UNIT-1 Sequences and Series

061

Sequence.

A Sequence is a function which is a domain set of natural numbers (N) to any set (S) it is denoted as $\{U_n\}$ or (U_n) or $\langle U_n \rangle$

Ex: $\{U_1, U_2, U_3, \ldots\ldots U_n\}$ where $U_1, U_2, \ldots\ldots U_n$ are the first, second and the end term of the Sequence respectively.

Real Sequence:

A sequence $\{U_n\}$ is said to be a real sequence if it is mapped from set of natural numbers to the set of real numbers (R).

Constant Sequence:

Here the range is singleton.

Ex: $\{2, 2, \ldots\ldots\}$

Null sequence:

A Sequence $\{U_n\}$ is said to be null sequence if every term of the Sequence is zero.

Ex: $\{0, 0, \ldots\ldots\}$

Finite Sequence:

A Sequence $\{U_n\}$ is said to be finite Sequence if the no. of terms in that Sequence is finite It is denoted as $\{U_n\}_{n=1}^{m}$, where 'm' is finite

Infinite Sequence:

A Sequence $\{U_n\}$ is said to be an infinite Sequence if the no. of terms in that sequence is infinite. It is denoted as $\{U_n\}_{n=1}^{\infty}$

Bounded Sequence

A sequence $\{U_n\}$ is said to be bounded if $\exists$ the numbers m & M $\ni$ $m < \{U_n\} < M \ \forall n$ otherwise it is unbounded

Monotonic Sequence

A Sequence $\{U_n\}$ is said to be

(i) Monotonically increasing sequence

if $U_{n+1} \geq U_n$

(ii) Monotonically decreasing Sequence

if $U_{n+1} \leq U_n$

(iii) Stricly monotonically increasing sequence

if $U_{n+1} > U_n$

(iv) Stricly Monotonically decreasing Sequence

if $U_{n+1} < U_n$

(v) Monotonic if it is either M·I or M·D Sequence

Limit of a Sequence

A no. l is said to be a limit of a sequence and it is denoted as

$$\lim_{n\to\infty} U_n = l$$

if for each $\epsilon > 0 \ \exists N \ni |U_n - l| < \epsilon \ \forall n \geq N$.

Convergent Sequence

A sequence $\{U_n\}$ is said to be convergent if it has a finite limit

Divergent Sequence

A sequence $\{U_n\}$ is said to be divergent sequence if it has an infinite limit.

Oscillatory of a Sequence

A sequence $\{U_n\}$ is said to be an Oscillatory sequence if its limit is not unique Here we have two cases

a). oscillates finitly

b) Oscillates infinitly

Finite Series

A series S_n is said to be a finite series if the number of terms in that series is finite. It is denoted as $\sum_{n=1}^{m} S_n$, where 'm' is finite

Infinite Series:

A series S_n is said to be an infinite series if the no. of terms in that series is infinite. It is denoted as $\sum_{n=1}^{\infty} S_n$.

Convergent Series:

A series $\sum S_n$ is said to be convergent if $Lt\ S_n = \text{Finite}$.

Divergent Series

A series $\sum S_n$ is said to be divergent series if $\underset{n\to\infty}{Lt} S_n =$ infinite.

Oscillatrey of a Series:

A series $\sum S_n$ is said to be an oscillatrey series if its limits is not unique. Here we have two cases:

a) Oscillates finitely

b) Oscillates infinitely

Standard Limits:-

1. $\underset{n\to\infty}{Lt} \frac{1}{n} = 0$

2. $\underset{n\to\infty}{Lt} x^{1/n} = 1 \quad ; x > 0$

3. $\underset{n\to\infty}{Lt} n^{1/n} = 1$

4. $\underset{n\to\infty}{Lt} \frac{\log n}{n} = 0$

5. $\underset{n\to\infty}{Lt} \left(1+\frac{x}{n}\right)^n = e^x \quad \forall x$

6. $\underset{n\to\infty}{Lt} x^n = 0 \quad$ if $|x|<1$ i.e, $-1<x<1$

Theorem-1

Prove that the Geometric Series $1+r+r^2+r^3+\cdots$

(i) Converges if $|r|<1$

(ii) Diverges if ~~|r|~~ $r \geq 1$

(iii) Oscillates if $r \leq -1$

Proof: let $S_n = 1+r+r^2+r^3+\cdots$ (1)

(i) Converges if $|r|<1$

w.k.t $S_n = \dfrac{a(1-r^n)}{1-r}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} \quad \text{Here } a=1.$$

$$= \lim_{n\to\infty} \frac{1-r^n}{1-r} = \lim_{n\to\infty} \frac{1}{1-r} - \lim_{n\to\infty} \frac{r^n}{1-r}$$

$$= \frac{1}{1-r} - 0 = \frac{1}{1-r} \text{ (finite)}$$

Hence the series converges

(ii) Diverges if $r \geq 1$

a) If $r=1$ in (1), we get

$S_n = 1+1+1+1+1+\cdots$

$$\lim_{n\to\infty} S_n = \infty$$

Hence the series diverges

b) If $r>1$ i.e, $S_n = \dfrac{a(r^n-1)}{r-1}$

As $n\to\infty$ then $r^n \to \infty$.

$$\lim_{n\to\infty} S_n = \frac{a(r^n-1)}{r-1}$$

$$= \lim_{n\to\infty} \frac{r^n}{r-1} - \lim_{n\to\infty} \frac{1}{r-1}$$

$$= \infty - \frac{1}{r-1} = \infty \text{ (Hence diverges)}$$

$\therefore$ The series diverges if $r \geq 1$.

(iii) Oscillates if $r \leq -1$

a) If $r = -1$ in (1), we get

$S_n = 1 - 1 + 1 - 1 + 1 \cdots$

$\lim_{n \to \infty} S_n = 0$ or 1. (Hence oscillates finetly)

b) If $r < -1$

let $r = -\rho \Rightarrow r^n = (-1)^n \rho^n$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{a(1-r^n)}{1-r}$$

$$= \lim_{n \to \infty} \frac{1-(-1)^n \rho^n}{1+\rho} = \pm \infty$$

Hence oscillates infinetly

$\therefore$ The given Series oscillates if $r \leq -1$

P-test or Harmonic Serio or Auxiliary Series

An infinite series $\sum \frac{1}{n^p}$ Converges if $p > 1$ & diverges if $p \leq 1$

Proof:- $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \quad (1)$

(i) Converges if $p > 1$

$$\frac{1}{n^p} = \frac{1}{1} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \cdots$$

$$< 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \cdots$$

$$< 1 + \frac{2}{2^p} + \frac{4}{4^p} + \cdots$$

$$1+\frac{1}{2^{p-1}}+\frac{1}{2^{2(p-1)}}+\cdots$$

which is a geometric Series with common ratio

$$r=\frac{1}{2^{p-1}}<1$$

which is possible only when $p>1$

Hence the given series converges if $p>1$

(ii) Diverges if $p\leq 1$

a) $p=1$ in ①, we get

$$\sum\frac{1}{n}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots$$

$$>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots$$

$$>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\infty$$

Hence diverges if $p=1$

b) When $p<1 \Rightarrow n^p<n^1$

$$\frac{1}{n^p}>\frac{1}{n} \Rightarrow \frac{1}{n}<\frac{1}{n^p}$$

$$\sum\frac{1}{n}<\sum\frac{1}{n^p}$$

Since L.H.S of the series diverges

$\therefore$ R.H.S also diverges.

Hence the series diverges if $p<1$

$\therefore$ The given series diverges if $p\leq 1$.

POINTS

1. If $\lim_{n\to\infty} U_n \neq 0$ then the given series diverges.
2. If $\lim_{n\to\infty} U_n = 0$ then the given series may or may not converges.

3) Comparison test-1

If $\sum U_n$ & $\sum V_n$ are the two series of positive terms such that

(i) $\sum V_n$ converges

(ii) If $U_n \leq V_n \; \forall n$ then $\sum U_n$ also converges

4) Comparison test-2

If $\sum U_n$ and $\sum V_n$ are the two series of positive term such that

(i) $\sum V_n$ diverges

(ii) If $U_n \geq V_n \; \forall n$ then $\sum U_n$ also diverges

5) Limit Comparison test (limit form)

If $\sum U_n$ and $\sum V_n$ are the two series of +ve terms such that $\lim_{n\to\infty} \frac{U_n}{V_n} = l$ {finite & non-zero}

$\sum U_n$ and $\sum V_n$ are either converge or diverge together.

6) An infinite series $\sum \frac{1}{n^p}$ converges if $p>1$ & diverges if $p \leq 1$

NOTE-1:

$\sum V_n$ is known as Auxiliary series.

NOTE 2

If $\sum U_n$ converges then $\lim_{n\to\infty} U_n = 0$ but the converse of these may not be true i.e, $\lim_{n\to\infty} U_n = 0$ then the series may or may not converges

NOTE: 3

A series of +ve terms may either converge or diverge but never oscillates.

Exercise 1

Comparison test problem:-

1. Test the converges of the following series.

(1) $\sum \frac{n+1}{2n+5}$

Given $\sum \frac{n+1}{2n+5}$ Here $U_n = \frac{n+1}{2n+5}$

$$\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{n+1}{2n+5}$$

$$= \underset{n\to\infty}{Lt} \frac{\cancel{n}(1+1/n)}{\cancel{n}(2+5/n)} = \frac{1}{2} \neq 0$$

Hence the series diverges.

(2) $\sum \frac{3n+4}{5n-8}$

Given $\sum \frac{3n+4}{5n-8}$; Here, $U_n = \frac{3n+4}{5n-8}$

$$\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{3n+4}{5n-8}$$

$$= \underset{n\to\infty}{Lt} \frac{\cancel{n}(3+4/n)}{\cancel{n}(5-8/n)} = \frac{3}{5} \neq 0$$

Hence the series diverges

(3) $\sum \frac{4n^3+2}{7n^3+2n}$

Given $\sum \frac{4n^3+2}{7n^3+2n}$, Here $U_n = \frac{4n^3+2}{7n^3+2n}$

$$\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{4n^3+2}{7n^3+2n}$$

$$= \underset{n\to\infty}{Lt} \frac{\cancel{n^3}\left(4+\frac{2}{n^3}\right)}{\cancel{n^3}\left(7+\frac{2}{n^2}\right)} = \frac{4}{7} \neq 0$$

Hence the series diverges

(4) $\sum \frac{n(n^3+4)}{(n^2+1)(n^2+4)}$

Given $\sum \frac{n(n^3+4)}{(n^2+1)(n^2+4)}$, here $U_n = \frac{n(n^3+4)}{(n^2+1)(n^2+4)}$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{n(n^3+4)}{(n^2+1)(n^2+4)}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^4}\left(1+\frac{4}{n^3}\right)}{\cancel{n^4}\left(1+\frac{1}{n^2}\right)\left(1+\frac{4}{n^2}\right)} = 1 \neq 0$$

Hence the series diverges.

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(5) $2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \cdots$

Given " Here $U_n = \frac{n+1}{n}$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{n+1}{n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{1}{n}\right)}{\cancel{n}} = 1 \neq 0$$

Hence the series diverges.

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(6) $\sum \left(1+\frac{1}{n}\right)^n$

Given ", Here $U_n = \left(1+\frac{1}{n}\right)^n$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$= e \neq 0.$$

Hence the series diverges.

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(7) $\sum \left(1+\frac{1}{n}\right)^{-n}$

Given ", Here $U_n = \left(1+\frac{1}{n}\right)^{-n}$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$= \frac{1}{e} \neq 0$

Hence the series diverges.

(8) $\sum \left(1+\frac{1}{2^n}\right)^{2^n}$

Given ", Here $U_n = \left(1+\frac{1}{2^n}\right)^{2^n}$

$\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \left(1+\frac{1}{2^n}\right)^{2^n} = e \neq 0$

Hence the series diverges.

(9) $\sum_{2}^{\infty} \frac{1}{\log n}$

Given ", Here $U_n = \frac{1}{\log n}$

let $V_n = \frac{1}{n}$, By p-test $P=1$

Hence the series diverges

(10) $\sum_{2}^{\infty} \frac{1}{n\log n}$

Given " Here $U_n = \frac{1}{n\log n}$

let $V_n = \frac{1}{n^2}$

Since $U_n > V_n$

Hence the series converges

(11) $\sum \frac{1}{n}$

Given ", Here $U_n = \frac{1}{n}$

$\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{1}{n} = 0$

May or may not converges

Consider an auxiliary series $\sum V_n \ni$

$V_n = \frac{1}{n}$ then $\underset{n\to\infty}{Lt} \frac{U_n}{V_n}$

$= \underset{n\to\infty}{Lt} \frac{1/n}{1/n} = 1$ [Finite & non-zero]

$\therefore \sum U_n$ & $\sum V_n$ are either converge or diverge together

But $\sum V_n = \sum \frac{1}{n}$

By p-test here $p = 1$

Hence the series $\sum V_n$ diverges.

$\therefore$ By comparison test $\sum U_n$ also diverges

(12) $\sum \frac{1}{n^2}$

Given " , Here $U_n = \frac{1}{n^2}$

$\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{1}{n^2} = 0.$

may or may not converges

• Consider an auxiliary series $\sum V_n \ni$

$V_n = \frac{1}{n^2}$ then $\underset{n\to\infty}{Lt} \frac{U_n}{V_n}$

$= \underset{n\to\infty}{Lt} \frac{1/n^2}{1/n^2} = 1$ [finite & non-zero]

$\therefore \sum U_n$ & $\sum V_n$ are either converge or diverge together

But $\sum V_n = \sum \frac{1}{n^2}$

By P-test here $P = 2 > 1$

Hence the series $\sum V_n$ converges.

$\therefore$ By comparison test $\sum U_n$ also converges

(13) $\sum \frac{1}{n^2+1}$

Given " Here $U_n = \frac{1}{n^2+1}$

$\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{1}{n^2+1} = \underset{n\to\infty}{Lt} \frac{1}{n^2(1+\frac{1}{n^2})} = 0$

May or may not converges

Consider an auxiliary series $\sum V_n \ni$

$V_n = \frac{1}{n^2}$ then $\underset{n\to\infty}{Lt} \frac{U_n}{V_n}$

$$= \underset{n\to\infty}{Lt} \frac{1}{n^2(1+\frac{1}{n^2})} \Big/ \frac{1}{n^2} = 1 \text{ [Finite \& non-zero]}$$

$\therefore \sum U_n$ & $\sum V_n$ are either converge or diverge together

But $\sum V_n = \frac{1}{n^2}$

By p-test here $P = 2 > 1$

Hence the series $\sum V_n$ converges.

$\therefore$ By comparison test $\sum U_n$ also converges

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(14) $\sum \frac{(n+1)(n+2)}{n^3\sqrt{n}}$

Given " , Here $U_n = \frac{(n+1)(n+2)}{n^3\sqrt{n}}$

$$\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{n^2(1+\frac{1}{n})(1+\frac{2}{n})}{n^{7/2}}$$

$$= \underset{n\to\infty}{Lt} \frac{(1+\frac{1}{n})(1+\frac{2}{n})}{n^{3/2}} = 0$$

may or may not converges

Consider an auxiliary series $\sum V_n \ni$

$V_n = \frac{1}{n^{3/2}}$ then $\underset{n\to\infty}{Lt} \frac{U_n}{V_n}$

$$\underset{n\to\infty}{Lt} \frac{(1+\frac{1}{n})(1+\frac{2}{n})}{n^{3/2}} \Big/ \frac{1}{n^{3/2}} = 1 \text{ [Finite \& non-zero]}$$

$\therefore \sum U_n$ & $\sum V_n$ are either converge or diverge together

But $\sum V_n = \sum \frac{1}{n^{3/2}}$

By p-test here $P = 3/2 > 1$

Hence the series $\sum V_n$ converges

$\therefore$ By comparison test $\sum U_n$ also converges

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(15) $\sum\left(\frac{1}{n^2+n}+\frac{1}{n}\right)$

Given ", Here $U_n = \frac{1}{n^2+n}+\frac{1}{n}$

$$= \frac{1}{n(n+1)}+\frac{1}{n} = \frac{n+2}{n(n+1)}$$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{n+2}{n(n+1)}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(1+2/n)}{n^{\cancel{2}}(1+1/n)} = 0$$

$\therefore$ may or may not converges

Consider an auxiliary series $\sum V_n \ni$

$V_n = \frac{1}{n}$ then $\lim_{n\to\infty} \frac{U_n}{V_n}$

$$\lim_{n\to\infty} \frac{(1+2/n)}{\cancel{n}(1+1/n)} \Big/ \cancel{1/n} = 1 \quad \text{[Finite and non-zero]}$$

$\therefore$ $\sum U_n$ & $\sum V_n$ are either converge or diverge together

But $\sum V_n = \sum \frac{1}{n}$

By P-test here $P = 1$

Hence the series $\sum V_n$ diverges

$\therefore$ By comparison test $\sum U_n$ also diverges

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(16) S.T $\frac{2}{1^q}+\frac{3}{2^q}+\frac{4}{3^q}+\ldots$ converges if $q>2$ & diverges if $q=2$

Given " Here $U_n = \frac{n+1}{n^q}$

$$\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{n+1}{n^q} = \underset{n\to\infty}{Lt} \frac{n(1+\frac{1}{n})}{n^q} = \underset{n\to\infty}{Lt} \frac{1+1/n}{n^{q-1}} = 0$$

may or may not converges

Consider an auxiliary series $\sum V_n \ni$

$V_n = \frac{1}{n^{q-1}}$, then $\underset{n\to\infty}{Lt} \frac{U_n}{V_n}$

$\underset{n\to\infty}{Lt} \left(\frac{1+1/n}{n^{q-1}}\right) \Big/ \frac{1}{n^{q-1}} = 1$ [Finite & non-zero.]

$\therefore \sum U_n$ & $\sum V_n$ are either converge or diverge together

But $\sum V_n = \frac{1}{n^{q-1}}$

By p-test here $P = q-1 > 1$ or $q > 2$

Hence the series $\sum V_n$ converges if $q > 2$

& diverges if $q \leq 2$

$\therefore$ By Comparison tes $\sum U_n$ also converges if $q > 2$

& Diverges if $q \leq 2$

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(17) $\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \cdots\cdots$

Given " , Here $U_n = \frac{2n-1}{n(n+1)(n+2)}$

$$\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{n(2-1/n)}{n^3(1+\frac{1}{n})(1+\frac{2}{n})} = 0$$

$\therefore$ May or may not converges

Consider an auxiliary series $\sum V_n \ni$

$V_n = \frac{1}{n^2}$ then $\underset{n\to\infty}{Lt} \frac{U_n}{V_n}$

$$\underset{n\to\infty}{Lt} \frac{2-1/n}{n^2(1+\frac{1}{n})(1+2/n)} \Big/ \frac{1}{n^2} = 2$$ [Finite & non-zero

Consider $\therefore \sum U_n$ & $\sum V_n$ are either converge or diverge together

But $\sum V_n = \sum \frac{1}{n^2}$

By p-test here $p=2>1$.

Hence the series $\sum V_n$ converges

$\therefore$ By Comparison test $\sum U_n$ also converges

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(18) $\sum \frac{1}{n} \sin \frac{1}{\sqrt{n}}$

Given " , Here $U_n = \frac{1}{n} \sin\frac{1}{\sqrt{n}}$

$$\left[\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots \right]$$

$$\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{1}{n}\left[\frac{1}{n^{1/2}} - \frac{1}{n^{3/2}3!} + \cdots\cdots\right]$$

$$= \underset{n\to\infty}{Lt} \frac{1}{n^{3/2}}\left[1 - \frac{1}{n\,3!} + \cdots\right] = 0$$

may or may not converges

Consider an auxiliary series $\sum V_n \ni$

$V_n = \frac{1}{n^{3/2}}$ then $\underset{n\to\infty}{Lt} \frac{U_n}{V_n}$

$$\underset{n\to\infty}{Lt} \frac{1}{n^{3/2}}\left[1 - \frac{1}{n3!} + \cdots\right] \Big/ \frac{1}{(n^{3/2})} = 1$$ [Finite & non-zero

$\therefore \sum U_n$ & $\sum V_n$ are either converge or diverge together.

But $\sum V_n = \sum \frac{1}{n^{3/2}}$

By P-test here $P = \frac{3}{2} > 1$

Hence the series $\sum V_n$ converges

$\therefore$ By comparison test $\sum U_n$ also converges

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(19) $\sum \frac{1}{n} \cos \frac{1}{n}$

Given $\therefore$, Here $U_n = \frac{1}{n} \cos \frac{1}{n}$

$\left[\because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \right]$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{n}\left[1 - \frac{1}{n^2 2!} + \frac{1}{n^4 4!} + \cdots \right] = 0$$

May or may not converges

consider an auxiliary series $\sum V_n \ni$

$V_n = \frac{1}{n}$ then $\lim_{n\to\infty} \frac{U_n}{V_n}$

$$\lim_{n\to\infty} \frac{\cancel{\frac{1}{n}}\left[1 - \frac{1}{n^2 2!} + \frac{1}{n^4 4!} + \cdots\right]}{\cancel{1/n}} = 1 \quad [\text{Finite \& non-zero}]$$

$\therefore \sum U_n$ & $\sum V_n$ are either converge or diverge together

But $\sum V_n = \sum \frac{1}{n}$

By P-test here $P = 1$

Hence the series $\sum V_n$ diverges

$\therefore$ By comparison test $\sum U_n$ also diverges

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(20) $\sum \frac{1}{\sqrt{n}} \tan \frac{1}{n}$

Given ", Here $U_n = \frac{1}{\sqrt{n}} \tan \frac{1}{n}$

$\left[\because \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \right]$

$$\underset{n\to\infty}{Lt}\, U_n = \underset{n\to\infty}{Lt}\, \frac{1}{\sqrt{n}}\left[\frac{1}{n} + \frac{1}{n^3 \cdot 3} + \frac{2}{15n^5} + \cdots\right] = 0$$

may or may not converges

consider an auxiliary series $\Sigma V_n \ni$

$V_n = \frac{1}{n^{3/2}}$ then $\underset{n\to\infty}{Lt} \frac{U_n}{V_n}$

$$\underset{n\to\infty}{Lt}\, \frac{1}{n^{3/2}}\left[1 + \frac{1}{n^2 \cdot 3} + \frac{2}{15n^4} + \cdots\right] \Big/ \frac{1}{n^{3/2}} = 1 \text{ [finite \& non zero}$$

$\therefore$ ΣU_n & ΣV_n are either converge or diverge together.

But $\Sigma V_n = \frac{1}{n^{3/2}}$

By p-test here $p = \frac{3}{2} > 1$

Hence the series ΣV_n converges

$\therefore$ By Comparison test ΣU_n also converges.

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(21) $\Sigma\left[\sqrt{n^4+1} - \sqrt{n^4-1}\right]$

Given " , Here $U_n = \left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)$

$$U_n = \left(\sqrt{n^4+1} - \sqrt{n^4-1}\right) \times \frac{\left(\sqrt{n^4+1} + \sqrt{n^4-1}\right)}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

$$\underset{n\to\infty}{Lt}\, U_n = \underset{n\to\infty}{Lt}\, \frac{n^4 + 1 - n^4 + 1}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

$$= \underset{n\to\infty}{Lt}\, \frac{2}{n^2\left[\sqrt{1+\frac{1}{n^2}} + \sqrt{1-\frac{1}{n^2}}\right]} = 0$$

may or may not converges

Consider an auxiliary series $\Sigma V_n \ni$.

$V_n = \frac{1}{n^2}$ then $\underset{n\to\infty}{Lt} \frac{U_n}{V_n}$

$$\underset{n\to\infty}{Lt} \frac{2}{\cancel{n^2}\left[\sqrt{1+\frac{1}{n^2}}+\sqrt{1-\frac{1}{n^2}}\right]} \Big/ \frac{1}{\cancel{n^2}} \doteq 1 \quad [\text{Finite \& non zero}]$$

$\therefore \Sigma U_n$ & ΣV_n are either converge or diverge together

But $\Sigma V_n = \Sigma \frac{1}{n^2}$

By P-test here $P = 2 > 1$

Hence the series ΣV_n converges

$\therefore$ By comparison test ΣU_n also converges

---

(22) $\Sigma\left[\sqrt{n^2+1} - \sqrt{n^2-1}\right]$

Given " , Here $U_n = \left[\sqrt{n^2+1} - \sqrt{n^2-1}\right]$

$$U_n = \left(\sqrt{n^2+1} - \sqrt{n^2-1}\right) \frac{\left(\sqrt{n^2+1} + \sqrt{n^2-1}\right)}{\left(\sqrt{n^2+1}\right) + \sqrt{n^2-1}}$$

$$\underset{n\to\infty}{Lt}\, U_n = \underset{n\to\infty}{Lt} \frac{\cancel{n^2}+1 - \cancel{n^2}+1}{\sqrt{n^2+1}+\sqrt{n^2-1}}$$

$$= \underset{n\to\infty}{Lt} \frac{2}{n\left(\sqrt{1+\frac{1}{n}} + \sqrt{1-\frac{1}{n}}\right]} = 0$$

May or may not converges

consider an auxiliary series $\Sigma V_n \ni$

$V_n = \frac{1}{n}$ then $\underset{n\to\infty}{Lt} \frac{U_n}{V_n}$

$$= \underset{n\to\infty}{Lt} \frac{2}{\cancel{n}\left(\sqrt{1+\frac{1}{n}} + \sqrt{1-\frac{1}{n}}\right)} \Big/ \frac{1}{\cancel{n}} = 1 \quad [\text{finite \& non-zero}]$$

$\therefore \Sigma U_n$ & ΣV_n are either converge or diverge together

But $\Sigma V_n = \Sigma \frac{1}{n}$

By P-test here $P = 1$.

Hence the series ΣV_n diverges

$\therefore$ By Comparison test ΣU_n also diverges

(23) $\sum \frac{\sqrt{n+1}-\sqrt{n}}{n^2}$

Given ", Here $U_n = \frac{\sqrt{n+1}-\sqrt{n}}{n^2}$

$U_n = \frac{(\sqrt{n+1}-\sqrt{n})}{n^2} \times \frac{(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})}$

$\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{n+1-n}{n^2(\sqrt{n+1}+\sqrt{n})}$

$= \underset{n\to\infty}{Lt} \frac{1}{n^4\left(\sqrt{1+\frac{1}{n^2}}+\sqrt{1}\right)} = 0$

May or may not converges

consider an auxiliary series $\sum V_n \ni$

$V_n = \frac{1}{n^4}$ then $\underset{n\to\infty}{Lt} \frac{U_n}{V_n}$

$= \underset{n\to\infty}{Lt} \frac{1}{\cancel{n^4}\left(\sqrt{1+\frac{1}{n^2}}+\sqrt{1}\right)} \Big/ \frac{1}{\cancel{n^4}} = \frac{1}{2}$ [finite & non-zero]

$\therefore \sum U_n$ & $\sum V_n$ are either converges or diverges together

But $\sum V_n = \sum \frac{1}{n^4}$

By P-test $p = 4 > 1$

Hence the series $\sum V_n$ converges

$\therefore$ By comparison test $\sum U_n$ also converges

(24) $\sum [\sqrt[3]{n^3+1} - n]$

Given ", Here $U_n = (\sqrt[3]{n^3+1} - n)$

$= (n^3)^{1/3}\left(1+\frac{1}{n^3}\right)^{1/3} - n$

$= n\left[\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right]$

$\left[\because (x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \cdots \right]$

i.e. $n\left[\left\{ 1(1)^{1/3}\left(\frac{1}{n^3}\right)^0 + \frac{1}{3}(1)^{1/3-1}\left(\frac{1}{n^3}\right)^1 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(1)^{1/3-2}\left(\frac{1}{n^3}\right)^2 + \cdots \right\} - 1\right]$

$= n\left[\left\{ 1 + \frac{1}{3n^3} - \frac{1}{9n^6} + \cdots \right\} - 1\right] = \frac{n}{n^3}\left[\frac{1}{3} - \frac{1}{9n^3} + \cdots\right]$

$\therefore \underset{n\to\infty}{Lt}\, U_n = \underset{n\to\infty}{Lt}\, \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9n^3} + \cdots\right] = 0$ May or may not converge

Consider an auxiliary series $\Sigma V_n \rightarrow V_n = \frac{1}{n^2}$

then $\underset{n\to\infty}{Lt}\, \frac{U_n}{V_n} = \underset{n\to\infty}{Lt}\, \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9n^3} + \cdots\right] = \frac{1}{3}$

[Finite & non-zero]

$\therefore \Sigma U_n$ & ΣV_n are either converge or diverge together

But $\Sigma V_n = \Sigma \frac{1}{n^2}$

$\therefore$ By P-test $P = 2 > 1$

Hence the series ΣV_n converges

$\therefore$ By comparison test, ΣU_n also converges.

D'Alembeets' Ratio test :-

If $\sum u_n$ is the series of +ve terms such that

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lambda$ then

(i) the given series converges if $\lambda > 1$

(ii) The given series diverges if $\lambda < 1$

(iii) Test fails if $\lambda = 1$

Test the converges of the following series:

① $\sum \frac{(10+5i)^n}{n!}$

Given ,, Here $u_n = \frac{(10+5i)^n}{n!}$.

$$u_{n+1} = \frac{(10+5i)^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(10+5i)^n}{n!} \times \frac{(n+1)!}{(10+5i)^{n+1}}$$

$$\lim_{n\to\infty} \frac{(n+1)\cancel{n!}}{\cancel{n!}(10+5i)} = \infty > 1 \qquad \boxed{n! = (n)(n-1)!}$$

$\therefore$ By Ratio test given series converges

② $\sum \frac{n!}{n^n}$

Given ,, , Here $u_n = \frac{n!}{n^n}$

$$u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n!}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{\cancel{n!}}{n^n} \times \frac{(n+1)^n(n+1)}{(n+1)\,n!}$$

$$\lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \frac{\cancel{n^n}(1+1/n)^n}{\cancel{n^n}} = e > 1$$

$\therefore$ By ratio test given series converges

---

③ $\sum \frac{n^2}{3^n}$

Given " , Here $u_n = \frac{n^2}{3^n}$

$$u_{n+1} = \frac{(n+1)^2}{3^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^2}{3^n} \times \frac{3^{n+1}}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}\cdot 3}{\cancel{n^2}(1+\frac{1}{n})^2} = 3 > 1$$

$\therefore$ By ratio test given series converges.

---

④ $\sum \frac{n^3}{3^n}$

Given " , Here $u_n = \frac{n^3}{3^n}$

$$u_{n+1} = \frac{(n+1)^3}{3^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^3}{3^n} \times \frac{(3^{n+1})}{(n+1)^3}$$

$$= \lim_{n\to\infty} \frac{n^3\cdot 3}{\cancel{n^3}(1+\frac{1}{n})^3} = 3 > 1$$

$\therefore$ By ratio test given series converges

⑤ $\sum \frac{n^3+\alpha}{2^n+\alpha}$

Given ", Here $u_n = \frac{n^3+\alpha}{2^n+\alpha}$

$$u_{n+1} = \frac{(n+1)^3+\alpha}{2^{n+1}+\alpha}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(\frac{n^3+\alpha}{2^n+\alpha}\right) \times \left(\frac{2^{n+1}+\alpha}{(n+1)^3+\alpha}\right)$$

$$= \lim_{n\to\infty} \frac{n^3\left(1+\frac{\alpha}{n^3}\right)}{\cancel{2^n}\left(1+\frac{\alpha}{2^n}\right)} \left[\frac{\cancel{2^n}\left(2+\alpha/2^n\right)}{\cancel{n^3}\left\{\left(1+\frac{1}{n}\right)^3+\frac{\alpha}{n^3}\right\}}\right] = 2>1$$

$\therefore$ By Ratio test given series converges

---

⑥ $\sum \sqrt{\frac{3^n+1}{2^n-1}}$

Given ", Here $u_n = \sqrt{\frac{3^n+1}{2^n-1}}$

$$u_{n+1} = \sqrt{\frac{3^{(n+1)}+1}{2^{n+1}-1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(\sqrt{\frac{3^n+1}{2^n-1}} \times \sqrt{\frac{2^{n+1}-1}{3^{n+1}+1}}\right)$$

$$= \lim_{n\to\infty} \left\{\frac{\cancel{3^{n/2}}}{\cancel{2^{n/2}}}\sqrt{\frac{\left(1+\frac{1}{3^n}\right)}{1-\frac{1}{2^n}}}\right\} \left\{\frac{\cancel{2^{n/2}}}{\cancel{3^{n/2}}}\sqrt{\frac{2-1/2^n}{3+1/3^n}}\right\}$$

$$= \sqrt{\frac{2}{3}} < 1$$

$\therefore$ By ratio test given series diverges

---

⑦ $\sum \left(\frac{1}{2^n}+\frac{1}{n}\right)$

$$\underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{Lt} \frac{1}{}$$

Given ,, . first term $u_n = \frac{1}{2^n}$, $u_{n+1} = \frac{1}{2^{n+1}}$.

$$\underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{Lt} \frac{1}{2^n} \times \frac{2^{n+1}}{1} = 2 > 1$$

Hence the series converges by Ratio test.

Second term $u_n = \frac{1}{n}$

Repeat problem (11) of C.T, we get diverges

$\therefore$ The given series diverges

---

(8) $\sum \left(\frac{1}{2^n} + \frac{1}{n^2}\right)$

First term $u_n = \frac{1}{2^n}$

from 7^{th} problem we get converges ..

Second term $u_n = \frac{1}{n^2}$

Repeat problem (12) of C.T, we get converges

$\therefore$ The given series converges

---

(9) $\sum \left(\frac{1}{2^n} + \frac{1}{n^{1/n}}\right)$.

Given ,,

First term $u_n = \frac{1}{2^n}$;

from 7^{th} problem converges.

Second term $u_n = \frac{1}{n^{1/n}}$

$$\underset{n\to\infty}{Lt} u_n = \underset{n\to\infty}{Lt} \frac{1}{n^{1/n}} = 1 \neq 0 .$$

Hence the series diverges

$\therefore$ The given series diverges

(10) $\sum 4x^n$

Given ", Here $U_n = 4x^n$.

$U_{n+1} = 4x^{n+1}$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{\cancel{4}x^n}{\cancel{4}x^{n+1}} = \frac{1}{x}$$

$\therefore$ By R.T given series converges if $\frac{1}{x} > 1$ i.e $x < 1$ & diverges if $\frac{1}{x} < 1$ i.e $x > 1$.

When $x = 1$ in $\lim_{n\to\infty} \frac{U_n}{U_{n+1}}$ [R.T fails)

$\therefore$ put $x=1$ in $\lim_{n\to\infty} 4(1)^n = 4 \neq 0$.

Hence the series diverges if $x=1$

$\therefore$ The given series converges if $x<1$ & diverges if $x \geq 1$

---

(11) $1 + 2x + 3x^2 + 4x^3 + \cdots$

Neglecting 1st term of the series.

Given ", Here $U_n = (n+1)x^n$.

$U_{n+1} = (n+2)x^{n+1}$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{1}{n}\right)x^n}{\cancel{n}\left(1+\frac{2}{n}\right)x^{n+1}} = \frac{1}{x}$$

$\therefore$ By R.T given series converges if $\frac{1}{x} > 1$ i.e $x < 1$ & diverges if $\frac{1}{x} < 1$ i.e, $x > 1$

When $x=1$ in $\lim_{n\to\infty} \frac{U_n}{U_{n+1}}$ [R.T fails)

$\therefore$ Put $x=1$ in $\underset{n\to\infty}{Lt}\,(n+1)1^n$
$= \infty \neq 0$.

Hence the series diverges if $x=1$

$\therefore$ The given series converges if $x<1$ & diverges if $x \geq 1$

---

(12) $\sum \frac{x^n}{n(n-1)(n-2)}$

Given " , Here $U_n = \frac{x^n}{n(n-1)(n-2)}$

& $U_{n+1} = \frac{x^{n+1}}{(n+1)(n)(n-1)}$

$\therefore \underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \underset{n\to\infty}{Lt} \frac{x^n}{n^3(1-\frac{1}{n})(1-\frac{2}{n})} \times \frac{n^3(1+\frac{1}{n})(1-\frac{1}{n})}{x^{n+1}} = \frac{1}{x}$.

$\therefore$ By Ratio test given series converges if $\frac{1}{x}>1$ i.e, $x<1$
and diverges if $\frac{1}{x}<1$ i.e, $x>1$

when $x=1$ [Ratio test fails]

$\therefore$ Put $x=1$ in $\underset{n\to\infty}{Lt}\,U_n$, we get $\underset{n\to\infty}{Lt} \frac{1}{n^3(1-\frac{1}{n})(1-\frac{2}{n})} = 0$

May or may not converges

consider an auxiliary series $\sum V_n \ni V_n = \frac{1}{n^3}$ then

$\underset{n\to\infty}{Lt} \frac{U_n}{V_n} = \underset{n\to\infty}{Lt} \frac{1}{n^3(1-\frac{1}{n})(1-\frac{2}{n})} \Big/ \frac{1}{n^3} = 1$ [Finite & non-zero]

$\therefore \sum U_n$ & $\sum V_n$ are either converge or diverge together

But $\sum V_n = \sum \frac{1}{n^3}$

$\therefore$ By P-test here $P = 3 > 1$

Hence the series $\sum V_n$ converges if $x=1$

$\therefore$ By C.T $\sum U_n$ also converges if $x=1$

Hence the given series converges if $x \leq 1$ and
diverges if $x>1$

(13) $\sum \frac{x^{2n-2}}{\sqrt{n}(n+1)}$

Given ", , Here $U_n = \frac{x^{2n-2}}{\sqrt{n}(n+1)}$

$U_{n+1} = \frac{x^{2n}}{\sqrt{n+1}(n+2)}$

$\therefore \lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{x^{2n-2}}{\cancel{\sqrt{n}}\cdot n(1+\frac{1}{n})} \times \frac{\cancel{\sqrt{n}}\left(\sqrt{1+\frac{1}{n}}\right) \cancel{n}(1+\frac{2}{n})}{\cancel{x^{2n}}} = \frac{1}{x^2}$

$\therefore$ By R.T given series converges if $\frac{1}{x^2} > 1$ i.e, $x^2 < 1$

and diverges if $\frac{1}{x^2} < 1$ i.e, $x^2 > 1$

when $x^2 = 1$ (Ratio test fails)

$\therefore$ Put $x^2 = 1$ in $\lim_{n\to\infty} U_n$, we get $\lim_{n\to\infty} \frac{1}{n^{3/2}(1+\frac{1}{n})} = 0$

May or may not converges

Consider an auxiliary series $\sum V_n \ni$; $V_n = \frac{1}{n^{3/2}}$ then

$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{1}{\cancel{n^{3/2}}(1+\frac{1}{n})} \Big/ \frac{1}{\cancel{n^{3/2}}} = 1$ [Finite & non-zero]

$\therefore$ $\sum U_n$ & $\sum V_n$ are either converges or diverges together

But $\sum V_n = \sum \frac{1}{n^{3/2}}$ $\therefore$ By P Test here $P = \frac{3}{2} > 1$

Hence the series $\sum V_n$ converges if $x^2 = 1$

By C.T $\sum U_n$ also converges if $x^2 = 1$

$\therefore$ Hence the given series converges if $x^2 \leq 1$

and diverges if $x^2 > 1$

(14) $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$

Given ", Here $U_n = \frac{\sqrt{n}}{\sqrt{n^2+1}} \cdot x^n$

& $U_{n+1} = \frac{\sqrt{n+1}}{\sqrt{n^2+2n+2}} \cdot x^{n+1}$

$\therefore \underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \frac{\sqrt{n}}{n\sqrt{1+\frac{1}{n^2}}} \cdot x^n \times \frac{n\sqrt{1+\frac{2}{n}+\frac{2}{n^2}}}{\sqrt{n}\sqrt{1+\frac{1}{n}} \cdot x^{n+1}} = \frac{1}{x}$

$\therefore$ By Ratio test given series converges if $\frac{1}{x} > 1$ i.e $x < 1$

and diverges if $\frac{1}{x} < 1$ i.e, $x > 1$

when $x = 1$ [Ratio test fails]

$\therefore$ Put $x=1$ in $\underset{n\to\infty}{Lt} U_n$ we get $\underset{n\to\infty}{Lt} \frac{1}{n^{1/2}(1+\frac{1}{n^2})} = 0$

May or may not converges

consider an auxiliary series $\sum V_n \ni V_n = \frac{1}{n^{1/2}}$ then

$\underset{n\to\infty}{Lt} \frac{U_n}{V_n} = \underset{n\to\infty}{Lt} \frac{1}{n^{1/2}(1+\frac{1}{n^2})} \Big/ \frac{1}{n^{1/2}} = 1$ [finite & non-zero]

$\therefore \sum U_n$ & $\sum V_n$ are either converge or diverge together

But $\sum V_n = \sum \frac{1}{n^{1/2}}$ $\therefore$ By P-test here $P = \frac{1}{2} < 1$

Hence the series $\sum V_n$ diverges if $x=1$

$\therefore$ By C.T $\sum U_n$ also diverges if $x=1$

Hence the given series converges if $x<1$ & diverges if $x \geq 1$

---

(15) $\sum \frac{x^{n-1}}{3^n . n}$

Given ", Here $U_n = \frac{x^{n-1}}{3^n n}$

& $U_{n+1} = \frac{x^n}{3^{n+1}(n+1)}$

$\therefore \underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \frac{x^{n-1}}{3^n \cdot n} \times \frac{3^{n+1} n(1+\frac{1}{n})}{x^n} = \frac{3}{x}$

$\therefore$ By ratio test given series converges if $\frac{3}{x} > 1$ i.e, $x < 3$ and diverge if $\frac{3}{x} < 1$ ie, $x > 3$

When $x = 3$ [Ratio test fails]

Put $x = 3$ in $\lim_{n\to\infty} u_n$ we get $\lim_{n\to\infty} \frac{3^{n-1}}{3^n \cdot n} = \lim_{n\to\infty} \frac{1}{3n} = 0$

May or may not converges

Consider an auxiliary series $\sum v_n \ni v_n = \frac{1}{n}$ then

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{3n} / \frac{1}{n} = \frac{1}{3}$ [finite & non-zero]

$\therefore \sum u_n$ & $\sum v_n$ are either converges or diverges together

But $\sum v_n = \sum \frac{1}{n}$ $\therefore$ By p-test here $p = 1$

Hence the series $\sum v_n$ diverges if $x = 3$

By comparision test $\sum u_n$ also diverges if $x = 3$

$\therefore$ Hence the given series converges if $x < 3$ & diverges if $x \geq 3$

---

(16) $\frac{x}{1 \cdot 2} + \frac{x^2}{3 \cdot 4} + \frac{x^3}{5 \cdot 6} + \cdots$

Given ", Here $u_n = \frac{x^n}{(2n-1)2n}$

& $u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$ $\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}}$

$\lim_{n\to\infty} \frac{x^n}{(2n-1)2n}$ $\lim_{n\to\infty} \frac{x^n}{4n^2\left(1-\frac{1}{2n}\right)} \times \frac{4n^2\left(1+\frac{1}{2n}\right)\left(1+\frac{1}{n}\right)}{x^{n+1}}$

$= \frac{1}{x}$

$\therefore$ By ratio test given series converges if $\frac{1}{x} > 1$ i e, $x < 1$ and diverges if $\frac{1}{x} < 1$ ie, $x > 1$

When $x = 1$ [Ratio test fails]

Put $x=1$ in $\lim_{n\to\infty} U_n$ we get $\lim_{n\to\infty} \frac{1}{4n^2\left(1-\frac{1}{2n}\right)} = 0$

May or may not converges.

Consider an auxiliary series $\sum V_n \ni V_n = \frac{1}{n^2}$ then

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{1}{4n^2\left(1-\frac{1}{2n}\right)} \Big/ \frac{1}{n^2} = \frac{1}{4} \text{ [finite \& non-zero]}$$

$\therefore$ $\sum U_n$ & $\sum V_n$ are either converge or diverge together

But $\sum U_n = \sum \frac{1}{n^2}$ $\therefore$ By P-test here $P = 2 > 1$

Hence the series $\sum V_n$ converges if $x=1$

$\therefore$ By C.T $\sum U_n$ also converges if $x=1$

Hence the given series converges if $x \le 1$ & diverges if $x > 1$

---

(17) $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots$

Neglecting first term

Given " , Here $U_n = \frac{x^n}{n^2+1}$

$$\sum U_{n+1} = \frac{x^{n+1}}{n^2+2n+2} \quad \therefore \lim_{n\to\infty} \frac{U_n}{U_{n+1}}$$

$$\lim_{n\to\infty} \frac{x^n}{n^2\left(1+\frac{1}{n^2}\right)} \times \frac{n^2\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}{x^{n+1}} = \frac{1}{x}$$

$\therefore$ By Ratio test given series converges if $\frac{1}{x} > 1$

i.e, $x < 1$

and diverges if $\frac{1}{x} < 1$ ie, $x > 1$

when $x = 1$ [Ratio test fail]

$\therefore$ Put $x=1$ in $\lim_{n\to\infty} U_n$ we get, $\lim_{n\to\infty} \frac{1}{n^2\left(1+\frac{1}{n^2}\right)} = 0$

May or may not converges

consider an auxiliary series $\sum V_n$ $\ni$ $V_n = \frac{1}{n^2}$,

then $\underset{n\to\infty}{Lt} \frac{U_n}{V_n} = \underset{n\to\infty}{Lt} \frac{1}{n^2(1+\frac{1}{n^2})^1} \Big/ \frac{1}{n^2} = 1$ [finite & non-zero]

$\therefore$ $\sum U_n$ & $\sum V_n$ are either converge or diverge together

But $\sum V_n = \sum \frac{1}{n^2}$ $\therefore$ By P-test here $P = 2 > 1$ $\therefore$

Hence the series $\sum V_n$ converges if $x = 1$

$\therefore$ By C.T $\sum U_n$ also converges if $x = 1$

Hence the given series converges if $x \le 1$

and diverges if $x > 1$

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(18) $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \cdots\cdots$ neglecting 1st term

Given ", Here $U_n = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$ &

$$U_{n+1} = \frac{x^{2(n+1)}}{(n+3)\sqrt{n+2}}$$

$$\therefore \underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \underset{n\to\infty}{Lt} \frac{x^{2n}}{\cancel{n}(1+\frac{2}{n})\cancel{\sqrt{n}}\sqrt{1+\frac{1}{n}}} \times \frac{\cancel{n}(1+\frac{3}{n})\cancel{\sqrt{n}}\sqrt{1+\frac{2}{n}}}{x^{2n+2}}$$

$$= \underset{n\to\infty}{Lt} \frac{(1+\frac{3}{n})\sqrt{1+\frac{2}{n}}}{(1+\frac{2}{n})\sqrt{1+\frac{1}{n}}\cdot x^2} = \frac{1}{x^2}$$

$\therefore$ By Ratio test given series converges if $\frac{1}{x^2} > 1$ i.e, $x^2 < 1$ and diverges if $\frac{1}{x^2} < 1$ i.e, $x^2 > 1$

When $x^2 = 1$ [Ratio test fails]

Put $x=1$ in $\lim_{n\to\infty} U_n$ we get $\lim_{n\to\infty} \frac{1}{n(1+\frac{2}{n})\sqrt{n}\sqrt{\frac{1}{n}+1}} = 0$

May or may not converges

Consider an auxiliary series $\sum V_n \ni V_n = \frac{1}{n^{3/2}}$

then $\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{1}{n^{3/2}(1+\frac{2}{n})\sqrt{\frac{1}{n}+1}} \Big/ \frac{1}{n^{3/2}} = 1$ [finite & non-zero)

$\therefore$ $\sum U_n$ & $\sum V_n$ are either converge or diverge together.

But $\sum V_n = \sum \frac{1}{n^{3/2}}$ $\therefore$ By P-test here $P = \frac{3}{2} > 1$

Hence the series $\sum V_n$ converges if $x^2=1$

$\therefore$ By C.T $\sum U_n$ also converges if $x^2=1$

Hence the given series converges if $x^2 \leq 1$ and diverges if $x^2 \geq 1$

Cauchy's Root test:-

If $\sum U_n$ is the series of +ve terms such that $\lim_{n\to\infty} (U_n)^{1/n} = \lambda$ then the given series

(i) Converges if $\lambda < 1$

(ii) Diverges if $\lambda > 1$

(iii) Test fails if $\lambda = 1$

Test the converges of the following series:-

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① $\sum \frac{1}{(\log n)^n}$

Given " Here $U_n = \frac{1}{(\log n)^n}$

$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{(\log n)^n}\right)^{1/n} = 0 < 1$

$\therefore$ By CRT series converges

② $\sum \frac{n^{n^2}}{(n+1)^{n^2}}$

Given " Here $U_n = \frac{n^{n^2}}{(n+1)^{n^2}}$

$$\underset{n\to\infty}{Lt} (U_n)^{1/n} = \underset{n\to\infty}{Lt} \left[\left(\frac{n}{(n+1)}\right)^{n^2}\right]^{1/n} = \underset{n\to\infty}{Lt} \frac{\cancel{n^n}}{\cancel{n^n}(1+\frac{1}{n})^n} = \frac{1}{e} < 1$$

$\therefore$ By C.R.T given series converges

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③ $\sum \left(\frac{n+1}{n}\right)^{n^2}$

Given " Here $U_n = \left(\frac{n+1}{n}\right)^{n^2}$

$$\underset{n\to\infty}{Lt} (U_n)^{1/n} = \underset{n\to\infty}{Lt} \left[\left(\frac{n+1}{n}\right)^{n^2}\right]^{1/n} = \underset{n\to\infty}{Lt} \frac{\cancel{n^n}(1+\frac{1}{n})^n}{\cancel{n^n}} = e > 1$$

$\therefore$ By C.R.T series diverges

---

④ $\sum \frac{1}{(1+\frac{1}{n})^{n^2}}$

Given " , Here $U_n = \frac{1}{(1+\frac{1}{n})^{n^2}}$

$$\underset{n\to\infty}{Lt} (U_n)^{1/n} = \underset{n\to\infty}{Lt} \left[\frac{1}{(1+\frac{1}{n})^{n^2}}\right]^{1/n} = \underset{n\to\infty}{Lt} \frac{1}{e} < 1$$

$\therefore$ By C.R.T series converges

---

⑤ $\sum \left(1+\frac{1}{\sqrt{n}}\right)^{-3/2}$

Given " , Here $U_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-3/2}$

$$\underset{n\to\infty}{Lt} (U_n)^{1/n} = \underset{n\to\infty}{Lt} \left(\frac{1}{(1+\frac{1}{\sqrt{n}})^{\sqrt{n}\cdot\sqrt{n}}}\right)^{1/n} = \frac{1}{e} < 1$$

$\therefore$ By C.R.T series converges

(6) $\sum \left(1+\frac{1}{n^p}\right)^{-n^{p+1}}$

Given " Here $U_n = \left(1+\frac{1}{n^p}\right)^{-n^{p+1}}$

$$\underset{n\to\infty}{Lt} (U_n)^{1/n} = \underset{n\to\infty}{Lt} \left[\frac{1}{\left(1+\frac{1}{n^p}\right)^{n^p n}}\right]^{1/n} = \frac{1}{e} < 1.$$

$\therefore$ By C.R.T series converges

(7) $\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \cdots\cdots$

Given " Here $U_n = \left(\frac{n}{2n+1}\right)^n$

$$\underset{n\to\infty}{Lt} (U_n)^{1/n} = \underset{n\to\infty}{Lt} \left[\left(\frac{n}{2n+1}\right)^n\right]^{1/n} = \underset{n\to\infty}{Lt} \frac{n}{n(2+\frac{1}{n})} = \frac{1}{2} < 1$$

$\therefore$ By C.R.T series converges

(8) $\left[\frac{2^2}{1^2} - \frac{2}{1}\right]^{-1} + \left[\frac{3^3}{2^3} - \frac{3}{2}\right]^{-2} + \cdots\cdots$

Given " , Here $U_n = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-n}$

$$\underset{n\to\infty}{Lt} (U_n)^{1/n} = \underset{n\to\infty}{Lt} \left[\frac{1}{\left\{\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right\}^n}\right]^{1/n}$$

$$= \underset{n\to\infty}{Lt} \frac{1}{\frac{(n+1)}{n}\left\{\left(\frac{n+1}{n}\right)^n - 1\right\}}$$

$$= \underset{n\to\infty}{Lt} \frac{1}{\frac{\cancel{n}(1+1/n)}{\cancel{n}}\left\{\frac{\cancel{n^n}(1+\frac{1}{n})^n}{\cancel{n^n}} - 1\right\}}$$

$$= \frac{1}{1(e-1)} = \frac{1}{e-1} < 1$$

$\therefore$ By C.R.T series converges

(9) $\sum \left(\frac{1+nx}{n}\right)^n$

Given ", Here $U_n = \left(\frac{1+nx}{n}\right)^n$

$$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \cancel{\ } = \lim_{n\to\infty} \left\{\left(\frac{1+nx}{n}\right)^n\right\}^{1/n} = \lim_{n\to\infty} \frac{n\left(\frac{1}{n}+x\right)}{n} = x$$

$\therefore$ By C.R.T given series converges if $x<1$ and diverges if $x>1$

when $x=1$ [C.R.T fails]

$\therefore$ Put $x=1$ in $\lim_{n\to\infty} U_n$ we get $\lim_{n\to\infty} \frac{n^n\left(\frac{1}{n}+1\right)^n}{n^n} = e \neq 0$

Hence the series diverges if $x=1$

$\therefore$ The given series converges if $x<1$ and diverges if $x \geq 1$

---

(10) $\sum \left(\frac{n+2}{n+3}\right)^n \cdot x^n$

Given " Here $U_n = \left(\frac{(n+2)x}{n+3}\right)^n$

$$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \left\{\left[\frac{(n+2)x}{(n+3)}\right]^n\right\}^{1/n} = \lim_{n\to\infty} \frac{n\left(1+\frac{2}{n}\right)x}{n\left(1+\frac{3}{n}\right)} = x$$

$\therefore$ By CRT given series converges if $x<1$ and diverges if $x>1$

when $x=1$ [C.R.T fails].

put $x=1$ in $\lim_{n\to\infty} U_n$ we get $= \lim_{n\to\infty} \frac{n^n\left(1+\frac{2}{n}\right)^n}{n^n\left(1+\frac{3}{n}\right)^n} = \frac{e^2}{e^3} = \frac{1}{e}$

Hence the series diverges if $x=1$

$\therefore$ The given series converges if $x<1$ and diverges if $x \geq 1$

* (11) $\sum \frac{[(n+1)x]^n}{n^{n+1}}$

Given ,, Here $U_n = \frac{((n+1)x)^n}{n^{n+1}}$;

$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \left(\frac{[(n+1)x]^n}{n^{n+1}}\right)^{1/n} = \lim_{n\to\infty} \frac{(n+1)x}{n \cdot n^{1/n}} = \lim_{n\to\infty} \frac{n(1+\frac{1}{n})x}{n \cdot n^{1/n}}$

$= x$.

