National University of Computer and Emerging Sciences, Lahore Campus

Course:	Analog and Digital Communications	Course Code:	EE3003
Program:	BS(Electrical Engineering)	Semester:	Fall 2024
Exam:	Assignment 1	Page(s):	2
Chapter(s):	2,3	Section:	ALL
Submission Date:		CLO:	02
Instructor(s):	Dr. S. M. Sajid, Mohsin Yousuf	Marks:	40

<u>Question#1 [CLO 02] [05]</u>

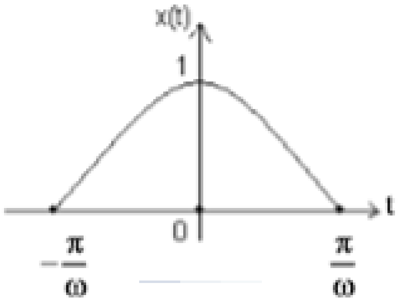
Check whether the following signals are energy or power signal and **find** there corresponding energy/power.

a) $x(t) = e^{-3t}$

b) $x(t) = cos^{2}\omega_{o}t$

c) $x(t) = u(t)$

d) $x(t) = \{\frac{1}{2}(\cos cos\, \omega t\ +\ 1), \qquad -\frac{\pi}{\omega}\le t\le\frac{\pi}{\omega}$ wise



e) $g_{1}(t) = 3\cos cos\,(t)\ +\ \cos cos\left(5t - \frac{2\pi}{3}\right)\ +\ \cos cos\left(8t + \frac{2\pi}{3}\right)$

<u>Question # 2 [CLO 02] [05]</u>

For the signal

$$g(t) = \frac{4a}{t^{2}+a^{2}}$$

Determine the essential bandwidth $B\ Hz$ of $g(t)$ such that the energy contained in the spectral components of $g(t)$ of frequencies below $B\ Hz$ is 92% of the signal energy E_{g}.

<u>Hint:</u> Determine $G(\omega)$ by applying the symmetry property.

<u>Question # 3 [CLO 02] [10]</u>

Figure 2 below shows a random binary pulse train g*(t)*. The pulse width is $T_b/2$, and one binary digit is transmitted every T_b seconds. A binary **1** is transmitted by the positive pulse, and a binary **0** is transmitted by the negative pulse. The two symbols are equally likely and occur randomly. **Determine i)** the autocorrelation function $R_x(\tau)$, **ii)** the PSD $S_x(\omega)$ and **iii)** the essential bandwidth of this signal.

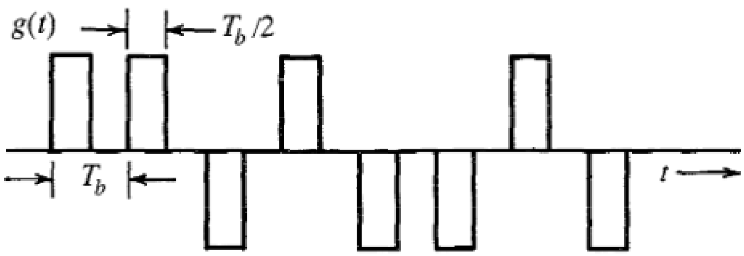


Figure 2. Random Binary Pulse Train

Question # 4 [CLO 02] [10]

The random binary signal $x(t)$ shown in Figure 3 below transmits one digit every T_b seconds. A binary **1** is transmitted by a pulse $p(t)$ of width $T_b/2$ and amplitude B; a binary **0** is transmitted by no pulse. The digits **1** and **0** are equally likely and occur randomly. **Determine** the **i)** autocorrelation function $R_x(\tau)$ and **ii)** the PSD $S_x(\omega)$.

