# Physics-II
# 10B11PH211

- Electromagnetic Theory
- Thermodynamics
- Solid State Physics
- Quantum Mechanics

*To understand the Electromagnetic, we must know basic vector algebra and coordinate systems. So let us start the coordinate systems.*

# COORDINATE SYSTEMS

- RECTANGULAR or Cartesian
- CYLINDRICAL
- SPHERICAL

Choice is based on symmetry of problem

Examples:

Sheets - RECTANGULAR

Wires/Cables - CYLINDRICAL

Spheres - SPHERICAL

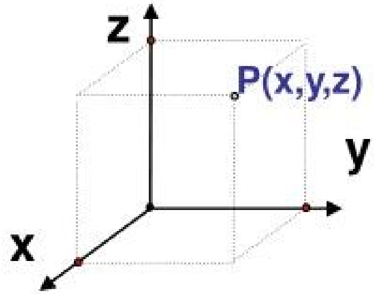
# *Orthogonal Coordinate Systems:*

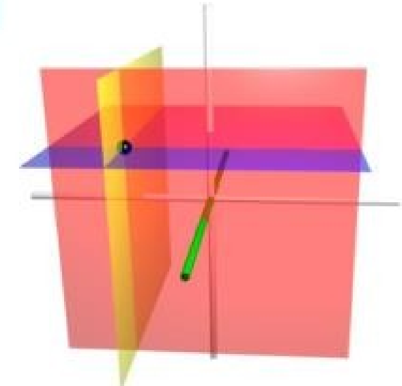
## 1. Cartesian Coordinates

Or

## Rectangular Coordinates

P (x, y, z)



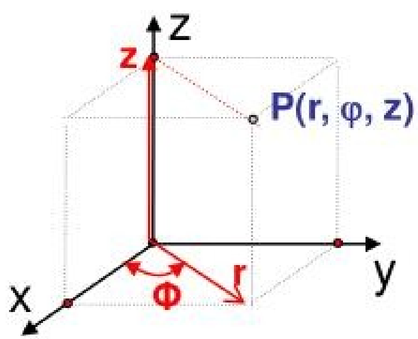
## 2. Cylindrical Coordinates

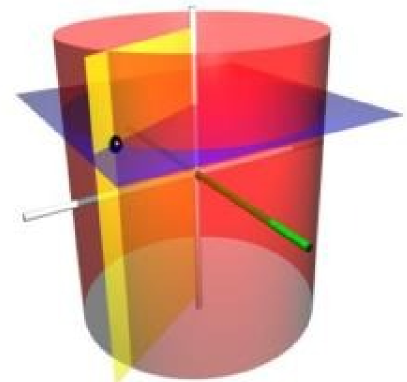
P (r, φ, z)

$$X = r\cos\varphi$$

$$Y = r\sin\varphi$$

$$Z = z$$



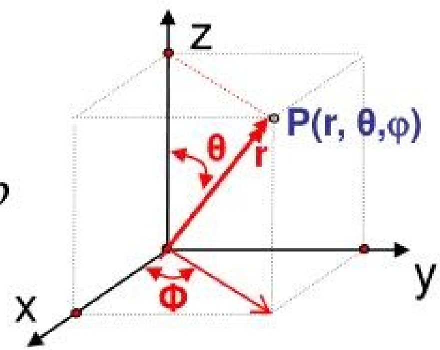
## 3. Spherical Coordinates

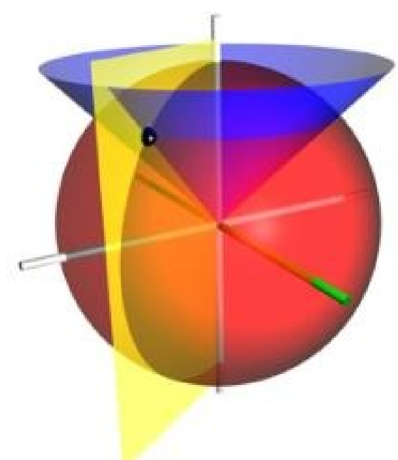
P (r, θ, φ)