$\therefore$ By C.R.T given series converges if $x<1$ and diverges if $x>1$

when $x=1$ [C.R.T fails]

$\therefore$ Put $x=1$ in $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{(n+1)^n}{n^{n+1}} = \lim_{n\to\infty} \frac{n^n(1+\frac{1}{n})^n}{n^n \cdot n} = 0$

May or may not converges

consider an auxiliary series $\sum V_n \ni V_n = \frac{1}{n}$ then

$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{e}{n} / \frac{1}{n} = e$ (Finite and non zero)

$\therefore \sum U_n$ & $\sum V_n$ are either converge or diverge together.

But $\sum V_n = \sum \frac{1}{n}$ By P-test here $P=1$

Hence the series $\sum V_n$ diverges if $x=1$

$\therefore$ By C.T $\sum U_n$ also diverges if $x=1$

Hence the given series converges if $x<1$ and diverges if $x \geq 1$.

Raabe's Test:- (Higher ratio test)

If $\sum u_n$ is the series of positive terms such that $\lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lambda$ then the given series

(i) Converges if $\lambda > 1$

(ii) Diverges if $\lambda < 1$

(iii) Test fails if $\lambda = 1$

Test the convergence of the following series:

① $\sum \frac{1^2 \cdot 5^2 \cdot 9^2 \cdots\cdots (4n-3)^2}{4^2 \cdot 8^2 \cdot 12^2 \cdots\cdots (4n)^2}$

Given, Here $u_n = \frac{1^2 5^2 9^2 \cdots (4n-3)^2}{4^2 \cdot 8^2 \cdot 12^2 \cdots (4n)^2}$

$$u_{n+1} = \frac{1^2 \cdot 5^2 \cdot 9^2 \cdots\cdots (4n-3)^2 (4n+1)^2}{4^2 \cdot 8^2 \cdot 12^2 \cdots\cdots (4n)^2 (4n+4)^2}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(4n+4)^2}{(4n+1)^2} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(4+\frac{4}{n}\right)^2}{\cancel{n^2}\left(4+\frac{1}{n}\right)^2} = 1$$

[Ratio test fails].

$$\therefore \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \lim_{n\to\infty} n\left[\frac{(4n+4)^2}{(4n+1)^2} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\cancel{16n^2} + 16 + 32n - \cancel{16n^2} - 1 - 8n}{16n^2 + 1 + 8n}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{24n+15}{16n^2+8n+1}\right] = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}}\left[\frac{24+15/n}{16+8/n+\frac{1}{n^2}}\right]$$

$$= \frac{3}{2} > 1$$

$\therefore$ By Raabe's test given series converges

② $1+\frac{1}{2}+\frac{1\cdot 3}{2\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}+\cdots$

Given " Here $u_n = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots (2n)}$

$$u_{n+1} = \frac{1\cdot 3\cdot 5 \cdots (2n-1)(2n+1)}{2\cdot 4\cdot 6 \cdots (2n)(2n+2)}$$

$$\therefore \underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{Lt} \frac{(2n+2)}{(2n+1)} = \underset{n\to\infty}{Lt} \frac{n(2+\frac{2}{n})}{n(2+\frac{1}{n})} = 1$$

[Ratio test fails].

$$\therefore \underset{n\to\infty}{Lt}\, n\left[\frac{u_n}{u_{n+1}}-1\right] = \underset{n\to\infty}{Lt}\, n\left[\frac{(2n+2)}{(2n+1)}-1\right]$$

$$= \underset{n\to\infty}{Lt}\, n\left[\frac{2n+2-2n-1}{2n+1}\right] = \underset{n\to\infty}{Lt} \frac{n}{n(2+\frac{1}{n})} = \frac{1}{2} < 1$$

$\therefore$ By Raabe's test given series diverges

③ $\sum \frac{4\cdot 7\cdot 10 \cdots (3n+1)}{1\cdot 2\cdot 3 \cdots n} \cdot x^n$

Given " Here $u_n = \frac{4\cdot 7\cdot 10 \cdots (3n+1)}{1\cdot 2\cdot 3 \cdots n} x^n$

$$u_{n+1} = \frac{4\cdot 7\cdot 10 \cdots (3n+1)(3n+4)}{1\cdot 2\cdot 3 \cdots n(n+1)} x^{n+1}$$

$$\therefore \underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{Lt} \frac{(n+1)}{(3n+4)x} = \underset{n\to\infty}{Lt} \frac{n(1+\frac{1}{n})}{n(3+\frac{4}{n})x} = \frac{1}{3x}$$

$\therefore$ By Ratio test given series converges if $\frac{1}{3x} > 1$

i.e, $3x < \frac{1}{3}$ and diverges if $\frac{1}{3x} < 1$ i.e $x > \frac{1}{3}$

when $x = \frac{1}{3}$ [R.T fails]

$\therefore \underset{n\to\infty}{Lt}\ n\left[\frac{u_n}{u_{n+1}}-1\right] = \underset{n\to\infty}{Lt}\ n\left[\frac{3n+3}{3n+4}-1\right]$ [put $x=\frac{1}{3}$]

$= \underset{n\to\infty}{Lt}\ n\left[\frac{\cancel{3n}+3-\cancel{3n}-4}{3n+4}\right] = \underset{n\to\infty}{Lt}\ n\left[\frac{-1}{3n+4}\right].$

$= \underset{n\to\infty}{Lt}\ \frac{\cancel{n}}{\cancel{n}}\left(\frac{-1}{3+\frac{4}{n}}\right) = \frac{-1}{3} < 1$

$\therefore$ By Raabes test given series diverges if $x=\frac{1}{3}$

Hence the given series converges if $x<\frac{1}{3}$ and diverges if $x\geq\frac{1}{3}$

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(4) $\sum \frac{3\cdot6\cdot9\cdots\cdots3n}{7\cdot10\cdot13\cdots(3n+4)}\cdot x^n$

Given " . Here $u_n = \frac{3\cdot6\cdot9\cdots3n}{7\cdot10\cdot13\cdots(3n+4)}\cdot x^n$

$u_{n+1} = \frac{3\cdot6\cdot9\cdots(3n)(3n+3)}{7\cdot10\cdot13\cdots(3n+4)(3n+7)}\cdot x^{n+1}$

$\therefore \underset{n\to\infty}{Lt}\ \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{Lt}\ \frac{(3n+7)}{(3n+3)\cdot x} = \underset{n\to\infty}{Lt}\ \frac{\cancel{n}\left(3+\frac{7}{n}\right)}{\cancel{n}\left(3+\frac{3}{n}\right)\cdot x} = \frac{1}{x}$

$\therefore$ By Ratio test given series converges if $\frac{1}{x}>1$ ie, $x<1$ and diverges if $\frac{1}{x}<1$ ie, $x>1$

When $x=1$ [R.T fails]

$\therefore \underset{n\to\infty}{Lt}\ n\left[\frac{u_n}{u_{n+1}}-1\right] = \underset{n\to\infty}{Lt}\ n\left[\frac{3n+7}{(3n+3)}-1\right]$

$= \underset{n\to\infty}{Lt}\ n\left[\frac{\cancel{3n}+7-\cancel{3n}-3}{3n+3}\right] = \underset{n\to\infty}{Lt}\ n\left[\frac{4}{3n+3}\right]$

$= \underset{n\to\infty}{Lt}\ \cancel{n}\left[\frac{4}{\cancel{n}\left(3+\frac{3}{n}\right)}\right] = \frac{4}{3} > 1$

$\therefore$ By Raabes test given series converges if $x=1$

Hence the given series converges if $x \le 1$ and diverges if $x > 1$

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(5) $\sum \frac{(n!)^2}{(2n)!} x^{2n}$

Given // Here $u_n = \frac{(n!)^2}{(2n)!} x^{2n}$

$$u_{n+1} = \frac{((n+1)!)^2}{(2n+2)!} x^{2n+2}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n!)^2}{(2n)!} \cancel{x^{2n}} \times \frac{(2n+2)!}{((n+1)!)^2 \cdot x^{2n+2}}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n!)^2}}{\cancel{(2n)!}} \times \frac{(2n+2)(2n+1)\cancel{(2n)!}}{(n+1)^2 \cancel{(n!)^2}\, x^2}$$

$$= \lim_{n\to\infty} \frac{4n^2+6n+2}{(n^2+2n+1)x^2} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(4+\frac{6}{n}+\frac{2}{n^2}\right)}{\cancel{n^2}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\cdot x^2}$$

$$= \frac{4}{x^2}$$

$\therefore$ By Ratio test given series converges if $\frac{4}{x^2} > 1$ i.e, $x^2 < 4$ & diverges if $\frac{4}{x^2} < 1$ ie, $x^2 > 4$

When $x = 4$ [R.T fails].

$$\therefore \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \lim_{n\to\infty} n\left[\frac{4n^2+6n+2}{4n^2+8n+4} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\cancel{4n^2}+6n+2-\cancel{4n^2}-8n-4}{4n^2+8n+4}\right] = \lim_{n\to\infty} n\left[\frac{-(2n+2)}{4n^2+8n+4}\right]$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}}\left[\frac{\left(-2-\frac{2}{n}\right)}{4+\frac{8}{n}+\frac{4}{n^2}}\right] = \frac{-1}{2} < 1$$

$\therefore$ By Raabes test given Series diverges if $x^2 = 4$

Hence the series converges if $x^2 < 4$ & diverges if $x^2 \geq 4$

⑥ $\sum \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}\cdot\frac{x^{2n}}{2n}$

Given " Here $u_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}\cdot\frac{x^{2n}}{2n}$

$$u_{n+1} = \frac{1\cdot3\cdot5\cdots(2n-1)(2n+1)}{2\cdot4\cdot6\cdots(2n)(2n+2)}\cdot\frac{x^{2n+2}}{2(n+1)}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{x^{2n}}{2n}\times\frac{(2n+2)(2n+2)}{(2n+1)x^{2n+2}}$$

$$= \lim_{n\to\infty}\frac{1}{2n}\times\frac{(2n+2)^2}{(2n+1)x^2} = \lim_{n\to\infty}\frac{\cancel{n^2}\left(4+\frac{8}{n}+\frac{4}{n^2}\right)}{2\cancel{n}\times\cancel{n}\left(2+\frac{1}{n}\right)}\cdot\frac{1}{x^2}$$

$$= \frac{1}{x^2}$$

∴ By Ratio test given series converges if $\frac{1}{x^2}>1$ i.e, $x^2<1$ and diverges if $\frac{1}{x^2}<1$ i.e, $x^2>1$

when $x^2=1$ [R.T. test fails]

$$\therefore \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right] = \lim_{n\to\infty} n\left[\frac{4n^2+8n+4}{4n^2+2n}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\cancel{4n^2}+8n+4-\cancel{4n^2}-2n}{4n^2+2n}\right] = \lim_{n\to\infty} n\left[\frac{6n+4}{4n^2+2n}\right]$$

$$= \lim_{n\to\infty}\frac{\cancel{n^2}}{\cancel{n^2}}\left[\frac{6+4/n}{4+2/n}\right] = \frac{3}{2}>1$$

∴ By Raabes test given series converges if $x^2=1$.

Hence the given Series converges if $x^2\le1$ and diveggges if $x^2>1$.

(7) Log Test:-

If $\sum u_n$ is the series of +ve terms such that $\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \lambda$. Then the given series

(i) Converges if $\lambda > 1$

(ii) Diverges if $\lambda < 1$

(iii) Test fails if $\lambda = 1$

Note: If $\lim_{n\to\infty} \frac{u_n}{u_{n+1}}$ involves the number 'e' then we use log test.

Test the converges of the following series:

(1) $\sum \frac{n!\, x^n}{(n+1)^n}$ (or) $1 + \frac{1!\,x}{2} + \frac{2!\,x^2}{3^2} + \frac{3!\,x^3}{4^3} + \cdots$

Given .., Here $u_n = \frac{n!\, x^n}{(n+1)^n}$

$$u_{n+1} = \frac{(n+1)!\, x^{n+1}}{(n+2)^{n+1}}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n!\, x^n}{(n+1)^n} \times \frac{(n+2)^{n+1}}{(n+1)!\, x^{n+1}} = \lim_{n\to\infty} \frac{n!\,(n+2)^{n+1}}{(n+1)^n \times (n+1)n! \cdot x}$$

$$= \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)^{n+1} \cdot \frac{1}{x} = \lim_{n\to\infty} \left(1 + \frac{1}{n+1}\right)^{n+1} \cdot \frac{1}{x} = \frac{e}{x}$$

$\therefore$ By Ratio test given series converges if $\frac{e}{x} > 1$ ie, $x < e$ and diverges if $\frac{e}{x} < 1$ ie, $x > e$

When $x = e$ [Ratio test fails]

$\log(a \cdot b) = \log a + \log b$

$\log\left(\frac{a}{b}\right) = \log a - \log b$

$$\therefore \lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n \log\left[\left(1 + \frac{1}{n+1}\right)^{n+1} \cdot \frac{1}{e}\right]$$

$$= \lim_{n\to\infty} n \left[\cancel{\log}\,(n+1) \log\left(1 + \frac{1}{n+1}\right) - \log e\right]$$

$\left[\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots \right]$

i.e, $\underset{n\to\infty}{Lt}\ n\left[(n+1)\left\{\frac{1}{n+1} - \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} \cdots\right\} - 1\right]$

$= \underset{n\to\infty}{Lt}\ n\left[\left\{1 - \frac{1}{2(n+1)} + \frac{1}{3(n+1)^2} \cdots\right\} - 1\right]$

$= \underset{n\to\infty}{Lt}\ n\left[\frac{-1}{2n(1+\frac{1}{n})} + \frac{1}{3n^2(1+\frac{1}{n})^2} + \cdots\right]$

$= \underset{n\to\infty}{Lt}\ \frac{n}{n}\left[\frac{-1}{2(1+\frac{1}{n})} + \frac{1}{3n(1+\frac{1}{n})^2} + \cdots\right]$

$= \frac{-1}{2} < 1$

$\therefore$ By log test given series diverges if $x = e$

Hence the given series converges if $x < e$ and diverges if $x \geq e$

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(2) $\sum \frac{n^n x^n}{n!}$ (or) $1 + \frac{x}{1!} + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \cdots$

Given " , Here $U_n = \frac{n^n \cdot x^n}{n!}$

$U_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$

$\therefore \underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \underset{n\to\infty}{Lt} \frac{n^n \cdot x^n}{n!} \times \frac{(n+1)!}{(n+1)^{n+1} \cdot x^{n+1}}$

$= \underset{n\to\infty}{Lt} \frac{n^n}{\cancel{n!}} \times \frac{\cancel{n!}(n+1)}{(n+1)^{(n+1)} x} = \underset{n\to\infty}{Lt} \frac{n^n \cancel{(n+1)}}{(n+1)^n \cancel{(n+1)} x}$

$= \underset{n\to\infty}{Lt} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{x} = \underset{n\to\infty}{Lt} \frac{\cancel{n^n}}{\cancel{n^n}(1+\frac{1}{n})^n x} = \frac{1}{ex}$

$\therefore$ By ratio test given series converges if $\frac{1}{ex} > 1$

i.e, $x < \frac{1}{e}$ and diverges if $\frac{1}{ex} < 1$ i.e, $x > \frac{1}{e}$

when $x = \frac{1}{e}$ [Ratio test fails]

$$\therefore \operatorname*{Lt}_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \operatorname*{Lt}_{n\to\infty} n \log\left[\frac{e}{\left(1+\frac{1}{n}\right)^n}\right]$$

$$= \operatorname*{Lt}_{n\to\infty} n\left[\log e - n\log\left(1+\frac{1}{n}\right)\right]$$

$$\left[\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right]$$

$$= \operatorname*{Lt}_{n\to\infty} n\left[1 - n\left\{\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots\right\}\right]$$

$$= \operatorname*{Lt}_{n\to\infty} n\left[\cancel{1} - \left\{\cancel{1} + \frac{1}{2n} - \frac{1}{3n^2} + \cdots\right\}\right]$$

$$= \operatorname*{Lt}_{n\to\infty} n\left[\frac{1}{2n} - \frac{1}{3n^2} + \cdots\right]$$

$$= \operatorname*{Lt}_{n\to\infty} \frac{\cancel{n}}{\cancel{n}}\left[\frac{1}{2} - \frac{1}{3n} + \cdots\right]$$

$$= \frac{1}{2} < 1$$

$\therefore$ By log test given series diverges if $x = \frac{1}{e}$

Hence the given series converges if $x < \frac{1}{e}$ and diverges if $x \geq \frac{1}{e}$

Alternating Series

The series of +ve and -ve terms is known as Alternating series.

Leibnitz's Test:-

An alternating series is to be convergent by Leibnitz's test if

(i) Each term of the series is numerically less than its preceding term.

(ii) $\lim_{n\to\infty} U_n = 0$

Note: If any of the condition is not satisfied then the alternating series oscillates.

Test the convergence of the following series:

1) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Given ,, Here $U_n = \frac{(-1)^{n-1}}{n}$

each term of the series is numerically less than its preceding term

i.e, $1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} \cdots$

$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{(-1)^{n-1}}{n} = 0$

$\therefore$ By Leibnitz's test given series converges

② $\sum \frac{(-1)^{n-1}}{2n-1}$

Given $1 - \frac{1}{3} + \frac{1}{5} \cdots\cdots$ Here $U_n = \frac{(-1)^{n-1}}{2n-1}$

each term of the series is numerically less than its preceding term.

i.e, $1 > \frac{1}{3} > \frac{1}{5} \cdots$

$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{(-1)^{n-1}}{n(2-\frac{1}{n})} = 0$ $\therefore$ By Leibnitz's test given series converges

(3) $\sum \frac{(-1)^n}{n^2}$

Given $-1+\frac{1}{4}-\frac{1}{9}+\cdots$, Here $u_n = \frac{(-1)^n}{n^2}$

Each term of the series is numerically less than its preceding term.

i.e, $1 > \frac{1}{4} > \frac{1}{9} \cdots$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0$.

$\therefore$ By Leibnitz's test given series converges.

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(4) $\sum \frac{\cos n\pi}{n^2+1}$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

Given $\frac{-1}{2}+\frac{1}{5}-\frac{1}{10}+\cdots$, Here $u_n = \frac{(-1)^n}{n^2+1}$

Each term of the series is numerically less than its preceding term

i.e, $\frac{1}{2} > \frac{1}{5} > \frac{1}{10} > \cdots$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{(-1)^n}{n^2\left(1+\frac{1}{n^2}\right)} = 0$

$\therefore$ By Leibnitz's test given series converges

---

(5) $\frac{x}{x+1} - \frac{x^2}{x^2+1} + \frac{x^3}{x^3+1} \cdots\cdots$; $(0 < x < 1)$

Given " , Here $u_n = \frac{(-1)^{n-1} \cdot x^n}{x^n+1}$

Each term of the series is numerically less than its preceding term.

i.e, $\frac{x}{x+1} > \frac{x^2}{x^2+1} > \frac{x^3}{x^3+1} > \cdots$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{(-1)^{n-1} x^n}{x^n+1} = 0$

By Leibnitz's test given series converges

⑥ $2 - \frac{3}{2} + \frac{4}{3} - \cdots$ $a_n = 2+(n-1)1$

$= 2+n-1 = (n+1)$

Given " Here $U_n = \frac{(-1)^{n-1}(n+1)}{n}$

Each term of the series is numerically less than its preceding term.

$2 > \frac{3}{2} > \frac{4}{3} \cdots$

$\lim_{n\to\infty} U_n = \lim_{n\to\infty} (-1)^{n-1} \frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}} \neq 0$

$\therefore$ By leibnitz's test given series oscillates.

---

⑦ $\sum (-1)^n \left(\frac{n}{n+1}\right)$

Given $-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} \cdots$ Here $U_n = \frac{(-1)^n . n}{(n+1)}$

Each term of the series is not numerically less than its preceding term.

$\therefore$ By leibnitz's test given series oscillates.

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⑧ $\sum (-1)^{n+1} \left(\frac{n}{n^3+8}\right)$

Given $\frac{1}{9} - \frac{2}{16} + \frac{3}{35} + \cdots$, Here $U_n = \frac{(-1)^{n+1} . n}{n^3+8}$

Each term of the series is not numerically less than its preceding term.

$\cancel{\frac{1}{9} > \frac{2}{16} > \frac{3}{35}} \cdots$

$\cancel{\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{(-1)^{n+1} . n}{n(n^2+\frac{8}{n})} = 0}$

$\therefore$ By leibnitz's test given series ~~converges~~ oscillates

1) Absolute Convergence:

A series $\sum U_n$ is said to be converges absolutely (or) absolute convergence if $\sum |U_n|$ converges.

Conditionally convergent:

A series $\sum U_n$ is said to be converges conditionally or conditionally convergent if $\sum |U_n|$ diverges and $\sum U_n$ converges.

state whether the following series are converges absolutely or conditionally.

1) $\frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \cdots$

Given " Here $|U_n| = \frac{1}{2^n}$

Which is a geometric series with common ratio

$r = \frac{1}{2} < 1$

Hence the series converges

$\therefore \sum |U_n|$ converges

$\therefore$ The given series converges absolutely.

2) $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \cdots$

Given " Here $|U_n| = \frac{1}{3^{n-1}}$

Which is a geometric series with common ratio

$r = \frac{1}{3} < 1$

Hence the series converges.

$\therefore$ The given series converges absolutely

③ $\sum \frac{(-1)^n}{n\cdot 2^n}$

Given ,, , Here $|U_n| = \frac{1}{n\cdot 2^n}$

$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n\cdot 2^n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{2\cdot n^{1/n}} = \frac{1}{2} < 1$

$\therefore$ By C.R.T given series converges

$\because$ $\sum U_n$ converges

Hence the series ~~at~~ converges absolutely

④ $\sum \frac{(-1)^n}{n\sqrt{n^2+1}}$

Given,, Here $|U_n| = \frac{1}{n\sqrt{n^2+1}}$

$\therefore \lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{n^2\sqrt{1+\frac{1}{n^2}}} = 0$

May or may not converges

Consider an auxiliary series $\sum |V_n| \ni |V_n| = \frac{1}{n^2}$ then

$\lim_{n\to\infty} \frac{|U_n|}{|V_n|} = \lim_{n\to\infty} \frac{1}{\cancel{n^2}\sqrt{1+\frac{1}{n^2}}} \Big/ \frac{1}{\cancel{n^2}} = 1$

[Finite & non-zero]

$\therefore$ $\sum U_n$ & $\sum V_n$ are either converges or diverges

But $\sum |V_n| = \frac{1}{n^2}$

$\therefore$ By P-test here $P = 2 > 1$

Hence the series $\sum |V_n|$ converges

$\therefore$ By C.T $\sum |U_n|$ also converges

$\because$ $\sum |U_n|$ converges

$\therefore$ The given series converges absolutely

⑤ $\sum \frac{(-1)^n}{n^2}$ (or) $-1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}+\cdots$

Given " , Here $|U_n|=\frac{1}{n^2}$

$\therefore \underset{n\to\infty}{Lt}\, U_n = \underset{n\to\infty}{Lt}\, \frac{1}{n^2} = 0$

May or may not converges

Consider an auxiliary series $\sum|V_n| \ni |V_n|=\frac{1}{n^2}$ then

$\therefore \underset{n\to\infty}{Lt}\, \frac{|U_n|}{|V_n|} = \underset{n\to\infty}{Lt}\, \frac{1}{n^2}\Big/\frac{1}{n^2} = 1$ [finite & non-zero]

$\therefore \sum U_n$ & $\sum V_n$ are either converges or diverges

But $\sum|V_n|=\frac{1}{n^2}$

$\therefore$ By P-test here $p=2>1$

Hence the series $\sum|V_n|$ converges

$\therefore$ By C.T $\sum|U_n|$ also converges.

$\because \sum|U_n|$ converges

$\therefore$ The given series converges absolutely.

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⑥ $\sum \frac{1}{n\sqrt{n}}$ (or) $1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\cdots$

Given " , Here $|U_n|=\frac{1}{n\sqrt{n}}$

$\therefore \underset{n\to\infty}{Lt}\, U_n = \underset{n\to\infty}{Lt}\, \frac{1}{n^{3/2}} = 0$

May or may not converges

Consider an auxiliary series $\sum|V_n| \ni |V_n|=\frac{1}{n^{3/2}}$ then

$\therefore \underset{n\to\infty}{Lt}\, \frac{|U_n|}{|V_n|} = \underset{n\to\infty}{Lt}\, \frac{1}{\cancel{n^{3/2}}}\Big/\frac{1}{\cancel{n^{3/2}}} = 1$ [finite & non-zero]

$\therefore \sum U_n$ & $\sum V_n$ are either converges or diverges

But $\sum|V_n| = \frac{1}{n^{3/2}}$

$\therefore$ By P-test here $p=3/2>1$.

Hence the series $\Sigma|V_n|$ converges

$\therefore$ By C·T $\Sigma|U_n|$ also converges

$\because$ $\Sigma|U_n|$ converges.

$\therefore$ The given series A·C

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(7) $\sum \frac{\cos n\pi}{n^2+1}$ $[\because \cos n\pi = (-1)^n]$

Given ,, Here $|U_n| = \frac{(+1)}{n^2+1}$

$\lim_{n\to\infty} |U_n| = \lim_{n\to\infty} \frac{1}{n^2(1+\frac{1}{n^2})} = 0$ May or may not converges

Consider an auxiliary series $\Sigma|V_n| \ni |V_n| = \frac{1}{n^2}$

then $\lim_{n\to\infty} \left|\frac{U_n}{V_n}\right| = \lim_{n\to\infty} \frac{1}{n^2(1+\frac{1}{n^2})} \Big/ \frac{1}{n^2} = 1$ (Finite & nonzero)

$\therefore$ ΣU_n & ΣV_n are either converge or diverge together.

But $\Sigma|V_n| = \Sigma\frac{1}{n^2}$

$\therefore$ By P-test here $p = 2 > 1$

Hence the series $\Sigma|V_n|$ converges

$\therefore$ By C·T $\Sigma|U_n|$ also converges

$\because$ $\Sigma|U_n|$ converges

$\therefore$ The given series A·C

(8) $\sum \frac{(-1)^{n-1}\sin nx}{n^3}$ $[\because |\sin nx| \text{ or } |\cos nx| \leq 1]$

Given ,, , Here $|U_n| = \frac{1}{n^3}$

$\therefore \lim_{n\to\infty} |U_n| = \lim_{n\to\infty} \frac{1}{n^3} = 0$

May or may not converges.

Consider an auxiliary series $\Sigma|V_n| \ni |V_n| = \frac{1}{n^3}$

then

$\lim_{n\to\infty}\left|\frac{U_n}{V_n}\right| = \lim_{n\to\infty} \frac{1}{n^3}\Big/\frac{1}{n^3} = 1$ [finite & non-zero].

$\therefore \sum U_n$ & $\sum V_n$ are either converge or diverge together

But $\sum |V_n| = \sum \frac{1}{n^3}$

$\therefore$ By P-test here $P = 3 > 1$

Hence the series $\sum |V_n|$ converges

$\therefore$ By C.T $\sum |U_n|$ also converges

$\because \sum |U_n|$ converges

$\therefore$ The given series A.C.

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9) $\sum \frac{\cos nx}{n^2}$

Given Here $|U_n| = \frac{1}{n^2}$

$\lim_{n\to\infty} |U_n| = \lim_{n\to\infty} \frac{1}{n^2} = 0$

May or may not converges

consider an auxiliary series $\sum |V_n| \ni |V_n| = \frac{1}{n^2}$

then $\lim_{n\to\infty}\left|\frac{U_n}{V_n}\right| = \lim_{n\to\infty} \frac{1}{n^2}\Big/\frac{1}{n^2} = 1$ [finite & non-zero]

$\therefore \sum U_n$ & $\sum V_n$ are either converge or diverge together.

But $\sum |V_n| = \sum \frac{1}{n^2}$

$\therefore$ By P-test here $P = 2 > 1$

Hence the series $\sum |V_n|$ converges.

$\therefore$ By C.T $\sum |U_n|$ also converges

$\because \sum |U_n|$ converges

$\therefore$ The given series A.C

(10) $\sum \frac{(-1)^n \cos^2 nx}{n\sqrt{n}}$

Given " Here $|U_n| = \frac{1}{n\sqrt{n}}$

$\underset{n\to\infty}{Lt} |U_n| = \underset{n\to\infty}{Lt} \frac{1}{n^{3/2}} = 0$

May or may not converges

Consider an auxiliary series $\sum |V_n| \ni |V_n| = \frac{1}{n^{3/2}}$

then $\underset{n\to\infty}{Lt} \left|\frac{U_n}{V_n}\right| = \underset{n\to\infty}{Lt} \cancel{\frac{1}{n^{3/2}}} / \cancel{\frac{1}{n^{3/2}}} = 1$ [finite & non-zero]

$\therefore \sum U_n$ & $\sum V_n$ are either converges or diverges together

But $\sum |V_n| = \sum \frac{1}{n^{3/2}}$

$\therefore$ By P-test here $P = 3/2 > 1$

Hence the series $\sum |V_n|$ converges

$\therefore$ By C-T $\sum |U_n|$ also converges

$\because \sum |U_n|$ converges

$\therefore$ The given series A-C

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(11) $\sum \frac{(-1)^{n-1}}{n}$

Given " Here $|U_n| = \frac{1}{n}$

$\underset{n\to\infty}{Lt} |U_n| = \underset{n\to\infty}{Lt} \frac{1}{n} = 0$

May or may not converges.

Consider an auxiliary series $\sum |V_n| \ni |V_n| = \frac{1}{n}$

then $\underset{n\to\infty}{Lt} \left|\frac{U_n}{V_n}\right| = \cancel{\frac{1}{n}} / \cancel{\frac{1}{n}} = 1$ (finite & non-zero]

$\therefore \sum U_n$ & $\sum V_n$ are either converges or diverges together

But $\sum |V_n| = \sum \frac{1}{n}$

$\therefore$ By P-test here $P = 1$

Hence the series $\sum |V_n|$ diverges.

$\therefore$ By C·T $\Sigma |U_n|$ also diverges

Here $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ which is an A·S

Here each term of the series is numerically less than its preceding term i·e, $1 > \frac{1}{2} > \frac{1}{3} > \cdots$

and $\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{(-1)^{n-1}}{n} = 0$

$\therefore$ By leibnitzs test given series converges.

$\because \Sigma |U_n|$ diverges & ΣU_n converges

$\therefore$ The given series C·C

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(12) $\Sigma \frac{(-1)^{n-1}}{\sqrt{n}}$

Given " Here $|U_n| = \frac{1}{\sqrt{n}}$

$\underset{n\to\infty}{Lt} |U_n| = \underset{n\to\infty}{Lt} \frac{1}{\sqrt{n}} = 0$

May or may not converges

consider an auxiliary series $\Sigma |V_n| \ni |V_n| = \frac{1}{n^{1/2}}$

then $\underset{n\to\infty}{Lt} \left|\frac{U_n}{V_n}\right| = \frac{1}{n^{1/2}} / \frac{1}{n^{1/2}} = 1$ [F & non-zero]

$\therefore \Sigma U_n$ & ΣV_n are either converges or diverges together

But $\Sigma |V_n| = \Sigma \frac{1}{n^{1/2}}$

$\therefore$ By p test here $p = \frac{1}{2} < 1$

Hence the series $\Sigma |V_n|$ diverges

$\therefore$ By C·T $\Sigma |U_n|$ also diverges

Here $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ which is an A·S

Here each term of the series is numerically less than its preceding term

i·e, $1 > \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} > \cdots$

& $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = 0$

$\therefore$ By leibnitzs test given series converges

$\therefore$ The given series C.C

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(13) $\sum (-1)^n \left[\sqrt{n^2+1} - n\right]$

$\frac{n}{\sqrt{3}} - \frac{n^2}{\sqrt{5}} + \frac{n^3}{\sqrt{7}}$

Given " Here $|U_n| = \sqrt{n^2+1} - n \times \left\{\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}\right\}$

$$= \frac{\cancel{n^2}+1-\cancel{n^2}}{\sqrt{n^2+1}+n} = \frac{1}{n\left[\sqrt{1+\frac{1}{n^2}}+1\right]}$$

$\lim_{n\to\infty} |U_n| = \lim_{n\to\infty} \frac{1}{n\left[\sqrt{1+\frac{1}{n^2}}+1\right]} = 0$

May or may not converges

Consider an auxiliary series $\sum |V_n| \ni |V_n| = \frac{1}{n}$

then $\lim_{n\to\infty} \left|\frac{U_n}{V_n}\right| = \lim_{n\to\infty} \frac{1}{n\left[\sqrt{1+\frac{1}{n^2}}+1\right]} \Big/ \frac{1}{n} = 1$ [Finite & non-zero]

$\therefore$ $\sum U_n$ & $\sum V_n$ are either converge or diverge together

But $\sum |V_n| = \sum \frac{1}{n}$

$\therefore$ By P-test here $P=1$

Hence the series $\sum |V_n|$ diverges

Here $-(\sqrt{2}-1)+(\sqrt{5}-2)-(\sqrt{10}-3)+ \ldots\ldots$ which is an A.S

Here each term of the series is numerically less than its preceding term i.e, $(\sqrt{2}-1)>(\sqrt{5}-2)> \ldots$

and $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{(-1)^n}{n\left[\sqrt{1+\frac{1}{n^2}}+1\right]} = 0$

$\therefore$ By Leibnitz's test given series converges

$\because$ $\sum |U_n|$ diverges & $\sum U_n$ converges

$\therefore$ The given series C.C

(14) $\sum \frac{(-1)^{n-1}}{n} x^n$

Given " Here $|U_n| = \frac{x^n}{n}$

$\underset{n\to\infty}{Lt} |U_n| = \underset{n\to\infty}{Lt}$ $\underset{n\to\infty}{Lt}$ $|U_{n+1}| = \frac{x^{n+1}}{n+1}$

$\underset{n\to\infty}{Lt} \left|\frac{U_n}{U_{n+1}}\right| = \underset{n\to\infty}{Lt} \frac{x^n}{n} \times \frac{n+1}{x^{n+1}} = \underset{n\to\infty}{Lt} \frac{(n+1)}{n \cdot x}$

$= \underset{n\to\infty}{Lt} \left[1 + \frac{1}{n}\right] \frac{1}{x} = \frac{1}{x}$

$\therefore$ The series converges if $\frac{1}{x} > 1$ i.e, $x < 1$ and diverges if $\frac{1}{x} < 1$ i.e, $x > 1$

when $x = 1$ [R.T fails]

put $x = 1$ in $\underset{n\to\infty}{Lt} |U_n|$

$\underset{n\to\infty}{Lt} \frac{1}{n} = 0$ May or may not converges.

Consider an auxiliary series $\sum V_n \ni |V_n| = \frac{1}{n}$

then $\underset{n\to\infty}{Lt} \left|\frac{U_n}{V_n}\right| = \underset{n\to\infty}{Lt} \frac{1/n}{1/n} = 1$ [Finite & non-zero]

$\therefore \sum U_n$ & $\sum V_n$ are either converge or diverge together.

But $\sum |V_n| = \frac{1}{n}$

By P-test, $P = 1$.

$\therefore$ The series $\sum |V_n|$ diverges

$\therefore$ The series converges if $x < 1$ and diverges if $x \geq 1$

(i) Each term of the series in numerically less than its preceding term.

$x > \frac{x^2}{2} > \frac{x^3}{3} > \frac{x^4}{4} \cdots$ and $\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{(-1)^{n-1} \cdot x^n}{n} = 0$

$\therefore$ By leibnitz's test given series converges

$\because \sum |U_n|$ diverges and $\sum U_n$ converges

$\therefore$ The given series conditionally converges

(15) $\frac{x}{\sqrt{3}} - \frac{x^2}{\sqrt{5}} + \frac{x^3}{\sqrt{7}} - \cdots$

Given $\frac{x}{\sqrt{3}} - \frac{x^2}{\sqrt{5}} + \frac{x^3}{\sqrt{7}} \cdots\cdots$ Here $U_n = \frac{(-1)^{n-1} \cdot x^n}{\sqrt{2n+1}}$

$|U_n| = \frac{x^n}{\sqrt{2n+1}}$; $|U_{n+1}| = \frac{x^{n+1}}{\sqrt{2n+3}}$

$\lim_{n\to\infty} \left|\frac{U_n}{U_{n+1}}\right| = \lim_{n\to\infty} \frac{x^n}{\sqrt{2n+1}} \times \frac{\sqrt{2n+3}}{x^{n+1}} = \lim_{n\to\infty} \frac{\sqrt{2n+3}}{\sqrt{2n+1}} \cdot \frac{1}{x}$

$\lim_{n\to\infty} \frac{n^{1/2}\sqrt{2+3/n}}{n^{1/2}\sqrt{2+1/n}} \cdot \frac{1}{x} = \frac{1}{x}$

$\therefore$ The series converges if $\frac{1}{x} > 1$ i.e, $x < 1$ and diverges if $\frac{1}{x} < 1$ i.e, $x > 1$

when $x = 1$ [Ratio test fails]

Put $x = 1$ in $\lim_{n\to\infty} |U_n| = \lim_{n\to\infty} \frac{1}{\sqrt{2n+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}\sqrt{2+\frac{1}{n}}} = 0$

May or may not converges

Consider an auxiliary series $\sum |V_n| \ni |V_n| = \frac{1}{n^{1/2}}$

then $\lim_{n\to\infty} \left|\frac{U_n}{V_n}\right| = \lim_{n\to\infty} \frac{1}{\cancel{n^{1/2}}\sqrt{2+\frac{1}{n}}} \Big/ \frac{1}{\cancel{n^{1/2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{2+\frac{1}{n}}} = \frac{1}{\sqrt{2}}$

[finite & non-zero)

$\therefore \sum U_n$ & $\sum V_n$ are either converge or diverge together But $\sum |V_n| = \frac{1}{n^{1/2}}$

By P-test $p = 1/2 < 1$

The series $\sum |V_n|$ diverges.

Hence by C.T $\sum |U_n|$ also diverges

The series converges if $x < 1$ and diverges if $x > 1$

Here $\frac{x}{\sqrt{3}} - \frac{x^2}{\sqrt{5}} + \frac{x^3}{\sqrt{7}} \cdots$ which is an A.S

(i) Each term of the series is numerically less than its preceding term

$$\frac{x}{\sqrt{3}} > \frac{x^2}{\sqrt{5}} > \frac{x^3}{\sqrt{7}} \cdots$$

and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{(-1)^{n-1} \cdot x^n}{\sqrt{2n+1}} = \lim_{n\to\infty} \frac{(-1)^{n-1} \cdot x^n}{n^{1/2}\sqrt{2+\frac{1}{n}}} = 0$

$\therefore$ By Leibnitz's test given series converges

$\therefore$ $\Sigma|u_n|$ diverges and Σu_n ~~also~~ converges

$\therefore$ Given series conditionally converges

UNIT- II Calculus of ~~the~~ One Variable

→ **Aggregate :-**

A non-empty subset S of real number R is called as ~~a~~ aggregate

Ex:- The set of all +ve Integers Z^+ is an aggregate

Ex:- An Empty set ϕ is not an aggregate.

→ **Upper bound :-**

A non-empty subset S of R is said to be bounded above if there exist the number $K_1 \in R \ni x \in S$ then $x \leq K_1$. Then the number K_1 is called upper bound of S.

Lower bound :-

A non- subset S of R is said to be bounded below if $\exists$ the number $K_2 \in R \ni x \in S$ then $x \geq K_2$. Then the number K_2 is called lower bound of S

Bounded ness :-

A non empty subset S of R is said to bounded if it is both bounded above and below.

Or an aggregate is said to be bounded $\exists$ $u, v \in R \ni v < x < u \ \forall \ x \in S$

Ex:- A finite set is bounded

Continious function :-

A function $f(x)$ is said to be continious at $x = c$ if

(i) $f(c)$ is defined

(ii) $\lim_{x \to c} f(x)$ exist

(iii) $\lim_{x \to c} f(x) = f(c)$

A function $f(x)$ is said to be right continious at $x = c$ if

(i) $f(c)$ is ~~df~~ defined

(ii) $\lim_{x \to c^+} f(x)$ exist

(iii) $\lim_{x \to c^+} f(x) = f(c)$

A function $f(x)$ is said to be left continious at $x = c$ if

(i) $f(c)$ is defined

(ii) $\lim_{x \to c^-} f(x)$ exist

(iii) $\lim_{x \to c^-} f(x) = f(c)$

A function $f(x)$ is said to be continous in $[a,b]$ then

(i) $f(x)$ is continious at $x \in [a,b]$

(ii) $f(x)$ is right continious at $x = a$

(iii) $f(x)$ is left continous at $x = b$

→ Geometrically if a function $f(x)$ is continious in $[a,b]$ then the graph of the function will be a continious curve at each $x = c \in [a,b]$

→ If $f(x)$ is continious in $[a,b]$:

(i) $f(x)$ is bounded and attains its bounds if M and m are the greatest and least value of function such that $M = f(c)$ and $m = f(d)$

(ii) $f(x)$ attains all values between $f(a)$ and $f(b)$

(iii) There Exist $c, d \in [a,b] \ni f'(c) = 0$ when ever $f(a)$ and $f(b)$ will have opposite signs.

→ **<u>Differentiable function</u>** :-

A function $f(x)$ is said to be differentiable or derivable at $x = c$

at at $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exist and finite

There is called the derivate of function $f(x)$

which is denoted as $f'(c)$

→ If a function $f(x)$ is derivable in ~~open~~ (a,b) then the function $f(x)$ is continous in the $[a,b]$ geometrically if a function $f(x)$ is derivable (a,b) then $\exists$ a unique tangent at $x = c \in (a,b)$

→ If a function $f(x)$ is derivable at $x=c$ then $f(x)$ is continious at $x=c$

→ If $f'(c) > 0$ then the function is increasing at c and if $f'(c) < 0$ then the function is decreasing at c

Rolle's Theorem :-

Let a function $f(x)$ be defined in $[a,b]$ such that

(i) $f(x)$ is continious in $[a,b]$

(ii) $f(x)$ is ~~diver~~ derivable in (a,b)

(iii) $f(a) = f(b)$ then $\exists$ $c \in (a,b) \ni f'(c) = 0$

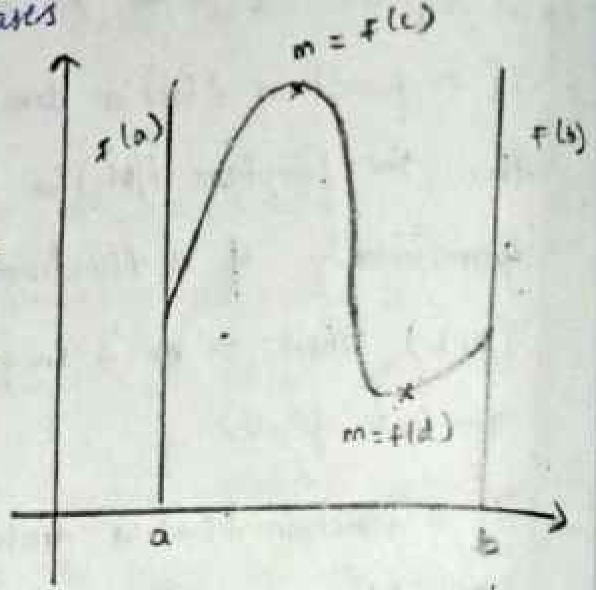
Proof:- Since $f(x)$ is continious in $[a,b]$

→ $f(x)$ is bounded in $[a,b]$

→ let M and m be the greatest and least value of the function $f(x)$

→ Then $\exists$ $c, d \in [a,b] \ni M = f(c)$, $m = f(d)$

Here we have 2 cases



Case (i)

When $M = m$

$f(x)$ is constant

$f'(x)$ is $= 0 \ \forall x$

$\therefore f'(c) = 0$

Case (ii)

When $M \neq m$

At least one of them should be different from equal values of $f(a)$ and $f(b)$

let $M = f(c)$ be different $f(a)$ and $f(b)$

$\because c \in (a, b)$ i.e. $c \neq a$ and $c \neq b$

$\therefore f(x)$ is derivable in (a, b)

$\therefore f(x)$ is derivable at $x = c$

ie $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exist

and which is same as $h \to 0$ through +ve and -ve values.

$f(c + h) \leq f(c)$ for $h > 0$ or $h < 0$

$\frac{f(c+h) - f(c)}{h} < 0 \; ; h > 0 \rightarrow (1)$ and $\frac{f(c+h) - f(c)}{h} > 0 \; ; h < 0 \rightarrow (2)$

Let $h \to 0$

through all +ve values of ① we get $f'(c) < 0$ ↳ ③

Let $h \to 0$

through all -ve values of ② we get $f'(c) > 0$ — ④

from ③ and ④ $f'(c) = 0$

Thus Rolle's Theorem.

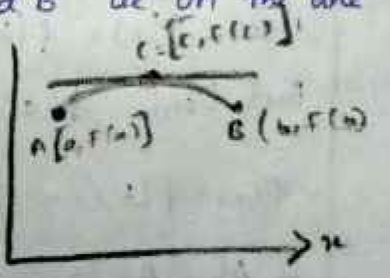
→ **Geometrical Interpretation of Rolle's Theorem:-**

Let A and B are the two points on the curve $y = f(x)$ corresponding to $x = a$ and $x = b$ the co-ordinates of A and B are $A = [a, f(a)]$ $B = [b, f(b)]$

Since $f(x)$ is continious in $[a, b]$ then the graph of the function will be a continious curve from A to B

Since $f(x)$ is derivable in (a, b) there exist a unique at each point between A and B

$f(a) = f(b)$ i.e the points A and B lie on the line parallel to x-axis



∴ by Rolle's Theorem

$\exists\ c \in (a, b) \ni f'(c) = 0$

There exist a point $C = [c, f(c)]$ where the tangent is parallel to x-axis

→ Lagrange's Mean value Theorm :-

Let a function f(x) be defined in [a,b] such that

(i) F(x) is continious in [a,b]

(ii) F(x) is derivable in (a,b) then ∃ c ∈ (a,b) ∋

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof :- Let us defined a new function

$$\phi x = f(x) + Ax \quad \text{--- ①}$$

where A is unknown constant which is to be determined.

Since f(x) is continous in [a,b]

∴ the new function ϕ(x) is also continues in [a,b].

Since f(x) is derivable in (a,b)

∴ The new function ϕ(x) is also derivable in (a,b)

Now f(a) = f(b)

But here ϕ(a) = ϕ(b)

$$f(a) + f(b) = f(b) + Ab$$

$$- Ab + Aa = f(b) - f(a)$$

$$- \cancel{Ab} A(b-a) = f(b) - f(a)$$

$$- A = \frac{f(b)-f(a)}{b-a} \quad \text{--- ②}$$

Hence by Rolle's Theorem

$\exists\, c \in (a,b) \ni f'(c) = 0$

But here $\phi'(c) = 0$

W.K.T

$\phi(c) = f(c) + Ac$

$\phi'(c) = f'(c) + A$

$-A = f'(c)$ — ③ $[\because \phi'(c) = 0]$

from ② and ③ we get

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Lagrange's Mean Value Theorem.

<u>Geometrical</u> <u>Interpretation</u> <u>of</u> <u>LMVT</u> :-

Let A and B are the two points on the curve $y=f(x)$ Corresponding to $x=a$ and $x=b$ the co-ordinates of A and B are $A=[a, f(a)]$, $B=[b, f(b)]$ respectively

$\therefore$ Slope of the chord AB $= \dfrac{f(b)-f(a)}{b-a}$

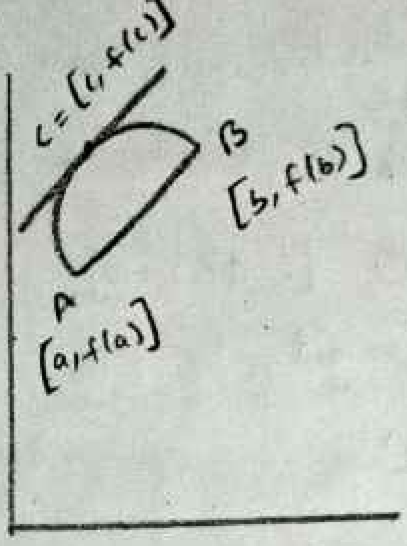
Since $f(x)$ is continues in $[a,b]$ then the graph of the function is also continues from A to B

Since $f(x)$ is derivable in (a,b) then there exist a unique tangent between A and B

$\therefore$ by LMVT there exist $c \in (a,b) \ni$

$$f'(c) = f(b)-f(a)/b-a$$

ie there exist a point $c = [c, f(c)]$ where the tangent is parallel to the chord AB



Cauchy's Mean Value Theorm :-

let $f(x)$ and $g(x)$ are the two functions be defined in $[a, b]$ such that

(i) $f(x)$ and $g(x)$ are continues in $[a, b]$

(ii) $f(x)$ and $g(x)$ are derivable in (a, b)

then for $g'(x) \neq 0$ for all x $\exists\, c \in (a, b) \ni$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof :- let us defined a new function

$$\phi(x) = f(x) + A\, g(x) \quad \text{—} ①$$

Now where A is an unknown constant which is to be determined

Since $f(x)$ and $g(x)$ are continuous in $[a,b]$

$\therefore$ the new function $\phi(x)$ is also continuous in $[a,b]$.

Since $f(x)$ and $g(x)$ are derivable in (a,b)

$\therefore$ The new function $\phi(x)$ is also derivable in (a,b).

Now $f(a) = f(b)$

$$f(a) + A\,g(a) = f(b) + A\,g(b)$$

$$-A\,g(b) + A\,g(a) = f(b) - f(a)$$

$$-A[g(b) - g(a)] = f(b) - f(a)$$

$$-A = \frac{f(b) - f(a)}{g(b) - g(a)} \quad \text{--- } ②$$

Hence by Rolle's Theorem

$\exists\, c \in (a,b) \ni f'(c) = 0$

But here $\phi'(c) = 0$

W.K.T

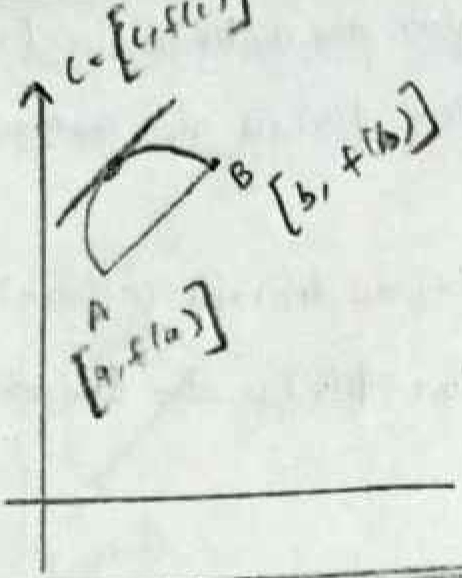
$$\phi(c) = f(c) + A\,g(c)$$

$$\phi'(c) = f'(c) + A\,g'(c)$$

$$-A = \frac{f'(c)}{g'(c)} \quad \text{--- } ③ \quad [\because \phi'(c) = 0]$$

from ② and ③ we get $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

C.M.V.T



Rolle's Theorm Problems:-

Verify rolle's theorm for the following function

① $x^2 - 2x$ is $[0, 2]$

sol let $f(x) = x^2 - 2x$

Since $f(x)$ is an algebraic function

$\therefore f(x)$ is continious in $[0, 2]$ and

derivable in $[0, 2]$

Now

$f(a) = f(b)$ ie $f(0) = f(2)$

$= 0 \quad = 0$

Hence it satisfies all the 3 conditions of Rolle's Theorm then $\exists$ $c \in (a, b) \ni f'(c) = 0$

Here

$f(c) = c^2 - 2c$

$f'(c) = 2c - 2$ $\qquad [f'(c) = 0]$

$2c-2=0$

$c=1 \in (0,2)$

Hence Rolle's theorm is verified

(2) x^3-4x in $[-2,2]$

Sol Let $f(x) = x^3-4x$

Since $f(x)$ is an algebric function

$\therefore$ $f(x)$ is continues ~~at~~ in $[-2,2]$ and

derivable in $(-2,2)$

Now

$f(a)=f(b) = f(-2)=f(2) = 0=0$

Hence ~~as~~ it satisfies all the

3 condition of Rolle's theorm then

$\exists\, c \in (a,b) \ni f'(c)=0$

Here $f(c) = c^3-4c$

$f'(c) = 3c^2-4$

$c = \pm \frac{2}{\sqrt{3}}$

$c = \pm 1.15 \in (-2,2)$

$\therefore$ Hence Rolle's theorm is verified.