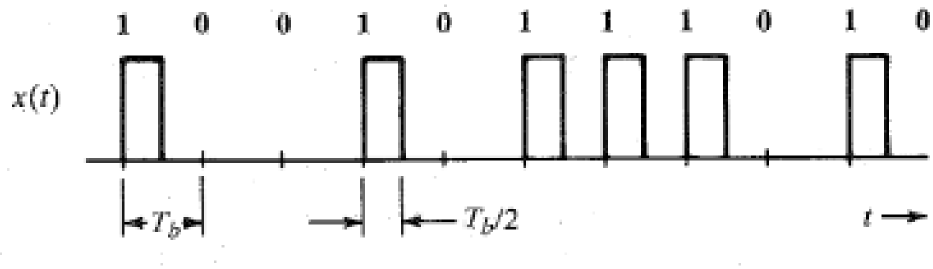


Figure 3. Random Binary Signal

Question # 5 [CLO 02] [10]

Find the mean square value (or power) of the output voltage $y(t)$ of the system shown in figure 4 below if the input voltage PSD $S_x(\omega) = 2\, rect\left(\frac{\omega}{2}\right)$. **Calculate** the power (mean square value) of the input signal $x(t)$.

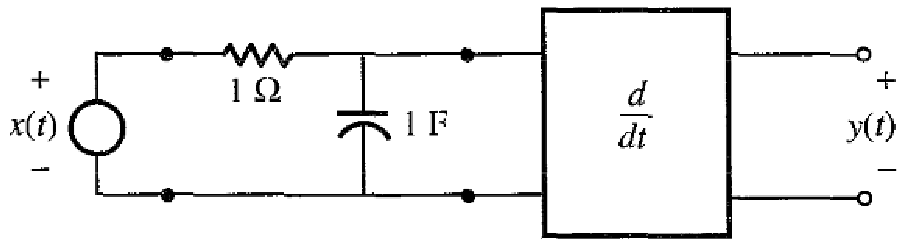


Figure 4. Circuit using differential system

Q.1 No.1

(a)

$$x(t) = e^{-3t}\, u(t)$$

$$E_g = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$$= \int_{-\infty}^{\infty} |e^{-3t}|^2\, dt$$

$$= \int_{0}^{\infty} e^{-6t}\, dt$$

$$= \frac{e^{-\infty} - e^{0}}{-6}$$

$$\boxed{E_g = -\frac{1}{6}}$$

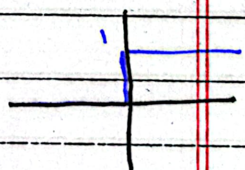
$\Rightarrow E_g = \int_{-\infty}^{\infty} |e^{-3t} u(t)|^2\, dt$

If $x(t) \to 0$ & $t \to \infty$, then signal is energy signal.

$\Rightarrow$ Finite energy $\Rightarrow$ $\boxed{\text{energy signal}}$

(c)

$$x(t) = u(t)$$



$$E_g = \int_{-\infty}^{\infty} 0^2\, dt + \int_{0}^{\infty} 1\, dt$$

$$= t \Big|_0^{\infty}$$

$$= \infty$$

$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$$

$\Rightarrow$ Not an energy signal

$$\Rightarrow P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |u(t)|^2 \, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} (1)^2 \, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} [t]_0^T$$

$$= \frac{1}{\cancel{T}} (\cancel{T})$$

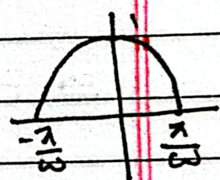
$$\boxed{P_x = 1}$$

$\Rightarrow$ $\boxed{\text{Power signal}}$

(d)

$$x(t) = \frac{1}{2}(\cos\omega t + 1) \quad ; \; -\frac{\pi}{\omega} < t < \frac{\pi}{\omega}$$

$$P_x(t) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[\frac{1}{2}(\cos\omega t + 1)\right]^2$$



$$= \lim_{T\to\infty} \frac{1}{T} \left[\frac{1}{4} \int_{-\pi/\omega}^{\pi/\omega} \cos^2\omega t \, dt + \int_{-\pi/\omega}^{\pi/\omega} dt \right.$$

$$\left. + \int_{-\pi/\omega}^{\pi/\omega} 2\cos\omega t \, dt \right]$$

$$= \lim_{T\to\infty} \frac{1}{4T} \left[\frac{1 + \cos 2\omega t}{2} \Big|_{-\pi/\omega}^{\pi/\omega} + [t]_{-\pi/\omega}^{\pi/\omega} + \left[\frac{2\sin\omega t}{\omega}\right]_{-\pi/\omega}^{\pi/\omega} \right]$$