$$X = r\sin\theta\cos\varphi$$

$$Y = r\sin\theta\sin\varphi$$

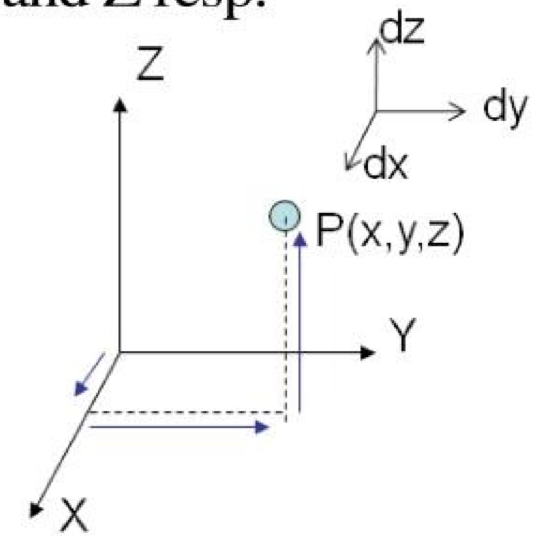
$$Z = r\cos\theta$$



# Cartesian Coordinates

$dx, dy, dz$ are infinitesimal displacements along X, Y and Z resp.




## *Differential quantities:*

### Length Element

$$d\vec{l} = \hat{x}dl_x + \hat{y}dl_y + \hat{z}dl_z$$

$$or,\ d\vec{l} = \hat{x}dx + \hat{y}dy + \hat{z}dz$$

### Area Element

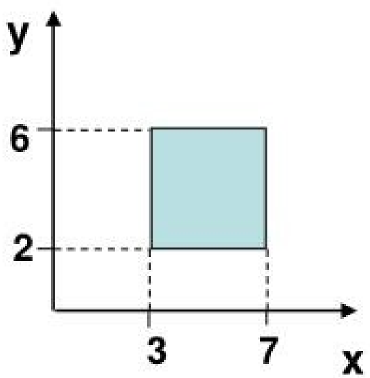
$$ds_x = dydz \qquad ds_y = dxdz$$

$$ds_z = dxdy$$

$$d\vec{s} = \hat{x}ds_x + \hat{y}ds_y + \hat{z}s_z$$

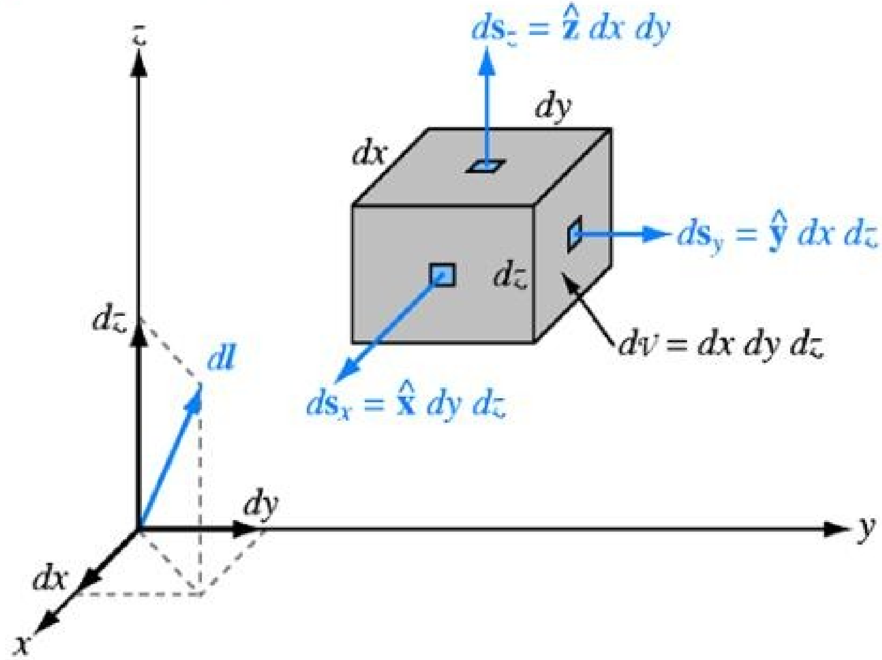
$$= \hat{x}dydz + \hat{y}dxdz + \hat{z}dxdy$$