(3) $(x+2)(x-3)$ in $[-2,3]$

sol Let $f(x) = (x+2)(x-3)$

$x^2 - x - 6$

Since $f(x)$ is an algebraic function

$\therefore f(x)$ is continues in $[-2,3]$ and

derivable in $(-2,3)$

Now

$f(a) = f(b)$ ie $f(-2) = f(3) = 0 = 0$

Hence it satisfies all 3 condition

of Rolle's theorm $\exists\, c \in (a,b) \ni f'(c) = 0$

Here $f(c) = c^2 - c - 6$

$f'(c) = 2c - 1$

$c = \frac{1}{2} \in (-2,3)$

Hence Rolle's theorm is verified.

(4) $x(x+3)e^{-x/2}$ in $[-3,0]$

sol Let $f(x) = (x^2 + 3x)e^{-x/2}$

Since $x^2 + 3x$ is an algebraic function and $e^{-x/2}$

is an exponential function.

$\therefore f(x)$ is continues in $[-3,0]$ and

derivable in $(-3,0)$

Now $f(a) = f(b) \Rightarrow f(-3) = f(0) = 0 = 0$

(5)

sol

Hence it satisfies all 3 conditions of Rolle's theorem then $\exists\ c \in (a,b) \ni f'(c) = 0$

Here $f(c) = (c^2+3c)e^{-c/2}$

$f'(c) = (c^2+3c)(-\frac{1}{2}e^{-c/2}) + e^{-c/2}(2c+3)$

$$e^{-c/2}\left[\frac{-(c^2+3c)+4c+6}{2}\right] = 0 \quad [\because f'(c)=0] \quad [e^{-c/2} \neq 0]$$

$c^2 - c - 6 = 0 \Rightarrow c = -2, 3$

$\therefore c = -2 \in (-3, 0)$

Hence Rolle's theorm is verified.

---

⑤ $\frac{\sin x}{e^x}$ in $[0, \pi]$

sol Let $f(x) = \frac{\sin x}{e^x}$

since $\sin x$ is trigonometric function and e^x is an exponential function

$\therefore f(x)$ is continues in $[0, \pi]$ and derivable in $(0, \pi)$

Now

$f(a) = f(b)$

$f(0) = f(\pi)$

$0 = 0$

Hence it satisfies all the 3 conditions of Rolle's theorm then $\exists\, c \in (a,b) \ni$ $f'(c) = 0$

Here $f(c) = \dfrac{\sin c}{e^c}$

$f'(c) = \dfrac{e^c(\cos c) - \sin c\, e^c}{e^{2c}}$ $\quad [f'(c) = 0]$

$e^c[\cos c - \sin c] = 0$ $\quad \therefore [e^c \neq 0]$

$\cos c = \sin c$

$c = \frac{\pi}{4} \in (0, \pi)$

Hence rolle's theorm is verified

(6) $(x-a)^m (x-b)^n$ in $[a,b]$ where m, n are +ve integers

sol let $f(x) = (x-a)^m (x-b)^n$

since $f(x)$ is algebric function

$\therefore f(x)$ is continues in $[a,b]$ and derivable in (a,b)

Now $f(a) = f(b)$

$0 = 0$

Hence it satisfies all the 3 condition of Rolle's theorm then $\exists\, c \in (a,b) \ni$

$f'(c) = 0$

Here $f(c) = (c-a)^m (c-b)^n$

$f'(c) = (c-a)^m n(c-b)^{n-1} + (c-b)^n m (c-a)^{n-1}$

$[\because f'(c) = 0]$

$(c-a)^m (c-b)^n \left[\frac{n}{c-b} + \frac{m}{c-a}\right] = 0$

$\left[\because c \in (a,b) \Rightarrow c \neq a, c \neq b\right]$

$nc - an + mc - bm = 0$

$c(m+n) = an + bm = 0$

$\Rightarrow c = \frac{an+bm}{m+n} \in (a,b)$

Hence Rolle's theorm verified.

---

(7) $\log\left[\frac{x^2+ab}{x(a+b)}\right]$ in $[a,b]$ where $a,b>0$

Sol: let $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$

since $f(x)$ is log function

$\therefore f(x)$ is continues in $[a,b]$ and derivable in (a,b)

Now $f(a) = f(b)$

$0 = 0$

Hence it satisfies all the 3 condition of Rolle's theorm then $\exists\, c \in (a,b) \ni f'(c) = 0$

Here $f(c) = \log\left[\frac{c^2+ab}{c(a+b)}\right] = \log(c^2+ab) - \log c - \log(a+b)$

$f'(c) = \frac{2c}{c^2+ab} - \frac{1}{c} - 0$

$\frac{2c}{c^2+ab} - \frac{1}{c} = 0 \Rightarrow 2c^2 - c^2 - ab = 0$

$c^2 - ab = 0 \Rightarrow c = \pm\sqrt{ab} \in (a,b)$

Rolle's theorm is verified.

(8) $\tan x$ in $[0, \pi]$

sol: Let $f(x) = \tan x$

since $f(x)$ is not continues at $x = \pi/2$

Hence rolle's theorm not applicable

(9) $|x|$ in $[-2, 2]$

sol: let $f(x) = |x|$

since $f(x)$ is continues in $[-2, 2]$

But it is not derivable in $(-2, 2)$

Hence Rolle's theorm is not applicable

(10) x^3 in $[1, 3]$

sol: let $f(x) = x^3$

Since $f(x)$ is continues in $[1, 3]$

It is derivable in $(1, 3)$

But $f(a) \neq f(b)$ ie $f(1) \neq f(3)$

Hence rolle's theorm is not applicable.

Lagrange's Mean value Theorm Problems :-

① $x^3 - 2x^2$ in $[2,5]$

sol: Let $f(x) = x^3 - 2x^2$

since $f(x)$ is algebraic function

$\therefore f(x)$ is continues in $[2,5]$

and derivable in $[2,5]$

$\therefore$ By L.M.V.T $\exists$ $c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

Here $a = 2$, $b = 5$, $f(a) = f(2) = 0$

$f(b) = 5 = 75$, $f'(c) = 3c^2 - 4c$

$3c^2 - 4c = \frac{75-0}{5-2} = \frac{75}{3} = 25$

$3c^2 - 4c - 25 = 0$

$c = 3.62$ and -2.29

$c = 3.62 \in (2,5)$

Hence L.M.V.T is verified

② $x^3 - 6x^2 + 11x - 6$ in $[0,4]$

sol: Let $f(x) = x^3 - 6x^2 + 11x - 6$

since $f(x)$ is an algebraic function

$\therefore f(x)$ is continues in $[0,4]$ and derivable in $(0,4)$

$\therefore$ By LMVT $\exists$ $c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

Here $a=0$, $b=4$. $f(a)=f(0)=-6$

$f(b)=f(4)=6$ $f'(c)=3c^2-12c+11$

$\therefore 3c^2-12c+11 = \frac{6+6}{4-0} = \frac{12}{4} = 3$

$3c^2-12c+8=0$

$c = 3.15$ and $0.84 \in (0,4)$

Hence LMVT is verified

③ $x(x-1)(x-2)$ in $[0, \frac{1}{2}]$

Sol Let $f(x) = x(x^2-3x+2)$

$= x^3-3x^2+2x$

since $f(x)$ is an algebraic function

$\therefore f(x)$ is continues in $[0, \frac{1}{2}]$ and derivable in $[0, \frac{1}{2}]$

$\therefore$ By LMVT $\exists$ $c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

Here $a=0$, $b=\frac{1}{2}$, $f(a)=f(0)=0$

$f(b)=f(\frac{1}{2})=\frac{3}{8}$, $f'(c)=3c^2-6c+2$

$\therefore 3c^2-6c+2 = \frac{\frac{3}{\cancel{8}_4}-0}{\frac{1}{\cancel{2}}-0}$

$12c^2-24c+5=0 \Rightarrow$

$c = 1.76$ and 0.23 $\therefore c = 0.23 \in (0, \frac{1}{2})$

Hence LMVT is verified

④ $x^3 - 3x - 1$ in $\left[\frac{-11}{7}, \frac{13}{7}\right]$

Sol: Let $f(x) = x^3 - 3x - 1$

Since $f(x)$ is algebric function

$\therefore f(x)$ is continues in $[-11/7, 13/7]$ and

derivable in $[-11/7, 13/7]$

$\therefore$ By LMVT $\exists\, c \in (a,b) \ni f'(c) = \frac{f(b) - f(a)}{b-a}$

Here $a = -11/7$, $b = \frac{13}{7}$ $f(a) = f\left(\frac{11}{7}\right) = -0.16$

$f(b) = f\left(\frac{13}{7}\right) = -\cancel{0.16}\ 0.16$, $f'(c) = 3c^2 - 3$

$\therefore 3c^2 - 3 = 0$

$c^2 = 1$

$\Rightarrow c = \pm 1 \in (-11/7, 13/7)$

Hence LMVT is verified

⑤ $\log x$ in $[1, e]$

Sol: $f(x) = \log x$

Since $f(x)$ is a log function

$\therefore f(x)$ is continues in $[1, e]$ and derivable in $(1, e)$

$\therefore$ By LMVT $\exists\, c \in (a,b) \ni f'(c) = \frac{f(b) - f(a)}{b-a}$

Here

$a = 1$ $b = e$, $f(a) = f(1) = 0$ $f(b) = f(e) = 1$

$f'(c) = 1/c$

$$\frac{1}{c} = \frac{1-0}{e-1}$$

$c = e-1 \in [1, e]$

Hence LMVT verified

---

⑥ $x^{1/3}$ in $[-1, 1]$

sol

let $f(x) = x^{1/3}$

Here the function is continious in $[-1, 1]$

But it is

not derivable in $(-1, 1)$

Hence LMVT is not applicable

---

⑦ P.T $|\cos b - \cos a| \leq |b - a|$

sol

let $f(x) = \cos x$

$f'(x) = -\sin x$

$f'(c) = -\sin c$

$\therefore$ By LMVT $\exists\ c \in (a, b) \Rightarrow f'(c) = \frac{f(b) - f(a)}{b-a}$

$$-\sin c = \frac{\cos b - \cos a}{b-a}$$

$$\Rightarrow |-\sin c| = \frac{|\cos b - \cos a|}{b-a}$$

$$\Rightarrow |\cos b - \cos a| \le |b-a| \quad \therefore [|\sin c| \le 1]$$

---

(5) P.T $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

Sol: $f(x) = \tan^{-1}x \Rightarrow f'(c) = \frac{1}{1+c^2}$

$\therefore$ By LMVT $\exists\ c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$\therefore c \in (a,b)$ ie $a < c < b$

$a^2 < c^2 < b^2$

$1+a^2 < 1+b^2 < 1+c^2$

$\frac{1}{1+a^2} > \frac{1}{1+b^2} > \frac{1}{1+b^2}$

$\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$

$\frac{1}{1+b^2} < f'(c) < \frac{1}{1+a^2}$

Put $a=1$ and $b = \frac{4}{3}$

$$\frac{1}{1+\frac{16}{9}} < \frac{\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(1)}{\frac{4}{3}-1} < \frac{1}{1+1}$$

$$\frac{9}{25} < \frac{\tan^{-1}(4/3) - \frac{\pi}{4}}{\frac{1}{3}} < \frac{1}{2}$$

$$\frac{3}{25} < \operatorname{Tan}^{-1}(4/3) - \pi/4 < \frac{1}{6}$$

$$\frac{\pi}{4} + \frac{3}{25} < \operatorname{Tan}^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$$

(ii) P.T $\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}b - \sin^{-1}a < \frac{b-a}{\sqrt{1-b^2}}$

sol Let $f(x) = \sin^{-1}x$

$$f'(c) = \frac{1}{\sqrt{1-c^2}}$$

$\therefore$ By LMVT $\exists\, c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$\therefore$ $c \in (a,b)$ ie $a < c < b$

$$a^2 < c^2 < b^2$$

$$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{1}{\sqrt{1-a^2}} < f'(c) < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1}(b) - \sin^{-1}(a)}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{1}{\sqrt{1-a^2}} \times b-a < \frac{\sin^{-1}(b) - \sin^{-1}(a)}{\cancel{b-a}} \times \cancel{b-a} < \frac{1}{1-b^2} \times b-a$$

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}(b) - \sin^{-1}(a) < \frac{b-a}{\sqrt{1-b^2}}$$

Hence Proved.

---

(9) If $f'(x) = \frac{1}{3-x^2}$ and $f(0) = 1$ then find an internal in which $f(1)$ lies

sol $f'(x) = \frac{1}{3-x^2}$ and $f(0) = 1$

w.r.t

$$\min_{a \le x \le b} f'(x) \le \frac{f(b)-f(a)}{b-a} \le \max_{a \le x \le b} f'(x)$$

Here $a = 0$ $b = 1$

$$\therefore \min_{0 \le x \le 1} \frac{1}{3-x^2} \le \frac{f(1)-f(0)}{1-0} \le \max_{0 \le x \le 1} \frac{1}{3-x^2}$$

$$\frac{1}{3} \le f(1) - 1 \le \frac{1}{2}$$

$\frac{4}{3} \le f(1) \le \frac{3}{2}$ which is an internal. for $f(1)$

---

(10) find a point at which tangent to the curve $y = \log x$ $\parallel^{el}$ to the chord Joining the Points $(1,0)$ and $(e,1)$

sol: Given $y = \log x \Rightarrow f'(x) = \frac{1}{x}$

By LMVT $\exists\ c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\frac{1}{x} = \frac{1-0}{e-1} \Rightarrow x = e-1$$

Hence $y = \log(e-1)$

$\therefore$ The point is $[e-1, \log(e-1)]$

CMVT (problem)

① x^2 and x^3 in $[1,4]$

sol: let $f(x) = x^2$ and $g(x) = x^3$

since $f(x)$ and $g(x)$ are algebraic function

$\therefore$ They are continous in $[1,4]$ and derivable in (1,

By CMVT $\exists\ c \in (a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

Here

$a = 1,\ b = 4,\ f(a) = 1 \quad f(b) = 16$

$g(a) = g(1) = 1, \quad g(b) = g(4) = 64 \quad f'(c) = 2c$

$g'(c) = 3c^2$

$$\therefore \frac{2c}{3c^2} = \frac{16-1}{64-1} \Rightarrow \frac{2}{3c} = \frac{15}{63}$$

$$c = \frac{\cancel{65}^{\cancel{13}\,2} \times 2}{\cancel{2} \times \cancel{18}\,5} = \frac{14}{5} = 2.8 \in (1,4)$$

$\therefore$ Hence CMVT is verified.

---

② $\log x$ and $\frac{1}{x}$ in $[1,e]$

Sol: Let $f(x) = \log x$ and $g(x) = \frac{1}{x}$

Since $f(x)$ is log function and $g(x)$ is Algebraic function

$\therefore$ They are continuous in $[1,e]$ and derivable in $(1,e)$

By CMVT $\exists$ $c \in (a,b)$ $\ni$ $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

Here $a=1$, $b=e$, $f(a)=f(1)=0$, $f(b)=f(e)=1$

$g(a)=g(1)=1$, $g(b)=g(e)=\frac{1}{e}$, $f'(c)=\frac{1}{c}$

$g'(c)=\frac{-1}{c^2}$

$$\therefore \frac{1/\cancel{c}}{-1/c^{\cancel{2}}} = \frac{1-0}{\frac{1}{e}-1} \Rightarrow -c = \frac{1}{\frac{1-e}{e}}$$

$$\therefore c = \frac{e}{e-1} \in (1,e)$$

Hence CMVT is verified.

---

③ $\frac{1}{x^2}$ and $\frac{1}{x}$ in $[a,b]$

Sol: Let $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x}$

Since $f(x)$ and $g(x)$ are algebraic

$\therefore$ They are continuous in $[a,b]$ and derivable in (a,b)

By CMVT $\exists$ $c \in (a,b)$ $\ni$ $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{b-a}$

$f(a) = \frac{1}{a^2}$, $f(b) = \frac{1}{b^2}$, $g(a) = \frac{1}{a}$.

$g(b) = \frac{1}{b}$, $f'(c) = \frac{-2}{c^3}$, $g'(c) = -\frac{1}{c^2}$

$\therefore \frac{\cancel{-}2/c^3}{\cancel{-}1/c^2} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}}$

$\frac{2}{c} = \frac{\frac{a^2-b^2}{a^2b^2}}{\frac{a-b}{ab}} \Rightarrow 2/c = \frac{a+b}{ab} \Rightarrow c = \frac{2ab}{a+b} \in (a,b)$

Hence CMVT is verified

④ $\sqrt{x}$ and $\frac{1}{\sqrt{x}}$ in $[a,b]$

sol let $f(x) = \sqrt{x}$ [redacted] and $g(x) = \frac{1}{\sqrt{x}}$

Since $f(x)$ and $g(x)$ are algebraic function

$\therefore$ They are continious in $[a,b]$ and derivable in (a,b)

By CMVT $\exists$ $c \in (a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

Here $f(a) = \sqrt{a}$, $f(b) = \sqrt{b}$, $g(a) = \frac{1}{\sqrt{a}}$.

$g(b) = \frac{1}{\sqrt{b}}$, $f'(c) = \frac{1}{2\sqrt{c}}$, $g'(c) = -\frac{1}{2\sqrt{c}}$

$\therefore \frac{1/\cancel{2\sqrt{c}}}{-1/\cancel{2c\sqrt{c}}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}} = -c = -\frac{(\cancel{\sqrt{a}-\sqrt{b}})}{\left(\frac{\cancel{\sqrt{a}-\sqrt{b}}}{\sqrt{ab}}\right)}$

$c = \sqrt{ab} \in (a,b)$

Hence CMVT is verified

(5) e^{-x} and e^{x} in $[a,b]$

sol: Let $f(x) = e^{-x}$ and $g(x) = e^{x}$

Since $f(x)$ and $g(x)$ are exponential function

They are continuous in $[a,b]$ and derivable in (a,b)

By CMVT $\exists$ $c \in (a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

Here $f(a) = e^{-a}$, $f(b) = e^{-b}$, $g(a) = e^{a}$, $g(b) = e^{b}$

$f'(c) = -e^{-c}$, $g'(c) = e^{c}$

$\therefore \frac{-e^{-c}}{e^{c}} = \frac{e^{-b}-e^{-a}}{e^{b}-e^{a}} = -\frac{1}{e^{2c}} = \frac{\frac{1}{e^{b}} - \frac{1}{e^{a}}}{e^{b}-e^{a}}$

$= \cancel{-}\frac{1}{e^{2c}} = \frac{\left(\frac{\cancel{e^{a}-e^{b}}}{e^{a+b}}\right)}{\cancel{-}(\cancel{e^{a}-e^{b}})}$

$= 2c = a+b \Rightarrow c = \frac{a+b}{2} \in (a,b)$

Hence CMVT is verified

(6) $\sin x$ and $\cos x$ in $[a,b]$

sol: Let $f(x) = \sin x$ and $g(x) = \cos x$

Since $f(x)$ and $g(x)$ are trigonometric function

$\therefore$ They are continious at $[a,b]$ and derivable at (a,b)

By CMVT $\exists c \in (a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

Here $f(a) = \sin a$, $f(b) = \sin b$, $g(a) = \cos a$

$g(b) = \cos b$ $f'(c) = \cos c$, $g'(c) = -\sin c$

$$\frac{\cos c}{-\sin c} = \frac{\sin b - \sin a}{\cos b - \cos a}$$

$$-\cot c = \frac{2\cos\left(\frac{b+a}{2}\right)\sin\left(\frac{b-a}{2}\right)}{2\sin\left(\frac{b+a}{2}\right)\sin\left(\frac{b-a}{2}\right)}$$

$$\cancel{\cot} c = \cancel{\cot}\left(\frac{a+b}{2}\right) \Rightarrow c = \frac{a+b}{2} \in (a,b)$$

$\therefore$ Hence CMVT is verified

→ **Taylors Series :-**

Taylor series for the function of one independent variable abo $x = a$ is given by

$$f(x) = f(a) + (x-a)\, f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

And when $x=0$ we get maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(a) + \dots$$

→ find the series of the following function (or) expand the following function in series.

1 (i) $\sinh x$

sol let $f(x) = \sinh x \Rightarrow f(0) = 0$

$f'(x) = \cosh x \Rightarrow f'(0) = 1$

$f''(x) = \sinh x \Rightarrow f''(0) = 0$

$f'''(x) = \cosh x \Rightarrow f'''(0) = 1$

$f^{iv}(x) = \sinh x \Rightarrow f^{iv}(0) = 0$

$f^{v}(x) = \cosh x \Rightarrow f^{v}(0) = 1$ .... so on

$$\sinh x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \cdots$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

(ii) $\sin^2 x$

Sol

$f(x) = \sin^2 x \Rightarrow f(0) = 0$

$f'(x) = 2\sin x \cos x$

$= \sin 2x \Rightarrow f'(0) = 0$

$f''(x) = 2\cos 2x \Rightarrow f''(0) = 2$

$f'''(x) = -4\sin 2x \Rightarrow f'''(0) = 0$

$f^{iv}(x) = -8\cos 2x \Rightarrow f^{iv}(0) = -8$

$\cdots\cdots$ so on

We have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

$$\sin^2 x = 0 + x(0) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(-8) + \cdots$$

$$= x^2 - \frac{x^4}{3} + \cdots$$

(iii) $\log(1+x)$

Sol

$f(x) = \log(1+x) \Rightarrow f(0) = 0$

$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$

$\cdots\cdots$ so on

we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots.$$

$$\log(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \dots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\dots$$

(iv) $\tan^{-1} x$

sol

$f(x) = \tan^{-1} x \Rightarrow f(0) = \tan^{-1}(0) = \tan^{-1}(\tan 0)$
$= 0$

$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(0) = 1$

$f''(x) = \frac{2x}{(1+x^2)^2} = f''(0) = 0$

$$f'''(x) = 2\left[\frac{(1+x^2)^2(1) - x\ 2(1+x^2)2x}{(1+x^2)^4}\right]$$

$\Rightarrow f'''(0) = 2$

.... so on

we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots.$$

$$\tan^{-1} x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \dots$$

$$= x + \frac{x^3}{3} + \dots.$$

(v) 2^x

sol

let $f(x) = 2^x \Rightarrow f(0) = 1$

$f'(x) = 2^x \log 2 \Rightarrow f'(0) = \log 2$

$f''(x) = 2^x (\log 2)^2 = f''(0) = (\log 2)^2$

...... so on

We have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

$$2^x = 1 + x(\log 2) + \frac{x^2}{2!}(\log 2)^2 + \cdots$$

(2) Find the Taylor series for the following function :-

(i) e^x about $x = 1$

sol. Let $f(x) = e^x \Rightarrow f(1) = e$

$f'(x) = e^x \Rightarrow f'(1) = e$

$f''(x) = e^x \Rightarrow f''(1) = e$

. . . . so on .

We have

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

$$e^x = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \cdots$$

$$= e + (x-1) e + \frac{(x-1)^2}{2!} e + \cdots$$

(ii) e^x about $x = 3$

sol. $f(x) = e^x \Rightarrow f(3) = e^3$

$f'(x) = e^x \Rightarrow f'(3) = e^3$

$f''(x) = e^x \Rightarrow f''(3) = e^3$

. . . . so on

we have

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

$$e^x = f(3) + (x-3) f'(3) + \frac{(x-3)^2}{2!} f''(3) + \dots\dots$$

$$e^3 + (x-3) e^3 + \frac{(x-3)^2}{2!} e^3 + \dots\dots$$

(iii) $x^2 \log x$ about $x = 1$

sol: let $f(x) = x^2 \log x \Rightarrow f(1) = 0$

$f'(x) = x^2 \cdot \frac{1}{x} + (\log x)(2x) \Rightarrow f'(1) = 1$

$f''(x) = 1 + 2\left[x \cdot \frac{1}{x} + \log x (1)\right]$

$\Rightarrow f''(1) = 3$

$\dots\dots$ so on

we have

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

$$x^2 \log x = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \dots\dots$$

$$= 0 + (x-1)(1) + \frac{(x-1)^2}{2!}(3) + \dots$$

$$= x-1 + \frac{(x-1)^2}{2}(3) + \dots$$

(iv) $\sin x$ but $x = \pi/4$

sol: let $f(x) = \sin x \Rightarrow f(\pi/4) = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x \Rightarrow f'(\pi/4) = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x \Rightarrow f''(\pi/4) = -\frac{1}{\sqrt{2}}$

$\dots\dots\dots$ so on

we have

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots\ldots$$

$$\sin x = f(\pi/4) + (x-\pi/4) f'(\pi/4) + \frac{(x-\pi/4)^2}{2!} f''(\pi/4) + \ldots\ldots$$

$$= \frac{1}{\sqrt{2}} + (x-\pi/4)\left(\frac{1}{\sqrt{2}}\right) - \frac{(x-\pi/4)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \ldots\ldots$$

(v) $\tan x$ but $x = \pi/4$

sol: let $f(x) = \tan x \Rightarrow f(\pi/4) = 2$

$f'(x) = \sec^2 x \Rightarrow f'(\pi/4) = 2$

$f''(x) = 2\sec^2 x \tan x \Rightarrow f''(\pi/4) = 4$

$\ldots\ldots$ so on

We have

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots\ldots$$

$$\tan x = f(\pi/4) + (x-\pi/4) f'(\pi/4) + \frac{(x-\pi/4)^2}{2!} f''(\pi/4) + \ldots\ldots$$

$$= 1 + (x-\pi/4)(2) + \frac{(x-\pi/4)^2}{2!}(\cancel{4}\,2) + \ldots$$

---

(3) find Taylor series for the following func :-

(1) $2x^3 + 7x^2 + 6x - 1$ in power of $(x-2)$

sol: let $f(x) = 2x^3 + 7x^2 + 6x - 1 \Rightarrow f(2) = 55$

$f'(x) = 6x^2 + 14x + 6 \Rightarrow f'(2) = 58$

$f''(x) = 12x + 14 \Rightarrow f''(2) = 38$

$f'''(x) = 12 \Rightarrow f'''(2) = 12$

We have

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^2}{3!} f'''(a)$$

$$2x^3 + 7x^2 + 6x - 1 = f(2) + (x-2) f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''(2)$$

$$= 55 + (x-2)(58) + \frac{(x-2)^2}{2!}(38) + \frac{(x-2)^2}{3!}(12)$$

(ii) $2x^3 + x^2 + 3x + 2$ in power of $(x-1)$

sol
$f(x) = 2x^3 + x^2 + 3x + 2 \Rightarrow f(1) = 8$

$f'(x) = 6x^2 + 2x + 3 \Rightarrow f'(1) = 11$

$f''(x) = 12x + 2 \Rightarrow f''(1) = 14$

$f'''(x) = 12 \Rightarrow f'''(1) = 12$

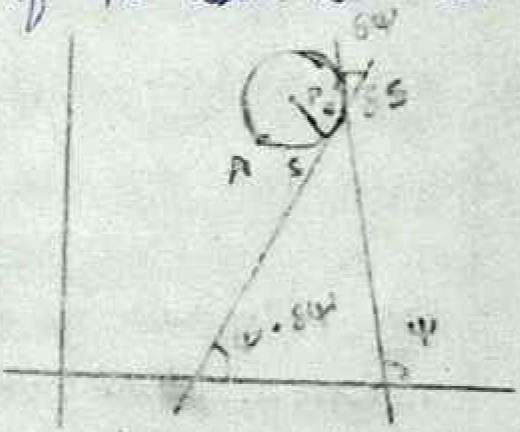
We have

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^2}{3!} f'''(a)$$

$$2x^3 + x^2 + 3x + 2 = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1)$$

$$= 8 + (x-1)(11) + \frac{(x-1)^2}{2!}(14) + \frac{(x-1)^3}{3!}(12)$$

Curvature:- In a continuous curve the rate of change of bending of a curve at a point is called as the curvature of the curve at that point.



Consider a curve and P is any point on it. Let Q be a neighbouring to P. Let A be a fixed point on the curve. Let AP = s, PQ = δs, let the tangents at P and Q makes an angle ψ and $\delta\psi$ with the x-axis. The angle b/w two tangent $\delta\psi$ where $\delta\psi$ denotes the angle through which the tangent turns as the point moves along the curve from P to Q through a distance δs

The total bending or total curvature of the arc PQ is equal to the angle $\delta\psi$. The avg. curvature of the arc PQ is equal to ratio $\frac{\delta\psi}{\delta s}$. The curvature of the curve at P is

$$\lim_{Q \to P} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds} = \text{Curvature } \frac{d\psi}{ds}$$

→ <u>Radius of curvature</u> :- R.O.C at a point P is defined as the ~~squared~~ reciprocal of the curvature at that point and it is denoted by ρ i.e. $\rho = \frac{ds}{d\psi}$

Three different form of R.OC are

(a) <u>Cartesian form</u> :- If $y = f(x)$ is any curve given then ROC is given by $\rho = \frac{[1+y_1^2]^{3/2}}{y_2}$

<u>Note</u> :- In case y_1 at a point becomes ∞ or indeterminate we consider the curve $x = f(y)$ and its ROC is given by $\rho = \frac{[1+x_1^2]^{3/2}}{x_2}$

(b) <u>Parametric form</u> :- If $x = f(\theta)$ and $y = g(\theta)$ then ROC is given by $\rho = \frac{[(f'(\theta))^2 + (g'(\theta))^2]^{3/2}}{f'(\theta)\, g''(\theta) - g'(\theta) f''(\theta)}$

(c) <u>polar form</u> :- If $r = f(\theta)$ be any curve given then ROC is given by $\rho = \frac{[r^2 + r_1^2]^{3/2}}{r^2 + 2r_1^2 - rr_2}$

where r_1 = first derivate w.r.to θ
r_2 = second derivative

<u>Center of curvature</u> :- In fig-1 let $c(\bar{x}, \bar{y})$ be the ~~x~~ center of circle. This center c is called the center of curvature at P whose co-ordinate

are given by :-

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} \quad , \quad \bar{y} = y + \frac{1+y_1^2}{y_2}$$

Evolute :- In fig-1 as the point P moves on the curve the center of curvature C will also trace a curve. This curve is called the evolute of the given curve. The original curve is called involute. By eliminating (x, y) from the co-ordinates of center of curvature we get evolute of given curve.

The eq of circle of curvature of a curve at a point P is given by

$$(x-\bar{x})^2 + (y-\bar{y})^2 = r^2$$

① $x^2+y^2=25$

sol: Given $x^2+y^2=25$ — ①

Diff ① we get

$2x+2y.y_1=0 \Rightarrow y_1=-\frac{x}{y}$ — ②

Diff ② we get

$$y_2=-\left[\frac{-y(1)-xy_1}{y^2}\right]=-\left[\frac{y+x^2/y}{y^2}\right]$$

$$y_2=-\left[\frac{x^2+y^2}{y^3}\right]$$

R.O.C $\rho=\frac{[1+y_1^2]^{3/2}}{y_2}=\frac{\left[1+\frac{x^2}{y^2}\right]^{3/2}}{-\left(\frac{x^2+y^2}{y^3}\right)}$

$$=\frac{-(x^2+y^2)^{3/2}}{(y^2)^{3/2}}\left(\frac{y^3}{x^2+y^2}\right)=-\sqrt{x^2+y^2}$$

$=\boxed{\rho=5}$

② $xy=c^2$

sol: Given $xy=c^2$ — ①

Diff ① we get

$xy_1+y(1)=0 \Rightarrow y_1=-\frac{y}{x}$ — ②

diff ① we get

$$y_2 = -\left[\frac{xy_1 - y(1)}{x^2}\right] = -\left[\frac{x(-y/x) - y}{x^2}\right]$$

$$= \frac{2y}{x^2}$$

$$\therefore \text{R.O.C.} = \rho = \frac{[1+y_1^2]^{3/2}}{y_2} = \frac{[1+y^2/x^2]^{3/2}}{2y/x^2}$$

$$= \frac{(x^2+y^2)^{3/2}}{(x^2)^{3/2}}\left(\frac{x^2}{2y}\right) = \frac{(x^2+y^2)^{3/2}}{2c^2} \quad [\because xy = c^2]$$

---

③ $x^2+y^2 = 4$ at $(1,2)$

Sol Given $x^2+y^2 = 4$ — ①

Diff ① we get

$2x + 2yy_1 = 0 \Rightarrow y_1 = -\frac{x}{y}$ — ②

y_1 at $(1,2)$ ie $y_1 = -\frac{1}{2}$

Diff ② we get

$$y_2 = -\left[\frac{y(1) - xy_1}{y^2}\right]$$

$$\therefore y_2 = -\left[\frac{2+1/2}{4}\right] = -\frac{5}{8}$$

$$\text{R.O.C} \quad \rho = \frac{[1+y_1^2]^{3/2}}{y_2} = \frac{[1+1/4]^{3/2}}{-5/8}$$

$$\rho = \left(\frac{5}{2^2}\right)^{3/2}\left(-\frac{8}{5}\right) = \sqrt{5}$$

④ $x^4+y^4=2$ at $(1,1)$ Also find curvature

sol Given $x^4+y^4=2$ — ①

diff ① we get

$4x^3+4y^3 \cdot y_1=0 \Rightarrow y_1=-\dfrac{x^3}{y^3}$ — ②

y_1 at $(1,1)$ ie $y_1=-1$

diff ② we get

$$y_2=-\left[\frac{y^3(3x^2)-x^3(3y^2y_1)}{y^6}\right]$$

y_2 at $(1,1)$ ie $y_2=-\left[\dfrac{3+3}{1}\right]=-6$

R.O.C. $\rho=\dfrac{[1+y_1^2]^{3/2}}{y_2}=\dfrac{[1+1]^{3/2}}{-6}=\dfrac{\cancel{2}\sqrt{2}}{-\cancel{6}\,3}$

$=\dfrac{\sqrt{2}}{3}$ $\therefore$ Curvature $\dfrac{3}{\sqrt{2}}$

⑤ $y=x^3+5x^2+6x$ at $(0,0)$

sol Given x^3+5x^2+6x — ①

diff ① we get

$y_1=3x^2+10x+6$ — ②

y_1 at $(0,0)$ ie $y_1=6$

diff ② we get

$y_2=6x+10$

y_2 at $(0,0)$ ie $y_2 = 10$

$$R.O.C = \rho = \frac{[1+y_1^2]^{3/2}}{y_2}$$

$$= \frac{[1+36]^{3/2}}{10} = \frac{37\sqrt{37}}{10}$$

(6) $y - x = x^2 + 2xy + y^2$ at $(0,0)$

sol Given $y - x = x^2 + 2xy + y^2$ — (1)

Diff (1) we get

$$y_1 - 1 = 2x + 2[x(y_1) + y(1)] + 2yy_1$$

$$y_1 [1 - 2x - 2y] = 2x + 2y + 1$$

$$y_1 = \frac{2x + 2y + 1}{1 - 2x - 2y} \quad \text{— (2)}$$

y_1 at $(0,0)$ ie $y_1 = 1$

$$y_2 = \frac{(1-2x-2y)(2+2y_1) - (2x+2y+1)(-2-2y_1)}{(1-2x-2y)^2}$$

y_2 at $(0,0)$ ie $y_2 = \frac{4+4}{1} = 8$

$$R.O.C \;\; \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{[1+1]^{3/2}}{8} = \frac{2\sqrt{2}}{8\,4}$$

$$\rho = \frac{1}{2\sqrt{2}}$$

(7) $y = x^2$ at $(1,1)$

sol Given $y = x^2$ — (1)

diff (1) we get

$y_1 = 2x$ — (2)

y_1 at $(1,1)$ ie $y_1 = 2$

Diff (2) we get

$y_2 = 2$

$$R.O.C \cdot \rho = \frac{[1+y_1^2]^{3/2}}{y_2} = \frac{[1+4]^{3/2}}{2} = \frac{5\sqrt{5}}{2}$$

(8) $y^2 = 4x$ at $(0,0)$

sol Given $y^2 = 4x$

diff (1) we get

$\cancel{2}yy_1 = \cancel{4}^{2} \Rightarrow y_1 = \frac{2}{y}$

y_1 at $(0,0)$ ie $y_1 = \infty$

Now diff (1) we get w.r. to 'y'

$\cancel{2}y = \cancel{4}^{2}x_1 \qquad x_1 = \frac{y}{2}$ — (3)

x_1 at $(0,0)$ ie $x_1 = 0$

diff (3) w.r. to 'y' we get

$x_2 = \frac{1}{2}$

$$R.O.C - \rho = \frac{[1+x_1^2]^{3/2}}{x_2} = \frac{[1+0]^{3/2}}{1/2}$$

$$\rho = \underline{\underline{2}}$$

(9) $x^2 = 8y$ at $(4,2)$

sol Given $x^2 = 8y$ — ①

diff ① we get

$2x = \cancel{8}^{4} y_1 \Rightarrow y_1 = \frac{x}{4}$

y_1 at $(4,2)$ is $y_1 = 1$

$y_2 = \frac{1}{4}$

$$ROC = \rho = \frac{[1+y_1^2]^{3/2}}{y_2} = \frac{[1+1]^{3/2}}{1/4} = 2\sqrt{2}(4)$$

$$= \underline{\underline{8\sqrt{2}}}$$

(10) $ay^2 = x^3$ at (a,a)

sol Given $ay^2 = x^3$ — ①

diff ① we get

$2ayy_1 = 3x^2 \Rightarrow y_1 = \frac{3x^2}{2ay}$ — ②

y_1 at (a,a) is $y_1 = \frac{3a^2}{2a^2} = \frac{3}{2}$

diff ② w.r.to 'x' we get

$$y_2 = \frac{3}{2a}\left[\frac{y(2x) - x^2 y_1}{y^2}\right] = \frac{3}{2a}\left[\frac{2a^2 - 3a^2/2}{a^2}\right]$$

$\frac{3}{2a}\left(\frac{a^2}{2a^2}\right) = \frac{3}{4a^2}$

R.O.C $= \rho = \frac{[1+y_1^2]^{3/2}}{y_2} = \frac{[1+9/4]^{3/2}}{}$

$= \frac{(13)^{3/2}}{(2^2)^{3/2}}\left(\frac{4a}{3}\right) = \frac{13\sqrt{13}\,a}{6}$

---

(11) $xy^2 = a^3 - x^3$ at $(a, 0)$

sol Given $xy^2 = a^3 - x^3$ —(1)

diff (1) we get

$x \cdot 2y \cdot y_1 + y^2(1) = -3x^2 \Rightarrow y_1 = \frac{-3x^2 - y^2}{2xy}$ —(2)

y_1 at $(a, 0)$ ie $y = \infty$

Now diff (1) w.r.to 'y' we get

$x(2y) + y^2(x_1) = 0 - 3x^2 \cdot x_1$

$x_1[y^2 + 3x^2] = -2xy \Rightarrow x_1 = \frac{-2xy}{3x^2+y^2}$ —(3)

x_1 at $(a, 0)$ ie $x_1 = 0$

diff. (3) w.r.to 'y'

$$x_2 = -2\left[\frac{(3x^2+y^2)(x(1)+yx_1) - xy(6xx_1+2y)}{(3x^2+y^2)^2}\right]$$

x_2 at $(a,0)$ ie $x_2 = \dfrac{-2\left[\cancel{3}a^{\cancel{5}}\right]}{\underset{3}{\cancel{9}}a^{\cancel{4}}} = -\dfrac{2}{3a}$

$$R.O.C = P = \frac{\left[1+x_1^2\right]^{3/2}}{x_2}$$

$$= \frac{\left[1+0\right]^{3/2}}{-2/3a} = \underline{-\frac{3a}{2}}$$

---

(12) $x^3+y^3-3axy = 0$ at $\left[\frac{3a}{2}, \frac{3a}{2}\right]$

sol Given $x^3+y^3-3axy = 0$ — (1)

diff (1) we get

$$3x^2+3y^2y_1 - 3a\left[xy_1 + y(1)\right] = 0$$

$$x^2+y^2y_1 - axy_1 - ay = 0$$

$$y_1(y^2-ax) = ay-x^2$$

$$y_1 = \frac{ay-x^2}{y^2-ax} \quad \text{— (2)}$$

y_1 at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

ie $$y_1 = \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}} = \frac{-\frac{\cancel{3a^2}}{\cancel{4}}}{\frac{\cancel{3a^2}}{\cancel{4}}} = -1$$

diff ① we get

$$y_2 = \frac{(y^2-ax)(ay_1-2x)-(ay-x^2)(2yy_1-a)}{(y^2-ax)^2}$$

y_2 at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ ie $y_2 =$

$$= \frac{\left(\frac{3a^2}{\cancel{4}}\right)(-\cancel{4}a) - \left(-\frac{3a^2}{\cancel{4}}\right)(-\cancel{4}a)}{\frac{63a^4}{16}}$$

$$= \frac{-3a^3-3a^3}{\frac{63a^4}{16}} = \frac{-6a^{\cancel{3}}}{\frac{63a^{\cancel{4}}}{16}} = \frac{-\cancel{96}^{32}}{\frac{63a}{\cancel{21}\,3}}$$

$$\rho = \frac{[1+1]^{3/2}}{-\frac{32}{3a}} = \frac{\cancel{2}\sqrt{2}\,(3a)}{-\frac{\cancel{32}}{16}}$$

$$\frac{3a(\sqrt{2})}{16} \;//$$

---

(13) $x = 2\cos\theta$, $y = 2\sin\theta$

sol Given — ①

$$\frac{dx}{d\theta} = -2\sin\theta \;:\; \frac{dy}{d\theta} = 2\cos\theta$$

$$\frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{2}\cos\theta}{-\cancel{2}\sin\theta} \Rightarrow y_1 = -\cot\theta$$

Now diff ② w.r.to x

$$y_2 = \operatorname{cosec}^2\theta \cdot \left(\frac{d\theta}{dx}\right)$$

$$= \frac{\operatorname{cosec}^2\theta}{dx/d\theta} = \frac{\operatorname{cosec}^2\theta}{-2\sin\theta} = \frac{-1}{2\sin^3\theta}$$

$$\therefore \text{R.O.C } \rho = \frac{\left[1+y_1^2\right]^{3/2}}{y_2} = \frac{\left[1+\cot^2\theta\right]^{3/2}}{-\frac{1}{2}\sin^3\theta}$$

$$= (\operatorname{cosec}^2\theta)^{3/2}\,(-2\sin^3\theta) = -2$$

$$\therefore \rho = 2$$

---

(14) $x = a(\theta+\sin\theta)\ ;\ y = a(1-\cos\theta)$

sol Given $x = a(\theta+\sin\theta)\ ;\ y = a(1-\cos\theta)$ — ①

$$\frac{dx}{d\theta} = a(1+\cos\theta)\ ;\ dy/d\theta = a\sin\theta$$

$$\frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{a}\sin\theta}{\cancel{a}(1+\cos\theta)} = \frac{\cancel{2}\sin\theta/2\ \cancel{\cos\theta/2}}{\cancel{2}\cos^2\theta/2}$$

~~diff~~ ②

$y_1 = \tan\theta/2$ — ②

Diff ② w.r.to x

$$y_2 = (\sec^2\theta/2)\left(\frac{1}{2}\cdot\frac{d\theta}{dx}\right) = \frac{\sec^2\theta/2}{2\,dx/d\theta} = \frac{\sec^2\theta/2}{2a(1+\cos\theta)}$$

$$= \frac{\sec^2\theta/2}{4a\cos^2\theta/2} = \frac{1}{4a\cos^4\theta/2}$$

$$R.O.C = \rho = \frac{[1+y_1^2]^{3/2}}{y_2} = [1+\tan^2\theta/_2]^{3/2}(4a\cos^4\tfrac{\theta}{2})$$

$$= (\sec^2\theta/_2)^{3/2}(4\cos^4\theta/_2) = 4\cos\theta/_2$$

(15) $x = a\cos^3\theta\ ,\ y = a\sin^3\theta$.

sol Given $x = a\cos^3\theta,\ y = a\sin^3\theta$ — ①

$dx/d\theta = -3a\cos^2\theta\sin\theta$

$dy/d\theta = 3a\sin^2\theta\cos\theta$

$$\frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta \qquad y_1 = -\tan\theta \to ②$$

Diff ② w.r.to x $\qquad y_2 = -\sec^2\theta\left(\frac{d\theta}{dx}\right)$

$$y_2 = \frac{-\sec^2\theta}{dx/d\theta} = \frac{-\sec^2\theta}{-3a\sin\theta\cos^2\theta} = \frac{1}{3a\sin\theta\cos^4\theta}$$

$$R.O.C = \rho = \frac{[1+y_1^2]^{3/2}}{y_2} = [1+\tan^2\theta]^{3/2}(3a\sin\theta\cos^4\theta)$$

$$= (\sec^2\theta)^{3/2}\,3a\sin\theta\cos^4\theta = 3a\sin\theta\cos\theta$$

(16) $x = a(\cos t + t\sin t)\ ;\ y = a(\sin t - t\cos t)$

sol Given $x = a(\cos t + t\sin t)\ ;\ y = a(\sin t - t\cos t)$ ↳ ①

$$\frac{dx}{dt} = a[-\cancel{\sin t} + t\cos t + \cancel{\sin t}(1)] = at\cos t$$

$$\frac{dy}{dt} = a[\cancel{\cos t} - \{t(-\sin t) + \cancel{\cos t}(1)\}] = at\sin t$$

$$\frac{dy/dt}{dx/dt} = \frac{a t \sin t}{a t \cos t} = y_1 = \tan t \quad —②$$

diff ② w.r.to x $\quad y_2 = \sec^2 t \cdot \frac{dt}{dx}$

$$y_2 = \frac{\sec^2 t}{dx/dt} = \frac{\sec^2 t}{a t \cos t} = \frac{1}{a t \cos^3 t}$$

$$\therefore \text{R.O.C} = \rho = \frac{[1+y_1^2]^{3/2}}{y_2} = [1+\tan^2 t]^{3/2} (a t \cos^3 \theta)$$

$$= (\sec^2 t)^{3/2} (a t \cos^3 t) = \underline{\underline{at}}$$

(17) $r = a\sin\theta + b\cos\theta$ at $\theta = \pi/2$

sol: $r = a\sin\theta + b\cos\theta \quad —①$

$\theta = \pi/2$ we get $r = a$

$r_1 = a\cos\theta - b\sin\theta \quad —②$

$\theta = \pi/2$ we get $r_1 = -b$

$r_2 = -a\sin\theta - b\cos\theta \quad —③$

$\theta = \pi/2$ we get $r_2 = -a$

$$\rho = \frac{[r^2 + r_1^2]^{3/2}}{r^2 + 2r_1^2 - r r_2} = \frac{(a^2+b^2)^{3/2}}{a^2+2b^2+a^2}$$

$$= \frac{(a^2+b^2)^{3/2}}{2(a^2+b^2)} = \underline{\underline{\frac{\sqrt{a^2+b^2}}{2}}}$$

(18) $r = a(1+\cos\theta)$

sol Given $r = a(1+\cos\theta)$ — ①

$r_1 = a(-\sin\theta)$ — ②

$r_2 = -a\cos\theta$ — ③

$\therefore \rho = \dfrac{[r^2+r_1^2]^{3/2}}{r^2+2r_1^2-rr_2}$

$\rho = \dfrac{[a^2(1+\cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta + a^2(1+\cos\theta)\cos\theta}$

$= \dfrac{[a^2+a^2\cos^2\theta + 2a^2\cos\theta + a^2\sin^2\theta]^{3/2}}{a^2+a^2\cos^2\theta + 2a^2\cos\theta + 2a^2\sin^2\theta + a^2\cos\theta + a^2\cos^2\theta}$

$= \dfrac{[2a^2 + 2a^2\cos\theta]^{3/2}}{3a^2 + 3a^2\cos\theta}$

$= \dfrac{(2)^{3/2}(a^2)^{3/2}(1+\cos\theta)^{3/2}}{3a^2(1+\cos\theta)}$

$= \dfrac{2\sqrt{2}\, a\sqrt{1+\cos\theta}}{3}$

$= \dfrac{2\sqrt{2ar}}{3}$

(19) $r^n = a^n \cos n\theta$

Sol Given $r^n = a^n \cos n\theta$ — ①

Apply log on B.S

$\log r^n = \log [a^n \cos n\theta]$

$n \log r = n \log a + \log \cos n\theta$ — ②

diff ② w.r.to θ

$$\frac{\cancel{n} r_1}{r} = 0 + \frac{(-\sin n\theta)}{\cos n\theta} (\cancel{n})$$

$r_1 = -r \tan n\theta \rightarrow$ ③

Diff ③ w.r.to θ

$r_2 = -[rn(\sec^2 n\theta) + \tan n\theta (r_1)]$

$r_2 = -nr \sec^2 n\theta + r \tan^2 n\theta$ — ④

$\therefore$ R.o.c $\rho = \dfrac{[r^2 + r_1^2]^{3/2}}{r^2 + 2r_1^2 - r r_2}$

$$\rho = \frac{[r^2 + r^2 \tan^2 n\theta]^{3/2}}{r^2 + 2r^2 \tan^2 n\theta + nr^2 \sec^2 n\theta - r^2 \tan^2 n\theta}$$

$$= \frac{(r^2)^{3/2} (1 + \tan^2 n\theta)^{3/2}}{r^2 (1 + \tan^2 n\theta) + nr^2 \sec^2 n\theta}$$

$$= \frac{r^3 (\sec^2 n\theta)^{3/2}}{r^2 \sec^2 n\theta (1+n)} = \frac{r}{\cos n\theta (1+n)}$$

$$= \frac{a^n}{r^{n-1} (1+n)}$$ //

(20) $r = a(1-\cos\theta)$

sol
Given $r = a(1-\cos\theta)$ — ①

$r_1 = a(\sin\theta)$ — ②

$r_2 = a(\cos\theta)$ — ③

$\rho = \dfrac{[r_1^2+r_2^2]^{3/2}}{r_1^2+2r_1^2-rr_2}$

$$\rho = \frac{[a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a(1-\cos\theta)\cdot a\cos\theta}$$

$$= \frac{[a^2(1-\cos^2\theta-2\cos\theta) + a^2\sin^2\theta]^{3/2}}{a^2(1+\cos^2\theta-2\cos\theta)+2a^2\sin^2\theta - a^2(\cos\theta-\cos^2\theta)}$$

$$= \frac{[a^2]^{3/2}[1+(\cos^2\theta+\sin^2\theta)-2\cos\theta]^{3/2}}{a^2[1+\cos^2\theta-2\cos\theta+2\sin^2\theta-\cos\theta+\cos^2\theta]}$$

$$= \frac{a^3[1+1-2\cos\theta]^{3/2}}{a^2[1+2\cos^2\theta+2\sin^2\theta-3\cos\theta]}$$

$$= \frac{\cancel{a^3}[2-2\cos\theta]^{3/2}}{\cancel{a^2}(1+2-3\cos\theta)} = \frac{[2(1-\cos\theta)]^{3/2}}{(3-3\cos\theta)}$$

$$= \frac{2^{3/2}a(1-\cos\theta)^{3/2}}{3(1-\cos\theta)} = \frac{2^1\times 2^{1/2}a(1-\cos\theta)^{3/2}}{3(1-\cos\theta)^1}$$

$$= \frac{2\sqrt{2}}{3}\cdot(1-\cos\theta)^{3/2-1} = \frac{2a\sqrt{2}}{3}(1-\cos\theta)^{1/2}$$

$$= \frac{2a\sqrt{2}}{3}\sqrt{1-\cos\theta}$$

Put $\theta = \pi/2$ $= \dfrac{2\sqrt{2}}{3}\sqrt{1-0} = \dfrac{2a\sqrt{2}}{3}$ //

Circle of curvature

(1) Find the eq of circle of curvature of the curve

$y = \sin x - \sin 2x$ at $x = \pi/2$

Sol: Given $y = 4\sin x - \sin 2x$ — (1)

$x = \pi/2, \quad y = 4$

$y_1 = 4\cos x - 2\cos 2x$ — (2)

At $x = \pi/2$, $y_1 = 2$

$y_2 = -4\sin x - 4\sin 2x$ — (3)

At $x = \pi/2$, $y_2 = -4$

$\therefore$ R.O.C $\rho = \dfrac{[1+y_1^2]^{3/2}}{y_2} = \dfrac{(1+4)^{3/2}}{-4} =$

$\bar{x} = x - \dfrac{y_1(1+y_1^2)}{y_2} = \dfrac{\pi}{2} - \dfrac{\cancel{2}(1+4)}{-\cancel{4}_2} = \dfrac{\pi+5}{2}$

$\bar{y} = y + \dfrac{1+y_1^2}{y_2} = 4 + \dfrac{1+4}{-4} = \dfrac{11}{4}$

$\therefore$ Eq of circle of curvature

$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

$\left[x - \left(\dfrac{\pi+5}{2}\right)\right]^2 + (y - 11/4)^2 = \dfrac{125}{16}$

(3) S.T eq of circle of curvature of the curve $x+y = ax^2+by^2+cx^3$ at $(0,0)$ is $(x^2+y^2)(a+b) = 2x+2y$

sol Given $x+y = ax^2+by^2+cx^3$

$1+y_1 = 2ax + 2byy_1 + 3cx^2$

$y_1[1-2by] = 2ax+3cx^2-1$

$$y_1 = \frac{2ax+3cx^2-1}{1-2by} \quad -(2)$$

At $(0,0)$ $y_1 = -1$

diff (2) w.r.to x

$$y_2 = \frac{(1-2by)(2a+6cx)-(2ax+3cx^2-1)(-2by_1)}{(1-2by)^2}$$

$y_2 =$ at $(0,0)$ $y_2 = \frac{2a+2b}{1} = 2(a+b)$

$$R.O.C \cdot \rho = \frac{[1+y_1^2]^{3/2}}{y_2} = \frac{[1+1]^{3/2}}{2(a+b)} = \frac{\cancel{2}\sqrt{2}}{\cancel{2}(a+b)}$$

$$= \frac{\sqrt{2}}{a+b}$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = 0 - \frac{(-1)(1+1)}{2(a+b)}$$

$$= \frac{\cancel{2}}{\cancel{2}(a+b)} = \frac{1}{a+b}$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = 0 + \frac{1+1}{2(a+b)} = \frac{1}{a+b}$$