$$= \lim_{T\to\infty} \frac{1}{4T} \left[\frac{2}{2\omega} - \frac{2}{2\omega} + \frac{2\pi}{\omega} + 0 \right]$$

$$= \lim_{T\to\infty} \frac{1}{4T} \left(\frac{2\pi}{\omega} \right)$$

$$= \infty$$

/ /20

⇒ Not Power signal

$$E_x(t) = \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} \left[\frac{1}{2}(\cos 2\omega t + 1)\right]^2 dt$$

$$= \frac{1}{4}\left[\frac{1+\cos^2\omega t}{2\omega}\Big|_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} + [t]\Big|_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}}\right.$$

$$\left. + \left[\frac{2\sin\omega t}{\omega}\right]\Big|_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}}\right.$$

$$= \frac{1}{4}\left[\frac{1}{2\omega}(1+\cos 2\pi - 1 - \cos 2\pi) + \frac{2\pi}{\omega}\right.$$

$$\left. + \frac{2}{\omega}(\sin\pi + \sin\pi)\right]$$

$$t = \frac{2\pi}{4\omega}$$

$$\boxed{E_x(t \;\; \frac{\pi}{2}\omega}$$

⇒ $\boxed{\text{Energy Signal}}$

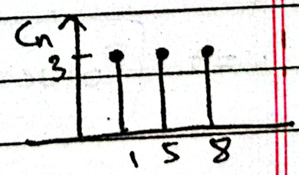
(e)

$$g(t) = 3\cos(t) + \cos\left(5t - \frac{2\pi}{3}\right) + \cos(8t)$$

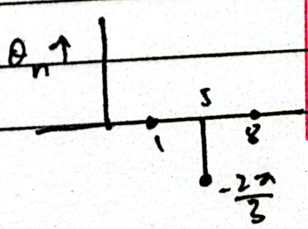
⇒ g(t) is periodic as it sinusoids.

⇒ Power signal

$$P_g = \frac{1}{2}\sum_{n=1}^{\infty} C_n^2$$



$$= \frac{3^2}{2} + \frac{1^2}{2} + \frac{1}{2}$$



$$\boxed{P_g(t) = \frac{11}{2}}$$

$\boxed{\Rightarrow \text{Power signal}}$

Date: __/__/20

(b)

$$x(t) = \cos^2 \omega_0 t$$

$$\text{As } P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} (\cos^2\omega_0 t)^2\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos^4\omega_0 t\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1+\cos 4\omega_0 t}{2}\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \frac{1}{2}\,dt - \int_{-\frac{T}{2}}^{T/2} \cos 4\omega_0 t\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\left[\frac{1}{2}t\Big|_{-\frac{T}{2}}^{\frac{T}{2}} - \frac{\sin 4\omega_0 t}{4\omega_0}\Big|_{-\frac{T}{2}}^{T/2} \right]$$

$$= \lim_{T\to\infty} \frac{1}{T}\left[\frac{1}{2}\left[\frac{2T}{2}\right] - \frac{\sin 4\omega_0 T/2 + \sin 4\omega_0 T/2}{4\omega_0} \right]$$

$$= \lim_{T\to\infty} \frac{1}{T}\left[\frac{T}{2} - \frac{1}{4\omega_0}\left[2\sin 4\omega_0 \frac{T}{2}\right]\right]$$

$$= \lim_{T\to\infty} \frac{1}{T}\left[\frac{T}{2} - \left(\frac{1}{2\omega_0}\sin 4\omega_0 \frac{T}{2}\right)\right]$$

$$\boxed{P_x = \lim_{T\to\infty} \frac{1}{2} - \frac{1}{2T\omega_0}\sin 4\omega_0 \frac{T}{2}}$$