Deduce the area of the lamina




$$\text{Area} = \int_3^7\int_2^6 dy\,dx = 16$$

Note: z is constant



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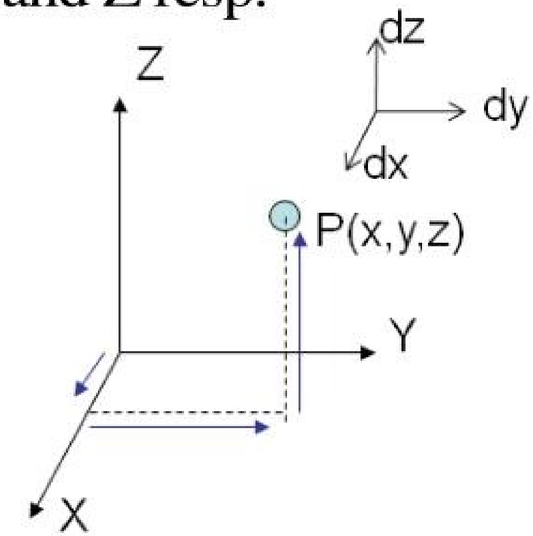


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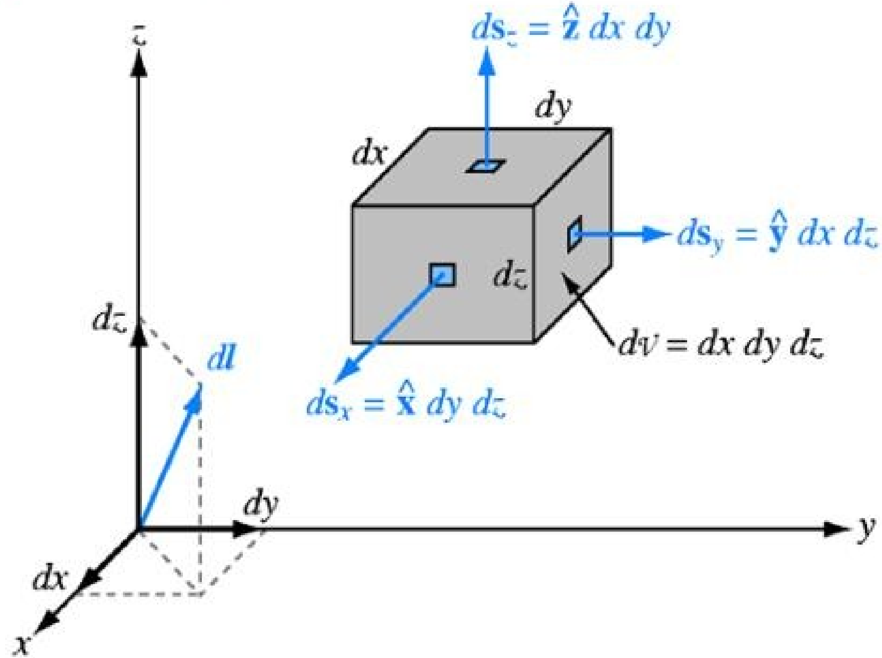
$$= \hat{x}dydz + \hat{y}dxdz + \hat{z}dxdy$$