Eq of circle of curvature

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$\left(x - \frac{1}{a+b}\right)^2 + \left(y - \frac{1}{a+b}\right)^2 = \frac{2}{(a+b)^2}$$

$$\left[\frac{(a+b)x - 1}{a+b}\right]^2 + \left[\frac{(a+b)y - 1}{a+b}\right] = \frac{2}{(a+b)^2}$$

$$(a+b)^2x^2 + 1 - 2x(a+b)(a+b)^2y^2 + 1 - 2y(a+b) = 2$$

$$(a+b)^2[x^2+y^2] - 2(a+b)(x+y) + \cancel{2} = \cancel{2}$$

$$(a+b)\left[(a+b)(x^2+y^2) - 2[x+y]\right] = 0$$

$$\Rightarrow (a+b)(x^2+y^2) = 2x+2y$$

$$\therefore (x^2+y^2)(a+b) = 2x+2y$$

(3) S.T the eq of circle of curvature of the curve $y = mx + \frac{x^2}{a}$ at $(0,0)$ is

$$x^2+y^2 = a(1+m^2)(y-mx)$$

sol

$$y = mx + \frac{x^2}{a}$$

diff w.r.to 'x'

$$y_1 = m + \frac{2x}{a} \quad -②$$

y_1 at $(0,0)$ i.e $y_2 = \frac{2}{a}$

ROC $$\rho = \frac{[1+m^2]^{3/2}}{2/a}$$

$$= a/_2 [1+m^2]^{3/2}$$

coc

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= 0 - \frac{m(1+m^2)}{2/a}$$

$$= \frac{-ma}{2}(1+m^2)$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= 0 + \frac{1+m^2}{2/a}$$

$$= a/_2 (1+m^2)$$

eq of O^{le} of curvature

$$(x-\bar{x})^2 + (y-\bar{y})^2 = r^2$$

$$\left[x + \frac{am}{2}(1+m^2)\right]^2 + \left[y - \frac{a}{2}(1+m^2)\right]^2 = \frac{a^2}{4}(1+m^2)^3$$

$$x^2 + amx(1+m^2) + \frac{a^2m^2}{4}(1+m^2)^2 +$$

$$y^2 - ay(1+m^2) + \frac{a^2}{4}(1+m^2)^2 = \frac{a^2}{4}(1+m^2)^3$$

$$x^2+y^2 = -amx(1+m^2) - \frac{a^2m^2}{4}(1+m^2)^2$$

$$+ ay(1+m^2) - \frac{a^2}{4}(1+m^2)^2 + \frac{a^2}{4}(1+m^2)^3$$

$$x^2+y^2 = a(1+m^2)\left[-mx - \frac{am^2}{4}(1+m^2)\right.$$

$$\left. + y - \frac{a}{4}(1+m^2) + \frac{a}{4}(1+m^2)^2\right]$$

$$= a(1+m^2)\left[y - mx + \frac{a}{4}(1+m^2)\right.$$

$$\left.\left[1+m^2-m^2-1\right]\right]$$

$$= a(1+m^2)(y-mx)$$

$$\therefore x^2+y^2 = a(1+m^2)(y-mx)$$

(6) S.T eq of circle of curvature of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at $[a/4, a/4]$ is

$$8(x^2+y^2)-12a(x+y)+5a^2=0$$

Given curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ — ①

diff ① w.r.to 'x'

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0 \;=\; \frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}\cdot\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=\frac{-\sqrt{y}}{\sqrt{x}}$$

at $(a/4, a/4)$ $\therefore$ $y_1=\dfrac{-\sqrt{a/4}}{\sqrt{a/4}}=-1$

diff again weget

$$\frac{d^2y}{dx^2}=-\left[\sqrt{x}\;\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}-\sqrt{y}\;\frac{1}{2\sqrt{x}}\right]$$

At $(a/4, a/4)$ = $y_2=\left[\sqrt{a/4}\cdot\frac{1}{2\sqrt{a/4}}(-1)-\sqrt{\frac{a}{4}}\cdot\frac{1}{2\sqrt{a/4}}\right]$

$$\Rightarrow y_2=-\frac{\left[-\frac{1}{2}-\frac{1}{2}\right]}{a/4}=-\frac{(-1)}{a/4}=\frac{4}{a}$$

Now

coc $(\bar{x},\bar{y})$ is given by

$$\bar{x}=x-\frac{y_1(1+y_1^2)}{y_2}=\frac{a}{4}-\frac{(-1)(1+1)}{4/a}$$

$$=\frac{a}{4}+\frac{2a}{4}=\frac{3a}{4}$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = \frac{a}{4} + \frac{1+1}{4/a} = \frac{a}{4} + \frac{2a}{4} = 3$$

Center of curvature $= \left(\frac{3a}{4}, \frac{3a}{4}\right) \Rightarrow (\bar{x}, \bar{y})$

Now ROC $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+1)^{3/2}}{4/a} = \frac{2^{3/2} a}{4} = \frac{\cancel{2}\sqrt{2}\, a}{\cancel{4}_2} = \frac{\cancel{\sqrt{2}}\, a}{\cancel{\sqrt{2}}\cdot\sqrt{2}} = \frac{a}{\sqrt{2}}$$

Hence the eq of coc is $(x-\bar{x})^2 + (y-\bar{y}_1)^2 = \rho^2$

$$(x - 3a/4)^2 + (y - 3a/4)^2 = \frac{a^2}{2}$$

$$\Rightarrow \left(\frac{4x-3a}{4}\right)^2 + \left(\frac{4y-3a}{4}\right)^2 = \frac{a^2}{2}$$

$$\Rightarrow 16x^2 + 9a^2 - 24ax + 16y^2 + 9a^2 - 24ay = \frac{\overset{8}{\cancel{16}}a^2}{\cancel{2}}$$

$$\Rightarrow 16(x^2+y^2) - 24a(x+y) + 18a^2 - 8a^2 = 0$$

$$\Rightarrow 16(x^2+y^2) - 24a(x+y) + 10a^2 = 0$$

$$\Rightarrow 8(x^2+y^2) - 12a(x+y) + 5a^2 = 0$$

$$\therefore 8(x^2+y^2) - 12a(x+y) + 5a^2 = 0$$

→ Evolute problem:-

① S.T the evolute of parabola $y^2 = 4ax$ is $27ay^2 = 4(x-2a)^3$

Sol Given $y^2 = 4ax$ — ①

Diff ① w.r to x we get $2yy_1 = 4a$

$y_1 = \frac{2a}{y}$ — ② diff ② w.r to x

$y_2 = \frac{-2a}{y^2}(y_1) = -\frac{4a^2}{y^3}$

$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = x - \frac{\cancel{2}a}{y}\left[1+\frac{4a^2}{y^2}\right]\left(\frac{-y^3}{\cancel{4}a^2}\right)$

$\bar{x} = x + \frac{y^2}{2a}\left[\frac{y^2+4a^2}{y^2}\right]$ $[\because y^2 = 4ax]$

$x = x + \frac{1}{2a}\left[4ax + 4a^2\right]$

$x = x + \frac{2\cancel{a}}{2\cancel{a}}\left[2x + 2a\right]$

$\bar{x} = 3x + 2a \Rightarrow x = \frac{x-2a}{3}$ — ④

$\bar{y} = y + \frac{1+y_1^2}{y_2} = y + \left[1+\frac{4a^2}{y^2}\right]\left(\frac{-y^3}{4a^2}\right)$

$\bar{y} = y - \frac{y^3}{4a^2}\left[\frac{y^2+4a^2}{y^2}\right]$ $[\because y^2 = 4ax]$

$\bar{y} = y - \frac{y}{4a^2}(4ax + 4a^2)$

$y = y - \frac{y(\cancel{4a})}{\cancel{4a}^2}(x+a)$

$$\bar{y} = \frac{\cancel{a y_1} - y_1 x - \cancel{a y_1}}{a} = \frac{-y_1 x}{a}$$

$$y = \frac{-x}{a} 2\sqrt{ax} = \frac{-2x^{3/2}}{\sqrt{a}} \quad \text{—⑤}$$

using ④ in ⑤ we get

$$\bar{y} = \frac{-2}{\sqrt{a}} \left(\frac{x-2a}{3} \right)^{3/2}$$

S. on B.S

$$\bar{y}^2 = \frac{4(\bar{x}-2a)^3}{27a}$$

$$27a\bar{y}^2 = 4(x-2a)^3$$

② S.T the evolute of cycloid $x = a(t-\sin t)$ $y = a(1-\cos t)$ from the another

sol Given $x = a(t-\sin t)$; $y = a(1-\cos t)$

$$\frac{dx}{dt} = a(1-\cos t) \quad ; \quad \frac{dy}{dt} = a\sin t$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cancel{a}\sin t}{\cancel{a}(1-\cos t)} = \frac{\cancel{2}\sin t/2 \cos t/2}{\cancel{2}\sin^2 t/2}$$

$$y_1 = \cot t/2 \quad \text{—②}$$

diff ② w.r.to x

$$y_2 = -\operatorname{cosec}^2 t/2 \left(\frac{1}{2} dt/dx \right)$$

$$y_2 = \frac{-\operatorname{cosec}^2 t/_2}{2a(1-\cos t)} = -\frac{\operatorname{cosec}^2 t/_2}{4a\sin^2 t/_2}$$

$$y_2 = \frac{-1}{4a\sin^4 t/_2}$$

$$\therefore \bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$x = x + \cot t/_2 \,(1+\cot^2 t/_2)\,(4a\sin^4 t/_2)$$

$$\bar{x} = x + \frac{\cos t/_2}{\sin t/_2}\,(\operatorname{cosec}^2 t/_2)\,(4a\sin^{\cancel{4}} t/_2)$$

$$\bar{x} = a(t-\sin t) + 4a\sin t/_2 \cos t/_2$$

$$\bar{x} = at - a\sin t + 2a\sin t$$

$$\bar{x} = at + \sin t = a(t+\sin t)$$

$$\bar{y} = y_1 + \frac{1+y_1^2}{y_2} = y - \left[1 - \cot^2 t/_2\right]\left[4a\sin^4 t/_2\right]$$

$$\bar{y} = y - (\operatorname{cosec}^2 t/_2)\; 4a\sin^{\cancel{4}\,2} t/_2$$

$$\bar{y} = a(1-\cos t) - \cancel{4}a\left(\frac{1-\cos t}{\cancel{2}}\right)$$

$$\bar{y} = a - a\cos t - 2a + 2a\cos t$$

$$\bar{y} = -a + a\cos t = -a(1-\cos t)$$

Hence a evolute of cycloid from the another.

③ Show that the evolute of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$

sol: $x=a\cos\theta$, $y=b\sin\theta$

$\frac{dx}{d\theta}=-a\sin\theta$ $\qquad \frac{dy}{d\theta}=b\cos\theta$

$\frac{dy/d\theta}{dx/d\theta}=\frac{b\cos\theta}{-a\sin\theta} \Rightarrow y_1=-\frac{b}{a}\cot\theta$ — ②

diff ② w.r.to 'x'

$y_2=-\frac{b}{a}(-\operatorname{cosec}^2\theta)\frac{d\theta}{dx}$

$y_2=\frac{\frac{b}{a}\operatorname{cosec}^2\theta}{-a\sin\theta}=-\frac{b}{a}\operatorname{cosec}^3\theta$ — ③

$\bar{x}=x-\frac{y_1(1+y_1^2)}{y_2}$

$\bar{x}=a\cos\theta-\frac{\left(-\frac{b}{a}\cot\theta\right)\left[1+\frac{b^2}{a^2}\cot^2\theta\right]}{-\frac{b}{a^2}\operatorname{cosec}^3\theta}$

$\bar{x}=a\cos\theta-\left(\frac{\cancel{b}}{a}\frac{\cos\theta}{\sin\theta}\right)\left(\frac{a^{\cancel{2}}}{\cancel{b}}\sin^{\cancel{3}2}\theta\right)\left[\frac{a^2\sin^2\theta+b^2\cos^2\theta}{a^2\sin^2\theta}\right]$

$\bar{x}=a\cos\theta-\frac{\cos\theta}{a}(a^2\sin^2\theta+b^2\cos^2\theta)$

$a\bar{x}=a^2\cos\theta-a^2\cos\theta\sin^2\theta-b^2\cos^3\theta$

$a\bar{x}=a^2\cos^2\theta-a^2\cos\theta+a^2\cos^3\theta-b^2\cos^3\theta$

$a\bar{x}=\cos^3\theta(a^2-b^2)$

$$(a\bar{x})^{2/3} = \cos^2\theta\,(a^2-b^2)^{2/3} \quad \text{---} \; 4$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$\bar{y} = b\sin\theta + \left[1 + \frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right]\left[\frac{-a^2}{b}\sin^3\theta\right]$$

$$\bar{y} = b\sin\theta - \frac{\cancel{a^2}}{b}\sin^{\cancel{3}\,1}\theta\left[\frac{a^2\sin^2\theta + b^2\cos^2\theta}{\cancel{a^2}\,\cancel{\sin^2\theta}}\right]$$

$$b\bar{y} = b^2\sin^2\theta - a^2\sin^3\theta - b^2\sin\theta\cos^2\theta$$

$$b\bar{y} = b^2\sin\theta - a^2\sin^3\theta - b^2\sin\theta\,(1-\sin^2\theta)$$

$$b\bar{y} = b^2\cancel{\sin\theta} - a^2\sin^3\theta - b^2\cancel{\sin\theta} + b^2\sin^3\theta$$

$$b\bar{y} = -a^2\sin^3\theta + b^2\sin^3\theta$$

$$b\bar{y} = -\sin^3\theta\,(a^2-b^2)$$

$$b\bar{y} = -\sin^2\theta\,(a^2-b^2)^{2/3} \quad \text{---} \; 5$$

Adding 4 and 5

$$(a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2-b^2)^{2/3} \quad (1)$$

$\therefore$ locus is

$$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}.$$

(4) S.T evolute of the hyperbola $2xy = a^2$ is $(x+y)^{2/3} - (x-y)^{2/3} = 2a^{2/3}$

sol Given $2xy = a^2$

$$y = \frac{a^2}{2x} \quad -(1) \qquad \text{diff (1) w.r. to 'x'}$$

$$y_1 = \frac{-a^2}{2x^2} \quad -(2) \qquad \text{diff (2) w.r. to 'x'}$$

$$y_2 = \frac{a^2}{x^3} \quad -(3)$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$\bar{x} = x - \left(\frac{-\cancel{a^2}}{2\cancel{x^2}}\right)\left[1 + \frac{a^4}{4x^4}\right] \cdot \frac{\cancel{x^3}}{\cancel{a^2}}$$

$$\bar{x} = x + \frac{\cancel{x}}{2}\left[\frac{4x^4 + a^4}{4x^{\cancel{4}}}\right]$$

$$\bar{x} = x + \frac{1}{2}\left[\frac{4x^4 + 4x^2y^2}{4x^3}\right] \qquad [2xy = a^2]$$

$$\bar{x} = x + \frac{\cancel{4}x^2}{2}\left[\frac{x^2+y^2}{\cancel{4}x^{\cancel{3}}}\right]$$

$$\bar{x} = \frac{2x^2 + x^2 + y^2}{2x} = \frac{3x^2+y^2}{2x} \quad -(4)$$

III)

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$\bar{y} = y + \frac{1+\frac{a^4}{4x^4}}{\frac{a^2}{x^3}} = \left[\frac{4y^4+a^4}{4x^4}\right]\cdot\frac{x^3}{a^2}$$

$$\bar{y} = \frac{a^2}{2x} + \left[\frac{4x^4+a^4}{4x^4}\right]\cdot\frac{x^{3}}{a^2}$$

$$\bar{y} = \frac{x}{2}\left[\frac{4x^4+a^4}{4x^{4}}\right] \qquad [2xy = a^2]$$

$$\bar{y} = \frac{1}{2}\left[\frac{4x^4+4x^2y^2}{4x^2}\right] = \frac{x^2+3y^2}{2y} \quad \text{(check)}$$

$$\bar{x}+\bar{y} = \frac{3x^2+y^2}{2x} + \frac{x^2+3y^2}{2y} = x+y = \frac{3x^2y+y^3+x^3+3y^2}{2xy}$$

$$\bar{x}+\bar{y} = \frac{(x+y)^3}{2xy} \quad (\bar{x}+\bar{y})^{2/3} = \frac{(x+y)^2}{(2xy)^{2/3}} \quad — ⑥$$

$$\bar{x}-\bar{y} = \frac{(x-y)^3}{2xy} = (\bar{x}-\bar{y})^{2/3} = \frac{(x-y)^2}{(2xy)^{2/3}}$$

$$(\bar{x}+\bar{y})^{2/3} - (\bar{x}-\bar{y})^{2/3} = \frac{(x+y)^2}{(2xy)^{2/3}} - \frac{(x-y)^2}{(2xy)^{2/3}}$$

$$= \frac{(x+y)^2-(x-y)^2}{(2xy)^{2/3}} = \frac{4xy}{(a^2)^{2/3}}$$

$$= \frac{2(2xy)}{a^{4/3}} = 2a^{2-4/3} = 2a^{2/3}$$

locus is

$$(x+y)^{2/3} - (x-y)^{2/3} = 2a^{2/3}$$

⑤ S.T evolute of asteroide $x = a\cos^3\theta$, $y = a\sin^3\theta$ is

$(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$

sol Given $x = a\cos^3\theta$; $y = a\sin^3\theta$

$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$; $\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = y_1 = -\tan\theta \quad — ②$$

diff ② w.r.to x

$$y_2 = -\sec^2\theta \cdot \frac{d\theta}{dx}$$

$$y_2 = \frac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta} = \frac{1}{3a\sin\theta\cos^4\theta} \quad — ③$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = x + \tan\theta(1+\tan^2\theta)\, 3a\sin\theta\cos^4\theta$$

$$\bar{x} = a\cos^3\theta + \frac{\sin\theta}{\cos\theta}(\sec^2\theta)\, 3a\sin\theta\cos^4\theta$$

$$\bar{x} = a\cos^3\theta + 3a\sin^2\theta\cos\theta \quad — ④$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = y + (1+\tan^2\theta)\, 3a\sin\theta\cos^4\theta$$

$$\bar{y} = a\sin^3\theta + \sec^2\theta\,(3a\sin\theta\cos^4\theta)$$

$$\bar{y} = a\sin^3\theta + 3a\sin\theta\cos^2\theta \quad — ⑤$$

Now

$\bar{x}+\bar{y} = a(\sin\theta+\cos\theta)^3$

$(\bar{x}+\bar{y})^{2/3} = a^{2/3}(\sin\theta+\cos\theta)^2$ — ⑥

$(\bar{x}+y)^{2/3} + (x-y)^{2/3} = a^{2/3}\, 2(1)$

Hence locus is

$$(\bar{x}+\bar{y})^{2/3} + (x-y)^{2/3} = 2a^{2/3}$$

(5) (ii) S.T evolute of parabola $x = 2at$

$y = at^2$ is $27ax^2 = 4(y-2a)^3$

Sol Given $x = 2at$; $y = at^2$ $\Rightarrow \frac{dx}{dt} = 2a$, $\frac{dy}{dt} = 2at$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2at}{2a}$$

$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = x - t(1+t^2)2a$

$\bar{x} = \cancel{2at} - \cancel{2at} - 2at^3 \Rightarrow \frac{\bar{x}}{2a} = -t^3$

SBS $= \frac{x^2}{4a^2} = t^6$ — ④

$\bar{y} = y + \frac{1+y_1^2}{y_2} = at^2 + (1+t^2)2a$

$\bar{y} = at^2 + 2a + 2at^2 = \bar{y} - 2a = 3at^2$

$\frac{y-2a}{3a} = t^2 \Rightarrow \frac{(\bar{y}-2a)^3}{27a^3} = t^6$ — ⑤

from ④ and ⑤ $\frac{\bar{x}^2}{4a^2} = \frac{(y-2a)^3}{27a^5}$.

$27a\bar{x}^2 = 4(\bar{y}-2a)^3$

$27ax^2 = 4(y-2a)^3$ is the locus

(ii) S.T the evolute of parabola $x^2 = 4ay$ is $27ax^2 = 4(y-2a)^3$

sol Given $x^2 = 4ay$ —①

diff ① w.r.to x

$2x = 4ay_1 \Rightarrow y_1 = \frac{x}{2a} \Rightarrow y_2 = \frac{1}{2a}$

$\bar{x} = x - y_1\frac{(1+y_1^2)}{y_2} = x - \frac{x}{2a}\left(1+\frac{x^2}{4a^2}\right)2a$

$\bar{x} = \cancel{x} - \cancel{x} - \frac{x^3}{4a^2} = x = \frac{x^3}{4a^2}$

$x^2 = \frac{x^6}{16a^4} \Rightarrow a\bar{x}^2 = \frac{x^6}{16a^3}$ —②

$\bar{y} = y + \frac{1+y_1^2}{y_2} = y + \left(1+\frac{x^2}{4a^2}\right)2a$

$\bar{y} = y + 2a + \frac{2ax^2}{4a^2} = \bar{y} - 2a = y + \frac{x^2}{2a}$

$\bar{y} - 2a = \frac{x^2}{4a} + \frac{x^2}{2a} = \frac{3x^2}{4a}$

$\frac{\bar{y}-2a}{3} = \frac{x^2}{4a} \Rightarrow \frac{(y-2a)^3}{27} = \frac{x^6}{64a^3}$

$\frac{4(y-2a)^3}{27} = \frac{x^6}{16a^3}$ —③ from ② and ③

$$\frac{4(y-2a)^3}{27} = a\bar{x}^2 \Rightarrow 27a\bar{x}^2 = 4(y-2a)^3$$

locus is $27ax^2 = 4(y-2a)^3$

---

(7) $xy = 1$ P.T $(x+y)^{2/3} - (x-y)^{2/3} = 4^{2/3}$

diff w.r.to x (uv)

$= xy_1 + y(1) = 0 \Rightarrow y_1 = -\frac{y}{x}$ $y_2 = -\left[\frac{xy_1 - y(1)}{x^2}\right]$

$$y_2 = -\left[\frac{x(-y/x) - y}{x^2}\right] = \frac{2y}{x^2}$$

$$\bar{x} = x + \frac{y}{x}\left[1 + \frac{y^2}{x^2}\right]\frac{x^2}{2y} = x + \frac{x}{2} + \frac{y^2}{2x}$$

$$\bar{x} = \frac{3x^2 + y^2}{2x}$$

$$\bar{y} = y + \left[1 + \frac{y^2}{x^2}\right]\frac{x^2}{2y} = y + \frac{x^2}{2y} + \frac{y}{2} = \frac{3y^2 + x^2}{2y}$$

$$(\bar{x} + \bar{y})^{2/3} - (x-y)^{2/3} = \frac{4xy}{2^{2/3}} = \frac{4}{2^{2/3}}$$

$$= 2^{2-2/3} = 2^{4/3}$$

$\therefore$ locus is

$$(x+y)^{2/3} - (x-y)^{2/3} = 4^{2/3}$$

Note

① if the eq of the given family of curve attains the quadratic form ie $a\alpha^2 + b\alpha + c = 0$ ~~where~~ which is quadratic in α then req envelope is given by its discriminant ie $b^2 - 4ac = 0$

② Evolute of the curve is the envelope of its normal.

<u>One parameter family problems</u> :-

→ Find the envelope of following family of curve.

① $y = ax + a^2$ (a)

sol Given $y = ax + a^2$ — ①

$a^2 + ax - y = 0$ which is quadratic in 'a'

$b^2 - 4ac = 0 \Rightarrow x^2 + 4y = 0$ is the req envelope

② $y = mx + \frac{a}{m}$ (m)

sol Given $my = m^2x + a$

$m^2x - my + a = 0$ which is quadratic.

$b^2 - 4ac = 0 \Rightarrow y^2 - 4ax = 0$ is the req envelope

③ $y = mx + \sqrt{a^2m^2 + b^2}$

sol Given $y = mx + \sqrt{a^2m^2 + b^2}$ — ①

$y - mx = \sqrt{a^2m^2 + b^2}$ S on B S

$y^2 + m^2x^2 - 2xym = a^2m^2 + b^2$

$m^2(x^2-a^2) - 2xym + (y^2-b^2) = 0$

which is quadratic in 'm'

$\therefore\ b^2 - 4ac = 0 \Rightarrow 4x^2y^2 - 4(x^2-a^2)(y^2-b^2)$

$= 4x^2y^2 - 4x^2y^2 + 4b^2x^2 + 4a^2y^2 - 4a^2b^2$

$= 4(b^2x^2 + a^2y^2 - a^2b^2) = 0$

$= b^2x^2 + a^2y^2 = a^2b^2 \qquad = \frac{b^2x^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

---

(4) $x^2+y^2 - 2xp + \frac{p^2}{2} = 0 \qquad (p)$

sol Given $x^2+y^2 - 2xp + \frac{p^2}{2} = 0 \quad -①$

$2(x^2+y^2) - 2xp + p^2 = 0$

which is quadratic in 'p'

$b^2 - 4ac = 0 \Rightarrow 4x^2 - 8(x^2+y^2) = 0$

is the req envelope.

---

(5) $y = (x-p)^2 \qquad (p)$

sol Given $y = (x-p)^2 \quad -①$

$y = x^2+p^2-2xp \Rightarrow p^2 - 2xp + (x^2-y) = 0$

$\therefore\ b^2 - 4ac = 0 \Rightarrow 4x^2 - 4(x^2-y) = 0$

$y = 0$ is the req envelope

(6) $y = 3cx - c^3$ (c)

Sol: Given $y = 3cx - c^3$ — (1) diff w.r.to 'c'

using (2) in (1) we get

$y = 3x^{3/2} - x^{3/2}$

$y = 2x^{3/2} \Rightarrow y^2 = 4x^3$ is the req envelope

(7) $x\cos\alpha + y\sin\alpha = p$ (α)

Sol: Given $x\cos\alpha + y\sin\alpha = p$ — (1)

diff w.r.to 'x'

$-x\sin\alpha + y\cos\alpha = 0$ — (2)

sq and add (1) and (2)

$x^2(1) + y^2(1) = p^2$

$\therefore x^2 + y^2 = p^2$ is the req envelope

(8) $\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$ (α)

Sol: Given $\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$ — (1)

diff (1) w.r to α

$-\frac{x}{a}\sin\alpha + \frac{y}{b}\cos\alpha = 0$ — (2)

Sq and add (1) and (2) we get

$\frac{x^2}{a^2}(1) + \frac{y^2}{b^2}(1) = 1 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(9) $x\tan\alpha + y\sec\alpha = 5$ (α)

sol Given $x\tan\alpha + y\sec\alpha = 5$ — ①

diff ① w.r.to α

$x\sec^2\alpha + y\sec\alpha\tan\alpha = 0$

$\sec\alpha\,[x\sec\alpha + y\tan\alpha] = 0$

$x\sec\alpha + y\tan\alpha = 0$ — ②

sq and subtract ① and ②

$x^2(-1) + y^2(1) = 25$

$y^2 - x^2 = 25$

is the req envelope

(10) $x^2+y^2 - 2ax\cos\alpha - 2ay\sin\alpha = c^2$ (α)

sol Given $x^2+y^2 - c^2 = 2ax\cos\alpha + 2ay\sin\alpha$

diff ① w.r.to α.

$0 = -2ax\sin\alpha + 2ay\cos\alpha$ ②

sq and add ① and ②

$(x^2+y^2-c^2)^2 = 4a^2x^2(1) + 4a^2y^2(1)$

$(x^2+y^2-c^2)^2 = 4a^2(x^2+y^2)$ is the

req envelope

Two parameter family :-

① $\frac{x}{a}+\frac{y}{b}=1$ where $ab=c^2$ (a,b)

sol Given $\frac{x}{a}+\frac{y}{b}=1$ — ① and $ab=c^2 \Rightarrow b=\frac{c^2}{a}$ — ②

using ② in ① we get

$\frac{x}{a}+\frac{y}{c^2/a}=1 \Rightarrow \frac{x}{a}+\frac{ay}{c^2}=1$

$xc^2+a^2y=ac^2 \Rightarrow a^2y-ac^2+xc^2=0$

which is quadratic in 'a'

$\therefore b^2-4ac=0 \Rightarrow c^4-4xyc^2=0$

$c^2-4xy=0$ is the req envelope

② $\frac{x}{a}+\frac{y}{b}=1$ where $a+b=c$ (a,b)

sol Given $\frac{x}{a}+\frac{y}{b}=1$ — ①

Also $a+b=c \Rightarrow b=c-a$ — ②

using ② in ① we get $\frac{x}{a}+\frac{y}{c-a}=1$ — ③

diff ③ w.r.to a we get

$-\frac{x}{a^2}-\frac{y(-1)}{(c-a)^2}=0 \Rightarrow \frac{x}{a^2}=\frac{y}{(c-a)^2}$

$\frac{(c-a)^2}{a^2}=\frac{y}{x} \Rightarrow \frac{c-a}{a}=\sqrt{\frac{y}{x}}$ — ④

$c-a=a\sqrt{\frac{y}{x}}$ — ⑤

Also from ④ $\frac{c}{a} = \sqrt{\frac{y}{x}} + 1$

$$c = a\left(\frac{\sqrt{y}+\sqrt{x}}{\sqrt{x}}\right) = a = \frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}} \quad — ⑥$$

using ⑥ in ⑤, we get

$$c-a = \frac{c\sqrt{y}}{\sqrt{x}+\sqrt{y}} \quad — ⑦$$

Using ⑥ in ⑦ in ③ we get

$$\frac{\frac{x\sqrt{x}}{c\sqrt{x}}}{\sqrt{x}+\sqrt{y}} + \frac{\frac{y\sqrt{y}}{c\sqrt{y}}}{\sqrt{x}+\sqrt{y}} = 1$$

$$\sqrt{x}(\sqrt{x}+\sqrt{y}) + \sqrt{y}(\sqrt{x}+\sqrt{y}) = c$$

$$(\sqrt{x}+\sqrt{y})\ t(\sqrt{x}+\sqrt{y}) = c$$

$$\Rightarrow (\sqrt{x}+\sqrt{y})^2 = c$$

$$\sqrt{x}+\sqrt{y} = \sqrt{c}$$

Unit-3 Multi-variable calculus [differentiation]

Funtions of Several Real Variables:

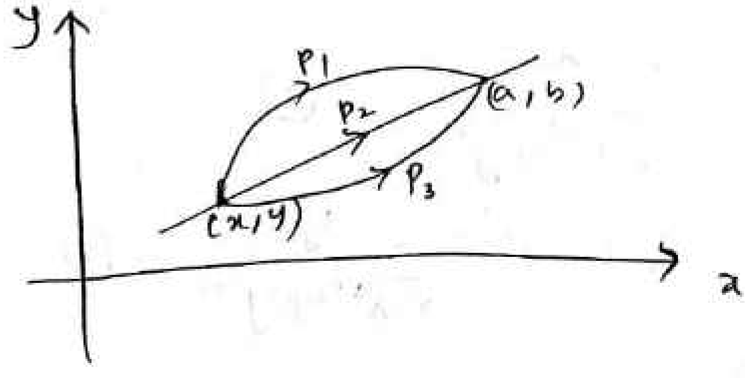
Let x, y be two independent variables and z be a dependant variable i.e, Z depends on x and y. It is denoted as $z = f(x,y)$. If the no. of I.v's are more than one then we say it is a function of S.v's.

Eg:① Volume of rectangular parallelopiped is a function of 3 variables length, breadth and height i.e, $V = lbh$. It is a function of 3 variable

Eg:② Area of Δ^{le} = $A = \frac{1}{2}xy$

x = base, y = altitude. It is a function of two variables.

Let the function $f(x,y)$ be define in a region R. Suppose (x,y) be a marking pointed (a,b) be a fixed point in the region R of the xy plane. The point (x,y) may approach (a,b) along different paths i.e, P_1, P_2, P_3



imit and continuity of function of 2 variables.

imit :- Let (i) $f(x,y)$ be defined in a region

(ii) (x,y) tend to (a,b) along a path

ii) $\epsilon > 0$ be given.

If $\exists\ \delta > 0 \ni |f(x,y) - l| < \epsilon$

$\forall\ |x-a| < \delta,\ |y-b| < \delta$ then

$f(x,y)$ is said to tend to l

$\therefore \displaystyle\lim_{\substack{x\to a\\ y\to b}} f(x,y) = l \quad \therefore \lim_{(x,y)\to(a,b)} f(x,y) = l$

Continuity: let (i) $\displaystyle\lim_{(x,y)\to(a,b)} f(x)$ exist

i) $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ then $f(x,y)$ is said to be continuous ~~in~~ at (a,b)

Note: If $f(x,y)$ is said to be C at every point of a region R, then it is said to be C in R

Note: Let $f(x,y)$ and $g(x,y)$ be C at (a,b) then $f \pm g$, fg and $\frac{f}{g}$ $(g \neq 0)$ are all C at (a,b)

P.D at origin :-

$$f_x = \left(\frac{\partial f}{\partial x}\right)_{(0,0)} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x}$$

$$f_y = \left(\frac{\partial f}{\partial y}\right)_{(0,0)} = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y}$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)_{(0,0)} = \lim_{\delta x \to 0} \frac{f_y(\delta x, 0) - f_y(0,0)}{\delta x}$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)_{(0,0)} = \underset{\delta y \to 0}{Lt} \frac{f_x(0,\delta y) - f_x(0,0)}{\delta y}$$

Partial Derivatives:-

P.D of a function of 2 variables:-

Let $u = f(x,y)$ be a function of two independent variables x and y. The derivative of u w.r.t 'x' if it ~~constant~~ exist keeping y as constant is called the P.D of u w.r.t x and is derivated by $\frac{\partial u}{\partial x}$ x $\frac{\partial f}{\partial x}$. It is also denoted by u_x & f_x

$$\therefore \frac{\partial u}{\partial x} = \underset{\delta x \to 0}{Lt} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

|||ly the P.D of 'u' w.r.t y keeping 'x' as constant is dendted by $\frac{\partial u}{\partial y}$ x $\frac{\partial f}{\partial y}$

$$\therefore \frac{\partial u}{\partial y} = \underset{\delta y \to 0}{Lt} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

Higher order P.D:- The P.D of higher order if exists, can be obtained from the P.D if ~~order~~ first order by using about def

Then $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$ is the second order Partial derivative of u w.r.t 'x' & is denoted by u_{xx} or

$$f_{xx} \text{ or } \frac{\partial^2 u}{\partial x^2} \text{ or } \frac{\partial f}{\partial x^2}$$

$|||^{ly}\ \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} = f_{yy}$

$\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial y}\right) = \frac{\partial^2 x}{\partial x \partial y} = f_{xy}$ & $\frac{\partial}{\partial y}\left(\frac{\partial x}{\partial x}\right) = \frac{\partial^2 x}{\partial y \partial x} = f_{yx}$

$$f_x = \underset{\delta x \to 0}{Lt} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

$$f_{xx} = \underset{\delta x \to 0}{Lt} \frac{f_x(x+\delta x, y) - f_x(x,y)}{\delta x}$$

$$f_{xy} = \underset{\delta x \to 0}{Lt} \frac{f_y(x+\delta x, y) - f_y(x,y)}{\delta x}$$

$$f_y = \underset{\delta y \to 0}{Lt} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

$$f_{yy} = \underset{\delta y \to 0}{Lt} \frac{f_y(x, y+\delta y) - f_y(x,y)}{\delta y}$$

$$f_{yx} = \underset{\delta y \to 0}{Lt} \frac{f_x(x, y+\delta y) - f_x(x,y)}{\delta y}$$

At $(0,0)$

$$f_x = \underset{\delta x \to 0}{Lt} \frac{f(\delta x, 0) - f(0,0)}{\delta x}$$

$$f_{xx} = \underset{\delta x \to 0}{Lt} \frac{f_x(\delta x, 0) - f_x(0,0)}{\delta x} \qquad f_{yx} = \underset{\delta y \to 0}{Lt} \frac{f_x(0,\delta y) - f_x(0,0)}{\delta y}$$

$$f_{xy} = \underset{\delta x \to 0}{Lt} \frac{f_y(\delta x, 0) - f_y(0,0)}{\delta x}$$

$$f_y = \underset{\delta y \to 0}{Lt} \frac{f(0, \delta y) - f(0,0)}{\delta y}$$

$$f_{yy} = \underset{\delta y \to 0}{Lt} \frac{f_y(0, \delta y) - f_y(0,0)}{\delta y}$$

Limits (Problem)

Determine whether the following limit exist or not.

① $\underset{\substack{x \to 1 \\ y \to -1}}{Lt} x^2 - y^2$

Sol (a) $\underset{\substack{x \to 1 \\ y \to -1}}{Lt} x^2 - y^2 = \underset{y \to -1}{Lt} \left\{ \underset{x \to 1}{Lt} (x^2 - y^2) \right\}$

$= \underset{y \to -1}{Lt} 1 - y^2 = 0$

(b) $\underset{\substack{y \to -1 \\ x \to 1}}{Lt} x^2 - y^2 = \underset{x \to 1}{Lt} \left\{ \underset{y \to -1}{Lt} x^2 - y^2 \right\}$

$= \underset{x \to 1}{Lt} x^2 - 1 = 0$

Hence the limit exist.

② $\underset{\substack{x \to 1 \\ y \to 1}}{Lt} x + y$

Sol Let a) $\underset{\substack{x \to 1 \\ y \to 1}}{Lt} x + y = \underset{y \to 1}{Lt} \left[\underset{x \to 1}{Lt} x + y \right]$

$= \underset{y \to 1}{Lt} 1 + y = 2$

b) $\underset{\substack{y \to 1 \\ x \to 1}}{Lt} x + y = \underset{x \to 1}{Lt} \left[\underset{y \to 1}{Lt} x + y \right]$

$= \underset{x \to 1}{Lt} x + 1 = 2$

Hence the limit exist

③ $\underset{\substack{x\to 1\\ y\to 1}}{Lt} \frac{x(y-1)}{y(x-1)}$

sol a) $\underset{\substack{x\to 1\\ y\to 1}}{Lt} \frac{x(y-1)}{y(x-1)} = \underset{y\to 1}{Lt}\left\{\underset{x\to 1}{Lt} \frac{x(y-1)}{y(x-1)}\right\}$

$= \underset{y\to 1}{Lt}\ \infty = \infty$

b) $\underset{\substack{y\to 1\\ x\to 1}}{Lt} \frac{x(y-1)}{y(x-1)} = \underset{x\to 1}{Lt}\left\{\underset{y\to 1}{Lt} \frac{x(y-1)}{y(x-1)}\right\}$

$= \underset{x\to 1}{Lt}\ 0 = 0$

Hence limit Does not exist

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④ $\underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{xy}{x^2+y^2}$

sol (a) $\underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{xy}{x^2+y^2} = \underset{y\to 0}{Lt}\left\{\underset{x\to 0}{Lt} \frac{xy}{x^2+y^2}\right\}$

$= \underset{y\to 0}{Lt}\ 0 = 0$

(b) $\underset{\substack{y\to 0\\ x\to 0}}{Lt} \frac{xy}{x^2+y^2} = \underset{x\to 0}{Lt}\left\{\underset{y\to 0}{Lt} \frac{xy}{x^2+y^2}\right\}$

$= \underset{x\to 0}{Lt}\ 0 = 0$

(c) $\underset{\substack{y\to mx\\ x\to 0}}{Lt} \frac{xy}{x^2+y^2} = \underset{x\to 0}{Lt}\left\{\underset{y\to mx}{Lt} \frac{xy}{x^2+y^2}\right\}$

$= \underset{x\to 0}{Lt} \frac{m\cancel{x^2}}{\cancel{x^2}(1+m^2)} = \frac{m}{m^2+1}$ which depends on the value of 'm'

Hence the limit is different for different 'm'.

$\therefore$ The limit does not exist

(5) $\underset{\substack{x\to 0\\ y\to 0}}{Lt} \dfrac{x^2+y^2}{x-y}$

Sol (a) $\underset{\substack{x\to 0\\ y\to 0}}{Lt} \dfrac{x^2+y^2}{x-y} = \underset{y\to 0}{Lt}\left\{\underset{x\to 0}{Lt} \dfrac{x^2+y^2}{x-y}\right\}$

$= \underset{y\to 0}{Lt} -y = 0$

(b) $\underset{\substack{y\to 0\\ x\to 0}}{Lt} \dfrac{x^2+y^2}{x-y} = \underset{x\to 0}{Lt}\left\{\underset{y\to 0}{Lt} \dfrac{x^2+y^2}{x-y}\right\}$

$= \underset{x\to 0}{Lt}\ x = 0$

(c) $\underset{\substack{y\to mx\\ x\to 0}}{Lt} \dfrac{x^2+y^2}{x-y} = \underset{x\to 0}{Lt}\left\{\underset{y\to mx}{Lt} \dfrac{x^2+y^2}{x-y}\right\}$

$= \underset{x\to 0}{Lt} \dfrac{x^{\cancel{2}}(1+m^2)}{\cancel{x}(1-m)} = 0$

(d) $\underset{\substack{y\to mx^2\\ x\to 0}}{Lt} \dfrac{x^2+y^2}{x-y} = \underset{x\to 0}{Lt}\left\{\underset{y\to mx^2}{Lt} \dfrac{x^2+y^2}{x-y}\right\}$

$= \underset{x\to 0}{Lt} \dfrac{x^{\cancel{2}}(1+m^2x^2)}{\cancel{x}(1-mx)} = 0$

(e) $\underset{\substack{y\to mx^3\\ x\to 0}}{Lt} \dfrac{x^2+y^2}{x-y} = \underset{x\to 0}{Lt}\left\{\underset{y\to mx^3}{Lt} \dfrac{x^2+y^2}{x-y}\right\}$

$= \underset{x\to 0}{Lt} \dfrac{x^{\cancel{2}}(1+m^2x^4)}{\cancel{x}(1-mx^2)} = 0$

Hence the limit exist

⑥ $\underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{xy}{\sqrt{x^2+y^2}}$

Sol (a) $\underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{xy}{\sqrt{x^2+y^2}} = \underset{y\to 0}{Lt} \left\{ \underset{x\to 0}{Lt} \frac{xy}{\sqrt{x^2+y^2}} \right\}$

$= \underset{y\to 0}{Lt} = 0$

(b) $\underset{\substack{y\to 0\\ x\to 0}}{Lt} \frac{xy}{\sqrt{x^2+y^2}} = \underset{x\to 0}{Lt} \left\{ \underset{y\to 0}{Lt} \frac{xy}{\sqrt{x^2+y^2}} \right\} = \underset{x\to 0}{Lt}\ 0 = 0$

(c) $\underset{\substack{y\to mx\\ x\to 0}}{Lt} \frac{xy}{\sqrt{x^2+y^2}} = \underset{x\to 0}{Lt} \left\{ \underset{y\to mx}{Lt} \frac{xy}{\sqrt{x^2+y^2}} \right\} = \underset{x\to 0}{Lt} \frac{mx^2}{\sqrt{x^2(1+m^2)}}$

$= \underset{x\to 0}{Lt} \frac{mx^2}{x\sqrt{1+m^2}} = 0$

(d) $\underset{\substack{y\to mx^2\\ x\to 0}}{Lt} \frac{xy}{\sqrt{x^2+y^2}} = \underset{x\to 0}{Lt} \left\{ \underset{y\to mx^2}{Lt} \frac{xy}{\sqrt{x^2+y^2}} \right\}$

$= \underset{x\to 0}{Lt} \frac{mx^2y}{\sqrt{x^2(1+m^2x^2)}} = 0$

(e) $\underset{\substack{y\to mx^3\\ x\to 0}}{Lt} \frac{xy}{\sqrt{x^2+y^2}} = \underset{x\to 0}{Lt} \left\{ \underset{y\to mx^3}{Lt} \frac{xy}{\sqrt{x^2+y^2}} \right\}$

$= \underset{x\to 0}{Lt} \frac{mx^3y}{\sqrt{x^2+m^2x^6}} = \underset{x\to 0}{Lt} \frac{x^{3}y\cdot m}{x\sqrt{1+m^2x^3}} = 0$

(f) $\underset{\substack{x\to my\\ y\to 0}}{Lt} \frac{xy}{\sqrt{x^2+y^2}} = \underset{y\to 0}{Lt} \left\{ \underset{x\to my}{Lt} \frac{xy}{\sqrt{x^2+y^2}} \right\}$

$= \underset{y\to 0}{Lt} \frac{my^2}{\sqrt{y^2(1+m^2)}} = \underset{y\to 0}{Lt} \frac{my^2}{y\sqrt{1+m^2}} = 0$

Henge the limit exist.

3) $\underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{xy^3}{x^2+y^6}$

Given $\underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{xy^3}{x^2+y^6}$

1) $\underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{xy^3}{x^2+y^6} = \underset{y\to 0}{Lt}\left[\underset{x\to 0}{Lt} \frac{xy^3}{x^2+y^6}\right] = \underset{y\to 0}{Lt}\, 0 = 0$

) $\underset{\substack{y\to 0\\ x\to 0}}{Lt} \frac{xy^3}{x^2+y^6} = \underset{x\to 0}{Lt}\left\{\underset{y\to 0}{Lt} \frac{xy^3}{x^2+y^6}\right\} = \underset{x\to 0}{Lt}\, 0 = 0$

$\underset{\substack{y\to mx\\ x\to 0}}{Lt} \frac{xy^3}{x^2+y^6} = \underset{x\to 0}{Lt}\left\{\underset{y\to mx}{Lt} \frac{xy^3}{x^2+y^6}\right\} = \underset{x\to 0}{Lt} \frac{x^{\cancel{4}\,2}\cdot m^3}{\cancel{x^2}(1+m^6x^4)} = 0$

) $\underset{\substack{y\to mx^2\\ x\to 0}}{Lt} \frac{xy^3}{x^2+y^6} = \underset{x\to 0}{Lt}\left\{\underset{y\to mx^2}{Lt} \frac{xy^3}{x^2+y^6}\right\} = \underset{x\to 0}{Lt} \frac{x^{\cancel{7}\,5}\cdot m^3}{x^2(1+x^{10}m^6)} = 0$

) $\underset{\substack{y\to mx^3\\ x\to 0}}{Lt} \frac{xy^3}{x^2+y^6} = \underset{x\to 0}{Lt}\left\{\underset{y\to mx^3}{Lt} \frac{xy^3}{x^2+y^6}\right\} = \underset{y\to 0}{Lt} \frac{m^3 \cancel{x^{10}}\,^8}{x^2(m^6x^{16}+1)} = 0$

) $\underset{\substack{x\to my\\ y\to 0}}{Lt} \left(\frac{xy^3}{x^2+y^6}\right) = \underset{y\to 0}{Lt}\left\{\underset{x\to my}{Lt} \frac{xy^3}{x^2+y^6}\right\} = \underset{y\to 0}{Lt} \frac{my^{\cancel{4}\,2}}{y^2(m^2+y^4)} = 0$

) $\underset{\substack{x\to my^2\\ y\to 0}}{Lt} \frac{xy^3}{x^2+y^6} = \underset{y\to 0}{Lt}\left\{\underset{x\to my^2}{Lt} \frac{xy^3}{x^2+y^6}\right\} = \underset{y\to 0}{Lt} \frac{y^5\cdot m}{y^4(m^2+y^2)} = 0$

) $\underset{\substack{x\to my^3\\ y\to 0}}{Lt} \frac{xy^3}{x^2+y^6} = \underset{y\to 0}{Lt}\left\{\underset{x\to my^3}{Lt} \frac{xy^3}{x^2+y^6}\right\} = \underset{y\to 0}{Lt} \frac{\cancel{y^6} m}{\cancel{y^6}(m^2+1)}$

$= \frac{m}{m^2+1}$

Which depends on value of 'm', Hence the limit is different for different value of 'm'.

$\therefore$ limit does not exist.

(9) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x+\sqrt{y}}{x^2+y^2}$

Sol (a) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x+\sqrt{y}}{x^2+y^2} = \lim\limits_{y\to 0}\left\{\lim\limits_{x\to 0}\dfrac{x+\sqrt{y}}{x^2+y^2}\right\}$

$= \lim\limits_{y\to 0}\dfrac{\sqrt{y}}{y^2} = 0$

(b) $\lim\limits_{\substack{y\to 0\\ x\to 0}} \dfrac{x+\sqrt{y}}{x^2+y^2} = \lim\limits_{x\to 0}\left\{\lim\limits_{y\to 0}\dfrac{x+\sqrt{y}}{x^2+y^2}\right\} = \lim\limits_{x\to 0}\dfrac{x}{x^2} = \infty$

c) $\lim\limits_{\substack{y\to mx\\ x\to 0}} \dfrac{x+\sqrt{y}}{x^2+y^2} = \lim\limits_{x\to 0}\left\{\lim\limits_{y\to mx}\dfrac{x+\sqrt{y}}{x^2+y^2}\right\} = \lim\limits_{x\to 0}\dfrac{x\cdot\frac{x}{\sqrt{x}}\cdot\sqrt{m}}{x^2(1+m^2)}$

$= \lim\limits_{x\to 0}\left\{\lim\limits_{y\to mx}\dfrac{x\left(1+\sqrt{\frac{m}{x}}\right)}{x^2(1+m^2)}\right\} = \lim\limits_{x\to 0}\infty = \infty$

d) $\lim\limits_{\substack{y\to mx^2\\ x\to 0}} \dfrac{x+\sqrt{y}}{x^2+y^2} = \lim\limits_{x\to 0}\left\{\lim\limits_{y\to mx^2}\dfrac{x+\sqrt{mx^2}}{x^2+m^2x^4}\right\}$

$= \lim\limits_{x\to 0}\left\{\lim\limits_{y\to mx^2}\dfrac{x(1+\sqrt{m})}{x^2(1+m^2x^2)}\right\} = \infty$

e) $\lim\limits_{\substack{y\to mx^3\\ x\to 0}} \dfrac{x+\sqrt{y}}{x^2+y^2} = \lim\limits_{x\to 0}\left\{\lim\limits_{y\to mx^2}\dfrac{x+\sqrt{mx^2\cdot x}}{x^2+m^2\cdot x^6}\right\}$

$= \lim\limits_{x\to 0}\left\{\lim\limits_{y\to mx^2}\dfrac{x\sqrt{1+mx}}{x(1+m^2x^4)}\right\} = \lim\limits_{x\to 0}\infty$

$= \infty$

Limit does not exist

Continuity (Problem)

Discuss the continuity of the following functions

① $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

a) $\underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{xy}{x^2+y^2} = \underset{y\to 0}{Lt} \left\{ \underset{x\to 0}{Lt} \frac{xy}{x^2+y^2} \right\} = \underset{y\to 0}{Lt}\ 0 = 0$

b) $\underset{\substack{y\to 0\\ x\to 0}}{Lt} \frac{xy}{x^2+y^2} = \underset{x\to 0}{Lt} \left\{ \underset{y\to 0}{Lt} \frac{xy}{x^2+y^2} \right\} = \underset{x\to 0}{Lt} = 0$

c) $\underset{\substack{y\to mx\\ x\to 0}}{Lt} \frac{xy}{x^2+y^2} = \underset{x\to 0}{Lt} \left\{ \underset{y\to mx}{Lt} \frac{xy}{x^2+y^2} \right\}$

$= \underset{x\to 0}{Lt} \frac{m\cancel{x^2}}{\cancel{x^2}(1+m^2)} = \frac{m}{m^2+1}$

which depends on the value of 'm'.

Hence the limit is different for different 'm'.

∴ The limit does not exist.

Hence the function is not continuous

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② $f(x,y) = \begin{cases} \frac{x^2+y^2}{x-y} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

Sol (a) $\underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{x^2+y^2}{x-y} = \underset{y\to 0}{Lt} \left\{ \underset{x\to 0}{Lt} \frac{x^2+y^2}{x-y} \right\}$

$= \underset{y\to 0}{Lt} -y = 0$

(b) $\underset{\substack{y\to 0\\ x\to 0}}{Lt} \frac{x^2+y^2}{x-y} = \underset{x\to 0}{Lt} \left\{ \underset{y\to 0}{Lt} \frac{x^2+y^2}{x-y} \right\} = \underset{x\to 0}{Lt}\ x = 0$

(c) $\underset{\substack{y\to mx\\ x\to 0}}{Lt} \frac{x^2+y^2}{x-y} = \underset{x\to 0}{Lt}\left\{\underset{y\to mx}{Lt} \frac{x^2+y^2}{x-y}\right\}$

$= \underset{x\to 0}{Lt} \frac{x^2(1+m^2)}{x(1-m)} = 0$

(d) $\underset{\substack{y\to mx^2\\ x\to 0}}{Lt} \frac{x^2+y^2}{x-y} = \underset{x\to 0}{Lt}\left\{\underset{y\to mx^2}{Lt} \frac{x^2+y^2}{x-y}\right\}$

$= \underset{x\to 0}{Lt} \frac{x^{\cancel{2}}(1+m^2x^2)}{\cancel{x}(1-mx^2)} = 0$

(e) $\underset{\substack{y\to mx^3\\ x\to 0}}{Lt} \frac{x^2+y^2}{x-y} = \underset{x\to 0}{Lt}\left\{\underset{y\to mx^3}{Lt} \frac{x^2+y^2}{x-y}\right\}$

$= \underset{x\to 0}{Lt} \frac{x^{\cancel{2}}(1+m^2x^4)}{\cancel{x}(1-mx^2)} = 0$

Since all the limits are same & it is equal to the limit of the function at the origin.