$\Rightarrow$ $\boxed{\text{Power signal}}$ as infinite energy

Date: __/__/20

Q.No.2

$$g(t) = \frac{4a}{t^2+a^2}$$

Using symmetric property,

$$g(t) \Leftrightarrow G(\omega)$$
$$G(t) \Leftrightarrow 2\pi g(-\omega)$$

As $\quad e^{-a|t|} \Leftrightarrow \frac{2a}{\omega^2+a^2}$

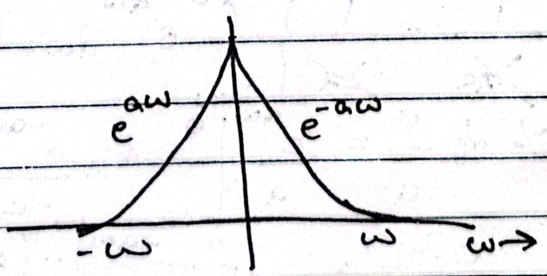
$$\Rightarrow \frac{4a}{t^2+a^2} = \frac{2(2a)}{t^2+a^2}$$

$$\Rightarrow 2\left(\frac{2a}{t^2+a^2}\right) \Leftrightarrow 2\pi e^{-a|-\omega|}$$

$$\frac{2a}{t^2+a^2} \Leftrightarrow \frac{2\pi}{2} e^{-a|\omega|}$$

$$\frac{2a}{t^2+a^2} \Leftrightarrow \pi e^{-a|\omega|}$$

$$\Rightarrow G(\omega) = \pi e^{-a|\omega|}$$
$$= \pi\left[e^{-a\omega} + e^{a\omega}\right]$$
$$= \pi\left[e^{-a\omega}u(\omega) + e^{a\omega}u(-\omega)\right]$$



$$\Rightarrow E_g = \int_{-\infty}^{\infty} g^2(t)\,dt$$

Using Parseval's theorem,

$$E_g = \frac{1}{2\pi}\left\{\int_{-\infty}^{0}(e^{a\omega})^2 d\omega + \int_{0}^{\infty}(e^{-a\omega})^2 d\omega\right\}$$

$$= \frac{1}{2\pi}\left\{\int_{-\infty}^{0} e^{2a\omega} d\omega + \int_{0}^{\infty} e^{-2a\omega} d\omega\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{e^{2a\omega}}{2a}\Big|_{-\infty}^{0} + \frac{e^{-2a\omega}}{-2a}\Big|_{0}^{\infty}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{e^{0}-e^{-\infty}}{2a} - \frac{e^{-\infty}-e^{0}}{2a}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{1}{2a} - \left(-\frac{1}{2a}\right)\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{\cancel{2}}{\cancel{2}a}\right\}$$

$$\Rightarrow E_g = \frac{1}{2\pi a}$$

Here $E_g = \frac{1}{2\pi} \times (\text{Area under ESD})$

$$E_g = \frac{1}{2\pi}\left\{\int_{-\omega}^{\omega} e^{-a|\omega|} d\omega\right\}$$

$$\Rightarrow \frac{1}{\cancel{2\pi}a} = \frac{1}{\cancel{2\pi}}\left\{\int_{-\omega}^{0} e^{a\omega} d\omega + \int_{0}^{\omega} e^{-a\omega} d\omega\right\}$$

$$\Rightarrow 92\% \text{ of } \frac{1}{a} = \frac{e^{a\omega}}{a}\Big|_{-\omega}^{0} + \frac{e^{-a\omega}}{-a}\Big|_{0}^{\omega}$$

$$\frac{0.92}{\cancel{a}} = \frac{(e^{0}-e^{-a\omega}) - (e^{-a\omega}-e^{0})}{\cancel{a}}$$

