P(n)$ be the equation

$$1+3+5+\ldots+(2n-1) = n^2$$

EXERCISE

1. Basis Step:

$P(1)$ is true

For $n = 1$,

L.H.S of $P(1) = 1$

and R.H.S $= 1^2 = 1$

Hence the equation is true for

$n = 1$

SOLUTION

2. **Inductive Step:**

Suppose $P(k)$ is true for some integer $k \geq 1$.

That is,

$$1 + 3 + 5 + \dots + (2k - 1) = k^2 \qquad (1)$$

To prove $P(k+1)$ is true.

$$1 + 3 + 5 + \dots + [2(k+1)-1] = (k+1)^2 \qquad (2)$$

SOLUTION

Consider L.H.S. of (2)

$$1+3+5+....+[2(k+1)-1]$$

$$=1+3+5+...+2k-1+(2k+1)$$

$$=1+3+5+...+(2k-1)+(2k+1)$$

$$=k^2+(2k+1) \quad \text{using R.H.S of (1)}$$

$$=(k+1)^2$$

$=$ R.H.S. of equation (2)

SOLUTION

Thus $P(k+1)$ is also true. Hence by mathematical induction, the given equation is true for all integers $n \geq 1$.

EXERCISE

Use **mathematical induction** to prove that

$$1+2+2^2 + \ldots + 2^n = 2^{n+1} - 1$$

for all integers $n \geq 0$

SOLUTION

Let

$P(n): 1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$

1. Basis Step: $P(0)$ is true.

   For $n = 0$

   L.H.S of $P(0) = 1$

   R.H.S of

   $P(0) = 2^{0+1} - 1 = 2 - 1 = 1$

   Hence $P(0)$ is true.

SOLUTION

2. **Inductive Step:** Suppose $P(k)$ is true for some integer $k \geq 0$.

$$1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1 \quad \ldots.(1)$$

We have to show $P(k+1)$ is true

$$1 + 2 + 2^2 + \ldots + 2^{k+1} = 2^{k+1+1} - 1 \quad \ldots.(2)$$

SOLUTION

Consider L.H.S of equation (2)

$$1+2+2^2+\ldots+2^{k+1}$$

$$= (1+2+2^2+\ldots+2^k) + 2^{k+1}$$

$$= (2^{k+1} - 1) + 2^{k+1} \text{ using R.H.S of (1)}$$

$$= 2 \cdot 2^{k+1} - 1$$

$$= 2^{k+1+1} - 1$$

= R.H.S of equation (2)

EXERCISE

Prove by mathematical induction

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all $n \geq 1$

SOLUTION

Inductive Step:

Suppose $P(k)$ is **true** for some integer $k \geq 1$.

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

$$\vdots$$

(1)

SOLUTION

for $n = k + 1$ we have equation (2)

$$1^2 + 2^2 + 3^2 + \dots + (k+1)^2$$

$$= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} \quad \dots.(2)$$

SOLUTION

Consider L.H.S of (2)

$$1^2 + 2^2 + 3^2 + \dots + (k+1)^2$$

$$= 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right]$$

SOLUTION

$$= (k+1)\left[\frac{2k^2 + k + 6k + 6}{6}\right]$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

EXERCISE

Prove by mathematical induction

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

for all integers $n \geq 1$.