### *Volume :*

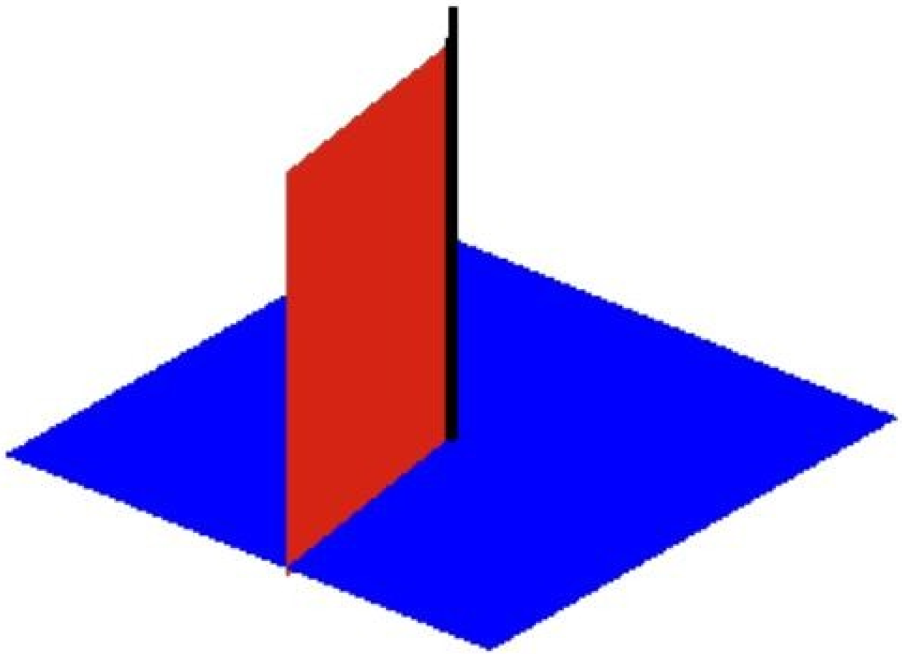
$$dv = dxdydz$$

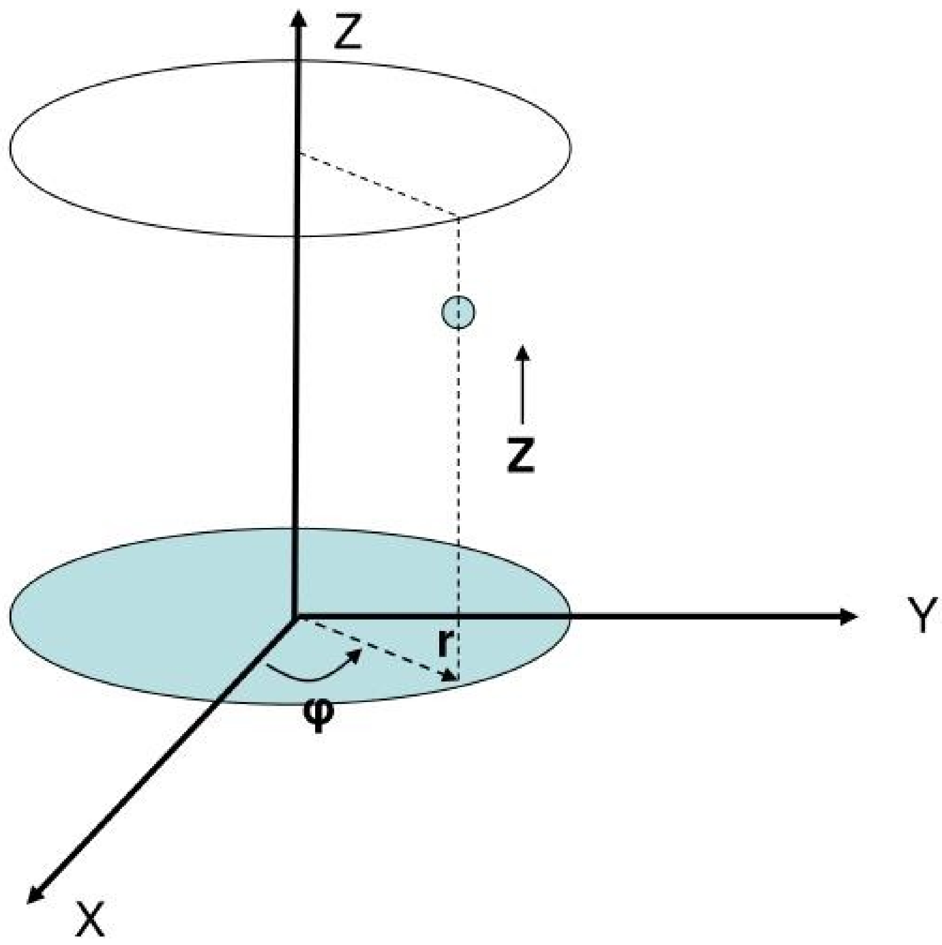
Ex: Show that volume of a cube of edge a is $a^3$.