Date: __/__/20

$$0.92 = 1 - e^{-a\omega} - e^{-a\omega} + 1$$

$$0.92 = 2 - 2e^{-a\omega}$$

$$\frac{0.92}{2} = 1 - e^{-a\omega}$$

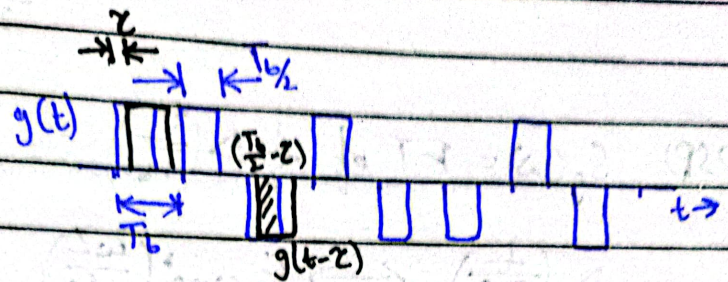
$$e^{-a\omega} = 1 - 0.46$$

$$\Rightarrow \ln(e^{-a\omega}) = \ln(0.54)$$

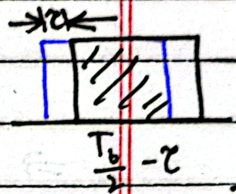
$$-a\omega = -0.616$$

$$\boxed{\Rightarrow \omega = \frac{0.616}{a} \text{ rad/s}}$$

Q. No. 3



(i) $R_g(\tau)$ (ii) $S_g(\omega)$ (iii) Essential Bandwidth



; $\tau < \frac{T_b}{2}$

Let, $T = NT_b$

$\Rightarrow T \to \infty \Rightarrow N \to \infty$

As $$R_g(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} g(t)\,g(t-\tau)\,dt$$

$$= \lim_{N\to\infty} \frac{1}{NT_b}\int_{-NT_b/2}^{NT_b/2} g(t)\,g(t-\tau)\,dt$$

$$= \lim_{N\to\infty} \frac{1}{\cancel{N}T_b}\left(\cancel{N}\left(\frac{T_b}{2}-\tau\right)\right)$$

$$= \frac{1}{2}\left(1-\frac{2\tau}{T_b}\right) \qquad ; |\tau| < \frac{T_b}{2}$$

$\Rightarrow R_g(\tau)$: even fnc of τ

As $\tau > \frac{T_b}{2}$; consequent pulses overlap having same or different polarity

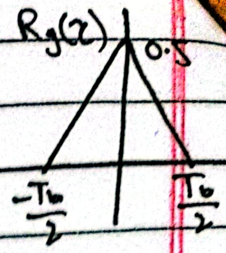
$\Rightarrow$ Pulse products on avg will be 1 and -1.

$\Rightarrow$ As $T \to \infty$,

$$R_g(\tau) = 0 \qquad ; |\tau| > \frac{T_b}{2}$$

Date: __/__/20

For this case, $R_g(\tau) = \frac{1}{2}\Delta\left(\frac{t}{T_b}\right)$

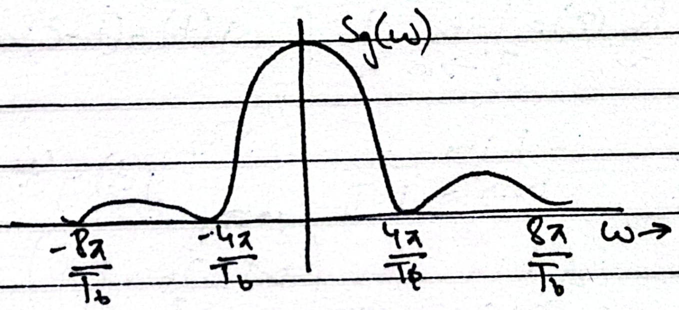


$\Rightarrow$ PSD $S_g(\omega) = FT$ of $\frac{1}{2}\Delta\left(\frac{t}{T_b}\right)$

As $\Delta\left(\frac{t}{T_b}\right) \Leftrightarrow \frac{\tau}{2}\,\mathrm{sinc}^2\left(\frac{\omega\tau}{4}\right)$

$\frac{1}{2}\Delta\left(\frac{t}{T_b}\right) \Leftrightarrow \frac{T_b}{4}\,\mathrm{sinc}^2\left(\frac{\omega T_b}{4}\right)$

$$\Rightarrow \boxed{S_g(\omega) = \frac{T_b}{4}\,\mathrm{sinc}^2\left(\frac{\omega T_b}{4}\right)}$$



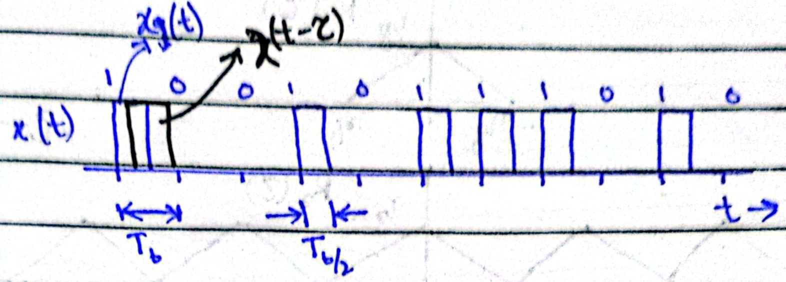
$\Rightarrow$ 90.28% of area of this spectrum is within the band from 0 to $\frac{4\pi}{T_b}$ rad/s or from 0 to $\frac{2}{T_b}$ Hz.