$\therefore$ The function is continuous at $(0,0)$

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③ $f(x,y) = \begin{cases} \frac{x^2+y^2+xy+x+y}{x+y} & ; (x,y) \neq (2,2) \\ 4 & ; (x,y) = (2,2) \end{cases}$ —— ①

Sol Given "

(a) $\underset{\substack{x\to 2\\ y\to 2}}{Lt} \frac{x^2+y^2+xy+x+y}{x+y} = \underset{y\to 2}{Lt}\left\{\underset{x\to 2}{Lt} \frac{x^2+y^2+xy+x+y}{x+y}\right\}$

$= \underset{y\to 2}{Lt} \frac{6+y^2+3y}{2+y} = \frac{16}{4} = 4$

(b) $\underset{\substack{y\to 2\\ x\to 2}}{Lt} \frac{x^2+y^2+xy+x+y}{x+y} = \underset{x\to 2}{Lt}\left\{\underset{y\to 2}{Lt} \frac{x^2+y^2+xy+x+y}{x+y}\right\}$

$$\underset{x\to 2}{Lt} \frac{x^2+3x+6}{x+2} = \frac{16}{4} = 4$$

Hence the limit exist

since all the limit are same & it is equal to the limit of the function at (2,2)

$\therefore$ The function is continuous at (2,2).

) S.T $f(x,y) = \begin{cases} (x+y)\sin\left(\frac{1}{x+y}\right) ; & x+y \neq 0 \\ 0 ; & x+y=0 \end{cases}$

continuous but f_x & f_y DNE at (0,0)

Given " ——— (1)

$$\underset{x+y\to 0}{Lt} (x+y)\sin\left(\frac{1}{x+y}\right) = 0$$

Hence the function is continuous.

$$f_x = \underset{\delta x\to 0}{Lt} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

At (0,0) $f_x = \underset{\delta x\to 0}{Lt} \frac{f(\delta x, 0) - f(0,0)}{\delta x}$

$$= \underset{\delta x\to 0}{Lt} \frac{\cancel{\delta x} \sin\frac{1}{\delta x} - 0}{\cancel{\delta x}}$$

Hence does not exist.

$$f_y = \underset{\delta y\to 0}{Lt} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

At (0,0) $f_y = \underset{\delta y\to 0}{Lt} \frac{f(0, \delta y) - f(0,0)}{\delta y}$

$$= \underset{\delta y\to 0}{Lt} \frac{\cancel{\delta y} \sin\frac{1}{\delta y} - 0}{\cancel{\delta y}}$$

Hence does not exist

⑤ $f(x,y) = \begin{cases} \frac{y(x^2-y^2)}{x^2+y^2}; & (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

find f_x & f_y at $(0,0)$.

Sol: Given " ——— ①

(a) $\underset{\substack{x\to 0 \\ y\to 0}}{Lt} \frac{y(x^2-y^2)}{x^2+y^2} = \underset{y\to 0}{Lt} \left\{ \underset{x\to 0}{Lt} \frac{y(x^2-y^2)}{x^2+y^2} \right\} = \underset{y\to 0}{Lt} -y = 0$

(b) $\underset{\substack{y\to 0 \\ x\to 0}}{Lt} \frac{y(x^2-y^2)}{x^2+y^2} = \underset{x\to 0}{Lt} \left\{ \underset{y\to 0}{Lt} \frac{y(x^2-y^2)}{x^2+y^2} \right. = \underset{x\to 0}{Lt}\, x = 0$

(c) $\underset{\substack{y\to mx \\ x\to 0}}{Lt} \frac{y(x^2-y^2)}{x^2+y^2} = \underset{x\to 0}{Lt} \left\{ \underset{y\to mx}{Lt} \frac{y(x^2-y^2)}{x^2+y^2} \right\} =$

$= \underset{x\to 0}{Lt} \frac{mx(x^2-m^2x^2)}{x^2+m^2x^2} = 0.$

(d) $\underset{\substack{y\to mx^2 \\ x\to 0}}{Lt} \frac{y(x^2-y^2)}{x^2+y^2} = \underset{x\to 0}{Lt} \left\{ \underset{y\to mx^2}{Lt} \frac{y(x^2-y^2)}{x^2+y^2} \right\} = 0$

(e) $\underset{\substack{y\to mx^2 \\ x\to 0}}{Lt} \frac{y(x^2-y^2)}{x^2+y^2} = \underset{x\to 0}{Lt} \left\{ \underset{y\to mx^3}{Lt} \frac{y(x^2-y^2)}{x^2+y^2} \right\} = 0$

Hence the limit exist

Since all the limits are same & it is equal to the limit of the function at the origin

∴ The function is continuous at $(0,0)$

$f_x = \underset{\delta x\to 0}{Lt} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$

At $(0,0)$ $f_x = \underset{\delta x \to 0}{Lt} \dfrac{f(\delta x, 0) - f(0,0)}{\delta x}$

$$= \underset{\delta x \to 0}{Lt} \frac{0-0}{\delta x} = 0$$

$$f_y = \underset{\delta y \to 0}{Lt} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

At $(0,0)$ $f_y = \underset{\delta y \to 0}{Lt} \dfrac{f(0, \delta y) - f(0,0)}{\delta y}$

$$= \underset{\delta y \to 0}{Lt} \frac{-\delta y - 0}{\delta y} = -1$$

1) $f(x,y) = \begin{cases} \dfrac{x^2 y(x-y)}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

& hence find f_{xy} & f_{yx}.

—— ①

Given $f(x,y) =$ ''

a) $\underset{\substack{x\to 0 \\ y \to 0}}{Lt} \dfrac{x^2 y(x-y)}{x^2+y^2} = \underset{y\to 0}{Lt} \left\{ \underset{x\to 0}{Lt} \dfrac{x^2 y(x-y)}{x^2+y^2} \right\} = \underset{y\to 0}{Lt}\, 0 = 0$

b) $\underset{\substack{y\to 0 \\ x \to 0}}{Lt} \dfrac{x^2 y(x-y)}{x^2+y^2} = \underset{x\to 0}{Lt} \left\{ \underset{y\to 0}{Lt} \dfrac{x^2 y(x-y)}{x^2+y^2} \right\} = \underset{x\to 0}{Lt}\, 0 = 0$

c) $\underset{\substack{y\to mx \\ x \to 0}}{Lt} \dfrac{x^2 y(x-y)}{x^2+y^2} = \underset{x\to 0}{Lt} \left\{ \underset{y\to mx}{Lt} \dfrac{x^2 y(x-y)}{x^2+y^2} \right\} =$

$$= \underset{x\to 0}{Lt} \frac{mx^4(1-m)}{x^2(1+m^2)} = 0.$$

d) $\underset{\substack{y \to mx^2 \\ x \to 0}}{Lt} \dfrac{x^2 y(x-y)}{x^2+y^2} = \underset{x \to 0}{Lt} \left\{ \underset{y \to mx^2}{Lt} \dfrac{x^2 y(x-y)}{x^2+y^2} \right\}$

$= \underset{x \to 0}{Lt} \dfrac{m x^5 (1-mx)}{x^2 (1+m^2 x^2)} = 0$

e) $\underset{\substack{y \to mx^3 \\ x \to 0}}{Lt} \dfrac{x^2 y(x-y)}{x^2+y^2} = \underset{x \to 0}{Lt} \left\{ \underset{y \to mx^3}{Lt} \dfrac{x^2 y(x-y)}{x^2+y^2} \right\}$

$= \underset{x \to 0}{Lt} \dfrac{m x^6 (1-mx)}{x^4 (1+m^2 x^2)} = 0$

Since all the values of the limit are same as the limit of given function at origin.

$\therefore$ The function is continuous at $(0,0)$

Now $f_x = \underset{\delta x \to 0}{Lt} \dfrac{f(x+\delta x, y) - f(x,y)}{\delta x}$

$f_x(0,0) = \underset{\delta x \to 0}{Lt} \dfrac{f(\delta x, 0) - f(0,0)}{\delta x}$

$f_x(0,0) = \underset{\delta x \to 0}{Lt} \dfrac{0-0}{\delta x} = 0$

Now $f_y = \underset{\delta y \to 0}{Lt} \dfrac{f(x, y+\delta y) - f(x,y)}{\delta y}$

$f_y(0,0) = \underset{\delta y \to 0}{Lt} \dfrac{f(0, \delta y) - f(0,0)}{\delta y}$

$f_y(0,0) = \underset{\delta y \to 0}{Lt} \dfrac{0-0}{\delta y} = 0$

$\therefore f_{xy} = \underset{\delta x \to 0}{Lt} \dfrac{f_y(x+\delta x, y) - f_y(x,y)}{\delta x}$

$f_{xy}(0,0) = \underset{\delta x \to 0}{Lt} \dfrac{f_y(\delta x, 0) - f_y(0,0)}{\delta x}$

$\therefore f_y(x,0) = \lim_{\delta y \to 0} \frac{f(x,\delta y) + f(x,0)}{\delta y}$

$$\lim_{\delta y \to 0} \frac{\frac{x^2 \delta y(x-\delta y)}{x^2+\delta y^2} - 0}{\delta y} = \frac{x^3}{x^2} = x$$

$\therefore f_{xy}(0,0) = \lim_{\delta x \to 0} \frac{\delta x - 0}{\delta x} = 1$

$|||^{ly}$ $f_{yx} = \lim_{\delta y \to 0} \frac{f_x(x, y+\delta y) - f_x(x,y)}{\delta y}$

$$f_{yx}(0,0) = \lim_{\delta y \to 0} \frac{f_x(0,\delta y) - f_x(0,0)}{\delta y}$$

$\therefore f_x(0,y) = \lim_{\delta x \to 0} \frac{f(\delta x, y) - f(0,y)}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{\frac{\delta x^2 y(\delta x - y)}{\delta x^2 + y^2} - 0}{\delta x} = 0$$

$\therefore f_{yx}(0,0) = \lim_{\delta y \to 0} \frac{0-0}{\delta y} = 0$

Hence $f_{xy} \neq f_{yx}$

⑦ $f(x,y) = \begin{cases} \dfrac{x^2 y(x-y)}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

then P.T $f_{xx} = f_{yy}$

W.K.T $f_x = \underset{\delta x \to 0}{Lt} \dfrac{f(x+\delta x, y) - f(x,y)}{\delta x}$

$f_x(0,0) = \underset{\delta x \to 0}{Lt} \dfrac{f(\delta x, 0) - f(0,0)}{\delta x}$

$= \underset{\delta}{Lt}$.

$f_{xx} = \underset{\delta x \to 0}{Lt} \dfrac{f_x(x+\delta x, y) - f_x(x,y)}{\delta x}$

$f_{xx}(0,0) = \underset{\delta x \to 0}{Lt} \dfrac{f_x(\delta x, 0) - f_x(0,0)}{\delta x}$ —①

$\because f_x(x,0) = \underset{\delta x \to 0}{Lt} \dfrac{f(x+\delta x, 0) - f(x,0)}{\delta x}$

$= \underset{\delta x \to 0}{Lt} \dfrac{0-0}{\delta x} = 0$

from ①

$\therefore f_{xx}(0,0) = \underset{\delta x \to 0}{Lt} \dfrac{0-0}{\delta x} = 0$

$f_{yy} = \underset{\delta y \to 0}{Lt} \dfrac{f_y(x, y+\delta y) - f_y(x,y)}{\delta y}$

$\because f_y(0,y) = \underset{\delta y \to 0}{Lt} \dfrac{f(0, y+\delta y) - f(0,y)}{\delta y}$

$= \underset{\delta y \to 0}{Lt} \dfrac{0-0}{\delta y} = 0$

$$f_{yy}(0,0) = \lim_{\delta y \to 0} \frac{f_y(0,\delta y) - f_y(0,0)}{\delta y} \quad \text{— ②}$$

from ②

$$\therefore f_{yy}(0,0) = \frac{0-0}{\delta y} = 0$$

Hence $f_{xx} = f_{yy}$

⑧ $f(x,y) = \begin{cases} \dfrac{xy(5x^2-4y^2)}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

& hence find f_{xy} & f_{yx}

Sol Given " ——— ①

a) $\underset{\substack{x\to 0\\ y\to 0}}{Lt} \dfrac{xy(5x^2-4y^2)}{x^2+y^2} = \underset{y\to 0}{Lt} \left\{ \underset{x\to 0}{Lt} \dfrac{xy(5x^2-4y^2)}{x^2+y^2} \right. = \underset{y\to 0}{Lt}\, 0 = 0$

b) $\underset{\substack{y\to 0\\ x\to 0}}{Lt} \dfrac{xy(5x^2-4y^2)}{x^2+y^2} = \underset{x\to 0}{Lt} \left\{ \underset{y\to 0}{Lt} \dfrac{xy(5x^2-4y^2)}{x^2+y^2} \right\} = \underset{x\to 0}{Lt}\, 0 = 0$

c) $\underset{\substack{y\to mx\\ x\to 0}}{Lt} \dfrac{xy(5x^2-4y^2)}{x^2+y^2} = \underset{x\to 0}{Lt} \left\{ \underset{y\to mx}{Lt} \dfrac{xy(5x^2-4y^2)}{x^2+y^2} \right\}$

$= \underset{x\to 0}{Lt} \dfrac{m\cancel{x^4}^{2}(5-4m^2)}{\cancel{x^2}(1+m^2)} = 0$

d) $\underset{\substack{y\to mx^2\\ x\to 0}}{Lt} \dfrac{xy(5x^2-4y^2)}{x^2+y^2} = \underset{x\to 0}{Lt} \left\{ \underset{y\to mx^2}{Lt} \dfrac{xy(5x^2-4y^2)}{x^2+y^2} \right\}$

$= \underset{x\to 0}{Lt} \dfrac{m\cancel{x^5}^{3}(5-4mx^2)}{\cancel{x^2}(1+m^2x^2)} = 0$

e) $\underset{\substack{y\to mx^3\\ x\to 0}}{Lt} \dfrac{xy(5x^2-4y^2)}{x^2+y^2} = \underset{x\to 0}{Lt} \left\{ \underset{y\to mx^3}{Lt} \dfrac{xy(5x^2-4y^2)}{x^2+y^2} \right\}$

$= \underset{x\to 0}{Lt} \dfrac{m\cancel{x^6}^{4}(5-4mx^4)}{\cancel{x^2}(1+m^2x^2)} = 0$

Hence the limit exist.

Since all the values of the limit are same as the limit of the function at origin.

$\therefore$ The function is continuous at $(0,0)$

Now, $f_x = \underset{\delta x \to 0}{Lt} \dfrac{f(x+\delta x, y) - f(x,y)}{\delta x}$

$f_x(0,0) = \underset{\delta x \to 0}{Lt} \dfrac{f(\delta x, 0) - f(0,0)}{\delta x}$

$f_x(0,0) = \underset{\delta x \to 0}{Lt} \dfrac{0-0}{\delta x} = 0$

Now $f_y = \underset{\delta y \to 0}{Lt} \dfrac{f(x, \delta y + y) - f(x,y)}{\delta y}$

$f_y(0,0) = \underset{\delta y \to 0}{Lt} \dfrac{f(0, \delta y) - f(0,0)}{\delta y}$

$f_y(0,0) = \underset{\delta y \to 0}{Lt} \dfrac{0-0}{\delta y} = 0$

$\therefore f_{xy} = \underset{\delta x \to 0}{Lt} \dfrac{f_y(x+\delta x, y) - f_y(x,y)}{\delta x}$

$f_{xy}(0,0) = \underset{\delta x \to 0}{Lt} \dfrac{f_y(\delta x, 0) - f_y(0,0)}{\delta x}$

$\because f_y(x,0) = \underset{\delta y \to 0}{Lt} \dfrac{f(x, \delta y) - f(x,0)}{\delta y}$

$= \underset{\delta y \to 0}{Lt} \dfrac{\dfrac{x \cdot \cancel{\delta y}(5x^2 - 4\delta y^2)}{x^2 + \delta y^2} - 0}{\cancel{\delta y}}$

$= \dfrac{5x^3}{x^2} = 5x$

$\therefore f_{xy}(0,0) = \underset{\delta x \to 0}{Lt} \dfrac{5\delta x - 0}{\delta x} = 5$

III^{ly} $f_{yx} = \underset{\delta y \to 0}{Lt} \dfrac{f_x(x, y+\delta y) - f_x(x,y)}{\delta y}$

$$f_{yx}(0,0) = \underset{\delta y \to 0}{Lt} \frac{f_x(0,\delta y) - f_x(0,0)}{\delta y}$$

$$\because f_x(0,y) = \underset{\delta x \to 0}{Lt} \frac{f(\delta x, y) - f(0,y)}{\delta x}$$

$$= \underset{\delta x \to 0}{Lt} \frac{\frac{\cancel{\delta x} \cdot y(5\delta x^2 - 4y^2)}{\delta x^2 + y^2} - 0}{\cancel{\delta x}} = \frac{-4y^3}{y^2} = -4y$$

$$\therefore f_{yx}(0,0) = \underset{\delta y \to 0}{Lt} \frac{-4\delta y - 0}{\delta y} = -4$$

Hence $f_{xy} \neq f_{yx}$

Partial Derivative Problems:

① If $f = \frac{x-y}{x+y}$ then find f_x at $(1,1)$.

Sol Given " ——— ①

$$f_x = \frac{(x+y)(1) - (x-y)(1)}{(x+y)^2}$$

At $(1,1)$ $f_x = \frac{2-0}{2^2} = \frac{2}{4} = \frac{1}{2}$

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② If $z = e^{ax+by} f(ax-by)$ then P.T.

$bz_x + az_y = 2abz$.

Sol Given $z = e^{ax+by} f(ax-by)$ — ①

$$z_x = a e^{ax+by} f'(ax-by) + f(ax-by) a e^{ax+by}$$

Now $z_y = -b e^{ax+by} f'(ax-by) + f(ax-by) b e^{ax+by}$

$$L.H.S = bz_x + az_y$$

$$\cancel{ab e^{ax+by} f'(ax-by)} + ab f(ax-by) e^{ax+by}$$

$$\cancel{- ab e^{ax+by} f'(ax-by)} + ab f(ax-by) e^{ax+by}$$

$= 2ab\, e^{ax+by} f(ax-by) = 2abz$

$= R.H.S$

If $f = x^3 \log\left(\frac{y}{x}\right)$, then P.T. $x f_x + y f_y = 3f$

Given $f = x^3 \log \frac{y}{x}$ — (1)

$f_x = x^3 \frac{1}{y/x}\left(\frac{-y}{x^2}\right) + \log\frac{y}{x}(3x^2)$

$f_x = -x^2 + 3x^2 \log\frac{y}{x}$

Now $f_y = x^3 \cdot \frac{1}{y/x}\left(\frac{1}{x}\right) = \frac{x^3}{y}$

$L.H.S = x f_x + y f_y$

$= \cancel{-x^3} + 3x^3 \log\frac{y}{x} + \cancel{y}\frac{x^3}{\cancel{y}} = 3x^3 \log\frac{y}{x}$

$= 3f = R.H.S.$

If $u = \log(x^3+y^3-x^2y-xy^2)$ then P.T

$u_x + u_y = \frac{2}{x+y}$

Given $u = \log(x^3+y^3-x^2y-xy^2)$ — (1)

$u_x = \frac{3x^2-2xy-y^2}{x^3+y^3-x^2y-xy^2}$

$u_y = \frac{3y^2-x^2-2xy}{x^3+y^3-x^2y-xy^2}$

$L.H.S = u_x + u_y = \frac{3x^2-2xy-y^2}{x^3+y^3-x^2y-xy^2} + \frac{3y^2-x^2-2xy}{x^3+y^3-x^2y-xy^2}$

$= \cancel{2x^2+2y^2}$

$= \frac{2x^2+2y^2-4xy}{x^3+y^3-x^2y-xy^2}$

$$= \frac{2(x^2+y^2-2xy)}{x^2(x-y)-y^2(x-y)}$$

$$= \frac{2(x-y)^2}{(x^2-y^2)(x-y)} = \frac{2(x-y)^2}{(x+y)(x-y)^2}$$

$$= \frac{2}{x+y} = R.H.S$$

**

⑤ If $u = \log(x^3+y^3+z^3-3xyz)$ then.

P.T $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

Sol Given $u = \log(x^3+y^3+z^3-3xyz)$ — ①

$$u_x = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$u_y = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$u_z = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$L.H.S = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}\right]$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left[\frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)}\right]$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left[\frac{3(\cancel{x^2+y^2+z^2-xy-yz-zx})}{(x+y+z)(\cancel{x^2+y^2+z^2-xy-yz-zx})}\right]$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left\{\frac{3}{(x+y+z)}\right\}$$ → partial Deri w.r.t x, y, z,

$$= \frac{-3}{(x+y+z)^2} + \frac{(-3)}{(x+y+z)^2} + \frac{(-3)}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2} = R.H.S$$

Higher Order Partial Derivative :- (Problems)

) If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ then P.T $u_{xx} + u_{yy} + u_{zz} = 0$

Given $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ — (1)

$$u_x = \frac{-1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}(\cancel{2}x) = -x(x^2+y^2+z^2)^{-3/2}$$

$$u_{xx} = -\left[x\left(\frac{-3}{\cancel{2}}\right)(x^2+y^2+z^2)^{-5/2}(\cancel{2}x) + (x^2+y^2+z^2)^{-3/2}(1)\right]$$

$$u_{xx} = 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \text{—(2)}$$

~~Hlly~~ $|||^{ly}$ $u_{yy} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$ —(3)

ly $u_{zz} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$ —(4)

L.H.S $= u_{xx} + u_{yy} + u_{zz}$

$$= \cancel{3}(= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$
$$+ 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$
$$+ 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$= 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2) - 3(x^2+y^2+z^2)^{-3/2}$

$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2} = 0$

$= R.H.S.$

② If $u = f(x+ay) + \phi(x-ay)$ then P.T $u_{yy} = a^2 u_{xx}$

Sol Given $u = f(x+ay) + \phi(x-ay)$ — ①

$u_x = f'(x+ay) + \phi'(x-ay)$ — ②

$u_{xx} = f''(x+ay) + \phi''(x-ay)$ — ③

$u_y = af'(x+ay) - a\phi'(x-ay)$ — ④

$u_{yy} = a^2 f''(x+ay) + a^2\phi''(x-ay)$

$= a^2[f''(x+ay) + \phi''(x-ay)]$ — ⑤

from ③ & ⑤, we get

$u_{yy} = a^2 u_{xx}$ [Hence proved]

③ If $u = e^x(x\cos y - y\sin y)$ then P.T $u_{xx} + u_{yy} = 0$.

Sol Given $u = e^x(x\cos y - y\sin y)$ — ①

$u_x = e^x\cos y + (x\cos y - y\sin y)e^x$

$u_x = e^x[\cos y + x\cos y - y\sin y]$ — ②

$u_{xx} = e^x(\cos y) + e^x[\cos y + x\cos y - y\sin y]$

$u_{xx} = e^x[2\cos y + x\cos y - y\sin y]$ — ③

$u_y = e^x[-x\sin y - \{1\cdot\sin y + y\cos y\}]$

$u_y = e^x[-y\cos y - x\sin y - \sin y]$ — ④

$u_{yy} = e^x[-\{1\cdot\cos y + y\sin y\} - x\cos y - \cos y]$

$U_{yy} = e^x[-x\cos y - 2\cos y + y\sin y]$ — ⑤

Adding ③ & ⑤,

$U_{xx} + U_{yy} = e^x(0) = 0$.

If. $U = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$ then P.T $U_{xy} = \frac{x^2-y^2}{x^2+y^2}$

Given $U = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$

$$U_y = x^{\cancel{2}}\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{\cancel{x}}\right) - \left[\cancel{y^2}\frac{1}{1+\frac{x^2}{y^2}}\left(\frac{-x}{\cancel{y^2}}\right) + 2y\tan^{-1}\frac{x}{y}\right]$$

$$U_y = \frac{x^3}{x^2+y^2} + \frac{xy^2}{y^2+x^2} - 2y\tan^{-1}\frac{x}{y} \quad (*)$$

$$U_y = \frac{x\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} - 2y\tan^{-1}\frac{x}{y}$$

$$U_y = x - 2y\tan^{-1}\frac{x}{y}$$

$$U_{xy} = 1 - 2\cancel{y}\frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{\cancel{y}}\right)$$

$$= 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2+y^2-2y^2}{x^2+y^2}$$

$$U_{xy} = \frac{x^2-y^2}{x^2+y^2}$$

If $z = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$ then P.T

$x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy} = 0$.

Given $z = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$ — ①

$$z_x = 1\cdot\phi\left(\frac{y}{x}\right) + \cancel{x}\left(\frac{-y}{x^{\cancel{2}}}\right)\phi'\left(\frac{y}{x}\right) + \left(\frac{-y}{x^2}\right)\psi'\left(\frac{y}{x}\right)$$

$$z_x = \phi\left(\frac{y}{x}\right) - \frac{y}{x}\phi'\left(\frac{y}{x}\right) - \left(\frac{y}{x^2}\right)\psi'\left(\frac{y}{x}\right)$$

$$\Rightarrow Z_x = -\frac{y}{x}\phi'\left(\frac{y}{x}\right) + \phi\left(\frac{y}{x}\right) - \frac{y}{x^2}\psi'\left(\frac{y}{x}\right) \quad -②$$

· Now diff ② p.w.r.t 'x' we get

$$Z_{xx} = -\left\{\frac{y}{x}\phi''\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + \cancel{\phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)}\right\} + \cancel{\phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)} - \left\{\frac{y}{x^2}\psi''\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + \psi'\left(\frac{y}{x}\right)\left(\frac{-2y}{x^3}\right)\right\}$$

$$Z_{xx} = \frac{y^2}{x^3}\phi''\left(\frac{y}{x}\right) + \frac{y^2}{x^4}\psi''\left(\frac{y}{x}\right) + \frac{2y}{x^3}\psi'\left(\frac{y}{x}\right) \quad -③$$

Now diff ② p.w.r.t 'y' we get

$$Z_{xy} = -\left\{\frac{y}{x}\phi''\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) + \cancel{\phi'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)}\right\} + \cancel{\phi'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)} - \left\{\frac{y}{x^2}\psi''\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) + \psi'\left(\frac{y}{x}\right)\left(\frac{1}{x^2}\right)\right\}$$

$$\Rightarrow Z_{xy} = \frac{-y}{x^2}\phi''\left(\frac{y}{x}\right) - \frac{y}{x^3}\psi''\left(\frac{y}{x}\right) - \frac{1}{x^2}\psi'\left(\frac{y}{x}\right) \quad -④$$

Diff ① p.w.r.t 'y'; we get

$$Z_y = \cancel{x}\phi'\left(\frac{y}{x}\right)\cancel{\left(\frac{1}{x}\right)} + \psi'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$Z_y = \phi'\left(\frac{y}{x}\right) + \frac{1}{x}\psi'\left(\frac{y}{x}\right) \quad -⑤$$

Now diff ⑤ p.w.r.t 'y' we get

$$Z_{yy} = \phi''\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) + \frac{1}{x}\psi''\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$Z_{yy} = \frac{1}{x}\phi''\left(\frac{y}{x}\right) + \frac{1}{x^2}\psi''\left(\frac{y}{x}\right) \quad -⑥$$

Using ③, ④ & ⑤ in

$$x^2 Z_{xx} + 2xy Z_{xy} + y^2 Z_{yy}$$

$$= \cancel{\frac{y^2}{x}\phi''\left(\frac{y}{x}\right)} + \cancel{\frac{y^2}{x^2}\psi''\left(\frac{y}{x}\right)} + \cancel{\frac{2y}{x}\psi'\left(\frac{y}{x}\right)} - \frac{2y^2}{x}\phi'\left(\frac{y}{x}\right) - \cancel{\frac{2y^2}{x^2}\psi''\left(\frac{y}{x}\right)} - \cancel{\frac{2y}{x}\psi'\left(\frac{y}{x}\right)} + \cancel{\frac{y^2}{x}\phi''\left(\frac{y}{x}\right)} + \cancel{\frac{y^2}{x^2}\psi''\left(\frac{y}{x}\right)} = 0 \text{ (Proved)}$$

PD (problem)

(6) If $x = r\cos\theta$, $y = r\sin\theta$, then P.T $(r_x)^2 + (r_y)^2 = 1$

Sol: Given $x = r\cos\theta$, $y = r\sin\theta$

$\therefore x^2 + y^2 = r^2$

$\cancel{2}x = \cancel{2}r \cdot \frac{\partial r}{\partial x} \Rightarrow r_x = \frac{x}{r}$

$\cancel{2}y = \cancel{2}r \frac{\partial r}{\partial y} \Rightarrow r_y = \frac{y}{r}$

$\therefore$ L.H.S $= (r_x)^2 + (r_y)^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2+y^2}{r^2}$

$= \frac{\cancel{r^2}}{\cancel{r^2}} = 1$

Total derivative:

Total differential or derivative of a function of three variable x, y, z is denoted by df and is defined as

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

~~where~~ whether or not x, y, z are independent of each other

Composite function:

If u is a function of two variables x & y and x & y are themselves functions of an another independent variable t, r and s,

then

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t};$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r};$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s};$$

Implicit function:

Let $u = f(x,y) = c$, where c is constant, such function are called implicit function. Since $du = 0$

$$\therefore \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

Divide with dx we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-u_x}{u_y} \text{ or } \frac{-f_x}{f_y}.$$

Total Derivative:-

① If $u = \tan^{-1}\left(\frac{y}{x}\right)$; $(x,y) \neq (0,0)$ then find du.

Sol Given $u = \tan^{-1}\frac{y}{x}$.

$du = u_x dx + u_y dy$

$du = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)dx + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)dy$

$du = \frac{-y}{x^2+y^2}dx + \frac{x\,dy}{x^2+y^2} = \frac{x\,dy - y\,dx}{x^2+y^2}$

② If $u = e^{xyz}$ then find du.

Sol Given $u = e^{xyz}$ —①

$\therefore du = u_x dx + u_y dy + u_z dz$

$= yz \cdot e^{xyz} \cdot dx + x \cdot z\, e^{xyz} dy + xy e^{xyz} \cdot dz$

$= e^{xyz}(zy\,dx + xz\,dy + xy\,dz]$

③ If $f = x^3y + xy^3$ then find df.

Sol Given $f = x^3y + xy^3$ —①

$df = f_x \cdot dx + f_y dy$

$df = (3x^2y + y^3) \cdot dx + (x^3 + 3xy^2) dy$

④ If $u = \left(xz + \frac{x}{z}\right)^y$ then find du.

Sol Given " —①

$du = u_x dx + u_y dy + u_z dz$

$du = y\left(xz + \frac{x}{z}\right)^{y-1}\left(z + \frac{1}{z}\right)dx + \left(xz + \frac{x}{z}\right)^y \log\left(xz + \frac{x}{z}\right)dy$

$+ y\left(xz + \frac{x}{z}\right)^{y-1}\left(x - \frac{x}{z^2}\right)dz$

⑤ If $u=\sin^{-1}(x-y)$, $x=3t$, $y=4t^3$. then P.T $\frac{du}{dt}=\frac{3}{\sqrt{1-t^2}}$

Sol Given " "

$$\frac{du}{dt}=u_x\cdot\frac{dx}{dt}+u_y\frac{dy}{dt}$$

$$=\frac{1}{\sqrt{1-(x-y)^2}}(3)+\frac{(-1)(12t^2)}{\sqrt{1}}$$

$$\frac{du}{dt}=\frac{3-12t^2}{\sqrt{1-(x-y)^2}}=\frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}$$

$$\frac{du}{dt}=\frac{3(1-4t^2)}{\sqrt{1-(9t^2+16t^6-24t^4)}}$$

$$\frac{du}{dt}=\frac{3(1-4t^2)}{\sqrt{-16t^6+24t^4-9t^2+1}}$$

$$\frac{du}{dt}=\frac{3(1-4t^2)}{\sqrt{(t-1)(t+1)(-16t^4+8t^2-1)}}$$

$$\begin{array}{r|rrrrrrr}
t=1 & -16 & 0 & 24 & 0 & -9 & 0 & 1 \\
 & 0 & -16 & -16 & 8 & -8 & -1 & -1 \\
\hline
t=-1 & -16 & -16 & 8 & 8 & -1 & -1 & \boxed{0} \\
 & 0 & 16 & 0 & -8 & 0 & 1 & \\
\hline
 & -16 & 0 & 8 & 0 & -1 & \boxed{0} &
\end{array}$$

$$\frac{du}{dt}=\frac{3(1-4t^2)}{\sqrt{\{-(1-t)(1+t)\}\{-(16t^4-8t^2+1)\}}}$$

$$\frac{du}{dt}=\frac{3\cancel{(1-4t^2)}}{\sqrt{(1-t^2)\cancel{(1-4t^2)^2}}}=\frac{3}{\sqrt{1-t^2}}$$

⑥ If $f=xy+yz+zx$, $x=t^2$, $y=te^t$ & $z=te^{-t}$ then find $\frac{df}{dt}$

Sol Given " " " "

$$\frac{df}{dt}=f_x\cdot\frac{dx}{dt}+f_y\frac{dy}{dt}+f_z\cdot\frac{dz}{dt}$$

$\frac{df}{dt} = (y+z)(2t) + (x+z)(te^t + e^t) + (x+y)(-te^{-t} + e^{-t})$

$\frac{df}{dt} = (te^t + te^{-t})(2t) + (t^2 + te^{-t})(te^t + e^t) +$
$(t^2 + te^t)(e^{-t} - te^{-t})$

(7) If $w = x^2 + y^2 + z^2$, $x = \cos t$, $y = \log(t+1)$, $z = e^t$ then $\frac{dw}{dt}$ at $t = 0$

Sol Given " " " " → (1)

$$\frac{dw}{dt} = w_x \frac{dx}{dt} + w_y \frac{dy}{dt} + w_z \frac{dz}{dt}$$

$$\frac{dw}{dt} = (2x)(-\sin t) + (2y)\left(\frac{1}{t+1}\right) + 2ze^t$$

$$\frac{dw}{dt} = -2\cos t \sin t + \frac{2\log(t+1)}{t+1} + 2e^{2t}$$

At $t = 0$ we get $\frac{dw}{dt} = 2$

Composite Functions (Problem)

(1) If $u = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$ then P.T
$(u_x)^2 + (u_y)^2 = (u_r)^2 + \frac{1}{r^2}(u_\theta)^2$.

Sol Given " " " "

$$u_r = u_x \cdot x_r + u_y \cdot y_r$$

$$u_r = u_x \cos\theta + u_y \sin\theta \quad \text{— (1)}$$

$$u_\theta = u_x x_\theta + u_y y_\theta$$

$$u_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta) \quad \text{— (2)}$$

$$R.H.S = (u_r)^2 + \frac{1}{r^2}(u_\theta)^2$$

$$= (u_x)^2(1) + (u_y)^2(1) = (u_x)^2 + (u_y)^2$$

$$= L.H.S$$

② If $x = u+v+w$, $y = uv+vw+uw$, $z = uvw$ and f is a function of x, y, z then P.T $uf_u + vf_v + wf_w$ $= xf_x + 2yf_y + 3zf_z$.

Sol Given " " "

$f_u = f_x x_u + f_y y_u + f_z z_u$

$f_u = f_x(1) + f_y(v+w) + f_z(vw)$

$u \cdot f_u = uf_x + (uv+uw)f_y + (uvw)f_z$ — ①

$f_v = f_x x_v + f_y y_v + f_z z_v = f_x(1) + f_y(u+w) + f_z(uw)$

$vf_v = vf_x + (uv+wv)f_y + (uvw)f_z$ — ②

$f_w = f_x x_w + f_y y_w + f_z z_w$

$f_w = f_x(1) + f_y(u+v) + f_z(uv)$

$wf_w = wf_x + (uw+vw)f_w + (uvw)f_z$ — ③

Adding ①, ② & ③, we get

$uf_u + vf_v + wf_w = (u+v+w)f_x + 2(uv+vw+uw)f_y + 3uvw\,f_z$

$= xf_x + 2yf_y + 3zf_z$

$=$ R.H.S

③ If $z = f(x,y)$, $x = e^{2u} + e^{-2v}$, $y = e^{-2u} + e^{2v}$ then P.T $z_u - z_v = 2(xz_x - yz_y)$

Given " " "

$z_u = z_x x_u + z_y y_u$

$z_u = z_x(2e^{2u}) + z_y(-2e^{-2u})$

$z_v = z_x x_v + z_y y_v$

$\therefore Z_v = Z_x(-2e^{-2v}) + Z_y(2e^{2v})$

$\therefore$ L.H.S $= Z_u - Z_v$

$= 2Z_x(e^{2u} + e^{-2v}) + 2Z_y(-e^{-2u} - e^{2v})$

$= 2[x \cdot Z_x - y Z_y] =$ R.H.S

4) If $u = f(x-y, y-z, z-x)$ then p.T $u_x + u_y + u_z = 0$

Sol: Given $u = f(x-y, y-z, z-x)$ — ①

let $p = x-y$, $q = y-z$, $r = z-x$

$u = f(p, q, r)$

$u_x = u_p p_x + u_q q_x + u_r r_x$

$u_x = u_p(1) + u_q(0) + u_r(-1)$

$u_x = u_p - u_r$ — ②

$u_y = u_p p_y + u_q q_y + u_r r_y$

$u_y = u_p(-1) + u_q(1) + u_r(0)$

$u_y = -u_p + u_q$ — ③

$u_z = u_p p_z + u_q q_z + u_r r_z$

$u_z = u_p(0) + u_q(-1) + u_r(1)$

$u_z = -u_q + u_r$ — ④

Adding ②, ③ & ④ we get

$u_x + u_y + u_z = 0$.

⑤ If $z = f(x,y)$, $x = u\cos\alpha + v\sin\alpha$, $y = u\sin\alpha + v\cos$
**
then P.T $Z_{uu} + Z_{vv} = Z_{xx} + Z_{yy}$

Sol Given " " "

$$Z_u = \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(\cos\alpha) + \frac{\partial z}{\partial y}(\sin\alpha)$$

$$\frac{\partial}{\partial u}(z) = \left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right)z.$$

$$Z_{uu} = \frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)$$

$$= \left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right)\left(\cos\alpha\frac{\partial z}{\partial x} + \sin\alpha\frac{\partial z}{\partial y}\right.$$

$$\frac{\partial^2 z}{\partial u^2} = \cos^2\alpha\frac{\partial^2 z}{\partial x^2} + 2\sin\alpha\cos\alpha\frac{\partial^2 z}{\partial x\cdot\partial y} + \sin^2\alpha\frac{\partial^2 z}{\partial y^2} \quad \text{—— (1)}$$

$$Z_v = \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-\sin\alpha) + \frac{\partial z}{\partial y}\cos\alpha$$

$$\frac{\partial}{\partial v}(z) = \left(-\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\cdot\frac{\partial}{\partial y}\right)z.$$

$$Z_{vv} = \frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)$$

$$= \left(-\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}\right)\left(-\sin\alpha\frac{\partial z}{\partial x} + \cos\alpha\frac{\partial z}{\partial y}\right)$$

$$= +\sin^2\alpha \frac{\partial^2 z}{\partial x^2} - 2\sin\alpha\cos\alpha \frac{\partial^2 z}{\partial x \partial y} + \cos^2\alpha \frac{\partial^2 z}{\partial y^2} \quad (2)$$

$$z_{uu} + z_{vv} = z_{xx}(1) + z_{yy}(1)$$

$$= z_{xx} + z_{yy}$$

Implicit Function:- (Problems)

① If $f = xe^{-y} - 2ye^{x} = 1$ then find $\frac{dy}{dx}$

Sol Let $f = xe^{-y} - 2ye^{x} - 1 = 0$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-(e^{-y} - 2ye^{x})}{-xe^{-y} - 2e^{x}}$$

$$= \frac{e^{-y} - 2ye^{x}}{xe^{-y} + 2e^{x}}$$

② If $(\cos x)^y - (\sin y)^x = 0$ then find $\frac{dy}{dx}$

Sol Let $f = (\cos x)^y - (\sin y)^x = 0$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-[y(\cos x)^{y-1}(-\sin x) - (\sin y)^x \log \sin y}{(\cos x)^y \log \cos x - x(\sin y)^{x-1}\cos y}$$

$$= \frac{\cancel{(\cos x)^y}\left[y\frac{\sin x}{\cos x} + \log \sin y\right]}{\cancel{(\cos x)^y}\left[\log\cos x - x\frac{\cos y}{\sin y}\right]}$$

$$= \frac{y\tan x + \log \sin y}{\log\cos x - x\cot y}$$

$$\frac{dy}{dx} = \frac{y\tan x + \log(\sin y)}{\log\cos x - x\cot y}$$

③ If $f(x,y)=0$ & $\phi(y,z)=0$ then P.T

~~dy~~ $f_y \phi_x \frac{dz}{dx} = f_x \phi_y$

Sol: Given $f(x,y)=0$ and $\phi(y,z)=0$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} \quad \& \quad \frac{dz}{dy} = \frac{-\phi_y}{\phi_z}$$

Multiplying

$$\frac{dy}{dx} \times \frac{dz}{dy} = \left(\frac{-f_x}{f_y}\right)\left(\frac{-\phi_y}{\phi_z}\right)$$

$$f_y \phi_z \frac{dz}{dx} = f_x \phi_y$$

Taylors Series for the function of 2 independent variables about (a,b)

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2}!\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)f_{yy}(a,b)\right] + \frac{1}{3}!\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \cdots$$

Taylor Series (Problems):-

① Find the taylor series of the following functions

(i) e^{2x+y} about $(0,0)$

sol:- Let $f = e^{2x+y} \Rightarrow f(0,0) = e^0 = 1$

$f_x = 2e^{2x+y} \Rightarrow f_x(0,0) = 2$

$f_{xx} = 4e^{2x+y} \Rightarrow f_{xx}(0,0) = 4$

$f_y = e^{2x+y} \Rightarrow f_y(0,0) = 1$

$f_{yy} = e^{2x+y} \Rightarrow f_{yy}(0,0) = 1$

$f_{xy} = 2e^{2x+y} \Rightarrow f_{xy}(0,0) = 2$

. . . . so on

We have from taylors series

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)f_{yy}(a,b) + \cdots$$

$$f(x,y) = f(0,0) + (x-0)f_x(0,0) + (y-0)f_y(0,0) + \frac{1}{2!}[(x-0)^2 f_{xx}(0,0) + 2(x-0)(y-0)f_{xy}(0,0) + (y-0)^2 f_{yy}(0,0) + \cdots$$

$$e^{2x+y} = 1 + x(2) + y(1) + \frac{1}{2!}[x^2(4) + 2xy(2) + y^2(1) + \cdots]$$

$$= 1 + 2x + y + \frac{1}{2!}[4x^2 + 4xy + y^2] + \cdots$$

(ii) $e^x \log(1+y)$ about $(0,0)$

Sol: let $f = e^x \log(1+y) \Rightarrow f(0,0) = 0$

$f_x = e^x \log(1+y) \Rightarrow f_x(0,0) = 0$

$f_{xx} = e^x \log(1+y) \Rightarrow f_{xx}(0,0) = 0$

$f_y = \dfrac{e^x}{1+y} \Rightarrow f_y(0,0) = 1$

$f_{yy} = \dfrac{-e^x}{(1+y)^2} \Rightarrow f_y(0,0) = -1$

$f_{xy} = \dfrac{e^x}{1+y} \Rightarrow f_{xy}(0,0) = 1$

.... so on

We have from Taylor series

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)f_{yy}(a,b)\right] + \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] + \cdots$$

$$f(x,y) = f(0,0) + (x-0)f_x(0,0) + (y-0)f_y(0,0) + \frac{1}{2!}\left[(x-0)^2 f_{xx}(0,0) + 2(x-0)(y-0)f_{xy}(0,0) + (y-0)^2 f_{yy}(0,0)\right] + \cdots$$

$$e^x \log y = 0 + x(0) + y(1) + \frac{1}{2!}\left[x^2(0) + 2xy(1) + y^2(-1)\right] + \cdots$$

$$= y + \frac{1}{2!}\left[2xy - y^2\right] + \cdots$$

(iii) $\tan^{-1}\frac{y}{x}$ about $(1,1)$

sol: let $f = \tan^{-1}\frac{y}{x} \Rightarrow f(1,1) = \pi/4$

$$f_x = \frac{1}{1+y^2/x^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} \Rightarrow f_x(1,1) = \frac{-1}{2}$$

$$f_{xx} = \frac{2xy}{(x^2+y^2)^2} \Rightarrow f_{xx}(1,1) = \frac{1}{2}$$

$$f_y = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} \Rightarrow f_y(1,1) = \frac{1}{2}$$

$$f_{yy} = \frac{-2xy}{(x^2+y^2)^2} \Rightarrow f_{yy}(1,1) = \frac{-1}{2}$$

$$f_{xy} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow f_{xy}(1,1) = 0$$

so on

We have from taylor series

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)f_{yy}(a,b)\right] + \dots$$

$$f(x,y) = f(1,1) + (x-1)f_x(1,1) + (y-1)f_y(1,1) +$$
$$\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right] + \dots$$

$$f(1,1) = \frac{\pi}{4} + (x-1)\left(\frac{-1}{2}\right) + (y-1)\frac{1}{2} +$$
$$\frac{1}{2!}\left[(x-1)^2\left(\frac{1}{2}\right) + 2(x-1)(y-1)(0) + (y-1)^2\left(\frac{-1}{2}\right)\right] + \cdots$$
$$= \frac{\pi}{4} - \frac{x}{2} + \frac{y}{2} + \frac{1}{2!}\left[\frac{x^2}{2} - \frac{y^2}{2}\right] + \cdots$$

2) If $f = \tan^{-1}xy$ then find the approximate value of function $f(1.1, 0.8)$ by using taylor's series of linear approximation.

<u>Sol</u>: Let $f = \tan^{-1}xy$, $h = 0.1$, $k = -0.2$

$$f(1,1) = \tan^{-1}(1)(1) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$$

$$f_x = \frac{1}{1+x^2y^2}(y) \Rightarrow f_x(1,1) = \frac{1}{2}$$

$$f_y = \frac{1}{1+x^2y^2}(x) \Rightarrow f_y(1,1) = \frac{1}{2}$$

We have Taylor's series linear approximation

$$f(x,y) = f(a,b) + f_x(a,b) + k f_y(a,b)$$
$$f(x,y) = f(1,1) + f_x(1,1) + k f_y(1,1)$$
$$\tan^{-1}xy = \frac{\pi}{4} + (0.1)\frac{1}{2} + (-0.2)\frac{1}{2}$$
$$= \frac{3.14}{4} + \frac{1}{20} - \frac{1}{10} = 0.735$$

(3) Find the taylor's series of linear approximation for the function $2x^3+3y^3-4x^2y$ about $(1,2)$

Sol Let $f = 2x^3+3y^3-4x^2y \Rightarrow f(1,2) = 18$

$f_x = 6x^2 - 8xy \Rightarrow f_x(1,2) = -10$

$f_y = 9y^2 - 4x^2 \Rightarrow f_y(1,2) = 32$

We have Taylor's series of linear approximation

$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$

$f(x,y) = f(1,2) + (x-1)f_x(1,2) + (y-2)f_y(1,2)$

$2x^3+3y^3-4x^2y = 18 + (x-1)(-10) + (y-2)(32)$

(4) Find the taylor's series of the function $x^2+3y^2-9x-9y+26$ about $(2,2)$ upto its max. order.

Sol Let $f = x^2+3y^2-9x-9y+26 \Rightarrow f(2,2) = 6$

$f_x = 2x-9 \Rightarrow f_x(2,2) = -5$

$f_y = 6y-9 \Rightarrow f_y(2,2) = 3$

$f_{xx} = 2$, $f_{xy} = 0$, $f_{yy} = 6$

We have Taylor's series of linear approximation

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$$f(x,y) = f(2,2) + (x-2)f_x(2,2) + (y-2)f_y(2,2) + \frac{1}{2!}\left[(x-2)^2 f_{xx}(2,2) + 2(x-2)(y-2)f_{xy}(2,2) + (y-2)^2 f_{yy}(2,2)\right]$$

$$x^2+3y^2-9x-9y+26 = 6+(x-2)(-5)+(y-2)(3)$$
$$+\frac{1}{2!}\left[(x-2)^2(2)+(y-2)^2(6)\right]$$

Jacobians:-

In order to convert one system of coordinates into another system of coordinates we used a determinant. This determinants called the Jacobian of transformation. If u & v are function of 2 independent varaibles x & y then the Jacobian of u,v w.r.t x,y is given by determinant $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ and is written as $J\left(\frac{u,v}{x,y}\right)$ (or) $\frac{\partial(u,v)}{\partial(x,y)}$ $\|^{ly}$, the Jacobian of u,v,w w.r.t x,y,z is given by $\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$ and is written as

$$J\left(\frac{u,v,w}{x,y,z}\right) \text{ (or) } \frac{\partial(u,v,w)}{\partial(x,y,z)}$$

Properties:

① If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$

② If u, v are function of r, s & r, s are function of x, y then $\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(u,v)} = \frac{\partial(u,v)}{\partial(x,y)}$

NOTE:- If the function u & v of the I.v's x, y are functionally dependent then the Jacobian $\frac{\partial(u,v)}{\partial(x,y)} = 0$ otherwise $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$

Jacobians:- (Problem)

① Find the Jacobian of polar coordinates

Sol W.k.t $x = r\cos\theta$, $y = r\sin\theta$ are polar coordinates

$$J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(1) = r.$$

② Find the Jacobian of cylindrical coordinates.

Sol. W.K.T $x = r\cos\theta$, $y = r\sin\theta$ & $z = z$ are C.C.

$$J\left(\frac{x,y,z}{r,\theta,z}\right) = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1(r\cos^2\theta + r\sin^2\theta) = r(1) = r$$

③ Find the Jacobian of spherical coordinates

Sol W.K.T $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ are S.C

$$J\left(\frac{x,y,z}{r,\theta,\phi}\right) = \begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix}$$

$\cancel{\cos\theta\sin\phi}$ $\cancel{-r\sin\theta\cos\phi}$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & -r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

④ Find the Jacobi If $u=x^2+y^2$, $v=x^2-y^2$ then find $J\left(\frac{u,v}{x,y}\right)$

Sol Given " "

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix}$$

$$= -4xy - 4xy = -8xy.$$

⑤ If $u = 2xy$, $v = x^2-y^2$ & $x = r\cos\theta$, $y = r\sin\theta$, then find $J\left(\frac{u,v}{r,\theta}\right)$.

Sol Given " " " "

$u = 2xy = 2r\cos\theta\, r\sin\theta = r^2\sin 2\theta$

$v = x^2-y^2 = r^2\cos^2\theta - r^2\sin^2\theta = r^2\cos 2\theta$

$$J\left(\frac{u,v}{r,\theta}\right) = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix} = \begin{vmatrix} 2r\sin 2\theta & 2r^2\cos 2\theta \\ 2r\cos 2\theta & -2r^2\sin 2\theta \end{vmatrix}$$

$$= -4r^3\sin^2(2\theta) - 4r^3\cos^2(2\theta)$$

$$= -4r^3(1) = -4r^3.$$

⑥ If $u = x+y+z$, $uv = y+z$ & $uvw = z$ then P.T $J\left(\frac{x,y,z}{u,v,w}\right) = u^2 v$.

Sol Given $u = x+y+z \Rightarrow u(1-v) = x$

$uv = y+z \Rightarrow uv(1-w) = y$

$uvw = z$

$$\therefore J\left(\frac{x,y,z}{u,v,w}\right) = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix} = u^2v \text{ Check}$$

$$= (1-v)\left\{[u^2v(1-w)] + [+u^2vw]\right\} + u\left[uv^2(1-w) + uv^2w\right]$$

$$= (1-v)\left\{u^2v - \cancel{u^2vw} + \cancel{u^2vw}\right\} + u\left[uv^2 - \cancel{uv^2w} + \cancel{uv^2w}\right]$$

$$= u^2v - \cancel{u^2v^2} + \cancel{u^2v^2} = u^2v.$$

pt

(7) If $x = r\cos\theta$, $y = r\sin\theta$, then $J \cdot J' = 1$

Sol Given $x = r\cos\theta$ —(1), $y = r\sin\theta$ —(2)

$$J\left(\frac{x, y}{r, \theta}\right) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(1) = r.$$

$$J'\left(\frac{r, \theta}{x, y}\right) = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix}$$

squaring & adding (1) & (2), we get

$$x^2 + y^2 = r^2$$

$$\cancel{2}x = \cancel{2}r\frac{\partial r}{\partial x} \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

|||ly $\cancel{2}y = \cancel{2}r\frac{\partial r}{\partial y} \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}.$

Dividing $\frac{(2)}{(1)}$ i.e., $\frac{y}{x} = \frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta}$

$$\frac{y}{x} = \tan\theta \Rightarrow \boxed{\theta = \tan^{-1}\frac{y}{x}}$$

$$\Rightarrow \theta_x = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} = \frac{-y}{r^2}$$

$\theta_y = \dfrac{1}{1+\frac{y^2}{x^2}}\left(\dfrac{1}{x}\right) = \dfrac{x}{x^2+y^2} = \dfrac{x}{r^2}$

i.e, $\begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{vmatrix} = \dfrac{x^2}{r^3} + \dfrac{y^2}{r^3} = \dfrac{x^2+y^2}{r^3} = \dfrac{r^2}{r^3} = \dfrac{1}{r}$

$\therefore JJ' = \cancel{r} \times \dfrac{1}{\cancel{r}} = 1$

(5) If $x = u(1-v)$ & $y = uv$ then P.T $JJ' = 1$

Sol Given $x = u(1-v) = u - uv = u - y \Rightarrow x+y = u$

$y = uv \Rightarrow \frac{y}{u} = v \Rightarrow \frac{y}{x+y} = v$

$J\left(\dfrac{x,y}{u,v}\right) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$

$= u - \cancel{uv} + \cancel{uv} = u$

$J'\left(\dfrac{u,v}{x,y}\right) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{-y}{u^2} & \frac{x}{u^2} \end{vmatrix}$

$v_x = \dfrac{-y}{(x+y)^2}$ & $v_y = \dfrac{(x+y)(1) - y(1)}{(x+y)^2} = \dfrac{x}{(x+y)^2} = \dfrac{x}{u^2}$

$= \dfrac{-y}{u^2}$

$\therefore J'\left(\dfrac{u,v}{x,y}\right) = \dfrac{x}{u^2} + \dfrac{y}{u^2} = \dfrac{x+y}{u^2} = \dfrac{u}{u^2} = \dfrac{1}{u}$

$\therefore J \cdot J' = \cancel{u} \cdot \dfrac{1}{\cancel{u}} = 1$

(9) If $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$ then find $J\left(\frac{u,v,w}{x,y,z}\right)$:

Sol Given " " "

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= yz(2y-2z) - xz(2x-2z) + xy(2x-2y$$

(10) If $u = x^2 - 2xy$, $V = x+y+z$, $W = x-2y+3z$ then find $J\left(\frac{u,v,w}{x,y,z}\right)$ at $(1,1,1)$.