$$V = \int_v dv = \int_0^a dx \int_0^a dy \int_0^a dz = a^3$$



# *Cylindrical coordinate system (r,φ,z)*






Limits of integration of $r, \varphi, z$ are

$0<r<\infty, 0<z<\infty, 0<\varphi<2\pi$

# Cylindrical Coordinates: Visualization of Volume element

## Differential quantities:

*Length Element:*

$$dl_r = dr, dl_\varphi = rd\varphi, dl_z = dz$$

$$d\vec{l} = \hat{r}dl_r + \hat{\varphi}dl_\varphi + \hat{z}dl_z$$

$$= \hat{r}dr + \hat{\varphi}rd\varphi + \hat{z}dz$$

*Area Element:*

$$ds_r = dl_\varphi dl_z = rd\varphi dz$$

$$ds_\varphi = dl_r dl_z = drdz$$

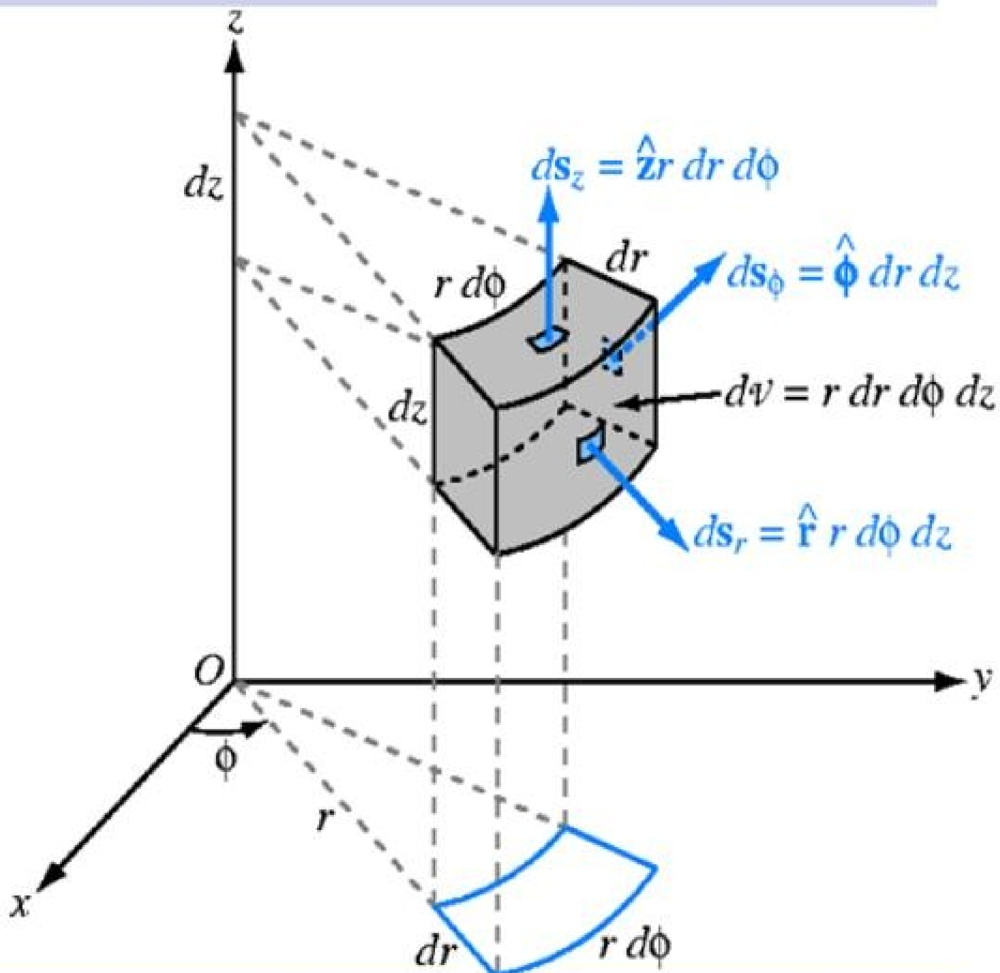
$$ds_z = dl_r dl_\varphi = rdrd\varphi$$

$$d\vec{s} = \hat{r}ds_r + \hat{\varphi}ds_\varphi + \hat{z}ds_z$$

$$d\vec{s} = \hat{r}\ rd\varphi dz + \hat{\varphi}\ drdz + \hat{z}\ rdrd\varphi$$

*Volume Element:*

$$dv = dl_r dl_\varphi dl_z = r\,dr\,d\varphi\,dz$$



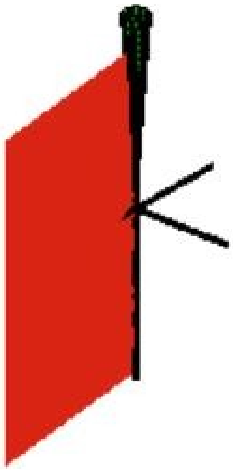

Find : 1) Curved surface area of cylinder

2) Base Area

3) Volume (radius 'R' ;height 'H')