$\Rightarrow$ Essential Bandwidth $\Rightarrow \frac{2}{T_b}$ Hz

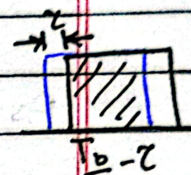
$$\boxed{\omega = \frac{2}{T_b}\ \text{Hz}}$$

Date: __/__/20

Q. No. 4



(i) $R_x(\tau)$ (ii) $S_x(\omega)$



Let, $T = NT_b$

$\Rightarrow T \rightarrow \infty \Rightarrow N \rightarrow \infty$

; $\tau < T_b/2$

As $R_x(\tau) = \lim\limits_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,x(t-\tau)\,dt$

$$= \lim_{N\to\infty} \frac{1}{NT_b}\int_{-NT_b/2}^{NT_b/2} x(t)\,x(t-\tau)\,dt$$

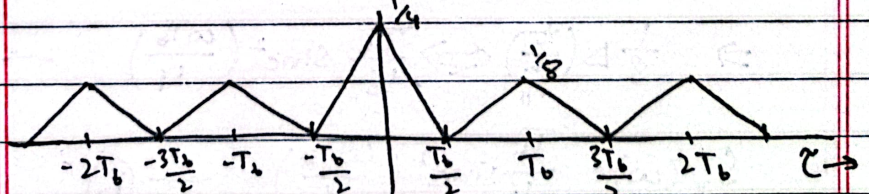
$$= \lim_{N\to\infty} \frac{1}{\cancel{N}T_b}\left(\frac{\cancel{N}}{2}\left(\frac{T_b}{2}-\tau\right)\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}-\frac{|\tau|}{T_b}\right) \quad ; |\tau| < \frac{T_b}{2}$$

For $\frac{T_b}{2} \le |\tau| \le T_b$; $R_x(\tau) = 0$: no overlap b/w pulses

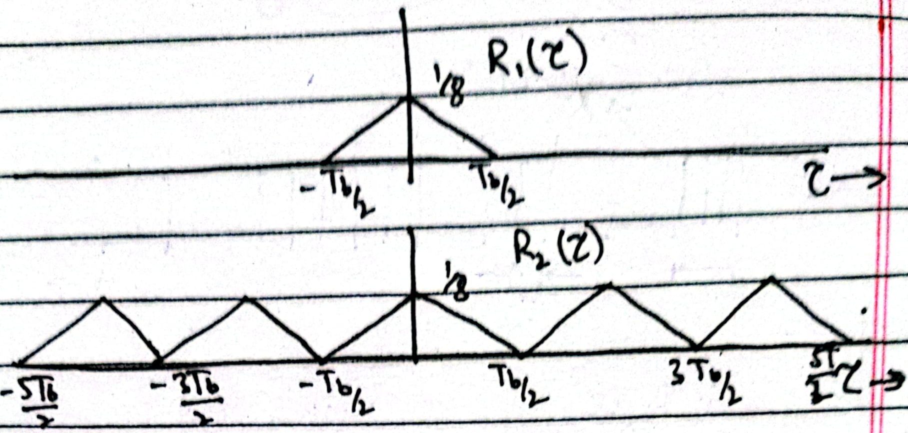
For $T_b \le |\tau| \le \frac{3T_b}{2}$; Pulses overlap but only half.

$\Rightarrow R_x(\tau)$ repeats every T_b seconds with half mgtde.