Sol Given " " "

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} u_x & u_y & u_z \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix} = \begin{vmatrix} 2x-2y & -2x & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= (2x-2y)(3+2) + 2x(3-1) + 0$$

$$= 5(2x-2y) + 4x$$

At $(1,1,1)$ $J\left(\frac{u,v,w}{x,y,z}\right) = 4$

For 3 inder if ask J' then: for $JJ' =$ $\Rightarrow J' = \frac{1}{J}$

Maxima & Minima of functions of two variables with & without constraints:- (in exam)

Let $u = f(x, y)$

(i) Calculate p, q, r, s and t where $p = u_x$; $q = u_y$; $r = u_{xx}$, $s = u_{xy}$, $t = u_{yy}$.

(ii) Equate p & q to zero & find x & y which gives stationary points.

(iii) At each stationary point find the values of r, s and t

(iv) If $rt - s^2 > 0$ & $r > 0$, then the function $f(x, y)$ will have a minima at that point.

(v) If $rt - s^2 > 0$ and $r < 0$, then the function $f(x, y)$ will have a maxima at that point

(vi) If $rt - s^2 < 0$, then the point is called saddle point or minimax.

(vii) If $rt - s^2 = 0$, then the case is undecided and requires further investigation

Discuss the maxima & minima of the following function

① x^3+y^3-3xy.

Sol Let $f=x^3+y^3-3xy$

$p=f_x=3x^2-3y$, $q=f_y=3y^2-3x$

$r=f_{xx}=6x$, $s=f_{xy}=-3$, $t=f_{yy}=6y$

Equate to zero p & q i.e, $x^2-y=0 \Rightarrow y=x^2$
$y^2-x=0 \Rightarrow x=y^2$

i.e, $y^4-y=0 \Rightarrow y(y^3-1)=0 \Rightarrow y=0$ (or) $y=1$

|||ly $x^4-x=0 \Rightarrow x(x^3-1)=0 \Rightarrow x=0$ (or) $x=1$

Hence $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ are the stationary point.

$rt-s^2=(6x)(6y)-9$

$rt-s^2_{(0,0)}=-9<0$

Hence $(0,0)$ is a saddle point

$rt-s^2_{(0,1)}=-9<0$ & $rt-s^2_{(1,0)}=-9<0$

Hence $(0,1)$ & $(1,0)$ are also saddle point.

$rt-s^2_{(1,1)}=27>0$ $r_{(1,1)}=6>0$

Hence the function is minimum at $(1,1)$

$\therefore f_{min}=-1$

(2) $x^3+3xy^2-3x^2-3y^2+4$

Sol Let $f = x^3+3xy^2-3x^2-3y^2+4$

$p = f_x = 3x^2+3y^2-6x$, $q = f_y = 6xy-6y$

$r = f_{xx} = 6x-6$, $s = f_{xy} = 6y$, $t = f_{yy} = 6x-6$

Equate to zero p & q i.e, $x^2+y^2-2x = 0$ — (1)

$y(x-1) = 0$

$\Rightarrow y = 0$ (or) $x = 1$

Put $y = 0$ in (1), we get

$x^2-2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0$ (or) $x = 2$

Put $x = 1$ in (1), we get

$y^2-1 = 0 \Rightarrow y = \pm 1$

Hence $(0,0), (2,0), (1,1), (1,-1)$ are the stationary points.

$rt-s^2 = (6x-6)^2 - 36y^2$

$rt-s^2_{(0,0)} = 36 > 0$ & $r_{(0,0)} = -6 < 0$

Hence the function is max at $(0,0)$

$f_{max} = 4$

$rt-s^2_{(2,0)} = 36 > 0$ & $r_{(2,0)} = 6 > 0$

Hence the function is min at $(2,0)$

$f_{min} = 0$

$rt-s^2_{(1,1)} = -36 < 0$ & $rt-s^2_{(1,-1)} = -36 < 0$

Hence $(1,1)$ & $(1,-1)$ are saddle points.

③ $2x^2-2y^2-x^4+y^4$

Sol Let $f = 2x^2-2y^2-x^4+y^4$

$p = f_x = 4x-4x^3$, $q = f_y = -4y+4y^3$

$r = f_{xx} = 4-12x^2$, $s = f_{xy} = 0$, $t = f_{yy} = -4+12y^2$

Equate p & q to zero i.e, $x(1-x^2) = 0$

$y(y^2-1) = 0$

$\Rightarrow x = 0$ (or) $x = \pm 1$

$\Rightarrow y = 0$ or $y = \pm 1$

Hence $(0,0)$, $(0,\pm 1)$, $(\pm 1, 0)$ $(\pm 1, \pm 1)$ are the stationary points

$rt-s^2 = (4-12x^2)(12y^2-4) - 0$

$rt-s^2_{(0,0)} = -16 < 0$

Hence $(0,0)$ is a saddlepoint.

$rt-s^2_{(0,\pm 1)} = 32 > 0$ and $r_{(0,\pm 1)} = 4 > 0$

Hence ~~(0,±1)~~ the function is minimum at $(0,\pm 1)$ $\therefore f_{min} = -1$

$rt-s^2_{(\pm 1, 0)} = +32 > 0$ & $r_{(\pm 1, 0)} = -8 < 0$.

Hence the function is maximum at $(\pm 1, 0)$

$\therefore f_{max} = 1$

$rt-s^2_{(\pm 1, \pm 1)} = -64 < 0$

Hence $(1,1)$, $(-1,-1)$; $(1,-1)$ & $(-1,1)$ are saddle points.

④ $x^3+y^3-12x-3y+20$.

Sol. let $f = x^3+y^3-12x-3y+20$

$p = f_x = 3x^2-12$; $q = f_y = 3y^2-3$

$r = f_{xx} = 6x$, $s = f_{xy} = 0$, $t = f_{yy} = 6y$

Equate p & q to zero i.e., $x^2-4=0$

$\Rightarrow x = \pm 2$ $y^2-1=0$

$\Rightarrow y = \pm 1$

Hence $(2,1), (2,-1), (-2,1)$ $(-2,-1)$ are the stationary points

$rt-s^2 = (6x)(6y) - 0 = 36xy$

$rt-s^2_{(2,1)} = 72 > 0$ & $r_{(2,1)} = 12 > 0$

Hence the function is min. at $(2,1)$ and min value i.e, $f_{min} = 2$.

$rt-s^2_{(2,-1)} = -72 < 0$ $rt-s^2_{(-1,2)} = -72 < 0$

Hence $(2,-1)$ & $(-1,2)$ are saddle points

$rt-s^2_{(-2,-1)} = 72 > 0$ & $r_{(-2,-1)} = -12 < 0$

Hence the function is maximum at $(-2,-1)$

$\therefore f_{max} = 38$

⑤ $x^3y^2(1-x-y)$

Sol Let $f = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

$p = f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$; $q = f_y = 2x^3y - 2x^4y - 3x^3y^2$

$f_x = x^2y^2(3-4x-3y)$ $\quad$ $q = f_y = x^3y(2-2x-3y)$

$r = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3 = xy^2(6-12x-6y)$

$s = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2 = x^2y(6-8x-9y)$

$t = f_{yy} = 2x^3 - 2x^4 - 6x^3y = x^3(2-2x-6y)$

Equate p & q to zero i.e, $x^2y^2(3-4x-3y)=0$

$\Rightarrow x=0$ (or) $y=0$ (or) $4x+3y=3$

$x^3y(2-2x-3y)=0$

$\Rightarrow x=0$ (or) $y=0$ (or) $2x+3y=2$

$4x+3y=3$ — ①

$2x+3y=2$ — ②

$2x=1 \Rightarrow x=1/2 \quad \therefore y=1/3$

Hence $(0,0)$, $\left(\frac{1}{2}, \frac{1}{3}\right)$, $\left(0, \frac{2}{3}\right)$, $(1,0)$, $(0,1)$, $\left(\frac{3}{4}, 0\right)$ are stationary points.

sub x=0 in ②   sub y=0 in ②   sub x=0 in ①   sub y=0 in ①

$rt-s^2 = x^4y^2(6-12x-6y)(2-2x-6y) - x^4y^2(6-8x-9y)^2$

$rt-s^2_{(0,0)} = 0$; $rt-s^2_{\left(0,\frac{2}{3}\right)} = 0$; $rt-s^2_{(1,0)} = 0$; $rt-s^2_{(0,1)} = 0$;

$rt-s^2_{\left(\frac{3}{4},0\right)} = 0$

Hence $(0,0)$, $\left(0,\frac{2}{3}\right)$, $(1,0)$, $(0,1)$ & $\left(\frac{3}{4},0\right)$ the cases were undecided

$rt-s^2_{\left(\frac{1}{2},\frac{1}{3}\right)} = \frac{1}{16}\times\frac{1}{9}\left(\cancel{6}-\cancel{6}-\frac{6}{3}\right)\left(\cancel{2}-1-\cancel{2}\right) - \frac{1}{16}\times\frac{1}{9}\left(6-4-3\right)^2$

$= 0.00694 > 0$ $\frac{}{144}$ $\frac{}{144}$

$r_{(\frac{1}{2},\frac{1}{3})} = \frac{-1}{9} < 0$ $\frac{1}{9} < 0$

Hence the function is max at $(\frac{1}{2}, \frac{1}{3})$

$\therefore f_{max} = \frac{1}{432}$ (check)

(6) $x^2+y^2+\frac{2}{x}+\frac{2}{y}$

Let $f = x^2+y^2+\frac{2}{x}+\frac{2}{y}$

$p = f_x = 2x - \frac{2}{x^2}$, $q = f_y = 2y - \frac{2}{y^2}$

$r = f_{xx} = 2+\frac{4}{x^3}$, $s = f_{xy} = 0$, $t = f_{yy} = 2+\frac{4}{y^3}$

Equate p & q to zero i.e, $2x - \frac{2}{x^2} = 0$

$\Rightarrow 2x^3 - 2 = 0$

$\Rightarrow x^3 - 1 = 0 \Rightarrow x = 1$

$2y - \frac{2}{y^2} = 0$

$\Rightarrow 2(y^3-1) = 0 \Rightarrow y = 1$

Hence $(1,1)$ is stationary points

$rt - s^2 = \left(2+\frac{4}{x^3}\right)\left(2+\frac{4}{y^3}\right) - 0$

$rt-s^2_{(1,1)} = 36 > 0$ & $r_{(1,1)} = 6 > 0$

Hence the function is minimum at $(1,1)$

and $f_{min} = 6$.

① $x^4 + 2x^2y - x^2 + 3y^2$

let $f = x^4 + 2x^2y - x^2 + 3y^2$

$p = f_x = 4x^3 + 4xy - 2x$, $q = f_y = 2x^2 + 6y$

$r = f_{xx} = 12x^2 + 4y - 2$, $s = f_{xy} = 4x$, $t = f_{yy} = 6$

Equate p & q to zero i.e,

$x(4x^2 + 4y - 2) = 0$

$2x(2x^2 + 2y - 1) = 0$

$x = 0$ (or) $2x^2 + 2y - 1 = 0$ — ①

$2x^2 + 6y = 0$ — ②

Solving ① and ②

$2x^2 + 2y - 1 = 0$

$-\cancel{2x^2} - 6y = 0$

$-4y - 1 = 0$

$\boxed{y = -1/4}$

$2x^2 - \frac{3}{2} = 0$

$x = \pm\frac{\sqrt{3}}{2}$

Sub $x = 0$ in ②

$2x^2 + 6y = 0$

$\boxed{y = 0}$

Hence $(0,0)$, $(0, -\frac{1}{4})$, $(\pm\frac{\sqrt{3}}{2}, -\frac{1}{4})$ are the stationary points

$rt - s^2 = (12x^2 + 4y - 2)(6) - 16x^2$

$= 72x^2 + 24y - 12 - 16x^2$

$= 56x^2 + 24y - 12$

$rt - s^2_{(0,0)} = -12 < 0$ & $rt - s^2_{(0,-\frac{1}{4})} = -18 < 0$

$\therefore (0,0), \left(0, -\frac{1}{4}\right)$ are saddle points

$$rt-s^2 \left(\pm\frac{\sqrt{3}}{2}, \frac{-1}{4}\right) = \cancel{56}\left(\frac{3}{\cancel{4}}\right) + \cancel{24}\left(\frac{-1}{\cancel{4}}\right) - 12 = 42-18 = 24 > 0$$

$$\text{and } r = \cancel{12}\left(\frac{3}{\cancel{4}}\right) + \cancel{4}\left(\frac{-1}{\cancel{4}}\right) - 2 = 9-3 = 6 > 0$$

Hence at $\left(\pm\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$ the function is minimum.

$$f_{min} = \frac{-3}{8}$$

Langrange's Method of Multipliers :-

Suppose it is required to find the extrema for a function $f(x,y,z)$ subject to the condition $\phi(x,y,z) = 0$ — ①

Now construct the Langrange's function

$$F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z) \quad \text{— ②}$$

where λ is called the L.M, which is determined by the condition.

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0$$

$$\therefore \left.\begin{aligned} f_x + \lambda \phi_x &= 0; \\ f_y + \lambda \phi_y &= 0 \\ f_z + \lambda \phi_z &= 0 \end{aligned}\right\} \text{— ③}$$

from ③, we find the value of λ and the Substitute it in each of ③ we get a stationary point of $f(x,y,z)$. Then find the maxima & minima of function.

① Find the point upto the plane $ax+by+cz=p$ at which the function $x^2+y^2+z^2$ is min. Also finds its min value.

Sol. let $f = x^2+y^2+z^2$, S.T.C $\phi = ax+by+cz-p=0$

construct a lagranges function as follows

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$

$$= x^2+y^2+z^2+\lambda(ax+by+cz-p)$$

$\therefore F_x=0$, $F_y=0$ & $F_z=0$

$F_x = 2x+a\lambda = 0 \Rightarrow x = -\frac{a\lambda}{2}$

$2y+b\lambda = 0 \Rightarrow y = \frac{-b\lambda}{2}$

$2z+b\lambda = 0 \Rightarrow z = \frac{-c\lambda}{2}$

Using all theree in S.T.C

i.e, $ax+by+cz-p=0$

$$-\frac{a^2\lambda}{2} - \frac{b^2\lambda}{2} - \frac{c^2\lambda}{2} = p$$

$$-\frac{\lambda}{2}(a^2+b^2+c^2) = p$$

$$-\frac{\lambda}{2} = \frac{p}{a^2+b^2+c^2}$$

$\therefore x = \frac{ap}{a^2+b^2+c^2}$, $y = \frac{bp}{a^2+b^2+c^2}$, $z = \frac{cp}{a^2+b^2+c^2}$.

$$\therefore f_{min} = \frac{a^2 \cdot p^2}{(a^2+b^2+c^2)^2} + \frac{b^2 \cdot p^2}{(a^2+b^2+c^2)^2} + \frac{c^2 \cdot p^2}{(a^2+b^2+c^2)^2}$$

$$= \frac{p^2(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2} = \frac{p^2}{a^2+b^2+c^2}$$

② Find the stationary points of the function $a^3x^2+b^3y^2+c^2z^2$ S.T.C $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

Sol Let $f=a^3x^2+b^3y^2+c^3z^2$, S.T.C $\phi=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

Construct a lagranges function as follows,

$$F(x,y,z)=f(x,y,z)+\lambda\phi(x,y,z)$$

$$=a^3x^2+b^3y^2+c^3z^2+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$$

$\because F_x=0,\ F_y=0,\ \&\ F_z=0$

$$2a^2x-\frac{\lambda}{x^2}=0 \Rightarrow a^3=\frac{\lambda}{2x^3} \Rightarrow x=\left(\frac{\lambda}{2}\right)^{1/3}\cdot\frac{1}{a}$$

$$2b^2x-\frac{\lambda}{y^2}=0 \Rightarrow b^3=\frac{\lambda}{2y^3} \Rightarrow y=\left(\frac{\lambda}{2}\right)^{1/3}\frac{1}{b}$$

$$2c^2z-\frac{\lambda}{z^2}=0 \Rightarrow c^3=\frac{\lambda}{2z^3} \Rightarrow z=\left(\frac{\lambda}{2}\right)^{1/3}\cdot\frac{1}{c}$$

Using all these S.T.C

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$$

$$\frac{a}{\left(\frac{\lambda}{2}\right)^{1/3}}+\frac{b}{\left(\frac{\lambda}{2}\right)^{1/3}}+\frac{c}{\left(\frac{\lambda}{2}\right)^{1/3}}=1$$

$$\Rightarrow a+b+c=\left(\frac{\lambda}{2}\right)^{1/3}$$

$$\therefore x=\frac{a+b+c}{a},\ y=\frac{a+b+c}{b},\ z=\frac{a+b+c}{c}$$

③ Find the min value of the function $x^2+y^2+z^2$

S.T.C $xyz=a^3$ (or) $xyz=8$

Sol Let $f=x^2+y^2+z^2$; S.T.C $\phi=xyz-a^3=0$

Construct a lagrange's function as follows

$$F(x,y,z)=f(x,y,z)+\lambda\phi(x,y,z)$$

$$=x^2+y^2+z^2+\lambda(xyz-a^3)$$

$\because F_x=0$, $F_y=0$ and $F_z=0$

$$2x+\lambda yz=0 \Rightarrow x=\frac{-\lambda yz}{2} \Rightarrow x^2=\frac{-\lambda yzx}{2}$$

$$2y+\lambda xz=0 \Rightarrow y=\frac{-\lambda xz}{2} \Rightarrow y^2=\frac{-\lambda xyz}{2}$$

$$2z+\lambda xy=0 \Rightarrow z=\frac{-\lambda xy}{2} \Rightarrow z^2=\frac{-\lambda xyz}{2}$$

$$\therefore x^2=y^2=z^2$$

S.T.C $xyz=a^3$ S.B.S

$$x^2y^2z^2=a^6$$

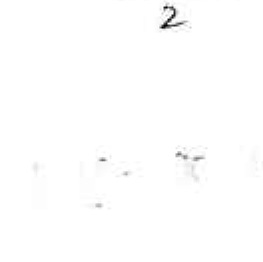
$$\therefore x=\pm a,\ y=\pm a,\ z=\pm a$$

$$\therefore f_{min}=a^2+a^2+a^2=3a^2$$

④ Find the point on the curve $x^2+xy+y^2=16$ which is nearest & farthest from the origin.

Dist b/w OP

Sol ~~Let f~~ $OP=\sqrt{x^2+y^2} \Rightarrow (OP)^2=x^2+y^2$



Let $f=x^2+y^2$; S.T.C $\phi=x^2+xy+y^2-16=0$ ——(1)

Construct a lagrange's function as follows

(2) $F(x,y) = f(x,y) + \lambda\phi(x,y)$

$= x^2+y^2+\lambda(x^2+xy+y^2-16)$

Sol $\because F_x = 0,\ F_y = 0$

$2x + 2\lambda x + \lambda y = 0 \Rightarrow 2x = -\lambda(2x+y)$

$2y + \lambda x + 2\lambda y = 0 \Rightarrow 2y = -\lambda(x+2y)$

$\therefore \dfrac{\cancel{2}x}{\cancel{2}y} = \dfrac{\cancel{-\lambda}(2x+y)}{\cancel{-\lambda}(x+2y)}$

$x^2 + \cancel{2xy} = \cancel{2xy} + y^2$

$x^2 = y^2 \Rightarrow x = \pm y$

Put $x = y$ in ①, we get. $y^2+y^2+y^2=16$

$3y^2 = 16$

$y = \pm\dfrac{4}{\sqrt{3}}$

Put $x=-y$ in ①, we get $y^2 - \cancel{y^2} + \cancel{y^2} = 16$

$\Rightarrow y^2 = 16$

$\Rightarrow y = \pm 4$

Hence $\left(\pm\frac{4}{\sqrt{3}}, \pm\frac{4}{\sqrt{3}}\right)$, $(-4,4)$ & $(4,-4)$ are the stationary point.

At $\left(\pm\frac{4}{\sqrt{3}}, \pm\frac{4}{\sqrt{3}}\right)$, $OP = \sqrt{x^2+y^2} = \sqrt{\frac{16}{3}+\frac{16}{3}} = \sqrt{\frac{32}{3}}$.

At $(-4,4)$ & $(4,-4)$, $OP = \sqrt{x^2+y^2} = \sqrt{16+16} = \sqrt{3}$

$(-4,4)$ & $(4,-4)$ is the farthest point from the origin to curve

$\left(\pm\frac{4}{\sqrt{3}}, \pm\frac{4}{\sqrt{3}}\right)$ is the Nearest point

5) Find the absolute max & min value of the function x^2-y^2-2y in the closed region $R: x^2+y^2\leq 1$.

Sol: Let $f=x^2-y^2-2y$; S.T.C $Q=x^2+y^2-1=0$

Construct a Langrange's function as follows

$F(x,y)=f(x,y)+\lambda Q(x,y)$

$=x^2-y^2-2y+\lambda(x^2+y^2-1)$

$\because F_x=0,\ F_y=0$

$2x+\lambda 2x=0 \Rightarrow 2x=-\lambda 2x$

$-2y-2+\lambda 2y=0 \Rightarrow 2y+2=\lambda 2y$

$\therefore \dfrac{\cancel{2}x}{\cancel{2}(y+1)}=\dfrac{-\cancel{\lambda}\cancel{2}x}{\cancel{\lambda}\cancel{2}y} \Rightarrow xy=-xy-x$

$2xy+x=0 \Rightarrow x(2y+1)=0$

$\Rightarrow x=0$ (or) $y=-1/2$

Put $x=0$ in $x^2+y^2=1$ i.e, $y=\pm 1$

put $y=\frac{-1}{2}$ in $x^2+y^2=1$ i.e, $x^2+\frac{1}{4}=1 \Rightarrow x=\pm\frac{\sqrt{3}}{2}$

Hence $(0,1)$ $(0,-1)$, $\left(\pm\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

At $\left(-\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$; $f=\frac{3}{4}-\frac{1}{4}+1=\frac{6}{4}=\frac{3}{2}$

At $(0,-1)$; $f=-1+2=1$

At $(0,1)$; $f_{min}=-1-2=-3$.

(6) Find the greatest volume of rectangular ||elopiped which is inscribed in ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Sol Let $2x, 2y, 2z$ be the l, b, h of the rectangular ||elopipped.

$$\therefore V = (2x)(2y)(2z) = 8xyz = f$$

S.T.C $\quad \phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Construct a lagranges function as follows:

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$$

$$= 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$\because F_x = 0, \; F_y = 0, \; F_z = 0$

$$\therefore 8yz + \frac{2\lambda x}{a^2} = 0 \Rightarrow 4yz = \frac{-\lambda x}{a^2}$$

$$\Rightarrow \frac{-4xyz}{\lambda} = \frac{x^2}{a^2}$$

$$8xz + \frac{2\lambda y}{b^2} = 0 \Rightarrow 4xz = \frac{-\lambda y}{b^2}$$

$$\Rightarrow \frac{-4xyz}{\lambda} = \frac{y^2}{b^2}$$

$$8xy + \frac{2\lambda z}{c^2} = 0 \Rightarrow 4xy = \frac{-\lambda z}{c^2}$$

$$\Rightarrow \frac{-4xyz}{\lambda} = \frac{z^2}{c^2}$$

$$\therefore \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Using them in S.T.C i.e, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

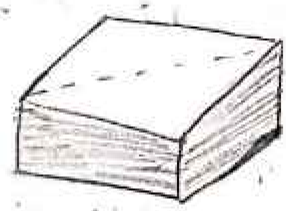
$$\frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1 \Rightarrow \frac{3x^2}{a^2} = 1 \Rightarrow x = \pm \frac{a}{\sqrt{3}}$$

III^{ly} $y = \frac{\pm b}{\sqrt{3}}$ & $z = \frac{\pm c}{\sqrt{3}}$

$$\therefore V = \frac{8abc}{3\sqrt{3}}$$

⑦ A rectangular box which is open at the top is to have volume 32 cu.ft. Find the dimension of the box requiring least material for its constructio

Sol: Let x, y, z be the l, b, h of the rectangular box



$V = xyz = 32$ (S.T.C) $\Rightarrow z = \frac{32}{xy}$

$S = xy + 2yz + 2zx$ (f) 5 faces not 6

$\therefore S = xy + \frac{64}{x} + \frac{64}{y}$

$p = S_x = y - \frac{64}{x^2}$; $q = S_y = x - \frac{64}{y^2}$; $r = S_{xx} = \frac{128}{x^3}$;

$s = S_{xy} = 1$, $t = S_{yy} = \frac{128}{y^3}$

Equate p & q to zero i.e, $y - \frac{64}{x^2} = 0$

$$\Rightarrow y = \frac{64}{x^2}$$

$$x - \frac{64}{y^2} = 0$$

$$\Rightarrow x = \frac{64}{y^2}$$

(2) $\therefore y = \frac{64}{(64)^2} \times y^4 \Rightarrow y^4 - 64y = 0$

$y(y^3 - 64) = 0 \Rightarrow y = 0$ (or) $y = 4$

Sol

III'ly we get $x = 0$ (or) $x = 4$

$\therefore (0,0)$ & $(4,4)$ are the stationary points

$rt-s^2$ At $(0,0)$ we get $z = \frac{32}{0} = \infty$ (absurd)

A$(4,4)$ we get $z = \frac{32}{16} = 2$

Hence, the dimensions required $x = 4ft$, $y = 4$,

and $z = 2ft$

Now $rt - s^2 = \left(\frac{128}{x^3}\right)\left(\frac{128}{y^3}\right) - 1$

$rt - s^2_{(4,4)} = (2)(2) - 1 = 3 > 0$, $r_{(4,4)} = 2 > 0$

Hence the function is min at $(4,4)$

$S_{min} = 48$

(8) Find the max distance of the point $(3,4,12)$ from the sphere $x^2 + y^2 + z^2 = 4$

Sol Let x, y, z be any point on the sphere

$\therefore (OP)^2 = \left(\sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}\right)^2 = f$

Constant Lagrange's function ~~F(x, y, z~~

$F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$

$F(x, y, z) = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 4)$

$\because f_x = 0,\ f_y = 0$ & $f_z = 0$

$2(x-3) + 2\lambda x = 0 \Rightarrow x - 3 = -\lambda x \Rightarrow x(1+\lambda) = 3$

$2(y-4) + 2\lambda y = 0 \Rightarrow y - 4 = -\lambda y \Rightarrow y(1+\lambda) = 4$

$2(z-12) + 2\lambda z = 0 \Rightarrow z - 12 = -\lambda z \Rightarrow z(1+\lambda) = 12$

$\therefore x = \dfrac{3}{1+\lambda}$; $y = \dfrac{4}{1+\lambda}$; $z = \dfrac{12}{1+\lambda}$ — (1)

Also $2x(x-3) + 2\lambda x^2 = 0 \Rightarrow 2x^2 - 6x + 2\lambda x^2 = 0$

$2y(y-4) + 2\lambda y^2 = 0 \Rightarrow 2y^2 - 4y + 2\lambda y^2 = 0$

$2z(z-12) + 2\lambda z^2 = 0 \Rightarrow 2z^2 - 24z + 2\lambda z^2 = 0$

Adding all we get $2(x^2+y^2+z^2) - 6x - 4y - 24z + 2\lambda(x^2+y^2+z^2) = 0$

$(x^2+y^2+z^2)[2 - 6x - 4y - 24z + 2\lambda] = 0$ $[\because x^2+y^2+z^2 \neq 0]$

$\therefore 1 - 3x - 2y - 12z + \lambda = 0 \Rightarrow 1 + \lambda = 3x + 4y + 12z$ — (2)

Using (1) in (2), we get

UNIT-4

Multivariable calculus (Integration)

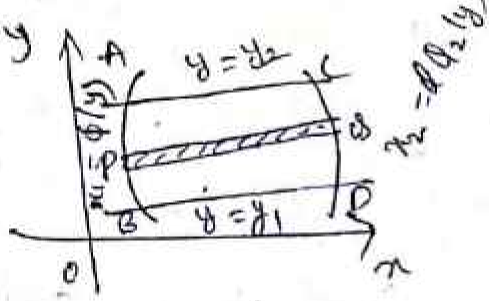
In multiple integrals we consider two types of integrals.

1. The double integrals ii) The triple integrals

Evaluation of Double integrals:

It depends upon the nature of the curves bounding the region. R. let the region R be bounded by the curves $x = x_1$, $x = x_2$ and $y = y_1$, $y = y_2$

Case (i):- when x_1 and x_2 are functions of y where as y_1 and y_2 are constants. Let the eqns of the curves AB and CD be $x = x_1 = \phi_1(y)$ and $x = x_2 = \phi_2(y)$ and let y_1 and y_2 be constants Hence AC and BD will be $\parallel^{le}$ to the x-axis.

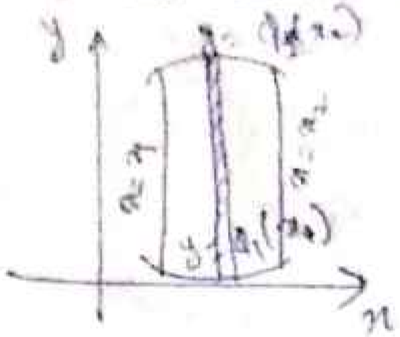


Take a H.S PQ of width δy. Then the D.I of the function $f(x,y)$ over the region R first evaluated w.r.t 'x' treating y as constant. The resulting expression which will be a function of y is integrated w.r.t 'y'. let the limits $y = y_1$ and $y = y_2$ $\therefore \iint_R f(x,y)\,dx\,dy$

$$= \int_{y_1}^{y_2} \boxed{\int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x,y)\,dx}\; dy$$

Here the I is carried out from the inner to the outer rectangle

<u>Case (ii)</u>: When y_1 and y_2 are functions of x when x_1 and x_2 are constants.



Let AB and CD be the curves $y_1 = \phi_1(x)$ and $y_2 = \phi_2(x)$ let x_1 and x_2 be constants as such AC and BD will be $\parallel^{el}$ to y-axis. Consider a vertical strip PQ of width δx. The D.I of $f(x,y)$ over the region R is first evaluated by integrating $f(x,y)$ w.r.t 'y' treating x as constant. let the limits $y = \phi_1(x)$ to $y = \phi_2(x)$

$$\therefore \iint_R f(x,y)\,dx\,dy = \int_{x=x_1}^{x_2} \boxed{\int_{y=\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy}\; dx$$

<u>Case: iii)</u> When x_1, x_2, y_1, y_2 are all constants

In this case, the region of integration ABDC i.e. R is a $\square^{le}$. Since x_1, x_2, y_1 and y_2 are all constants.

It is immaterial whether we integrate first along the H.S PQ and then slide it from AB to CD or integrate first along the V.S P'Q' and then slide the same from AC to BD

$$\therefore \iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x,y)\,dx\right]dy$$

$$= \int_{x_1}^{x_2}\left[\int_{y_1}^{y_2} f(x,y)\,dy\right]dx$$

change of order of integration:

In evaluation of a double integral by successive integration we may first integrate w.r.t 'y' and then w.r.t 'x' & vice versa. Given the region of integration R, we determine the limits of integration in the former case by taking a strip $\parallel^{el}$ to y-axis and in later case by taking a strip $\parallel^{el}$ to x-axis.

Note: changing from cartesian coordinates (x,y) to polar coordinates (r,θ) .. The elementary area dx dy in the Cartesian system is to be replaced by $r\,dr\,d\theta$ in the polar coordinates

Evaluation of triple integrals:

case(i): If $x_1, x_2, y_1, y_2, z_1, z_2$ are all constants then, the order of integration is immaterial provided the limits of integration are changed accordingly.

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_{y_1}^{y_2}\int_{z_1}^{z_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dz\,dy$$

$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx$$

case(ii):- If z_1, z_2 are the functions of x and y and y_1, y_2 are the functions of 'x' while x_1 and x_2 are constants, then the integration must be perform first w.r.t 'z' then w.r.t 'y' and finally w.r.t 'x'

$$\therefore \iiint f(x,y,z)\,dx\,dy\,dz$$

$$= \int_{x=a}^{b} \boxed{\int_{y=Q_1(x)}^{Q_2(x)} \boxed{\int_{z=g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz}\,dy}\,dx$$

where the integration is carried out from the inner most rectangle to the outer most rectangle

Note: If $f(x,y,z) = 1$, then $\iiint\limits_V dx\,dy\,dz$

$$= \iiint\limits_V dv$$

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Double Integral Problems

① $\int_1^a \int_1^b \frac{dy\,dx}{xy}$

Sol: Given,

$$\int_1^a \int_1^b \frac{dy\,dx}{xy}$$

$$\int_{x=1}^{a} \int_{y=1}^{b} \frac{1}{xy}\,dy\,dx$$

$$\int_{x=1}^{a} \frac{1}{x}\left[\log y\right]_1^b dx.$$

$$\int_{x=1}^{a} \frac{1}{x}\left[\log b\right] dx$$

$$\log b\left[\log x\right]_1^a$$

$$\log b \log a$$

② $\int_0^1 \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-y^2)}}\,dy\,dx$

Sol: Given,

$$\int_0^1 \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-y^2)}}\,dy\,dx$$

$$\int_{x=0}^{1} \int_{y=0}^{1} \frac{1}{\sqrt{(1-x^2)(1-y^2)}}\,dy\,dx$$

$$\int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}}\left[\sin^{-1} y\right]_0^1 dx$$

$$\int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}}\left[\frac{\pi}{2}-0\right] dx$$

$$\frac{\pi}{2}\int_{x=0}^{1}\frac{1}{\sqrt{1-x^2}}\,dx$$

$$\frac{\pi}{2}\left(\sin^{-1}x\right)_0^1 \Rightarrow \frac{\pi}{2}\times\frac{\pi}{2} = \frac{\pi^2}{4}$$

③ $$\int_0^2\int_1^2 (x^3+y^3)\,dx\,dy$$

Sol: Given, $$\int_{y=0}^{2}\int_{x=1}^{2}(x^3+y^3)\,dx\,dy$$

$$\int_{y=0}^{2}\left(\frac{x^4}{4}+y^3x\right)_1^2 dy$$

$$\int_{y=0}^{2}\left(\frac{16}{4}+2y^3\right)-\left(\frac{1}{4}+y^3\right)\,dy$$

$$\int_{y=0}^{2}\left(\frac{15}{4}+y^3\right)dy$$

$$\left(\frac{15}{4}y+\frac{y^4}{4}\right)_0^2$$

$$\frac{30}{4}+\frac{16}{4} = \frac{46}{4} = \frac{23}{2}$$

④ $$\int_0^{\pi/4}\int_{\sin x}^{\cos x} dy\,dx$$

Sol: Given,

$$\int_{x=0}^{\pi/4}\int_{y=\sin x}^{\cos x} dy\,dx$$

$$\int_{x=0}^{\pi/4}\left[y\right]_{\sin x}^{\cos x} dx$$

$$\int_{x=0}^{\pi/4} (\cos x - \sin x)\, dx$$

$$\Rightarrow \left(\sin x + \cos x\right)_0^{\pi/4} \Rightarrow \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \sin 0 - \cos 0$$

$$\Rightarrow \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1$$

$$\Rightarrow \sqrt{2} - 1$$

(5) $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$

Sol: Given

$$\int_{x=0}^{1} \int_{y=0}^{x^2} e^{y/x}\, dy\, dx$$

$$= \int_{x=0}^{1} \left(\frac{e^{y/x}}{1/x}\right)_0^{x^2} dx$$

$$= \int_{x=0}^{1} \left(x e^{x} - x\right) dx$$

$$= \left(e^{x}(x-1) - \frac{x^2}{2}\right)_0^1$$

$$= \left[\left(e(0) - \frac{1}{2}\right) - \left(1(0-1) - 0\right)\right] = -\frac{1}{2} + 1 = \frac{1}{2}$$

(6) $\int_0^1 \int_0^{1-x} e^{2x+3y}\, dy\, dx$

Sol: Given $\int_{x=0}^{1} \int_{y=0}^{1-x} e^{2x+3y}\, dy\, dx$

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$$= \int_{x=0}^{1} \left(\frac{e^{2x+3y}}{3} \right)_0^{1-x} dx$$

$$= \int_{x=0}^{1} \left(\frac{e^{2x+3(1-x)}}{3} - \frac{e^{2x}}{3} \right) dx$$

$$= \int_{x=0}^{1} \left(\frac{e^{3-x}}{3} - \frac{e^{2x}}{3} \right) dx$$

$$= \left(\frac{-e^{3-x}}{3} - \frac{e^{2x}}{6} \right)_0^1$$

$$= \left(-\frac{e^2}{3} - \frac{e^2}{6} \right) - \left(-\frac{e^3}{3} - \frac{1}{6} \right)$$

$$= -\frac{e^2}{3} - \frac{e^2}{6} + \frac{e^3}{3} + \frac{1}{6}$$

$$= \frac{-2e^2 - e^2 + 2e^3 + 1}{6} = \frac{2e^3 - 3e^2 + 1}{6}$$

(7) $\displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

Sol: Given,

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$$

$$= \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{(\sqrt{1+x^2})^2 + y^2}$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left(\tan^{-1} \frac{y}{\sqrt{1+x^2}} \right)_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left(\tan^{-1}(1) - \tan^{-1} 0 \right) dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left(\frac{\pi}{4}\right) dx$$

$$= \frac{\pi}{4} \left[\log|x + \sqrt{1+x^2}| \right]_0^1$$

$$= \frac{\pi}{4} \left[\log(1+\sqrt{2}) \right]$$

$$= \frac{\pi}{4} \log(1+\sqrt{2})$$

(3) $$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$$

Sol: Given,

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{(\sqrt{a^2-x^2})^2 - y^2}\, dy\, dx$$

$$\boxed{\because \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C}$$

i.e $$\int_0^a \left(\frac{y}{2}\sqrt{a^2-x^2-y^2} + \left(\frac{a^2-x^2}{2}\right)\sin^{-1}\frac{y}{\sqrt{a^2-x^2}} \right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left\{ \left\{ 0 + \left(\frac{a^2-x^2}{2}\right)\sin^{-1}(1) \right\} - \{0+0\} \right\} dx$$

$$= \int_0^a \frac{a^2-x^2}{2} \frac{\pi}{2}\, dx$$

$$= \frac{\pi}{4} \int_0^a a^2 - x^2\, dx$$

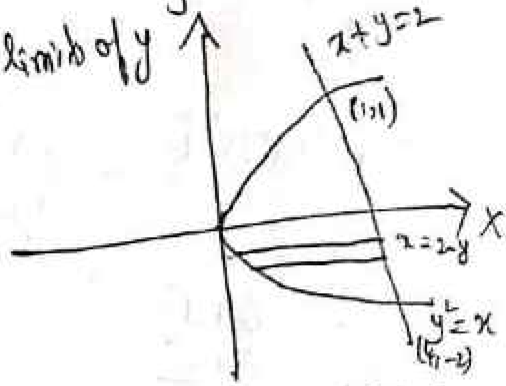
$$= \frac{\pi}{4} \left[a^2x - \frac{x^3}{3} \right]_0^a = \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right]$$

$$= \frac{\cancel{2} a^3 \pi}{\cancel{12}_6} = \frac{\pi a^3}{6}$$

D.I (problems).

9) Find the area of the region bounded by the curves $x = y^2$ and $x + y = 2$.

Sol: Given, $x = y^2$, which is a parabola about x-axis and $x + y = 2$, which is a line which cuts x-axis. & limits of y varies from -2 to 1.



$$\therefore \int_{-2}^{1}\int_{x=y^2}^{2-y} dx\, dy$$

$$= \int_{-2}^{1} \left\{x\right\}_{y^2}^{2-y} dy$$

$$= \int_{-2}^{1} (2 - y - y^2)\, dy$$

$$= \left(2y - \frac{y^2}{2} - \frac{y^3}{3}\right)_{-2}^{1}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - \frac{4}{2} + \frac{8}{3}\right)$$

$$= \frac{7}{2}.$$

Put $x = y^2$ in $x + y = 2$

we get $y^2 + y = 2$

$y^2 + y - 2 = 0$

$y = 1$ or -2

Put $y = 1$ in $x + y = 2$

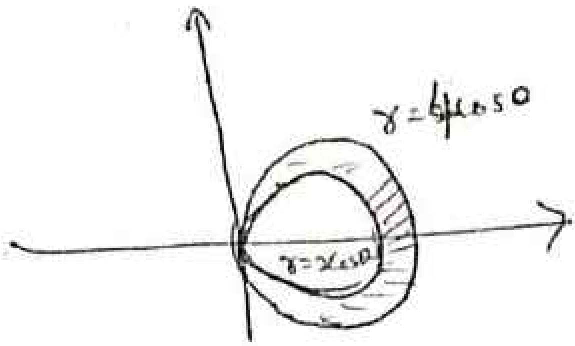
$\boxed{x = 1}$

Put $y = -2$ in $x + y = 2$

$\boxed{x = 4}$

(10) Evaluate $\iint_R r^3\,dr\,d\theta$ over the area between the circles. $r = 2\cos\theta$ and $r = 4\cos\theta$

Sol: Given $\int_{-\frac{\pi}{2}}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^3\,dr\,d\theta$



$$= \int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^3\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2}\left(256\cos^4\theta - 16\cos^4\theta\right)d\theta$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2} 240\cos^4\theta\,d\theta$$

$$= 60\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta$$

$$= 60\times 2\int_{0}^{\pi/2}\cos^4\theta\,d\theta$$

$$= \cancel{120}^{\cancel{30}\,15}\left[\frac{3\cdot 1}{\cancel{4}\cdot\cancel{2}_{1}}\frac{\pi}{2}\right]$$

$$= \frac{45\pi}{2}$$

Triple Integration problems.

① P.T $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz = (e-1)^3$

Sol: Given, $\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} e^{x+y+z}\,dx\,dy\,dz$

$$= \int_0^1\int_0^1 \left(e^{x+y+z}\right)_0^1 dy\,dz$$

$$= \int_0^1\int_0^1 \left(e^{1+y+z} - e^{y+z}\right) dy\,dz$$

$$= \int_0^1 \left[e^{1+y+z} - e^{y+z}\right]_0^1 dz$$

$$= \int_0^1 \left[\left(e^{2+z} - e^{1+z}\right) - \left(e^{1+z} - e^{z}\right)\right] dz$$

$$= \int_0^1 \left(e^{2+z} - 2e^{1+z} + e^{z}\right) dz$$

$$= \left[e^{2+z} - 2e^{1+z} + e^{z}\right]_0^1$$

$$= \left(e^3 - 2e^2 + e\right) - \left(e^2 - 2e + 1\right)$$

$$= e^3 - 3e^2 + 3e - 1$$

$$= (e-1)^3 = RHS$$

② $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$

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Sol: $$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx$$

$$\int_0^1\int_0^{\sqrt{1-x^2}} xy\left(\frac{z^2}{2}\right)_0^{\sqrt{1-x^2-y^2}} dy \, dx$$

$$= \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} xy(1-x^2-y^2) \, dy \, dx$$

$$= \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} \left(x(1-x^2)y - xy^3\right) dy \, dx$$

$$= \frac{1}{2}\int_0^1 \left[x(1-x^2)\frac{y^2}{2} - x\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}}$$

$$= \frac{1}{2}\int_0^1 \frac{x(1-x^2)(1-x^2)}{2} - \frac{x(1-x^2)^2}{4} \, dx$$

$$= \frac{1}{2}\int_0^1 \frac{x(1-x^2)^2}{2} - \frac{x(1-x^2)^2}{4} \, dx$$

$$= \frac{1}{2}\int_0^1 \frac{x(1-x^2)^2}{4} \, dx$$

$$= \frac{1}{8}\int_0^1 x(1-x^2)^2 \, dx$$

$$= \frac{1}{8}\int_0^1 x(1+x^4-2x^2) \, dx$$

$$= \frac{1}{8}\int_0^1 (x+x^5-2x^3) \, dx$$

$$= \frac{1}{8}\left[\frac{x^2}{2}+\frac{x^6}{6}-\frac{2x^4}{4}\right]_0^1$$

$$= \frac{1}{8}\left[\left(\frac{1}{2}+\frac{1}{6}-\frac{2}{4}\right)-0\right]$$

$$= \frac{1}{8}\left[\frac{\cancel{6}+2-\cancel{6}}{12}\right] \Rightarrow \frac{1}{48}$$ //

(3) $\int_{1}^{3}\int_{1/x}^{1}\int_{0}^{\sqrt{xy}} xyz \; dz\, dy\, dx$

Sol: Given,

$$\int_{x=1}^{3}\int_{y=1/x}^{1}\int_{z=0}^{\sqrt{xy}} xyz \; dz\, dy\, dx$$

$$\int_{1}^{3}\int_{1/x}^{1} xy\left(\frac{z^2}{2}\right)_0^{\sqrt{xy}} dy\, dx$$

$$= \frac{1}{2}\int_{1}^{3}\int_{1/x}^{1} xy(xy)\; dy\, dx$$

$$= \frac{1}{2}\int_{1}^{3}\int_{1/x}^{1} x^2y^2\; dy\, dx$$

$$= \frac{1}{2}\int_{1}^{3} x^2\left[\frac{y^3}{3}\right]_{1/x}^{1} dx$$

$$= \frac{1}{6}\int_{1}^{3} x^2\left[1-\frac{1}{x^3}\right] dx$$

$$= \frac{1}{6}\int_{1}^{3}\left(x^2-\frac{1}{x}\right) dx$$

$$= \frac{1}{6}\left[\frac{x^3}{3}-\log x\right]_1^3$$

$$= \frac{1}{6}\left[(9-\log 3)-\left(\frac{1}{3}-0\right)\right]$$

$$= \frac{1}{6}\left[\frac{26}{3}-\log 3\right]$$

(11) $\int_0^1 \int_0^x \int_0^{x+y} (x+y+z)\, dz\, dy\, dx$

Sol: Given,

$$= \int_{x=0}^{1} \int_{y=0}^{x} \int_{z=0}^{x+y} (x+y+z)\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^x \left(xz + yz + \frac{z^2}{2}\right)_0^{x+y} dy\, dx$$

$$= \int_0^1 \int_0^x \left[x(x+y) + y(x+y) + \frac{(x+y)^2}{2}\right] dy\, dx$$

$$= \int_0^1 \int_0^x \left[(x+y)^2 + \frac{(x+y)^2}{2}\right] dy\, dx$$

$$= \int_0^1 \int_0^x \frac{3(x+y)^2}{2}\, dy\, dx$$

$$= \frac{3}{2} \int_0^1 \int_0^x (x+y)^2\, dy\, dx$$

$$= \frac{3}{2} \int_0^1 \int_0^x (x^2+y^2+2xy)\, dy\, dx$$

$$= \frac{3}{2} \int_0^1 \left(x^2y + \frac{y^3}{3} + \frac{2xy^2}{2}\right)_0^x dx$$

$$= \frac{3}{2} \int_0^1 \left(x^3 + \frac{x^3}{3} + \frac{\cancel{2}x^3}{\cancel{2}}\right) dx$$

$$= \frac{3}{2} \int_0^1 \frac{7x^3}{3}\, dx$$

$$= \frac{\cancel{3}}{2} \times \frac{7}{\cancel{3}} \left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{7}{2} \times \frac{1}{4} = \frac{7}{8}$$

(5) $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$

Sol: Given;

$$= \int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$$

$$= \int_{-1}^{1}\int_{0}^{z} \left[xy + \frac{y^2}{2} + yz\right]_{x-z}^{x+z} dx\, dz$$

$$= \int_{-1}^{1}\int_{0}^{z} \left[x(x+z) + \frac{(x+z)^2}{2} + z(x+z)\right] - \left[x(x-z) + \frac{(x-z)^2}{2} + (x-z)z\right] dx dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left[\cancel{x^2} + xz + \frac{(x+z)^2}{2} + zx + z^2 - \cancel{x^2} + \cancel{xz} - \frac{(x-z)^2}{2} - \cancel{zx} + z^2\right] dx dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left[2xz + \frac{\overset{2}{\cancel{4}}xz}{\cancel{2}} + 2z^2\right] dx\, dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left[4xz + 2z^2\right] dx\, dz$$

$$= 2\int_{-1}^{1}\left(\frac{\overset{2}{\cancel{4}}zx^2}{\cancel{2}} + 2z^2x\right)_0^z dz$$

$$4\int_{-1}^{1}\left(z^3 + z^3\right) dz$$

$$4\int_{-1}^{1} 2z^3\, dz$$

$$8\int_{-1}^{1} z^3 dz \Rightarrow \overset{2}{\cancel{8}}\left(\frac{z^4}{\cancel{4}}\right)_{-1}^{1} \Rightarrow \underline{\underline{0}}$$

(6) $$\int_1^e \int_1^{\log y} \int_1^{e^x} (\log z)\, dz\, dx\, dy$$

<u>Sol:</u> Given,

$$= \int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz\, dx\, dy$$

$$= \int_1^e \int_1^{\log y} z(-1+\log z)\Big|_1^{e^x} dx\, dy$$

$$= \int_1^e \int_1^{\log y} (z\log z - z)\Big|_1^{e^x} dx\, dy$$

$$= \int_1^e \int_1^{\log y} \left[(e^x \log e^x - e^x) - (-1)\right] dx\, dy$$

$$= \int_1^e \int_1^{\log y} (e^x \log e^x - e^x + 1)\, dx\, dy$$

$$= \int_1^e \int_1^{\log y} (x \cancel{\log}\, e^x - e^x + 1)\, dx\, dy$$

$$= \int_1^e \left(e^x \frac{x^2}{2} - e^x - e^x + x\right)\Big|_1^{\log y} dy$$

$$= \boxed{\cancel{\int_1^e \left[\left(y\frac{\log^2 y}{2} - 2y + \log y\right) - \left(e\frac{e}{2} - 2e + 1\right)\right] dy}}$$

$$= \int_1^e \int_1^{\log y} \left[(x-1)e^x + 1\right] dx\, dy$$

$$= \int_1^e \left[(x-1)e^x - e^x + x\right]_1^{\log y} dy$$

$$= \int_1^e \left[\left\{(\log y - 1)y - y + \log y\right\} - \left\{1 - e\right\}\right] dy$$

$$= \int_1^e (y\log y - 2y + \log y - 1 + e)\, dy$$

$$= \left[\left(\log y \frac{y^2}{2} - \frac{y^2}{4}\right) - \frac{2y^2}{2} + y(\log y - 1) - y + ey\right]_1^e$$

$$= \left[\left(\frac{e^2}{2} - \frac{e^2}{4} - \frac{2e^2}{2} - e + e^2\right) - \left(\frac{1}{4} - 1 + (-1) - 1 + e\right)\right]$$

$$= \left[\frac{e^2}{4} - e + \frac{11}{4} - e\right]$$

$$= \left[\frac{e^2 + 11}{4} - 2e\right]$$

$$= \frac{e^2 - 8e + 11}{4}$$

Change of variables:

(1) $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\,dy$

Sol: Given, $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\,dy$

let $x = r\cos\theta$ $y = r\sin\theta$ then 'θ' varies from 0 to $\frac{\pi}{2}$ and r varies from 0 to ∞

hence $dx\,dy = r\,dr\,d\theta$

i.e $\int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r\,dr\,d\theta$

put $r^2 = t$

$2r\,dr = dt$

$r\,dr = \frac{dt}{2}$

i.e $\int_0^{\pi/2} \int_0^\infty e^{-t} \frac{dt}{2} \, d\theta$

$= \frac{1}{2}\int_0^{\pi/2} (-e^{-t})_0^\infty \, d\theta$

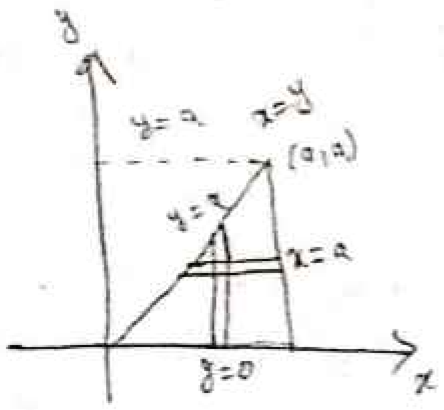
$= \frac{1}{2}\int_0^{\pi/2} (-e^{-\infty} + e^{0}) \, d\theta$

$= \frac{1}{2}\int_0^{\pi/2} d\theta$

$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{4}$

*** Using change of order of integration, evaluate the following.

① $\int_0^a \int_y^a f(x,y)\,dx\,dy$



Sol: Given, $\int_{y=0}^{a} \int_{x=y}^{a} f(x,y)\,dx\,dy$

Here the region of integration lies from $x=y$ which is a st. line to $x=a$ which is a line parallel to y-axis & the limits for y varies from $y=0$ to $y=a$.