Date: __/__/20

Hence, $R_x(\tau) = R_1(\tau) + R_2(\tau)$ — ①



As PSD $S_x(\omega)$ = Fourier Transforms of $R_1(\tau) + R_2(\tau)$ — Ⓐ

$$\Rightarrow R_{x_1}(\tau) = \frac{1}{8}\Delta\left(\frac{t}{T_b}\right) \quad — ②$$

$$\Rightarrow R_{x_2}(\tau) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0\tau} \quad ; \omega_0 = \frac{2\pi}{T_0}$$

$$\text{where } D_n = \frac{1}{16}\operatorname{sinc}^2\left(\frac{n\pi}{2}\right)$$

$$R_{x_2}(\tau) = \sum_{n=-\infty}^{\infty} \frac{1}{16}\operatorname{sinc}^2\left(\frac{n\pi}{2}\right) e^{jn\omega_0\tau} \quad — ③$$

By ①, ② & ③

$$\boxed{R_x(\tau) = \frac{1}{8}\Delta\left(\frac{t}{T_b}\right) + \sum_{n=-\infty}^{\infty} \frac{1}{16}\operatorname{sinc}^2\left(\frac{n\pi}{2}\right) e^{jn\omega_0\tau}}$$

FT of $R_{x_1}(\tau)$:-

$$\text{As } \Delta\left(\frac{t}{\tau}\right) \Longleftrightarrow \frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right)$$

$$\Rightarrow \frac{1}{8}\Delta\left(\frac{t}{T_b}\right) \Longleftrightarrow \frac{T_b}{16}\operatorname{sinc}^2\left(\frac{\omega T_b}{4}\right) \quad — \cancel{Ⓑ}$$

$$S_{x_1}(\omega) = \frac{T_b}{16}\operatorname{sinc}^2\left(\frac{\omega T_b}{4}\right) \quad — Ⓑ$$

__/__/20

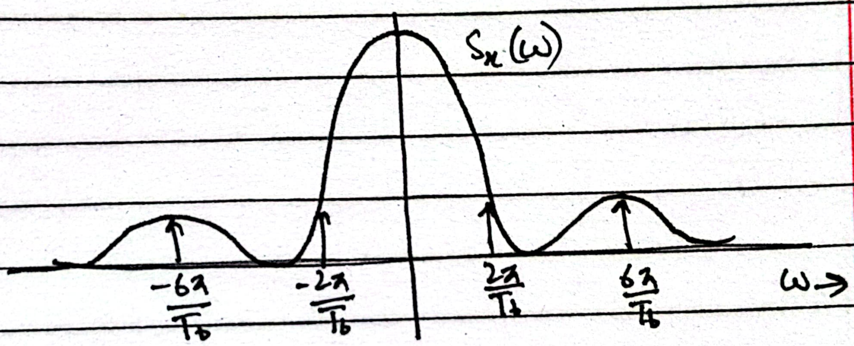
FT of $R_{x_2}(\tau)$:-

As $e^{j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega-\omega_0)$

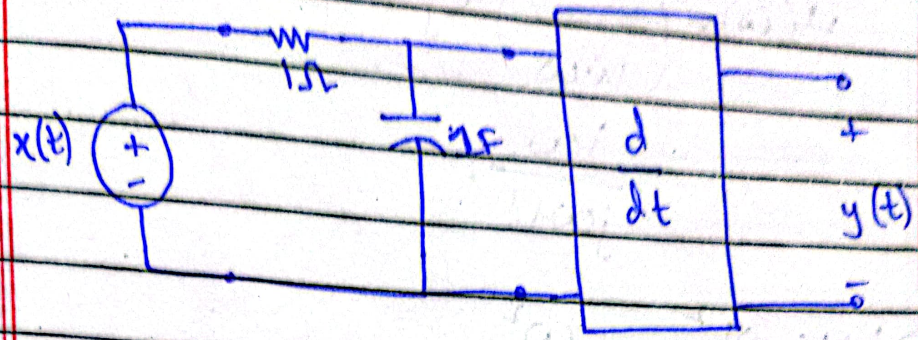
$$S_{x_2}(\omega) = \frac{\pi}{8}\sum_{n=-\infty}^{\infty} \text{sinc}^2\left(\frac{n\pi}{2}\right)\delta(\omega - n\omega_0) \; ; \omega_0 = \frac{2\pi}{T_b} \quad \text{— (C)}$$

From (A), (B) & (C)

$$\boxed{S_x(\omega) = \frac{T_b}{16}\,\text{sinc}^2\left(\frac{\omega T_b}{4}\right) + \frac{\pi}{8}\sum_{n=-\infty}^{\infty}\text{sinc}^2\left(\frac{n\pi}{2}\right)\delta(\omega - n\omega_0) \; ; \omega_0 = \frac{2\pi}{T_b}}$$

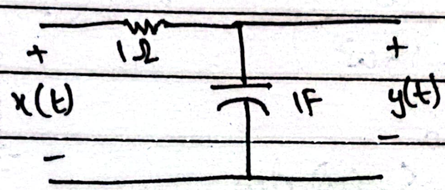


Q. No. 5



$S_x(\omega) = 2\,\text{rect}\left(\frac{\omega}{2}\right)$

⇒ Ideal diferentiator transfer fnc ⇒ $j\omega$



$$H(s) = \frac{y(s)}{x(s)}$$

$$= \frac{\frac{1}{sC}}{R + \frac{1}{sC}}$$

where $s = j\omega$

$$H(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}$$

$$= \frac{1}{1 + j\omega RC}$$

$$\Rightarrow H(\omega) = \frac{1}{1 + \frac{j\omega}{a}}$$

$$H(\omega) = \frac{1}{1 + j\omega}$$

$\Rightarrow a = \frac{1}{RC} = \frac{1}{(1)(1)} = 1$

Date: __/__/20

The tranfer function of entire system is

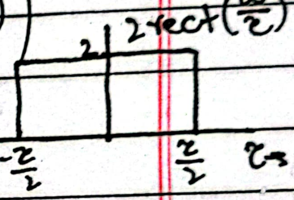
$$H(\omega) = \left(\frac{1}{1+j\omega}\right)(j\omega)$$

$$= \frac{j\omega}{j\omega+1}$$

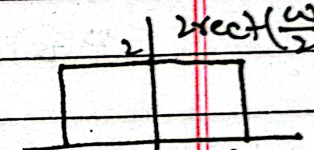
$$\Rightarrow |H(\omega)|^2 = \frac{\omega^2}{1+\omega^2}$$

As $S_y(\omega) = |H(\omega)|^2 S_x(\omega)$

$$= \left(\frac{\omega^2}{1+\omega^2}\right)\left(2\,\text{rect}\left(\frac{\omega}{2}\right)\right)$$



As $P_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(\omega)\, d\omega$



$$\Rightarrow y(t) = \frac{1}{2\pi}\int_{-1}^{1}\left(\frac{\omega^2}{1+\omega^2}\right)\left(2\,\text{rect}\left(\frac{\omega}{2}\right)\right)d\omega$$

$$= \frac{\cancel{2}}{\cancel{2}\pi}\int_{-1}^{1}\left(\frac{\omega^2}{1+\omega^2}\right)(1)\, d\omega$$

$$= \frac{1}{\pi}\int_{-1}^{1}\frac{1+\omega^2-1}{1+\omega^2}\, d\omega$$

$$= \frac{1}{\pi}\left[\int_{-1}^{1}\frac{\cancel{1+\omega^2}}{\cancel{1+\omega^2}}\, d\omega - \int_{-1}^{1}\frac{1}{1+\omega^2}\, d\omega\right]$$

$$= \frac{1}{\pi}\left[\omega\Big|_{-1}^{1} - \tan^{-1}(\omega)\Big|_{-1}^{1}\right]$$

$$= \frac{1}{\pi}\left[(1-(-1)) - (\tan^{-1}(1) - \tan^{-1}(-1))\right]$$

$$= \frac{1}{\pi}\left[(1+1) - \left(\frac{\pi}{4} + \frac{\pi}{4}\right)\right]$$

$$= \frac{1}{\pi}\left(2 - \frac{2\pi}{4}\right)$$

$$\cancel{= \frac{1}{\pi}\left(\frac{8-\pi}{4}\right)}$$

$$\Rightarrow \boxed{y(t) = 0.137}$$; Power of output signal

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1} 2\,\text{rect}\left(\frac{\omega}{2}\right) d\omega$$

$$= \frac{\cancel{2}}{\cancel{2}\pi}\int_{-1}^{1} (1)\, d\omega$$

$$= \frac{1}{\pi}\left[\omega\right]_{-1}^{1}$$

$$= \frac{1}{\pi}(1+1)$$

$$\Rightarrow \boxed{x(t) = \frac{2}{\pi}}$$; Power of input signal