For this we need to consider a horizontal strip. Now by change of order of integration this H.S is change into V.S & the limits of integration will also changes accordingly

∴ Now y varies from $y=0$ to $y=x$

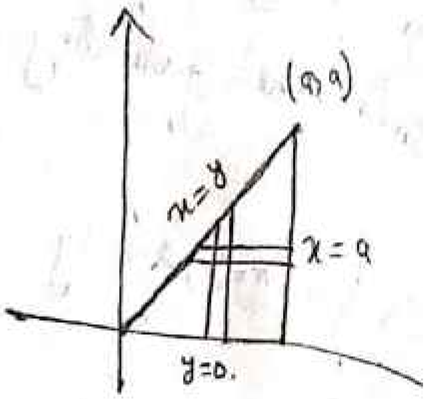
& the limits for x varies from 0 to a

$$\therefore \int_{x=0}^{a} \int_{y=0}^{x} f(x,y)\,dy\,dx$$

(2) $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$

Sol: Given;

$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$$



Here the region of integration lies ~~of~~ from $x=y$ which is a st line to $x=a$ which is a line parallel to y-axis & the limit for y varies from $y=0$ to $y=a$

For this we need to consider a horizontal strip. Now by change of order of integration the H.S is change to V.S and the limits of integration will also changes accordingly.

i.e Now y varies from $y=0$ to $y=x$ and the limits for x varies from $x=0$ to $x=a$

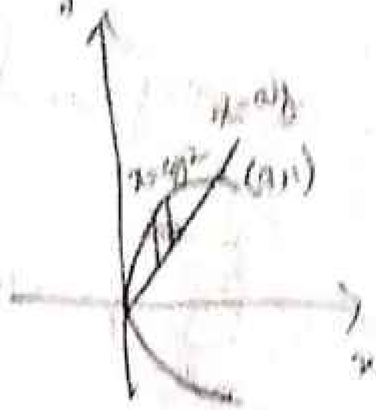
$$\therefore \int_{x=0}^{a} \int_{y=0}^{x} \frac{x}{x^2+y^2}\, dy\, dx$$

$$\int_{x=0}^{a} \cancel{x} \times \frac{1}{\cancel{x}} \left[\tan^{-1}\left(\frac{y}{x}\right)\right]_0^{x} dx$$

$$\int_0^a \tan^{-1}(1)\, dx = \int_0^a \frac{\pi}{4}\, dx$$

$$= \frac{\pi}{4}[x]_0^a = \frac{\pi a}{4}$$

(3) $\int_{0}^{a}\int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dy\,dx$



Sol: Given,

$$\int_{0}^{a}\int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dy\,dx$$

Here the region of integration lies from $y = \frac{x}{a}$ which is a st. line to $y = \sqrt{\frac{x}{a}}$ or $y^2 = \frac{x}{a}$ which is a parabola about x-axis and the limits for x varies from 0 to a.

For this we need to consider a vertical strip. Now by change of order of integration the V.S is change to H.S and the limits of integration will also changes accordingly.

i.e now x varies from $x = ay^2$ to $x = ay$ and the limits for y varies from $y = 0$ to $y = 1$

$$\therefore \int_{0}^{1}\int_{ay^2}^{ay} (x^2+y^2)\,dx\,dy$$

$$= \int_{0}^{1}\left(\frac{x^3}{3} + xy^2\right)_{ay^2}^{ay} dy$$

$$= \int_{0}^{1}\left[\left(\frac{a^3y^3}{3} + ay^3\right) - \left(\frac{a^3y^6}{3} + ay^4\right)\right] dy$$

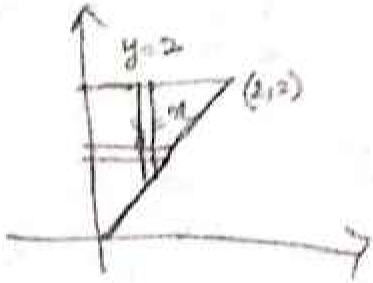
$$= \int_0^1 \left(\frac{a^3y^3}{3} + ay^3 - \frac{a^3y^6}{3} - ay^4 \right) dy$$

$$= \left(\frac{a^3y^4}{12} + \frac{ay^4}{4} - \frac{a^3y^7}{21} - \frac{ay^5}{5} \right)_0^1$$

$$= \left(\frac{a^3}{12} + \frac{a}{4} - \frac{a^3}{21} - \frac{a}{5} \right)$$

$$= \frac{a^3}{28} + \frac{a}{20}$$

(11) $\int_0^2 \int_x^2 2y^2 \sin xy \, dy \, dx$



Sol: Given's $\int_{y=x}^{2} \int^{2} 2y^2 \sin xy \, dy \, dn$

Here the region of integration lies from $y=x$ which is a st. line to $y=2$ which is a line parallel to x-axis and the limits of x varies from 0 to 2.

For this we need to consider a vertical strip. Now by change of order of integration this V.S is change to H.S and the limits of integration will also changes accordingly.

i.e Now ~~y~~ x varies from $x=0$ to $x=y$ and the limits for y varies from 0 to 2

Here point of intersection is $(2,2)$

$$\therefore \int_0^2 \int_{x=0}^{y} 2y^2 \sin xy \, dx \, dy.$$

$$= \int_0^2 2y^{\cancel{2}} \left(\frac{-\cos xy}{\cancel{y}}\right)_0^y dy$$

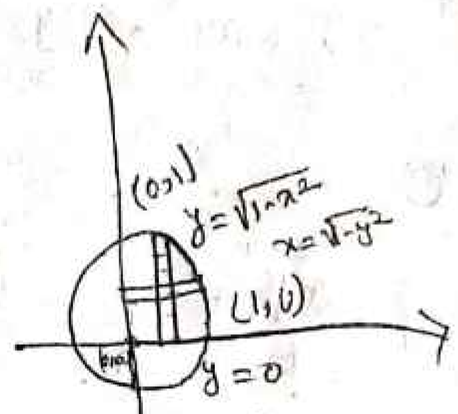
$$= \int_0^2 \left(2y(-\cos y^2) - 2y\right) dy = -2 \int_0^2 \left(y \cos y^2 + y\right) dy$$

$$= -2 \left[\left(\frac{\sin t}{2}\right)_0^4 - \left(\frac{\cancel{x}^2}{\cancel{2}_1} - 0\right) \right]$$

$y^2 = t$
$2ydy = dt$
$ydy = \frac{dt}{2}$

$$= -\left[\sin 4 - 4\right]$$

⑤ $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx$



Sol: Given,

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx$$

Here the region of integration lies from $y=0$ on x-axis to $y=\sqrt{1-x^2} \Rightarrow y^2=1-x^2 \Rightarrow x^2+y^2=1$ which is eqn of circle. & the limits for x varies from $x=0$ to $x=1$

For this we need to consider a vertical strip. Now by change of order of integration, this vertical strip is change to H.S, and the limit of integration will also changes accordingly.

i.e. Now x varies from $x=0$ to $x=\sqrt{1-y^2}$ and y varies from $y=0$ to $y=1$

$$\therefore \int_0^1 \int_0^{\sqrt{1-y^2}} y^2 \, dx \, dy$$

$$\int_0^1 \left[y^2 x\right]_0^{\sqrt{1-y^2}} dy$$

$$\int_0^1 y^2\left(\sqrt{1-y^2}\right) dy$$

Put $y=\sin\theta$

$dy=\cos\theta \, d\theta$

If $y=0 \Rightarrow \theta=0$

$y=1 \quad \theta=\pi/2$

$$\int_0^{\pi/2} \sin^2\theta \,(\cos\theta)(\cos\theta)\, d\theta$$

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$$= \int_0^{\pi/2} \sin^2\theta \,(1 - \sin^2\theta)\, d\theta$$

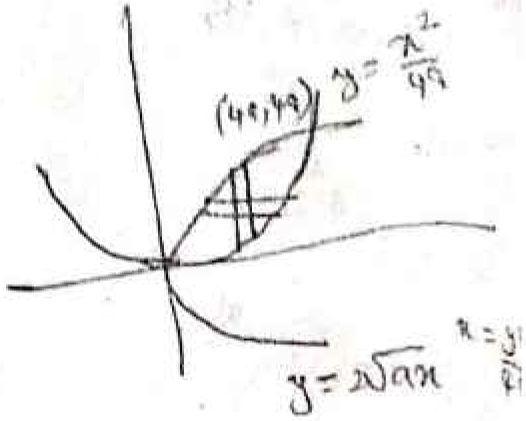
$$= \int_0^{\pi/2} \sin^2\theta \, d\theta - \int_0^{\pi/2} \sin^4\theta \, d\theta$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi}{2}\left[\frac{1}{2} - \frac{3}{8}\right]$$

$$= \frac{\pi}{\cancel{2}}\left[\frac{\cancel{2}}{16}\right] = \underline{\underline{\frac{\pi}{16}}}$$

(6) $\int_{0}^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$



Sol: Given,

$$\int_{0}^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$$

Here the region of integration lies from $y = \frac{x^2}{4a}$ which is a parabola about y-axis, to $y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$, which is also a parabola about x-axis & the limits for x varies from 0 to 4a.

For this we need to consider a vertical strip. Now by change of order of integration this V.S is change to H.S, and the limit of integration also changes accordingly.

i.e Now x varies from $x = \frac{y^2}{4a}$ to $x = 2\sqrt{ay}$

& y varies from $y = 0$ to $y = 4a$

$$\therefore \int_{0}^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\, dy$$

$$= \int_0^{4a} [x]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a}\left[2\sqrt{ay} - \frac{y^2}{4a}\right] dy$$

$$= \int_0^{4a} \left(2\sqrt{a}\sqrt{y} - \frac{y^2}{4a} \right) dy$$

$$= \left[2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a} \right]_0^{4a}$$

$$= \frac{2\sqrt{a}(4a)^{3/2}}{3/2} - \frac{(4a)^3}{12a}$$

$$= 2 \times \frac{2\sqrt{a}\, 4a\, 2\sqrt{a}}{3} - \frac{64a^2}{12}$$

$$= \frac{32a^2}{3} - \frac{64a^2}{12}$$

$$= \frac{128a^2 - 64a^2}{12} = \frac{\overset{16}{\cancel{64}}a^2}{\cancel{12}\,3} = \frac{16a^2}{3}$$

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9/02/2022

Unit-V Vector Calculus

Vector function :- If 't' is a scalar variable and if to each value of 't' in same interval there corresponds a value of a vector $\vec{v}$, we say that $\vec{v}$ is a vector function of t. It is denoted as $\vec{v} = \vec{v}(t)$

or

$\vec{v} = \vec{f}(t)$

Derivative of a vector function :- Let $\vec{v} = \vec{F}(t)$ be a vector function of a scalar 't' & $\delta\vec{v}$ be the increment in $\vec{v}$ corresponding to a small increment δt in 't'. Then

$$\frac{d\vec{v}}{dt} = \underset{\delta t \to 0}{Lt} \frac{\delta\vec{v}}{\delta t} = \underset{\delta t \to 0}{Lt} \frac{\vec{f}(t+\delta t)-\vec{f}(t)}{\delta t}.$$

Note ① :- If $\vec{v}(t) = v_1(t)\hat{i} + v_2(t)\hat{j} + v_3(t)\hat{k}$ then

$$\frac{d\vec{v}}{dt} = \frac{dv_1}{dt}\hat{i} + \frac{dv_2}{dt}\hat{j} + \frac{dv_3}{dt}\hat{k}$$

Note ② :- Since $\frac{d\vec{v}}{dt}$ is itself a vector function of 't', it can be differentiable ~~respor~~ again w.r.t 't' and thereby obtain its derivatives called second derivative of $\vec{v}(t)$ denoted by $\frac{d^2\vec{v}}{dt^2}$. $|||^{ly}$ higher order derivative of $\vec{v}(t)$ can be obtained.

Rules for vector differentiation :-

let $\vec{u}, \vec{v}, \vec{w}$ be three vector functions and ϕ is a scalar function of 't' then we have the following

(i) $\frac{d}{dt}(\alpha\vec{v}) = \alpha\frac{d\vec{v}}{dt}$, α being a constant

(ii) $\frac{d}{dt}(\vec{u} \pm \vec{v}) = \frac{d\vec{u}}{dt} \pm \frac{d\vec{v}}{dt}$

(iii) $\frac{d}{dt}(\phi\vec{v}) = \frac{d\phi}{dt}(\vec{v}) + \phi\frac{d\vec{v}}{dt}$

(iv) $\frac{d}{dt}(\vec{u}\cdot\vec{v}) = \frac{d\vec{u}}{dt}\cdot\vec{v} + \vec{u}\cdot\frac{d\vec{v}}{dt}$

(v) $\frac{d}{dt}(\vec{u}\times\vec{v}) = \frac{d\vec{u}}{dt}\times\vec{v} + \vec{u}\times\frac{d\vec{v}}{dt}$

(vi) $\frac{d}{dt}(\vec{u}\,\vec{v}\,\vec{w}) = \frac{d\vec{u}}{dt}(\vec{v}\cdot\vec{w}) + \vec{u}\left(\frac{d\vec{v}}{dt}\cdot\vec{w}\right) + \vec{u}\left(\vec{v}\cdot\frac{d\vec{w}}{dt}\right)$

(vii) $\frac{d}{dt}(\vec{u}\times\vec{v}\times\vec{w}) = \frac{d\vec{u}}{dt}\times(\vec{v}\times\vec{w}) + \vec{u}\times\left(\frac{d\vec{v}}{dt}\times\vec{w}\right) + \vec{u}\times\left(\vec{v}\times\frac{d\vec{w}}{dt}\right)$

Note ③ :- The derivative of constant vector is zero

Note ④ :- If OP represents the position vector $\vec{r}$ and 't' denotes the time then $\frac{d\vec{r}}{dt}$ & $\frac{d^2\vec{r}}{dt^2}$ represents the velocity and acceleration respectively at a point P.

POINT FUNCTION :- A variable quality whose value at any point depends on the position of the point in a region of space is called a point function. There are two types of point function

① Scalar point function :- If to each point $P(x,y,z)$ of a region R, their corresponds a scalar quantity $\phi(x,y,z)$ then $\phi(x,y,z)$ is said to be SPF

② Vector point function :-
If to each point $p(x,y,z)$ of a region R, their corresponds a vector quality $\vec{F}(x,y,z)$ then $\vec{F}(x,y,z)$ is saidtobe V.P.F .

FIELD:- When a point function is defined at each point of a region R of space, then the region is called field. There are two types of fields.

① Scalar field:- A point function that assigns a scalar to every point in same region in the plane or space is called S.F

Eg: ① Temp. distribution throughout a body

② Potential energy of a particle

② Vector field :- A point function that assigns a vector to every point in same region in the plane or space is called V.F

Eg: ① Electric field.

② Magnetic field.

③ The velocity of flow of liquid.

Note⑤ :- A vector operator ∇ (called as del) is defined as

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

GRADIENT(SPF):-

If the SPF $\phi(x,y,z)$ is continuous and differentiable then the gradient of the function ϕ is denoted by grad ϕ & is defined as

$$\text{grad}\,\phi \text{ or } \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Note ⑥:- $\nabla\phi$ cannot be written as $\phi\nabla$.

Greatest value (or) Max. value :-

The G.V (or) Max.V of the function $f(x,y,z)$ is $|\nabla f|$

UNIT normal vector :-

The UNV $\vec{N}$ normal to the surface $\phi(x,y,z)=C$ in the direction of grad ϕ is called the UNV at $P(x,y,z)$ i.e, $\vec{N} = \dfrac{\nabla\phi}{|\nabla\phi|}$

Angle between two surfaces :-

The angle between two surfaces $\phi_1 = C_1$ and $\phi_2 = C_2$ at a common point P is the angle between the normals $\nabla\phi_1$ and $\nabla\phi_2$ to the surfaces at the point.

$$\theta = \cos^{-1}\frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

Directional Derivative:-

The D·D of the function $f(x,y,z)$ at a point $P(x,y,z)$ in the direction of normal to the surfaces $\phi(x,y,z)=c$ is given by

$$D\cdot D = \nabla f \cdot \frac{\nabla\phi}{|\nabla\phi|} \text{ (or) } \nabla f \cdot \frac{\overline{PQ}}{|PQ|} \text{ [For line PQ]}$$

Divergence (VPF):-

If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ then the divergence of $\vec{F}$ is denoted as $\nabla\cdot\vec{F}$ or $\operatorname{div}\vec{F}$ and it is defined as

$$\operatorname{div}\vec{F} \text{ or } \nabla\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Solenoidal Vector:-

If $\operatorname{div}\vec{F} = 0$ then $\vec{F}$ is said to be solenoidal vector.

CURL (VPF):-

If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ then the curl of $\vec{F}$ is denoted as $\nabla\times\vec{F}$ or $\operatorname{curl}\vec{F}$ and it is defined as

$$\operatorname{curl}\vec{F} \text{ or } \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Irrotational vector:-

If curl $\vec{F} = 0$ then $\vec{F}$ is said to be irrotational vector

Scalar potential function:-

If $\vec{F}$ is a VPF if $\exists$ a S.P.F $\phi \Rightarrow \vec{F} = \nabla\phi$ then ϕ is said to be S.P.F of $\vec{F}$. In such a case curl $\vec{F} = 0$ i.e, $\vec{F}$ is irrotational.

The S.P.F ϕ is determined from the above relations by using exact differential

$$d\phi = \phi_x\,dx + \phi_y\,dy + \phi_z\,dz.$$

1) Find the gradient of the function $\log(x, y, z)$ at $(1, 2, -1)$.

Sol) let $\phi = \log(x, y, z)$

grad ϕ or $\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\log(x, y, z)$

$$= \frac{\hat{i}}{x+y+z} + \frac{\hat{j}}{x+y+z} + \frac{\hat{k}}{x+y+z}$$

$$= \frac{\hat{i}+\hat{j}+\hat{k}}{x+y+z}$$

$$\nabla\phi_{(1,2,-1)} = \frac{\hat{i}+\hat{j}+\hat{k}}{2}$$

2) Find the max. or greatest value of the functions x^2yz^3 at $(2, 1, -1)$

Sol) Let $f = x^2yz^3$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)x^2yz^3$$

$$\nabla f = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$$

$$\nabla f_{(2,1,-1)} = -4\hat{i} - 4\hat{j} + 12\hat{k}$$

$\therefore$ Max. value or greatest value

$$|\nabla f| = \sqrt{16+16+144} = \sqrt{176} = 4\sqrt{11}$$

3) Find the unit normal vector to the surface $x^3+y^3+3xyz=3$ at a point $(1, 2, -1)$

Sol) Let $\phi = x^3+y^3+3xyz-3=0$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3+y^3+3xyz-3)$$

$(1,2,-1)$

$$= (3x^2 + 3yz)\hat{i} + (3y^2 + 3xz)\hat{j} + 3xy\hat{k}$$

$-3\hat{i} - 6\hat{j} + 6\hat{k}$

$3 - 6\ \hat{i}\ +$ $12 - 6$

$\nabla\phi_{(1,2,-1)} = -3\hat{i}+9\hat{j}+6\hat{k}$

$|\nabla\phi| = \sqrt{9+81+36} = 3\sqrt{14}$

Unit normal vector $\vec{N} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$\vec{N} = \frac{-3\hat{i}+9\hat{j}+6\hat{k}}{3\sqrt{14}} = \frac{-\hat{i}+3\hat{j}+2\hat{k}}{\sqrt{14}}$$

(4) Find the angle between 2 surfaces $x^2+y^2+z^2=9$ and $x^2+y^2-z=3$ at $(2,-1,2)$

Sol Let $\phi_1 = x^2+y^2+z^2-9=0$ and

$\phi_2 = x^2+y^2-z-3=0$

$$\nabla\phi_1 = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9)$$

$$= 2x\hat{i}+2y\hat{j}+2z\hat{k}.$$

$\nabla\phi_{1(2,-1,2)} = 4\hat{i}-2\hat{j}+4\hat{k}$

$|\nabla\phi_1| = \sqrt{16+4+16} = \sqrt{36} = 6$

$$\nabla\phi_2 = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2-z-3)$$

$\nabla\phi_2 = 2x\hat{i}+2y\hat{j}-\hat{k}$

$\nabla\phi_{2(2,-1,2)} = 4\hat{i}-2\hat{j}-\hat{k}$

$|\nabla\phi_2| = \sqrt{16+4+1} = \sqrt{21}$

Angle b/w 2 surfaces is given by

$$\theta = \cos^{-1}\left[\frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}\right]$$

$$= \cos^{-1}\left[\frac{(4\hat{i}-2\hat{j}+4\hat{k})(4\hat{i}-2\hat{j}-\hat{k})}{6\sqrt{21}}\right]$$

$$\theta = \cos^{-1}\left[\frac{\cancel{18}^{\,24}}{\cancel{6}_{\,3}\sqrt{21}}\right] = \cos^{-1}\left[\frac{8}{3\sqrt{21}}\right]$$

11/02/2022

(5) Find the directional derivative of the following functions:

(a)(i) x^2+y^2 at $(1,1)$ in the direction of vector $2\hat{i}-4\hat{j}$

let $f = x^2+y^2$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2)$$

$$= 2x\hat{i} + 2y\hat{j}$$

$\nabla f_{(1,1)} = 2\hat{i}+2\hat{j}$ Here $\nabla\phi = 2\hat{i}-4\hat{j}$

$\therefore |\nabla\phi| = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$

$\therefore$ Directional derivative $= \nabla f \cdot \dfrac{\nabla\phi}{|\nabla\phi|}$

$$= (2\hat{i}+2\hat{j})\frac{(2\hat{i}-4\hat{j})}{2\sqrt{5}} = \frac{4-8}{2\sqrt{5}} = \frac{-2}{\sqrt{5}}$$

(ii) $x^2+y^2+z^2$ at $(1,2,3)$ in the direction of vector $2\hat{i}+3\hat{j}+6\hat{k}$

let $f = x^2+y^2+z^2$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$\nabla f_{(1,2,3)} = 2\hat{i}+4\hat{j}+6\hat{k}$

Here $\nabla \phi = 2\hat{i}+3\hat{j}+6\hat{k}$

$\therefore |\nabla \phi| = \sqrt{4+9+36} = 7.$

$$\therefore D\cdot D = \nabla f \cdot \frac{\nabla \phi}{|\nabla \phi|} = (2\hat{i}+4\hat{j}+6\hat{k})\frac{(2\hat{i}+3\hat{j}+6\hat{k})}{7}$$

$$= \frac{4+12+36}{7} = \frac{52}{7}$$

(b) $x^2-y^2+2z^2$ at $(1,2,3)$ in the direction of the line PQ where $Q=(5,0,4)$.

Sol let $f = x^2-y^2+2z^2$

$$\nabla f = \left(\hat{i}\frac{d}{dx}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2-y^2+2z^2)$$

$$= 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$$

$$\nabla f_{(1,2,3)} = 2\hat{i}-4\hat{j}+12\hat{k}$$

$\bar{PQ} = OQ-OP = 4\hat{i}-2\hat{j}+\hat{k}$ $\quad \therefore |PQ| = \sqrt{16+4+1} = \sqrt{21}$

$$\therefore D\cdot D = \nabla f \cdot \frac{\bar{PQ}}{|\bar{PQ}|}$$

$$= (2\hat{i}-4\hat{j}+12\hat{k})\frac{(4\hat{i}-2\hat{j}+\hat{k})}{\sqrt{21}}$$

$$= \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

(c) $\cancel{x^2}+xyz^2+xz$ at $(1,1,1)$ in the direction of normal to the surface $3xy^2+y=z$ at $(0,1,1)$

Sol Let $f = xyz^2 + xz$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xyz^2 + xz)$$

$$= (yz^2 + z)\hat{i} + xz^2\hat{j} + (2xyz + x)\hat{k}$$

$$\nabla f_{(1,1,1)} = 2\hat{i} + \hat{j} + 3\hat{k}$$

Here $\phi = 3xy^2 + y - z = 0$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3xy^2 + y - z)$$

$$= 3y^2\hat{i} + (6xy + 1)\hat{j} - \hat{k}$$

$$\nabla\phi_{(0,1,1)} = 3\hat{i} + \hat{j} - \hat{k}$$

$$\therefore |\nabla\phi| = \sqrt{9+1+1} = \sqrt{11}$$

$$\therefore D.D = \nabla f \cdot \frac{\nabla\phi}{|\nabla\phi|}$$

$$= (2\hat{i} + \hat{j} + 3\hat{k})\frac{(3\hat{i} + \hat{j} - \hat{k})}{\sqrt{11}} = \frac{6+1-3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

(6)(i) P.T $\vec{F} = (x + 3y)\hat{i} + (y - 3z)\hat{j} + (x - 2z)\hat{k}$ is solenoidal

Sol Given $\vec{F} = (x + 3y)\hat{i} + (y - 3z)\hat{j} + (x - 2z)\hat{k}$

If $\vec{F}$ is solenoidal then $\nabla \cdot \vec{F} = 0$

$$L.H.S = \nabla \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left\{(x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}\right\}$$

$$= 1 + 1 - 2 = 0 = R.H.S$$

Hence $\vec{F}$ is solenoidal

(ii) Find 'a' if $\vec{F} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is solenoidal

sol Given $\vec{F} =$ "

If $\vec{F}$ is solenoidal then $\vec{F} = 0$

$$\therefore \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left\{(x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}\right\} = 0$$

$$1 + 1 + a = 0 \Rightarrow \boxed{a = -2}$$

(7) Prove the foll:

(i) $\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^2}$

W.K.T $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r^2 = x^2 + y^2 + z^2$$

$$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$|||^{ly}$ $\frac{\partial r}{\partial y} = \frac{y}{r}$ & $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$L.H.S = \nabla\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\frac{1}{r}$$

$$= \sum \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \sum i\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right)$$

$$= \sum -i\left(\frac{1}{r^2}\cdot\frac{x}{r}\right) = \sum \frac{-x\hat{i}}{r^3}$$

$$= -\frac{x\hat{i}}{r^3} - \frac{y\hat{j}}{r^3} - \frac{z\hat{k}}{r^3}$$

$$= \frac{-(x\hat{i} + y\hat{j} + z\hat{k})}{r^3} = \frac{-\vec{r}}{r^3}$$

$= R.H.S$

(ii) $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$

Sol let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$L.H.S = \nabla(\vec{a}\cdot\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\{(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})\}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a} = R.H.S$$

(iii) $(\vec{u}\cdot\nabla)\vec{r} = \vec{u}$

let $\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ &

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$L.H.S = (\vec{u}\cdot\nabla)\vec{r}$$

$$= \left\{(u_1\hat{i}+u_2\hat{j}+u_3\hat{k})\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\right\}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= \left(u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y} + u_3\frac{\partial}{\partial z}\right)(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= u_1\hat{i} + u_2\hat{j} + u_3\hat{k} = \vec{u} = R.H.S$$

(iv) $\nabla^2(\log r) = \frac{1}{r^2}$

Sol w.r.t $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$|||^{ly}\ \frac{\partial r}{\partial y} = \frac{y}{r} \ \& \ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$L.H.S = \nabla^2 \log r = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\log r$$

$$= \sum\frac{\partial^2}{\partial x^2}(\log r) = \sum\frac{\partial}{\partial x}\left(\frac{1}{r}\cdot\frac{\partial r}{\partial x}\right)$$

$$= \sum\frac{\partial}{\partial x}\left(\frac{1}{r}\cdot\frac{x}{r}\right)$$

$$= \sum\frac{\partial}{\partial x}\left(\frac{x}{r^2}\right)$$

$$= \sum \left[\frac{r^2(1) - x\, 2r\, \partial r/\partial x}{r^4} \right]$$

$$= \sum \left[\frac{r^2 - 2x^2}{r^4} \right] = \sum \left(\frac{1}{r^2} - \frac{2x^2}{r^4} \right)$$

$$= \frac{3}{r^2} - \frac{2}{r^4}(x^2 + y^2 + z^2)$$

$$= \frac{3}{r^2} - \frac{2}{r^4}(r^2)$$

$$= \frac{3}{r^2} - \frac{2}{r^2} = \frac{1}{r^2} = R.H.S$$

(V) $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

sol W.K.T $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r^2 = x^2 + y^2 + z^2$$

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$|||^{ly}$ $\frac{\partial r}{\partial y} = \frac{y}{r}$ & $\frac{\partial r}{\partial z} = \frac{z}{r}$

$L.H.S = \nabla^2 f(r)$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f(r)$$

$$= \sum \frac{\partial^2}{\partial x^2} f(r)$$

$$= \sum \frac{\partial}{\partial x} \left\{ f'(r) \frac{\partial r}{\partial x} \right\}$$

$$= \sum \frac{\partial}{\partial x} \left\{ f'(r) \cdot \frac{x}{r} \right\}$$

$$= \sum \left[f'(r) \left\{ \frac{r(1) - x \cdot \partial r/\partial x}{r^2} \right\} + \frac{x}{r} f''(r) \frac{\partial r}{\partial x} \right]$$

$$= \sum \left[f'(r)\left(\frac{r - x^2/r}{r^2}\right) + \frac{x^2}{r^2} f''(r) \right]$$

$$= \sum \left[f'(r)\left(\frac{1}{r} - \frac{x^2}{r^3}\right) + \frac{x^2}{r^2} f''(r) \right]$$

$$= \frac{3}{r} f'(r) - \frac{f'(r)}{r^3} \times r^2 + \frac{f''(r)}{r^2} \cdot r^2$$

$$= \frac{2}{r} f'(r) + f''(r) = R \cdot H \cdot S$$

(Vi) $\nabla^2 r^n = n(n+1) r^{n-2}$

W.K.T $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r^2 = x^2 + y^2 + z^2$$

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$|||^{ly}$ $\frac{\partial r}{\partial y} = \frac{y}{r}$ & $\frac{\partial r}{\partial z} = \frac{z}{r}$.

$$L \cdot H \cdot S = \nabla^2 r^n$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^n$$

$$= \sum \frac{\partial^2}{\partial x^2} \cdot r^n$$

$$= \sum \frac{\partial}{\partial x} n \cdot r^{n-1} \cdot \frac{\partial r}{\partial x}$$

$$= \sum \frac{\partial}{\partial x} n \cdot r^{n-1} \left(\frac{x}{r}\right)$$

$$= \sum n \frac{\partial}{\partial x} (x \cdot r^{n-2})$$

$$= \sum n \left\{ x (n-2) r^{n-3} \frac{\partial r}{\partial x} + r^{n-2}(1) \right\}$$

$$= \sum n \left\{ x^2 (n-2) r^{n-4} + r^{n-2} \right\}$$

$= n(n-2) r^{n-4} (x^2 + y^2 + z^2) + 3n r^{n-2}$

$= n(n-2) r^{n-2} + 3n r^{n-2}$

$= n r^{n-2} (n-2+3)$

$= n(n+1) r^{n-2}$

$=$ R.H.S

(vii) Find 'n' if $\vec{r} r^n$ is solenoidal

sol W.K.T $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r^2 = x^2 + y^2 + z^2$$

$$2r \frac{\partial r}{\partial x} = \cancel{2}x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$|||^{ly} \quad \frac{\partial r}{\partial y} = \frac{y}{r} \quad \& \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

If $\vec{r} \cdot r^n$ is solenoidal then $\nabla (\vec{r} r^n) = 0$

$$\left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \{ (x\hat{i} + y\hat{j} + z\hat{k}) r^n \} = 0$$

$$\sum \frac{\partial}{\partial x} (x r^n) = 0$$

$$\sum \{ x n r^{n-1} \frac{\partial r}{\partial x} + r^n (1) \} = 0$$

$$\sum (n x^2 r^{n-2} + r^n) = 0$$

$$n r^{n-2} (x^2 + y^2 + z^2) + 3 r^n = 0$$

$$n r^n + 3 r^n = 0 \quad [\because r^n \neq 0]$$

$$r^n (n+3) = 0$$

$$\Rightarrow n = -3$$

⑧ If $\vec{F} = (x+y+z)\hat{i} + 2\hat{j} - (x+y)\hat{k}$ then P.T $\vec{F} \cdot \text{curl}\,\vec{F} = 0$

Sol Given $\vec{F} = (x+y+z)\hat{i} + 2\hat{j} - (x+y)\hat{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+z & 2 & -(x+y) \end{vmatrix}$$

$$= \hat{i}(-1-0) + \hat{j}(0+1) + \hat{k}(0-1)$$

$$= -\hat{i} + \hat{j} - \hat{k}$$

$$\text{L.H.S} = \vec{F} \cdot \text{curl}\,\vec{F} = \{(x+y+z)\hat{i} + 2\hat{j} - (x+y)\hat{k}\} \cdot (-\hat{i} + \hat{j} - \hat{k})$$

$$= -(x+y+z) + 2 + x + y = 0 = \text{R.H.S}$$

***

⑨ If $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ & $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ then P.T $\text{curl}(\vec{A} \times \vec{R}) = 2\vec{A}$

Sol Given $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ & $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

Consider
$$\vec{A} \times \vec{R} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(A_2 z - A_3 y) + \hat{j}(A_3 x - A_1 z) + \hat{k}(A_1 y - A_2 x)$$

Now L.H.S = $\text{curl}(\vec{A} \times \vec{R})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_2 z - A_3 y & A_3 x - A_1 z & A_1 y - A_2 x \end{vmatrix}$$

$$= (A_1 + A_1)\hat{i} + (A_2 + A_2)\hat{j} + (A_3 + A_3)\hat{k}$$

$$= 2A_1\hat{i} + 2A_2\hat{j} + 2A_3\hat{k} = 2(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}]$$

$$= 2\vec{A} = \text{R.H.S}$$

(10) (i) P.T $\vec{F} = (x^2-yz)\hat{i} + (y^2-xz)\hat{j} + (z^2-xy)\hat{k}$ is irrotational. Also find its scalar potential function

sol Given $\vec{F} =$

If $\vec{F}$ is irrotational then $\text{curl}\,\vec{F} = \vec{0}$

$$\text{L.H.S} = \text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$$

$$= (-x+x)\hat{i} + (-y+y)\hat{j} + (-z+z)\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0} = \text{R.H.S}$$ Hence $\vec{F}$ is irrotational

For scalar potential function we have $\vec{F} = \nabla\phi$

$$(x^2-yz)\hat{i} + (y^2-xz)\hat{j} + (z^2-xy)\hat{k}$$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}.$$

$$\therefore d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz.$$

$$d\phi = (x^2-yz)\,dx + (y^2-xz)\,dy + (z^2-xy)\,dz$$

$$d\phi = x^2dx + y^2dy + z^2dz - (yz\,dx + xz\,dy + xy\,dz)$$

$$d\phi = x^2dx + y^2dy + z^2dz - d(xyz)$$

Integration

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + C$$

(ii) Find a, b, c if $\vec{F} = (3x+ay+z)\hat{i} + (2x-y+bz)\hat{j} + (x+cy+z)\hat{k}$ is irrotational. Also find its S.P.F

Sol Given $\vec{F} =$ "

If $\vec{F}$ is irrotational then $\nabla \times \vec{F} = \vec{0}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x+ay+z & 2x-y+bz & x+cy+z \end{vmatrix} = \vec{0}$$

$$(c-b)\hat{i} + (0-0)\hat{j} + (2-a)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$c-b=0 \Rightarrow \boxed{c=b}$ & $2-a=0$ i.e, $\boxed{a=2}$

When $a=2$ & $c=b$ given vector is irrotational.

For S.P.F we have $\vec{F} = \nabla\phi$

$$(3x+2y+z)\hat{i} + (2x-y+bz)\hat{j} + (x+by+z)\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\therefore d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$d\phi = (3x+2y+z)\,dx + (2x-y+bz)\,dy + (x+by+z)\,dz$$

$$d\phi = 3x\,dx - y\,dy + z\,dz + 2y\,dx + z\,dx + 2x\,dy + bz\,dy + x\,dz + by\,dz$$

$$d\phi = 3x\,dx - y\,dy + z\,dz + 2d(x,y) + d(x,z) + b\,d(yz)$$

integrating

$$\phi = \frac{3x^2}{2} - \frac{y^2}{2} + \frac{z^2}{2} + 2xy + xz + byz + K$$

VECTOR INTEGRATION:

If $\vec{F}(t)$ and $\vec{f}(t)$ be two vectors functions of a scalar variable t $\ni$ $\frac{d}{dt}\vec{F}(t)=\vec{f}(t)$, then $\vec{F}(t)$ as called an integral of $\vec{F}(t)$ w.r.t 't' and written as $\int \vec{f}(t)\,dt = \vec{F}(t)+C$, where $\vec{c}$ is a constant vector. This is called indefinite interval. The definite interval of $\vec{f}(t)$. but the limit $t=a$ and $t=b$ is written as $\int_a^b \vec{f}(t)\,dt = [\vec{F}(t)]_a^b = \vec{F}(b)-\vec{f}(a)$

LINE INTEGRAL:-

Any integral which is evaluated along a curve is called a line integral.

If $\vec{F} = F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$, $\vec{R}=x\hat{i}+y\hat{j}+z\hat{k}$ then

$$\int_c \vec{F}\cdot d\vec{R} = \int_c F_1\,dx + F_2\,dy + F_3\,dz$$

The other two types of line integrals are $\int \vec{F}\times d\vec{R}$ and $\int_c \vec{Q}\,d\vec{R}$ which are both vectors. If c is a closed curve, then the integral sign $\int_c$ is replaced by $\oint_c$

CIRCULATION:

If $\vec{V}$ represents the velocity of a fluid particle and c is closed curve, then the integral $\oint_c \vec{V}\cdot d\vec{R}$ is called the circulation of $\vec{V}$ round the curve C.

If the circulation of $\vec{V}$ round every closed curve in a regions R varies, then $\vec{V}$ is said to be irrotational in R.

WORK DONE BY A FORCE:

If $\vec{F}$ represents the force vector acting on the particle moving along an arc AB, then the work done during a small displacement $\delta\vec{R}$ is $\vec{F}\cdot\delta\vec{R}$

Hence the total work done by $\vec{F}$ during the displacement from A to B is given by the line integral

$$\boxed{\int_A^B \vec{F}\cdot d\vec{R}}$$

If the force $\vec{F}$ is conservative i.e, if $\vec{F} = \nabla\phi$, then the work done is independent of the path and vice versa i.e, $\text{curl}\,\vec{F} = \text{curl grad}\,\phi = 0$

NOTE: If $\vec{F}$ is an irrotational, it is conservative

SURFACE INTEGRALS:

Any integral which is to be evaluated over a surface is called surface integral. (The other types of surface integral are $\int_S \vec{F}\times d\vec{s}$ or $\int_S \phi\, d\vec{s}$)

NOTE:- To evaluate any surface integral, it is convenient to evalute the double integral of its projection in the xy (or) yz (or) zx.. Plane $\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_S \vec{F}\cdot d\vec{s} = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$ where R is the projection of S in the xy plane $ds = \frac{dy\cdot dz}{|\hat{n}\cdot\hat{i}|}$

H^{ly} if the projection of S is taken in yz plane and if the projection of S is taken in xy plane $ds = \frac{dz\cdot dx}{|\hat{n}\cdot\hat{j}|}$

Flux :- If $\vec{F}(x,y,z)$ is a functions, continous over S and $\hat{n}$ be the unit normal vector at any point P then the integral $\iint_S \vec{F}\cdot\hat{n}\,ds$ is called the flux of $\vec{F}$ and S. Since $d\vec{s} = \hat{n}\,ds$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S \vec{F}\cdot d\vec{s}.$$

Volume integrals:-

Any integral which is evaluated over a volume is called a volume integral

If V is the volume bounded by a surface S then the triple integral $\iiint_V \phi \, dv$ is a scalar and $\iiint_V \vec{F} \cdot dv$ is a vector are called the volume integral

Line integrals Problems:-

① S.T the force field vector $\vec{F} = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$ is conservative. Also find work done for moving a particle from $(1,-1,2)$ to $(3,1,-1)$

Sol Given $\vec{F} = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$

If $\vec{F}$ is conservative then $\text{curl}\,\vec{F} = 0$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$= (3x^2z^2 - 3x^2z^2)\hat{i} + \hat{j}(6xyz^2 - 6xyz^2)\hat{j} + \hat{k}(2xz^3 - 2xz^3)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

Hence $\vec{F}$ is conservative.

Now workdone $\int_C \vec{F}\cdot d\vec{r}$ $\quad [\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}]$

$$= \int_C (2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k})(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C 2xyz^3\,dx + x^2z^3\,dy + 3x^2yz^3\,dz$$

$$= \int_{(1,-1,2)}^{(3,1,-1)} d(x^2yz^3) = [x^2yz^3]_{(1,-1,2)}^{(3,1,-1)} = -17$$

$9(1)(1) - (1)(-1)(2)$

$9 +$

② If $\phi = x^2yz^3$ then evaluate $\int_C \phi \, d\vec{r}$ along the paths $x=t, y=2t, z=3t$ from $t=0$ to $t=1$

Sol Given $\phi = x^2yz^3$

$$\int_C \phi \, d\vec{r} = \int_C x^2yz^3(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$\because \; x=t \; ; \; y=2t \; ; \; z=3t$

$dx = dt \; ; \; dy = 2dt \; ; \; dz = 3dt$

i.e, $\int_C t^2(2t)(27t^3)(dt\,\hat{i} + 2dt\,\hat{j} + 3dt\,\hat{k})$

$= 54(\hat{i} + 2\hat{j} + 3\hat{k}) \int_{t=0}^{1} t^6 \, dt$

$= 54(\hat{i} + 2\hat{j} + 3\hat{k}) \left[\frac{t^7}{7}\right]_0^1$

$= \frac{54}{7}(\hat{i} + 2\hat{j} + 3\hat{k})$

③ If $\vec{F} = (5x^2 - 6xy)\hat{i} + (2y - 4x)\hat{j}$ then evaluate $\int_C \vec{F} \cdot d\vec{r}$ where 'C' is the curve $y = x^3$ in the xy plane from $(1,1)$ to $(2,8)$

Sol Given $\vec{F} = (5x^2 - 6xy)\hat{i} + (2y - 4x)\hat{j}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (5x^2 - 6xy)\,dx + (2y - 4x)\,dy$$

$\because \; y = x^3 \Rightarrow dy = 3x^2 dx$ & x varies from 1 to 2

i.e, $\int_{x=1}^{2} (5x^2 - 6x^4)\,dx + (2x^3 - 4x)\,3x^2\,dx$

$= \int_1^2 (5x^2 - 6x^4 + 6x^5 - 12x^3)\,dx$

$$= \left[5\frac{x^3}{3} - \frac{6x^5}{5} + \frac{\cancel{6}x^6}{\cancel{6}} - \cancel{12}^{3}\frac{x^4}{\cancel{4}} \right]_1^2$$

$$= \left[\frac{40}{3} - \frac{32\times 6}{5} - 64 - 3(16) \right] - \left[\frac{5}{3} - \frac{6}{3} + 1 - 3 \right]$$

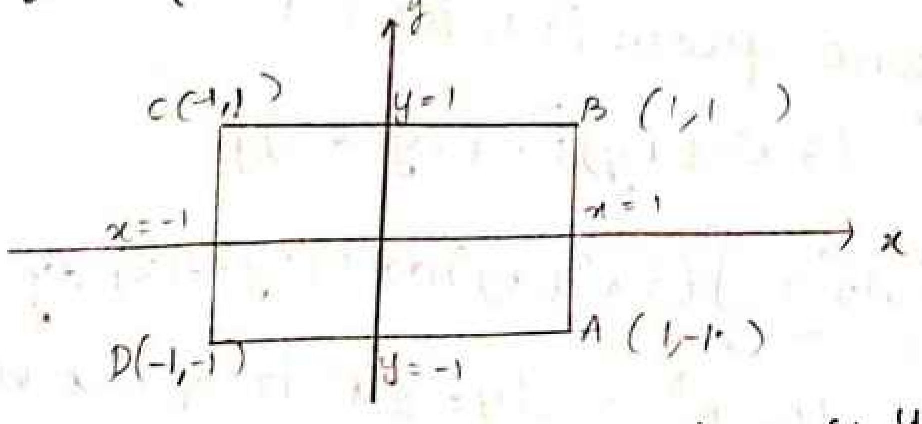
$$= \left(\frac{200-576+240}{15} \right) - \left(\frac{25-18-30}{15} \right) = \frac{-136+23}{15}$$

$$= \frac{-113}{15} = -7.53$$

(4) Compute the line integral $\int_C \vec{F}\cdot d\vec{r}$ if $\vec{F} = (x^2+xy)\,\hat{i} + (x^2+y^2)\,\hat{j}$ where C is the square formed by the lines $x = \pm 1$, $y = \pm 1$

Sol Given $\vec{F} = (x^2+xy)\,\hat{i} + (x^2+y^2)\,\hat{j}$

$\therefore \vec{F}\cdot d\vec{r} = (x^2+xy)\,dx + (x^2+y^2)\,dy$ — (*)



Along AB :- We have $x = 1 \Rightarrow dx = 0$ & y varies from -1 to 1

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{y=-1}^{1} (1+y^2)\,dy = \left[y + \frac{y^3}{3} \right]_{-1}^{1}$$

$$= \left(1+\frac{1}{3}\right) - \left(-1-\frac{1}{3}\right) = \frac{8}{3}$$

Along BC :- We have $y=-1 \Rightarrow dy=0$ and 'x' varies from 1 to -1

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_{x=1}^{-1} (x^2+x)\,dx = \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_1^{-1}$$

$$= \left(\frac{-1}{3}+\frac{1}{2}\right) - \left(\frac{1}{3}+\frac{1}{2}\right) = \frac{-2}{3}$$

Along CD :- We have $x=-1 \Rightarrow dx=0$ & 'y' varies from 1 to -1

$$\int_{CD} \vec{F}\cdot d\vec{r} = \int_{y=1}^{-1} (1+y^2)\,dy = \left[y+\frac{y^3}{3}\right]_1^{-1}$$

$$= \left(-1-\frac{1}{3}\right) - \left(1+\frac{1}{3}\right) = \frac{-8}{3}$$

Along DA :- We have $y=-1 \Rightarrow dy=0$ and 'x' varies from -1 to 1

$$\int_{DA} \vec{F}\cdot d\vec{r} = \int_{x=-1}^{1} (x^2-x)\,dx = \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{-1}^{1}$$

$$= \left(\frac{1}{3}-\frac{1}{2}\right) - \left(\frac{-1}{3}-\frac{1}{2}\right) = \frac{2}{3}$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r} + \int_{DA} \vec{F}\cdot d\vec{r}$$

$$= \frac{8}{3} - \frac{2}{3} - \frac{8}{3} + \frac{2}{3} = 0$$

(5) If $\vec{F} = (2x+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$ then evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the path $x=2t^2$, $y=t$ and $z=t^3$ joining the points $(0,0,0)$ to $(2,1,1)$.

sol Given $\vec{F}$ =

$$\int_C \vec{F}\cdot d\vec{r} = \int_C [(2x+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}][dx\hat{i}+dy\hat{j}+dz\hat{k}]$$

$$= \int (2x+3)\,dx + (xz)\,dy + (yz-x)\,dz$$

$$\because x = 2t^2;\ y = t;\ z = t^3$$

$$dx = 4dt;\ dy = dt;\ dz = 3t^2 dt$$

Here 't' varies from 0 to 1

$$\text{i.e., } \int_{t=0}^{1} (4t^2+3)(4t\,dt) + 2t^5\,dt + (t^4-2t^2)3t^2\,dt$$

$$= \int_0^1 (16t^3+12t+2t^5+3t^6-6t^4)\,dt$$

$$= \left[\overset{4}{\cancel{16}}\frac{t^4}{\cancel{4}} + \overset{6}{\cancel{12}}\frac{t^2}{\cancel{2}} + \cancel{2}\frac{t^6}{\cancel{6}_3} + 3\frac{t^7}{7} - 6\frac{t^5}{5}\right]_0^1$$

$$= \left[4 + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5}\right] - 0$$

$$= \left[10 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5}\right]$$

$$= \left[\frac{1050+35+45-126}{105}\right] = \frac{1004}{105} = 9.56$$

⑥ If $\vec{F} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz\hat{k}$ then find $\int_C \vec{F}\cdot d\vec{r}$ where C is a st. line joining the points (0,0,0) to (1, 1, 1)

Sol Given $\vec{F}$ = "

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (3x^2+6y)dx - (14yz)dy + 20xz^2\,dz.$$

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0}$$

let $\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$

$\therefore x = t;\ y = t$ and $z = t$

$dx = dt;\ dy = dt$ & $dz = dt$

$$\therefore \int_{t=0}^{1} (3t^2+6t)dt - 14t^2dt + 20t^3\,dt$$

$$= \int_0^1 (3t^2+6t-14t^2+20t^3)\,dt$$

$$= \int_0^1 (6t - 11t^2 + 20t^3)\,dt$$

$$= \left[6\frac{t^2}{2} - 11\frac{t^3}{3} + 20\frac{t^4}{4}\right]_0^1$$

$$= \left[3 - \frac{11}{3} + 5\right] = \frac{13}{3}$$

Surface Integrals (Problems)

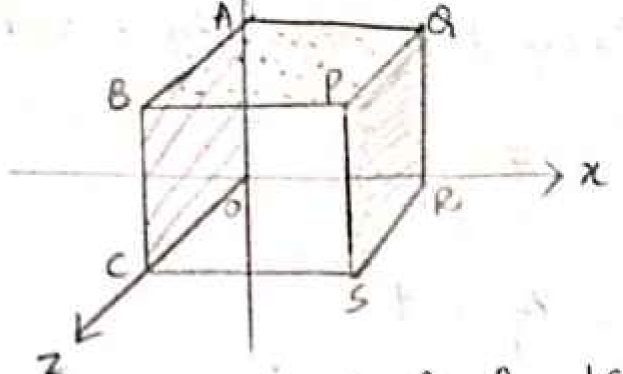
① If $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ then evaluate $\int_S \vec{F}\cdot\hat{n}\,ds$ where S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

Sol Given $\vec{F} =$

$$\int_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_1} \vec{F}\cdot\hat{n}_1 ds_1 + \iint_{R_2} \vec{F}\cdot\hat{n}_2 dS_2 + \iint_{R_3} \vec{F}\cdot\hat{n}_3 ds_3$$

$$+ \iint_{R_4} \vec{F}\cdot\hat{n}_4 dS_4 + \iint_{R_5} \vec{F}\cdot\hat{n}_5 dS_5 + \iint_{R_6} \vec{F}\cdot\hat{n}_6 dS_6 \longrightarrow (*)$$

Γ



Face PQRS :- we have $x = 1$, $\hat{n}_1 = \hat{i}$, $dS_1 = \dfrac{dy\,dz}{|\hat{n}_1 \cdot \hat{i}|}$

$$\int_{R_1} \vec{F} \cdot \hat{n}_1 ds = \int_{R_1} (4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}) \cdot \hat{i}\ \frac{dy\,dz}{|\hat{i} \cdot \hat{i}|}$$

$$= \int_{z=0}^{1} \int_{y=0}^{1} 4z \cdot dy \cdot dz = \int_0^1 4z\,[y]_0^1\,dz$$

$$= \int_0^1 4z \cdot dz = \overset{2}{\cancel{4}}\left[\frac{z^2}{\cancel{2}}\right]_0^1 = 2$$

Face OABC :- We have $x = 0$, $\hat{n}_2 = -\hat{i}$, $dS_2 = \dfrac{dy\,dz}{|\hat{n}_2 \cdot \hat{i}|}$

$$\int_{R_2} \vec{F} \cdot \hat{n}_2 \cdot dS_2 = 0$$

Face ABPQ :- we have $y = 1$, $\hat{n}_3 = \hat{j}$, $dS_3 = \dfrac{dx\,dz}{|\hat{n}_3 \cdot \hat{j}|}$

$$\int_{R_3} \vec{F} \cdot \hat{n}_3 \cdot dS_2 = \int_0^1 \int_0^1 (-1)\,d\cancel{y}x \cdot dz$$

$$= \int_0^1 (-x)_0^1 \cdot dz$$

$$= \int_0^1 -1\,dz = -1$$

Face OCSR :- we have $y = 0$, $\hat{n}_4 = -\hat{j}$, $dS_4 = \dfrac{dx\,dz}{|\hat{n}_4 \cdot \hat{j}|}$

$$\int_{R_4} \vec{F} \cdot \hat{n}_4\,dS_4 = 0$$

Face BCSP :- we have $z=1$, $\hat{n}_5 = \hat{k}$, $dS_5 = \dfrac{dx\,dy}{|\hat{n}_5\cdot\hat{k}|}$

$$\int_{R_5} \vec{F}\cdot\hat{n}_5\cdot dS_5 = \int_0^1\int_0^1 (4x\hat{i} - y^2\hat{j} + y\hat{k})\cdot\hat{k}\,dx\cdot dy$$

$$= \int_0^1\int_0^1 y\cdot dx\cdot dy$$

$$= \int_0^1 [yx]_0^1 = \int_0^1 y\cdot dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$$

Face AQRO :- we have $z=0$, $\hat{n}_6 = -\hat{k}$ and $dS_6 = \dfrac{dx\cdot dy}{|\hat{n}_6\cdot\hat{k}|}$

$$\int_{R_6} \vec{F}\cdot\hat{n}_6\cdot dS_6 = \iint (-y^2\cdot\hat{j})\cdot\hat{k}\cdot dx\cdot dy$$

$$= 0$$

Using all these in (x), we get

$$\int_S \vec{F}\cdot\hat{n}\,ds = 2+0-1+0+\frac{1}{2}+0 = 1+\frac{1}{2} = \frac{3}{2}$$

(2) Evaluate $\int_S \vec{F}\cdot\hat{n}\,ds$ if $\vec{F} = 18\hat{i}z - 12\hat{j} + 3y\hat{k}$ where s is the plane $2x+3y+6z=12$ located in the first octant.

Sol Given $\vec{F} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$

$\because \phi = 2x+3y+6z = 12 \Rightarrow z = \dfrac{12-2x-3y}{6} \Rightarrow y = \dfrac{12-2x}{3}$

$\nabla\phi = 2\hat{i} + 3\hat{j} + 6\hat{k}$ & $|\nabla\phi| = \sqrt{4+9+36} = 7$

$\therefore \hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2\hat{i}+3\hat{j}+6\hat{k}}{7}\cdot\hat{k}$

$$\int_S \vec{F}\cdot\hat{n}\,ds = \int_S (18z\hat{i} - 12\hat{j} + 3y\hat{k})\frac{(2\hat{i}+3\hat{j}+6\hat{k})}{7}\,ds$$

$$= \frac{1}{7}\iint_S (36z - 36 + 18y)\, ds$$

$$= \frac{1}{7}\iint_S \left[\cancel{36}^{6} \frac{(12-2x-3y)}{\cancel{6}} - 36 + 18y \right] ds$$

$$= \frac{1}{7}\iint_S (72 - 12x - \cancel{18y} - 36 + \cancel{18y})\, ds$$

$$= \frac{1}{7}\iint_S (36 - 12x)\, \frac{dx \cdot dy}{|\hat{n}\cdot\hat{k}|} \qquad \left[\because ds = \frac{dx \cdot dy}{|\hat{n}\cdot\hat{k}|} \right]$$

$$= \frac{\cancel{12}^{2}}{\cancel{7}_{1}} \iint_S (3-x)\, \frac{dx \cdot dy}{\cancel{6}/\cancel{7}} = 2\int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (3-x)\, dy \cdot dx$$

$$= 2\int_0^6 \left[(3-x)y \right]_0^{\frac{12-2x}{3}} dx$$

$$= 2\int_0^6 (3-x)\left(\frac{12-2x}{3}\right) dx$$

$$= \frac{2}{3}\int_0^6 (36 - 18x + 2x^2)\, dx$$

$$= \frac{2}{3}\left[36x - \cancel{18}^{9}\frac{x^2}{\cancel{2}} + 2\frac{x^3}{3} \right]_0^6$$

$$= \frac{2}{3}\left[216 - 324 + \frac{\cancel{432}^{144}}{\cancel{3}} \right]$$

$$= \frac{2}{\cancel{3}}\left[\cancel{36}^{12} \right] = 24$$

Volume Integral (Problems)

1) Evalute $\int_V \vec{F}\cdot dv$ if $\vec{F} = 2xz\,\hat{i} - x\hat{j} + y^2\hat{k}$ taken over the region bounded by surfaces $x=0,\ y=0,\ y=6,\ z=x^2$ & $z=4$

Sol. Given $\vec{F} = 2xz\,\hat{i} - x\hat{j} + y^2\hat{k}$

$$\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} (2xz\,\hat{i} - x\hat{j} + y^2\hat{k})\, dz\cdot dy\cdot dx$$

$$= \int_0^2\int_0^6 \left[2x\frac{z^2}{2}\hat{i} - xz\hat{j} + y^2 z\hat{k}\right]_{x^2}^{4} dy\cdot dx$$

$2x(16)$

$$= \int_0^2\int_0^6 \left(\left[16x\hat{i} - 4x\hat{j} + 4y^2\hat{k}\right] - \left[x^5\hat{i} - x^3\hat{j} + x^2y^2\hat{k}\right]\right) dy\, dx$$

$$= \int_0^2\int_0^6 \left((16x - x^5)\hat{i} - (4x - x^3)\hat{j} + (4y^2 - x^2y^2)\hat{k}\right) dy\, dx$$

$$= \int_0^2 \left\{(16x - x^5)y\,\hat{i} - (4x - x^3)y\,\hat{j} + \left(4\frac{y^3}{3} - x^2\frac{y^3}{3}\right)\hat{k}\right\}_0^6 dx$$

$$= \int_0^2 \left\{(96x - 6x^5)\hat{i} - (24x - 6x^3)\hat{j} + (288 - 72x^2)\hat{k}\right\} dx$$

$$= \left[\left(96\frac{x^2}{2} - 6\frac{x^6}{6}\right)\hat{i} - \left(24\frac{x^2}{2} - 6\frac{x^4}{4}\right)\hat{j} + \left(288x - 72\frac{x^3}{3}\right)\hat{k}\right]_0^2$$

$$= 128\hat{i} - 24\hat{j} + 384\hat{k}$$

(2) Evalute $\int_V \phi\, dv$ if $\phi = 45x^2y$ over the closed region bounded by the plane $4x+2y+z=8$, $x=0$ $y=0$ & $z=0$.

Sol Here z varies from 0 to $8-4x-2y$

y varies from 0 to $\frac{8-4x}{2} = 4-2x$

and x varies from 0 to 2

i.e $$\int_V \phi\, dv = \int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-4x-2y} 45x^2y \cdot dz\, dy\, dx.$$

$$= 45\int_{x=0}^{2}\int_{y=0}^{4-2x} \left(x^2yz\right)_0^{8-4x-2y} dy \cdot dx$$

$$= 45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2y(8-4x-2y)\, dy \cdot dx$$

$$= 45 \int_{x=0}^{2} \int_{y=0}^{4-2x} (8x^2y - 4x^3y - 2x^2y^2)\, dy \cdot dx$$

$$= 45 \int_{x=0}^{2} \left[\cancel{8}^{4} x^2 \frac{y^2}{\cancel{2}} - \cancel{4}^{2} x^3 \frac{y^2}{\cancel{2}} - 2x^2 \frac{y^3}{3} \right]_0^{4-2x} dx$$

$$= 45 \int_{x=0}^{2} \left[4x^2(4-2x)^2 - 2x^3(4-2x)^2 - \frac{2}{3}x^2(4-2x)^3 \right] dx$$

$$= 45 \int_{x=0}^{2} \left\{ 4x^2(16+4x^2-16x) - 2x^3(16+4x^2-16x) - \frac{2}{3}x^2\left(64-8x^3-3(4)^2(2x) + 3(4)(2x)^2\right) \right\} dx$$

$$= 45 \times 2 \int_{x=0}^{2} \left\{ \frac{32x^2 + 8x^4 - 32x^3 - 16x^3 - 4x^5 + 16x^4 - 64x^2 + 8x^5 + 96x^3 - 48x^4}{3} \right\} \cdot dx$$

$$= \frac{\cancel{90}^{30}}{3} \int_{x=0}^{2} \left\{ 32x^2 + 24x^4 - 48x^3 - 4x^5 \right\} dx$$

$$= 30 \left[32\frac{x^3}{3} + 24\frac{x^5}{5} - \cancel{48}^{12}\frac{x^4}{\cancel{4}} - \cancel{4}^{2}\frac{x^6}{\cancel{6}_{3}} \right]_0^2$$

$$= 30 \left[\frac{256}{3} + \frac{768}{5} - 192 - \frac{128}{3} \right]$$

$$= \cancel{30}^{2} \left[\frac{1280 + 2304 - 2880 - 640}{\cancel{15}} \right]$$

$$= 2[64] = 128.$$

Gauss Divergence Theorem:- (This theorem shows the relation b/w volume & the Surface integral)

If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is a vector point function which is continuous and having its first order partial derivative over the surface 'S' inclosing the volume 'V', then

$$\int_V \nabla \cdot \vec{F}\, dv = \int_S \vec{F} \cdot \hat{n}\, ds$$

where $\hat{n}$ is the unit outward drawn normal to the surface 'S'.

Green's theorem in a plane :- (This theorem shows the relation b/w line & the surface integral)

If 'R' is the closed region bounded by the surface 's' of a simple closed curve 'C'. If 'M' & 'N' are the functions of 'x', 'y', which are continous and having its first order partial derivatives then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \cdot dy$$

where 'C' is ~~triverse~~ traverse in comton close wise direction

Stoke's theorem : (This theorem also shows the relation b/w line & the surface integral)

Let 's' be an open surface of a simple closed curve 'C'. If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is a vector point function which is continuous and having its first order partial derivatives then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot \hat{n} \cdot ds$$

where $\hat{n}$ is the unit outward drawn normal at a point 'p' over a surface 'S'.

G.D.T (Problem)

1) S.T $\int_S (ax\hat{i} + by\hat{j} + cz\hat{k})\hat{n}\,ds = \frac{4\pi}{3}(a+b+c)$ where S is the surface of sphere $x^2+y^2+z^2=1$

Sol: Here $\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$

By G.D.T we have.

$$\int_S \vec{F}\cdot\hat{n}\,ds = \int_V \nabla\cdot\vec{F}\,dV$$

$$\int_S (ax\hat{i} + by\hat{j} + cz\hat{k})\hat{n}\,ds$$

$$= \int_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(ax\hat{i} + by\hat{j} + cz.\hat{k})\,dV$$

$$= \int_V (a+b+c)\,dV$$

$$= (a+b+c)\int_V dV = (a+b+c)V$$

$$= (a+b+c)\frac{4}{3}\pi r^3$$

$$= \frac{4}{3}\pi(a+b+c) \quad \because r=1$$

$$= R.H.S$$

2) Verify Gauss Divergence theorem if $\vec{F} = (x^2-yz)\hat{i} + (y^2-xz)\hat{j} + (z^2-xy)\hat{k}$ taken over a rectangular parallelopiped $0\le x\le a$, $0\le y\le b$ & $0\le z\le c$

Sol Given $\vec{F} =$ "

By G.D.T we have $\int_V \nabla\cdot\vec{F}\,dv = \int_S \vec{F}\cdot\hat{n}\,ds$

L.H.S: $\int_V \nabla\cdot\vec{F}\,dv = \int_V (2x+2y+2z)\,dv$

$$= 2\int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a} (x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^c\int_0^b \left[\frac{x^2}{2} + xy + zx\right]_0^a dy\,dz.$$

$$= 2\int_0^c\int_0^b \left(\frac{a^2}{2} + ay + az\right) dy \cdot dz$$

$$= 2\int_0^c \left(\frac{a^2}{2}y + a\frac{y^2}{2} + azy\right)_0^b dz.$$

$$= 2\int_0^c \left(\frac{a^2 b}{2} + \frac{ab^2}{2} + abz\right) dz$$

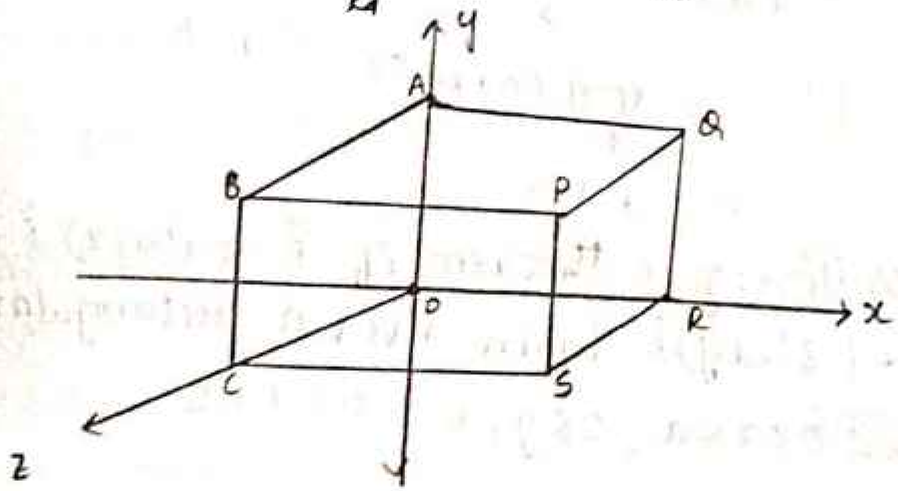
$$= \cancel{2}\left[\frac{a^2bz}{\cancel{2}} + \frac{ab^2z}{\cancel{2}} + \frac{abz^2}{2}\right]_0^c$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc(a+b+c) \quad \text{—— (i)}$$

$$R.H.S = \int_S \vec{F}\cdot\hat{n}\cdot ds = \iint_{R_1}\vec{F}\cdot\hat{n}_1 ds_1 + \iint_{R_2}\vec{F}\cdot\hat{n}_2 dS_2 + \iint_{R_3}\vec{F}\,\hat{n}_3\, ds_3$$

$$+ \iint_{R_4}\vec{F}\cdot\hat{n}_4\, ds_4 + \iint_{R_5}\vec{F}\cdot\hat{n}_5 dS_5 + \iint_{R_6}\vec{F}\cdot\hat{n}_6 ds_6 \quad \text{—— (1)}$$



<u>Face PQRS</u> :- we have $x = a$, $\hat{n}_1 = \hat{i}$, $dS_1 = \frac{dy\,dz}{|\hat{n}_1\cdot\hat{i}|}$

$$= \iint_{R_1}\vec{F}\cdot\hat{n}_1 ds = \int_{z=0}^{c}\int_{y=0}^{b}(a^2 - yz)\frac{dy\cdot dz}{|\hat{i}\cdot\hat{i}|}$$

$$= \int_0^c \left(a^2y - \frac{y^2 z}{2}\right)_0^b dz = \int_0^c \left(a^2b - \frac{b^2z}{2}\right) dz$$

$$= \left[a^2bz - \frac{b^2}{2}\cdot\frac{z^2}{2}\right]_0^c = a^2bc - \frac{b^2c^2}{4}.$$

Face OABC :- we have $x=0$, $\hat{n}_2 = -\hat{i}$, $ds_2 = \frac{dy\,dz}{|\hat{n}_2 \cdot \hat{i}|}$

$$\iint_{R_2} \vec{F}\cdot\hat{n}_2\, dS_2 = \int_{z=0}^{c}\int_{y=0}^{b} yz\,\frac{dy\,dz}{|-\hat{i}\cdot\hat{i}|}$$

$$= \int_{z=0}^{c}\left[\frac{y^2}{2}z\right]_0^b dz = \int_0^c \frac{b^2 z}{2}\,dz = \left[\frac{b^2z^2}{4}\right]_0^c$$

$$= \frac{b^2c^2}{4}$$

Face ABPQ :- we have $y=b$, $\hat{n}_3 = \hat{j}$ and $dS_3 = \frac{dx\,dz}{|\hat{n}_3\cdot\hat{j}|}$

$$\iint_{R_3} \vec{F}\cdot\hat{n}_3\, dS_3 = \int_{z=0}^{c}\int_{x=0}^{a} (b^2 - xz)\,\frac{dx\cdot dz}{|\hat{j}\cdot\hat{j}|}$$

$$= \int_0^c \left[b^2x - \frac{x^2 z}{2}\right]_0^a dz$$

$$= \int_0^c \left[ab^2 - \frac{a^2 z}{2}\right] dz$$

$$= \left[ab^2 z - \frac{a^2z^2}{4}\right]_0^c$$

$$= ab^2c - \frac{a^2c^2}{4}$$

Face OCSR :- we have $y=0$, $\hat{n}_4 = -\hat{j}$, $dS_4 = \frac{dx\,dz}{|\hat{n}_4\cdot\hat{j}|}$

$$\iint_{R_4} \vec{F}\cdot\hat{n}_4\cdot dS_4 = \int_0^c\int_0^a xz\; dx\cdot dz$$

$$= \int_0^c \left[\frac{x^2}{2}z\right]_0^a dz = \int_0^c \frac{a^2 z}{2}\,dz$$

$$= \left[\frac{a^2z^2}{4}\right]_0^c = \frac{a^2c^2}{4}$$

Face BCSP :- we have $z = c$, $\hat{n}_5 = \hat{k}$, $dS_5 = \dfrac{dx\,dy}{|\hat{n}_5 \cdot \hat{k}|}$

$$\iint_{R_5} \vec{F} \cdot \hat{n}_5 \, dS_5 = \int_0^b \int_0^a (c^2 - xy) \cdot dx \cdot dy$$

$$= \int_0^b \left[c^2 x - \frac{x^2 y}{2} \right]_0^a dy$$

$$= \int_0^b \left(ac^2 - \frac{a^2 y}{2} \right) \cdot dy$$

$$= \left[ac^2 y - \frac{a^2 y^2}{4} \right]_0^b$$

$$= abc^2 - \frac{a^2 b^2}{4}$$

Face AORQ :- we have $z = 0$, $\hat{n}_6 = -\hat{k}$, $dS_6 = \dfrac{dx\,dy}{|\hat{n}_6 \cdot \hat{k}|}$

$$\iint_{R_6} \vec{F} \cdot \hat{n}_6 \cdot dS_6 = \int_0^b \int_0^a xy \cdot dx \cdot dy$$

$$= \int_0^b \left[\frac{x^2 y}{2} \right]_0^a dy = \int_0^b \left[\frac{a^2 y}{2} \right] \cdot dy$$

$$= \left[\frac{a^2 y^2}{4} \right]_0^b = \frac{a^2 b^2}{4}$$

Using all these in (i), we get

$$\int_S \vec{F} \cdot \hat{n} \, ds = a^2 bc - \cancel{\frac{b^2 c^2}{4}} + \cancel{\frac{b^2 c^2}{4}} + ab^2 c - \cancel{\frac{a^2 c^2}{4}} + \cancel{\frac{a^2 c^2}{4}}$$

$$+ abc^2 - \cancel{\frac{a^2 b^2}{4}} + \cancel{\frac{a^2 b^2}{4}}$$

$$= abc(a + b + c) \quad \text{— (ii)}$$

$\because$ (i) = (ii)

Hence G.D.T is verified.

① Verify G.D.T if $\vec{F} = 4xy\hat{i} - y^2\hat{j} + xz\hat{k}$ taken over a cube $x=0, x=1, y=0, y=1, z=0 \& z=1$.

Given $\vec{F} =$

By G.D.T we have $\int_V \nabla \cdot \vec{F} \, dV = \int_{S} \vec{F} \cdot \hat{n} \, ds$

$$L.H.S = \int_V \nabla \cdot \vec{F} \, dV = \int_V (4y - 2y + x) \, dV$$

$$= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} (4y - 2y + x) \, dx \cdot dy \cdot dz$$

$$= \int_0^1\int_0^1\int_0^1 (2y + x) \, dx \, dy \, dz$$

$$= \int_0^1\int_0^1 \left[2xy + \frac{x^2}{2}\right]_0^1 dy \cdot dz$$

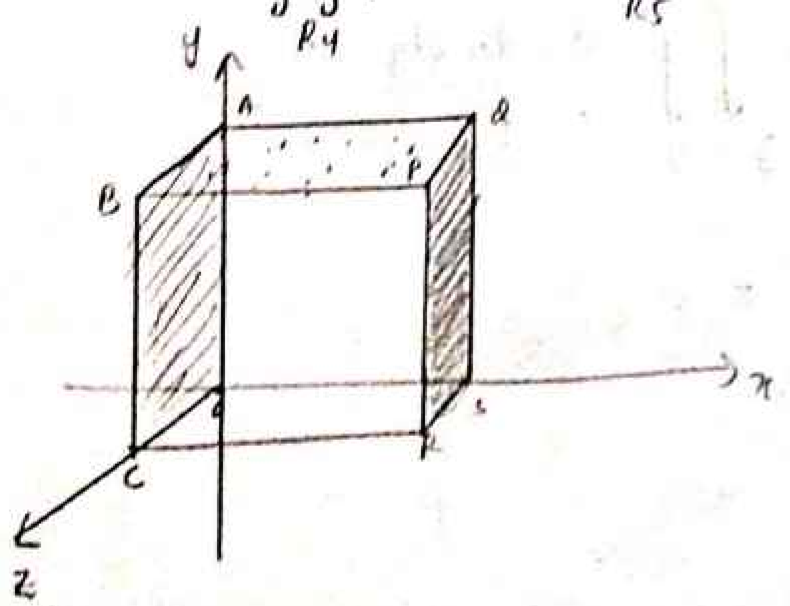
$$= \int_0^1\int_0^1 (2y + \tfrac{1}{2}) \, dy \, dz = \int_0^1 \left[y^2 + \frac{y}{2}\right]_0^1 dz$$

$$= \int_0^1 (1 + \tfrac{1}{2}) \, dz = \int_0^1 \frac{3}{2} dz = \frac{3}{2}\int_0^1 dz$$

$$= \frac{3}{2}\left[z\right]_0^1 = \frac{3}{2}(1) = \frac{3}{2} \quad \text{— ①}$$

$$R.H.S = \int_S \vec{F} \cdot \hat{n} \, ds = \iint_{R_1} \vec{F} \hat{n}_1 \, ds_1 + \iint_{R_2} \vec{F} \hat{n}_2 \, ds_2 + \iint_{R_3} \vec{F} \hat{n}_3 \, ds_3$$

$$+ \iint_{R_4} \vec{F} \hat{n}_4 \, ds_4 + \iint_{R_5} \vec{F} \hat{n}_5 \, ds_5 + \iint_{R_6} \vec{F} \hat{n}_6 \, ds_6 \quad \text{— ②}$$



Face PQRS :- $x = 1$; $\hat{n}_1 = \hat{i}$, $dS_1 = \dfrac{dy \cdot dz}{|\hat{n}_1 \cdot \hat{i}|}$

$$\iint_{R_1} \vec{F} \hat{n}_1 \, dS_1 = \int_{z=0}^{1} \int_{y=0}^{1} 4y \cdot \frac{dy \cdot dz}{|\hat{i} \cdot \hat{i}|} = \int_0^1 (2y^2)_0^1 \, dz$$

$$= \int_0^1 2 \, dz = 2 [z]_0^1 = 2$$

Face OABC:- $x = 0$; $\hat{n}_2 = -\hat{i}$; $dS_2 = \dfrac{dy \, dz}{|\hat{n}_2 \cdot \hat{i}|}$

$$\iint_{R_2} \vec{F} \cdot \hat{n}_2 \, dS_2 = \int_{z=0}^{1} \int_{y=0}^{1} 4(0) y \cdot \frac{dy \, dz}{|-\hat{i} \cdot \hat{i}|}$$

$$= 0$$

Face ABPQ :- $y = 1$; $\hat{n}_3 = \hat{j}$; $dS_3 = \dfrac{dx \, dz}{|\hat{n}_3 \cdot \hat{j}|}$

$$\iint_{R_3} \vec{F} \cdot \hat{n}_3 \, dS_3 = \int_{z=0}^{1} \int_{x=0}^{1} \frac{-dx \, dz}{|\hat{j} \cdot \hat{j}|} = \int_0^1 [-x]_0^1 dz$$

$$= \int_0^1 -dz = [-z]_0^1 = -1$$

Face OCSR :-

$y = 0$; $\hat{n}_4 = -\hat{j}$, $dS_4 = \dfrac{dx \, dz}{|\hat{n}_4 \cdot \hat{j}|}$

$$\iint_{R_4} \vec{F} \, \hat{n}_4 \, dS_4 = \int_{z=0}^{1} \int_{y=0}^{1} 0 \cdot \frac{dx \, dz}{|-\hat{j} \cdot \hat{j}|}$$

$$= 0$$

Face BCSP :-

$z=1$; $\hat{n}_5 = \hat{k}$, $ds_5 = \dfrac{dx \cdot dy}{|\hat{n}_5 \cdot \hat{k}|}$

$$\iint_{S} \vec{F} \cdot \hat{n}_5 \, ds_5 = \int_{y=0}^{1} \int_{x=0}^{1} x \, dx \cdot dy$$

$$= \int_0^1 \left[\frac{x^2}{2} \right]_0^1 dy$$

$$= \frac{1}{2} \int_0^1 dy$$

$$= \frac{1}{2} [y]_0^1 = \frac{1}{2}$$

FACE AORQ :-

$z=0$, $\hat{n}_6 = -\hat{k}$, $ds_6 = \dfrac{dx \cdot dy}{|\hat{n}_6 \cdot \hat{k}|}$

$$\iint_{R_5} \vec{F} \hat{n}_6 \cdot ds_6 = \int_{y=0}^{1} \int_{x=0}^{1} x(0) \, \hat{i} \cdot -\hat{k} \, \frac{dx \, dy}{|\hat{k} \cdot \hat{i}|}$$

$$= 0$$

Substitute all these in (A) eq^n, we get

$$\int_S \vec{F} \hat{n} \, ds = 2 + 0 - 1 + 0 + \frac{1}{2}$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2} \quad \leftarrow (2)$$

$eq^n ① = eq^n ②$

$\therefore$ Hence Gauss divergence theorem is verified.

Green's theorem (Problem)

① Evaluate $\oint_C (2xy - y^2)dx + (x^2 + y^2)dy$ where C is the boundary of region defined by $y = x^2$ and $x = y^2$.

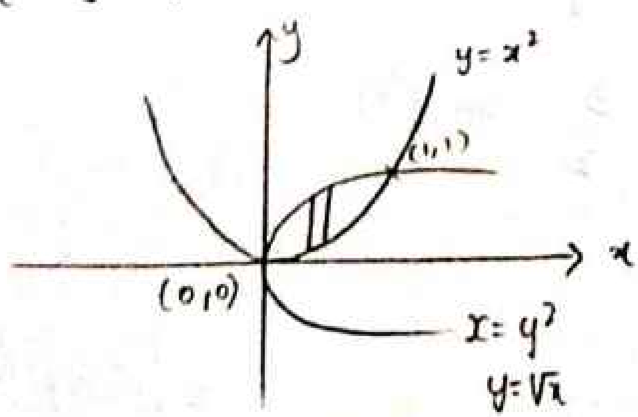
Sol. By G.T, we have

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = 2xy - y^2$ and $N = x^2 + y^2$.

$\frac{\partial M}{\partial y} = 2x - 2y$ and $\frac{\partial N}{\partial x} = 2x$

$$\therefore \oint_C (2xy - y^2)dx + (x^2 + y^2)dy = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (2x - 2x + 2y)\,dy\,dx$$



Put $y=x^2$ in $x=y^2$

$x=x^4 \Rightarrow x^4-x=0 \Rightarrow x(x^3-1)=0$

$x=0$ (or) $x=1$

$|||^{ly}$ $y=0$ (or) $y=1$

i.e $\int_0^1 \left[\frac{xy^2}{2}\right]_{x^2}^{\sqrt{x}} dx$

$= \int_0^1 (x-x^4)\,dx = \left[\frac{x^2}{2}-\frac{x^5}{5}\right]_0^1$

$= \frac{1}{2}-\frac{1}{5} = \frac{3}{10}$

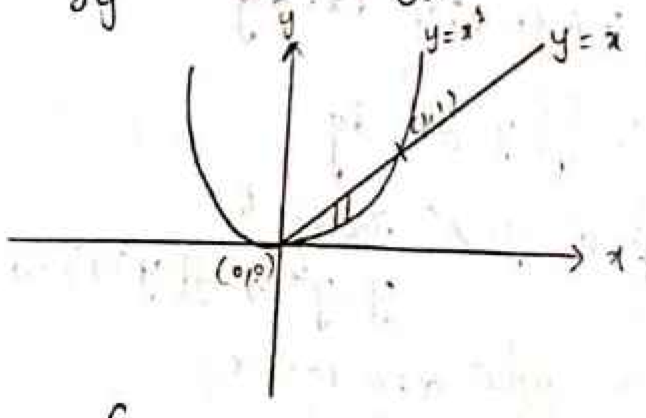
② Evalute $\oint_C (xy+x^2)dx+(x^2+y^2)dy$ where c is the region defined by $y=x$ & $y=x^2$

Sol By G.T, we have

$$\oint_C M\,dx+N\,dy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

Here $M=xy+x^2$ & $N=x^2+y^2$

$\frac{\partial M}{\partial y}=x$ & $\frac{\partial N}{\partial x}=2x$



i.e, $\oint_C (xy+x^2)dx+(x^2+y^2)dy$

$= \int_{x=0}^{1}\int_{y=x^2}^{x} (2x-x)\,dy.dx$

Put $y=x$ in $y=x^2 \Rightarrow x^2-x=0$

$x(x-1)=0 \Rightarrow x=0$ (or) $x=1$

$$= \int_0^1 \int_{x^2}^{x} x \, dy \, dx = \int_0^1 (xy)_{x^2}^{x} dx$$

$$= \int_0^1 x(x - x^2) dx = \int_0^1 (x^2 - x^3) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

22/02/2022

③ Verify G.T $\oint_C (2x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of region defined as $y = \sqrt{x}$ and $y = x^2$

Sol. By G.T we have $\oint_C M \cdot dx + N \cdot dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \cdot dy$

Here $M = 2x^2 - 8y^2$ and $N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y$ & $\frac{\partial N}{\partial x} = -6y$

$$R.H.S = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \cdot dy$$

$$= \iint (-6y + 16y) dx \cdot dy$$

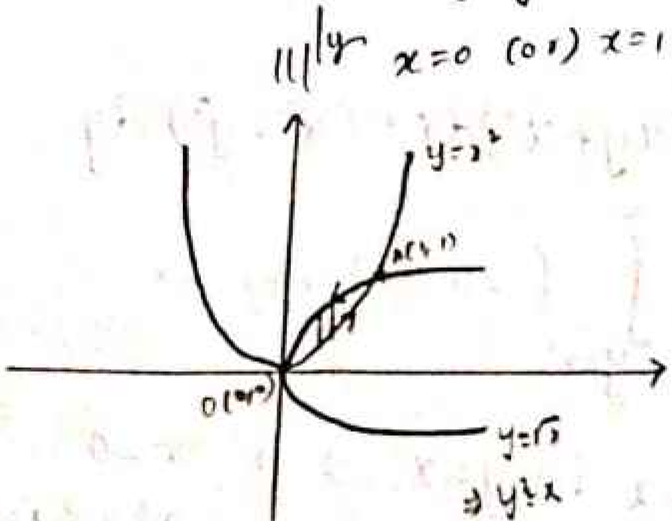
$$= \iint_R (-6y + 16y) dx \cdot dy$$

$$= 10 \iint_R y \cdot dx \cdot dy$$

Put $x = y^2$ in $y = x^2$ we get

$y = y^4 \Rightarrow y(y^3 - 1) = 0 \Rightarrow y = 0$ (or) $y = 1$

IIIly $x = 0$ (or) $x = 1$



i.e, $+10\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} y\cdot dy\cdot dx = 10^{5}\int_{0}\left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} dx$

$= 5\int_{0}^{1}(x-x^4)\,dx = 5\left(\frac{x^2}{2}-\frac{x^5}{5}\right)_0^1$

$= 5\left(\frac{1}{2}-\frac{1}{5}\right) = \cancel{5}\left(\frac{3}{\cancel{10}_{2}}\right) = \frac{3}{2}$ —(i)

Now L.H.S $= \oint_C (2x^2-8y^2)\,dx + (4x-6xy)\,dy$

$= \int_{OA}\{(2x^2-8y^2)\,dx+(4x-6xy)\,dy\} + \int_{AO}\{(2x^2-8y^2)\,dx+(4x-6xy)\,dy\}$

Along OA we have $y=x^2 \Rightarrow dy = 2x\cdot dx$ and 'x' varies from 0 to 1.

Along AO we have $x=y^2 \Rightarrow dx = 2y\cdot dy$ and 'y' varies from 1 to 0

i.e $\int_{x=0}^{1}(2x^2-8x^4)\,dx+(4x^2-6x^3)\,2x\cdot dx$

$+\int_{y=1}^{0}(2y^4-8y^2)\,2y\,dy+(4y-6y^3)\,dy$

$= \int_0^1(2x^2-8x^4+8x^3-12x^4)\,dx + \int_1^0(4y^5-16y^3+4y-6y^3)\,dy$

$= \int_0^1(2x^2-20x^4+8x^3)\,dx + \int_1^0(4y^5-22y^3+4y)\,dy$

$= \left[\frac{2x^3}{3}-\frac{\cancel{20}^{4}x^5}{\cancel{5}}+\frac{\cancel{8}^{2}x^4}{\cancel{4}}\right]_0^1 + \left[\frac{\cancel{4}^{2}y^6}{\cancel{6}_{3}}-\frac{\cancel{22}^{11}y^4}{\cancel{4}_{2}}+\frac{\cancel{4}^{2}y^2}{\cancel{2}}\right]_1^0$

$= \left[\cancel{\frac{2}{3}}-4+\cancel{2}\right]+\left[-\cancel{\frac{2}{3}}+\frac{11}{2}-\cancel{2}\right]$

$= \frac{3}{2}$ —(ii)

$\therefore$ (i) = (ii)

Hence G.T is verified

④ Verify G.T $\oint_C (3x^2-8y^2)dx+(4y-6xy)dy$ where C is the boundary of region defined as $x=0$, $y=0$, $x+y=1$

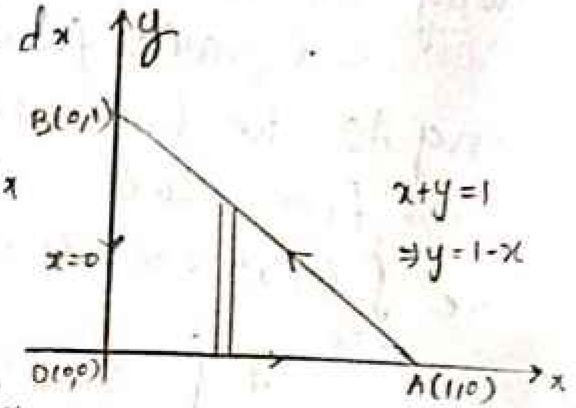
Sol By G.T we have $\oint_C M\,dx+N\,dy=\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$

Here $M=3x^2-8y^2$ and $N=4y-6xy$

$\frac{\partial M}{\partial y}=-16y$ and $\frac{\partial N}{\partial x}=-6y$

$$R.H.S=\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy=\iint_R(-6y+16y)\,dx\,dy$$

$$=10\int_{x=0}^{1}\int_{y=0}^{1-x} y\,dy\,dx$$



$$=\overset{5}{\cancel{10}}\int_0^1\left[\frac{y^2}{\cancel{2}}\right]_0^{1-x}dx$$

$$=5\int_0^1(1-x)^2\,dx$$

$$=5\int_0^1(1+x^2-2x)\,dx$$

$$=5\left[x+\frac{x^3}{3}-\frac{\cancel{2}x^2}{\cancel{2}}\right]_0^1$$

$$=5\left[1+\frac{1}{3}-1\right]=\frac{5}{3}\text{ ---(i)}$$

$$L.H.S=\oint_C(3x^2-8y^2)dx+(4x-6xy)dy$$

$$=\int_{OA}\{(3x^2-8y^2)dx+(4x-6xy)dy\}+\int_{AB}\{(3x^2-8y^2)dx+(4x-6xy)dy\}+\int_{BO}\{(3x^2-8y^2)dx+(4x-6xy)dy\}$$

Along OA we have $y=0\Rightarrow dy=0$ & 'x' varies from 0 to 1.

Along AB we have $y=1-x\Rightarrow dy=-dx$ and 'x' varies from 1 to 0

Along BO we have $x=0 \Rightarrow dx=0$ and y varies from 1 to 0

i.e, $\int_{x=0}^{1} 3x^2\,dx + \int_{x=1}^{0} \{3x^2 - 8(1-x)^2\}\,dx + \{4(1-x) - 6x(1-x)\}(-dx)$

$+ \int_{y=1}^{0} 4y\,dy$

$= \left[\frac{3x^3}{3}\right]_0^1 + \int_1^0 \{3x^2 - 8(1+x^2-2x) + \{-4+4x+6x-6x^2\}\,dx$

$+ \left[4\frac{y^2}{2}\right]_0^1$

$= 1 + \int_1^0 \{3x^2 - 8 - 8x^2 + 16x - 4 + 10x - 6x^2\}\,dx + 2(0-1)$

$= 1 + \int_1^0 (-11x^2 + 26x - 12)\,dx - 2$

$= -1 + \left[-\frac{11x^3}{3} + 26\frac{x^2}{2} - 12x\right]_1^0$

$= -1 + \frac{11}{3} - 13 + 12 = \frac{11}{3} - 2 = \frac{5}{3}$ — (ii)

$\therefore$ (i) = (ii)

Hence G.T is verified.

) Verify G.T $\oint_C (x^2 - 2xy)\,dx + (x^2y + 3)\,dy$ where C is the boundary of region defined as $y^2 = 8x$ & $x=2$

A By G.T $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$

Here $M = x^2 - 2xy$ and $N = x^2y + 3$

$\frac{\partial M}{\partial y} = -2x$ and $\frac{\partial N}{\partial x} = 2xy$

$$R.H.S = \int_{x=0}^{2}\int_{y=-2\sqrt{2x}}^{2\sqrt{2x}} (2xy+2x)\,dy\,dx$$

$$= \int_{x=0}^{2} \left[2x\frac{y^2}{2} + 2xy\right]_{-2\sqrt{2x}}^{2\sqrt{2x}} \cdot dx$$

$$= \int_{x=0}^{2} \left\{\left[2x\frac{(8x)}{\cancel{2}} + 4x\sqrt{2x}\right] - \left[+2x\frac{(8x)}{\cancel{2}} - 4x\sqrt{2x}\right]\right\}$$

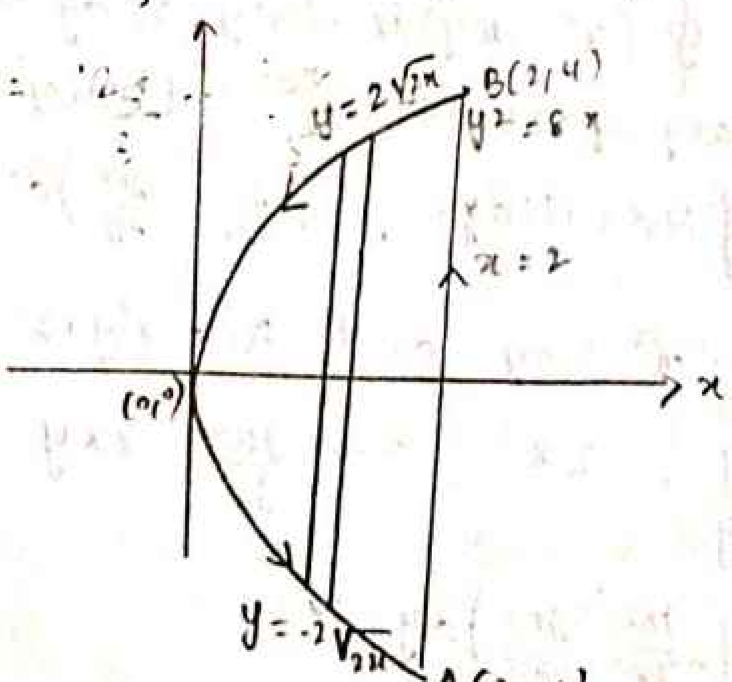
$$= \int_{x=0}^{2} \left(\cancel{8x^2} + 4x\sqrt{2x} - \cancel{8x^2} + 4x\sqrt{2x}\right) dx$$

$$= \int_{x=0}^{2} 8x\sqrt{2x}\,) \cdot dx$$

$$= \left(8\sqrt{2}\,\frac{x^{5/2}}{5/2}\right)_0^2$$

$$= \left(\frac{16\sqrt{2}}{5}\,x^{5/2}\right)_0^2$$

$$= \frac{16\sqrt{2}\times\sqrt{2}\cdot 4}{5} = \frac{128}{5} \quad -(i)$$



$$L.H.S = \oint_C (x^2-2xy)\,dx + (x^2y+3)\,dy$$

$$= \int_{OA} \{(x^2-2xy)dx + (x^2y+3)dy\} + \int_{AB} \{(x^2-2xy)dx + (x^2y+3)dy\}$$

$$+ \int_{BO} \{(x^2-2xy)dx + (x^2y+3)dy$$

Along OA we have $y = -2\sqrt{2x} \Rightarrow dy = -2\sqrt{2}\frac{1}{2\sqrt{x}}dx = -\sqrt{\frac{2}{x}}dx$
and x varies from 0 to 2

Along AB we have $x=2 \Rightarrow dx=0$ and y varies from -4 to 4

Along BO we have $y = 2\sqrt{2x} \Rightarrow dx = \sqrt{\frac{2}{x}}dx$ and x varies from 2 to 0

$$= \int_{x=0}^{2} \left\{ (x^2+4x\sqrt{2x})dx + (-2x^2\sqrt{2x}+3)\left(-\sqrt{\frac{2}{x}}dx\right) \right\}$$

$$+ \int_{-4}^{4} (4y+3)\,dy + \int_{2}^{0} (x^2-4x\sqrt{2x})dx + (2x^2\sqrt{2x}+3)\sqrt{\frac{2}{x}}dx$$

$$= \int_{x=0}^{2} (x^2+4x\sqrt{2x})dx + \left(4x^2 - \frac{3\sqrt{2}}{\sqrt{x}}\right)dx$$

$$+ \left[\frac{4y^2}{2}+3y\right]_{-4}^{4} + \int_{2}^{0} (x^2-4x\sqrt{2x})dx + \left(4x^2+\frac{3\sqrt{2}}{\sqrt{x}}\right)dx$$

$$= \int_{x=0}^{2} \left(5x^2+4x\sqrt{2x} - \frac{3\sqrt{2}}{\sqrt{x}}\right)dx + \left\{\left[\frac{4\times16}{2}+12\right] - \left[\frac{4\times16}{2}-12\right]\right\}$$

$$+ \int_{2}^{0} \left(5x^2 - 4x\sqrt{2x} + \frac{3\sqrt{2}}{\sqrt{x}}\right)dx$$

$$= \left(5\frac{x^3}{3} + 4\sqrt{2}\cdot x\left(\frac{2}{5}x^{5/2}\right) - 3\sqrt{2}\,2\sqrt{x}\right)_0^2 + 24$$

$$+ \left[\frac{5x^3}{3} - 4\sqrt{2}\left(\frac{2}{5}x^{5/2}\right) + 3\sqrt{2}\,2\sqrt{x}\right]_2^0$$

$$= \frac{40}{3} + \frac{64}{5} - 12 + 24 - \frac{40}{3} + \frac{64}{5} - 12$$

$$= \frac{128}{5} \quad \text{--- (ii)}$$

(i) = (ii) Hence G.T is verified

Stoke's Theorem (Problems)

① P.T $\oint e^x dx + 2y\,dy - dz = 0$ where C is a curve $x^2+y^2=9$ and $z=2$?

Sol Here $\vec{F} = e^x \hat{i} + 2y \hat{j} - \hat{k}$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

By stoke's theorem, we have

$$\oint_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \hat{i}(0) + (0-0)\hat{j} + (0-0)\hat{k} = \vec{0}$$

$$\therefore \oint_C e^x dx + 2y\,dy - dz = \int_S \vec{0}\cdot\hat{n}\,ds = 0$$

$= R.H.S$

② Evaluate $\oint_C (x+z)dx + (2x-y)dy + (y+z)dz$ where C is the vertices of Δ^{le} $(2,0,0)$, $(0,3,0)$ & $(0,0,6)$

(or) Equation of Surface passing through

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$$

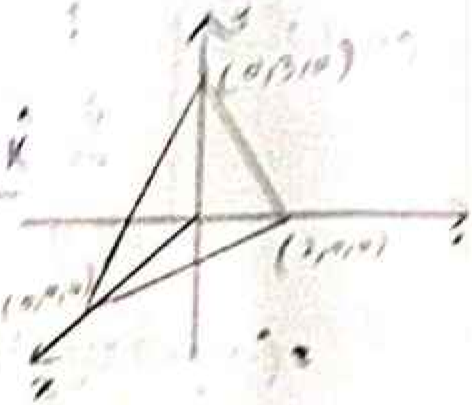
Sol: By S.T we have $\oint_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$

Here $\vec{F} = (x+z)\hat{i} + (2x-y)\hat{j} + (y+z)\hat{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+z & 2x-y & y+z \end{vmatrix}$$

$$= (1-0)\hat{i} + (1-0)\hat{j} + (2-0)\hat{k} = \hat{i} + \hat{j} + 2\hat{k}$$

Here $\phi = \frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$



$\nabla\phi = \frac{\hat i}{2} + \frac{\hat j}{3} + \frac{\hat k}{6} = \frac{3\hat i+2\hat j+\hat k}{6}$

$|\nabla\phi| = \sqrt{\frac{9+4+1}{36}} = \frac{\sqrt{14}}{6}$

$\therefore \hat n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3\hat i+2\hat j+\hat k}{\cancel{6}} \Big/ \frac{\sqrt{14}}{\cancel{6}}$

Now $\oint_C (x+z)dx + (2x-y)dy + (y+z)dz$

$= \int_S (\hat i+\hat j+2\hat k)\cdot\left(\frac{3\hat i+2\hat j+\hat k}{\sqrt{14}}\right) ds$

$= \int_S \left(\frac{3+2+2}{\sqrt{14}}\right) ds = \frac{7}{\sqrt{14}}\int_S \frac{dx\,dy}{|\hat n\cdot \hat k|}$

area of Δle

$= \frac{7}{\cancel{\sqrt{14}}}\int_S \frac{dx\,dy}{1/\cancel{\sqrt{14}}} = 7\cdot\frac{1}{2}(xy) = 7\cdot\frac{1}{\cancel{2}}(\cancel{2}\times 3) = 21$

③ Verify S.T if $\vec F = (2x-y)\hat i - yz^2\hat k - y^2z\hat k$ where S is the upper half of the sphere $x^2+y^2+z^2=1$ bounded by its projection in xy-plane

By S.T we have $\oint_C \vec F\cdot d\vec r = \int_S \text{curl}\vec F\cdot\hat n\, ds$

L.H.S $= \oint_C (2x-y)\,dx \quad [\because z=0]$

Put $x=\cos\theta$, $y=\sin\theta$
$dx = -\sin\theta\, d\theta$ & θ varies from 0 to 2π

i.e, $\int_{\theta=0}^{2\pi}(2\cos\theta - \sin\theta)(-\sin\theta\, d\theta)$

$= \int_0^{2\pi}(-\sin 2\theta + \sin^2\theta)\cdot d\theta$

$= \left[\frac{\cos 2\theta}{2}\right]_0^{\pi} + 4\int_0^{\pi/2}\sin^2\theta\, d\theta$

$= \left[\frac{1-1}{2}\right] + 4\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \pi$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= (-2yz + 2yz)\hat{i} + (0-0)\hat{j} + (0+1)\hat{k} = \hat{k}$$

Here $\hat{n} = \hat{k}$ [$\because$ xy plane]

$$R.H.S = \int_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \int_S \hat{k}\cdot\hat{k}\,ds = \int_S ds$$ area of O^{le}

$$= \pi r^2 = \pi(1) = \pi \text{ — (ii)}$$

$\therefore$ (i) = (ii)

Hence S.T is verified

④ Verified S.T if $\vec{F} = x^3\hat{i} - x^2y\hat{j}$ where c is the boundary of rectangle $x=0$, $x=3$, $y=0$ & $y=4$ in in the plane $z=0$

Sol By S.T we have $\oint_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & -x^2y & 0 \end{vmatrix}$$

$$= (0-0)\hat{i} + (0-0)\hat{j} + (-2xy - 0)\hat{k} = -2xy\,\hat{k}$$

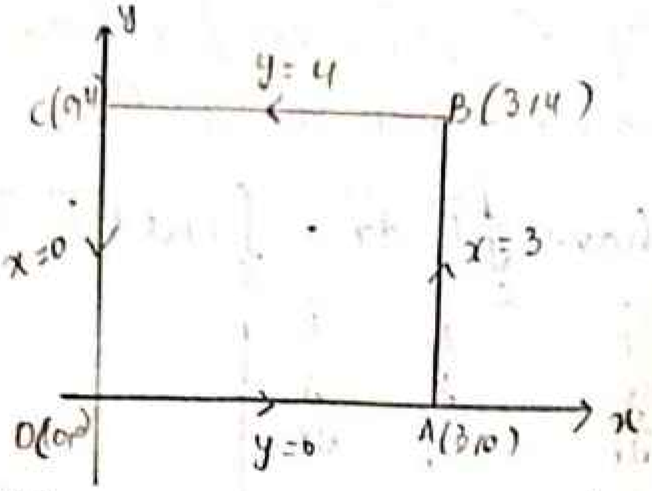
$$R.H.S = \int_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

Here $\hat{n} = \hat{k}$ [$\because$ xy plane]

$$= \int_S -2xy\,\hat{k}\cdot\hat{k}\,\frac{dx\,dy}{|\hat{k}\cdot\hat{k}|} = \int_S -2xy\,dx\,dy$$

$$= -2\int_0^4\int_0^3 xy\,dx\cdot dy = -2\int_0^4 \left(\frac{x^2}{2}y\right)_0^3 dy$$

$$= -9\left[\frac{y^2}{2}\right]_0^4 = -9\left(\frac{16}{2}\right) = -72 \text{ — (i)}$$



$$L.H.S = \oint_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F}\, d\vec{r} + \int_{BC} \vec{F}\, d\vec{r} + \int_{CO} \vec{F}\, d\vec{r} \quad \text{—(i)}$$

Here $\vec{F} \cdot d\vec{r} = x^3 dx - x^2 y\, dy$

Along OA we have $y = 0 \Rightarrow dy = 0$ and x varies from 0 to 3

Along AB we have $x = 3 \Rightarrow dx = 0$ and y varies from 0 to 4

Along BC we have $y = 4 \Rightarrow dy = 0$ and y varies from 3 to 0

Along CO we have $x = 0 \Rightarrow dx = 0$ and y varies from 4 to 0

i.e, $\int_{OA} \vec{F} \cdot d\vec{r} = \int_{x=0}^{3} x^3 dx = \left[\frac{x^4}{4}\right]_0^3 = \frac{81}{4}$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_{y=0}^{4} -9y\, dy = -9\left[\frac{y^2}{2}\right]_0^4 = -9\left(\frac{16}{2}\right) = -72$$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_{x=3}^{0} x^3 dx = \left[\frac{x^4}{4}\right]_3^0 = \frac{-81}{4}$$

$$\int_{CO} \vec{F} \cdot d\vec{r} = 0$$

Using all these in (i), we get

$$\oint_C \vec{F}\, d\vec{r} = -72 \quad \text{—(ii)}$$

$\therefore$ (i) = (ii)

Hence Stokes theorem is verified

(5) Verify S.T if $\vec{F} = x^2\hat{i} + xy\hat{j}$ where C is square formed by $x=0$, $x=a$, $y=0$, $y=a$ in the xy plane

Sol By S.T we have $\oint_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= (0)\hat{i} + \hat{j}(0-0) + \hat{k}(y-0)$$

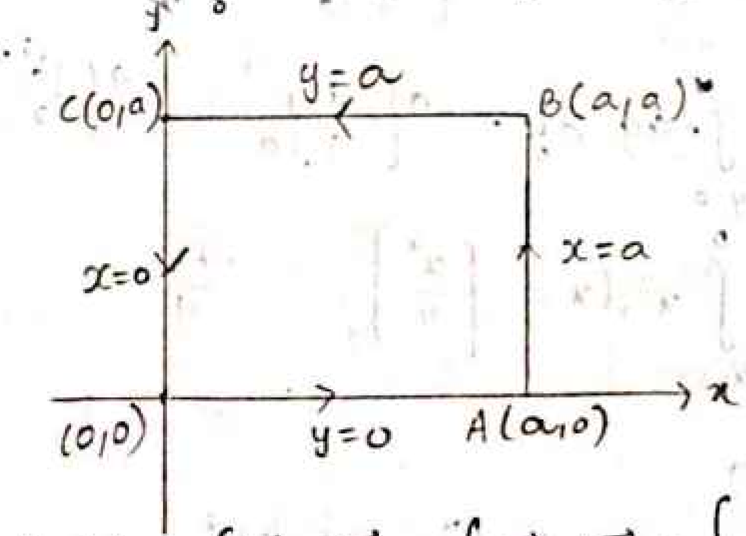
$$= y\hat{k}$$

R.H.S $= \int_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$

$$= \int_S y\,\hat{k}\cdot\hat{k}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \int_S y\cdot dx\cdot dy$$

$$= \int_{y=0}^{a}\int_{x=0}^{a} y\cdot dx\cdot dy = \int_0^a [xy]_0^a\,dy$$

$$= \int_0^a ay\,dy = \left[\frac{ay^2}{2}\right]_0^a = \frac{a^3}{2} \quad \text{—(i)}$$



$$\text{L.H.S} = \oint_C \vec{F}\cdot d\vec{r} = \int_{OA}\vec{F}\cdot d\vec{r} + \int_{AB}\vec{F}\cdot d\vec{r} + \int_{BC}\vec{F}\cdot d\vec{r}$$

$$+ \int_{CO}\vec{F}\cdot d\vec{r} \quad \text{—(*)}$$

Here $\vec{F}\cdot d\vec{r} = x^2 dx + xy\,dy$

Along OA we have $y=0 \Rightarrow dy=0$ & x varies from 0 to a

Along AB we have $x=a \Rightarrow dx=0$ & y varies from 0 to a

Along BC we have $y=a \Rightarrow dy=0$ & x varies from a to 0

Along CO we have $x=0 \Rightarrow dx=0$ & y varies from a to 0

i.e $\int_{OA} \vec{F}\cdot d\vec{r} = \int_{x=0}^{a} x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$

$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{y=0}^{a} ay\cdot dy = \left[a\frac{y^2}{2}\right]_0^a = \frac{a^3}{2}$

$\int_{BC} \vec{F}\cdot d\vec{r} = \int_{x=a}^{0} x^2\, dx = \left[\frac{x^3}{3}\right]_a^0 = \frac{-a^3}{3}$

$\int_{CO} \vec{F}\cdot d\vec{r} = \int_{y\,a}^{0} (0)\cdot dx = 0$

Using all these in (*), we get

$\oint_C \vec{F}\cdot d\vec{r} = \frac{a^2}{2}$ —— (2)

eq (1) = eq (2)

Hence stokes theorem is verified