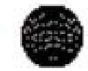
Limits of integration of r, φ,z are $0<r<\infty$ , $o<\varphi<2\pi$ ,$0<z<\infty$

# Spherical coordinate system (r, θ,φ)



**Radius=r**

$0<r<\infty$



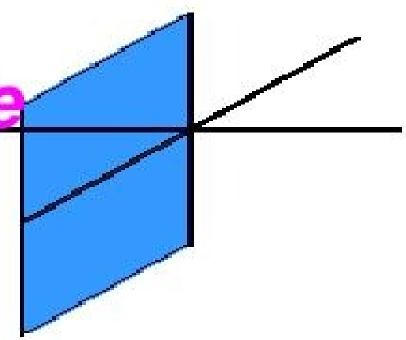
**θ-Zenith angle**

$0<\theta<\pi$

*( starts from +Z reaches up to –Z) ,*

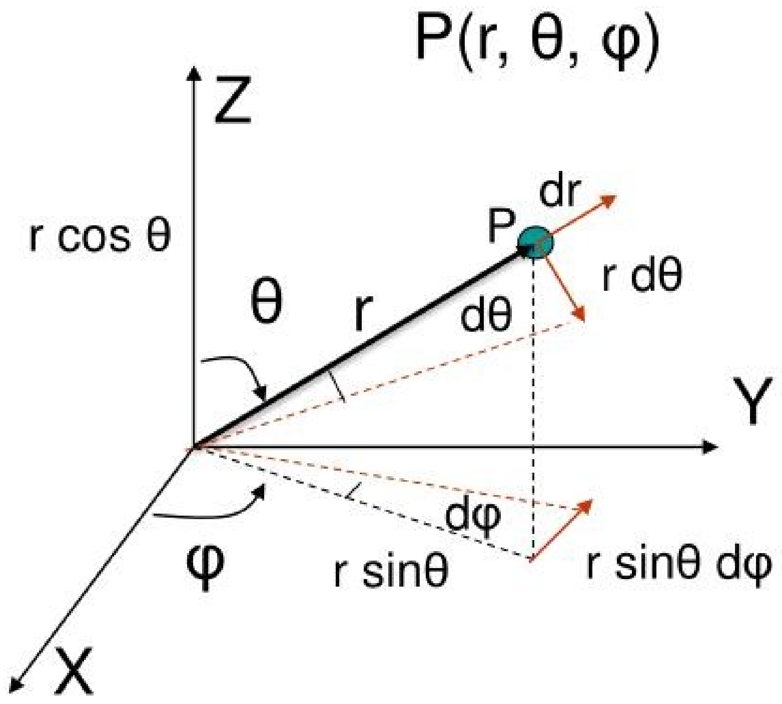
***φ -Azimuthal Angle***

$0<\varphi<2\pi$



*(starts from +X direction and lies in x-y plane only)*

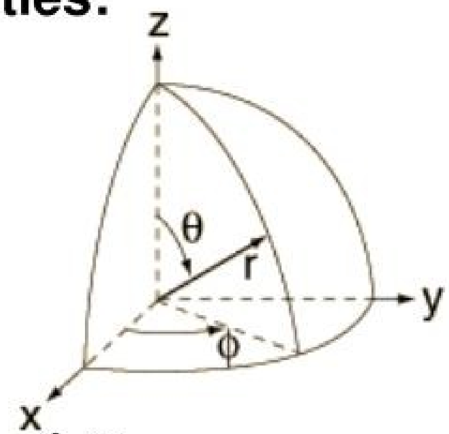
# Spherical Coordinates



# Spherical Coordinates

**Differential quantities:**

*Length Element*




$$dl_r = dr$$

$$dl_\theta = r d\theta$$

$$dl_\varphi = r \sin\theta d\varphi$$

$$d\vec{l} = \hat{r} dl_r + \hat{\theta} dl_\theta + \hat{\varphi} dl_\varphi$$

*Area Element:*

$$ds_r = dl_\theta dl_\varphi = r^2 \sin\theta d\theta d\varphi$$

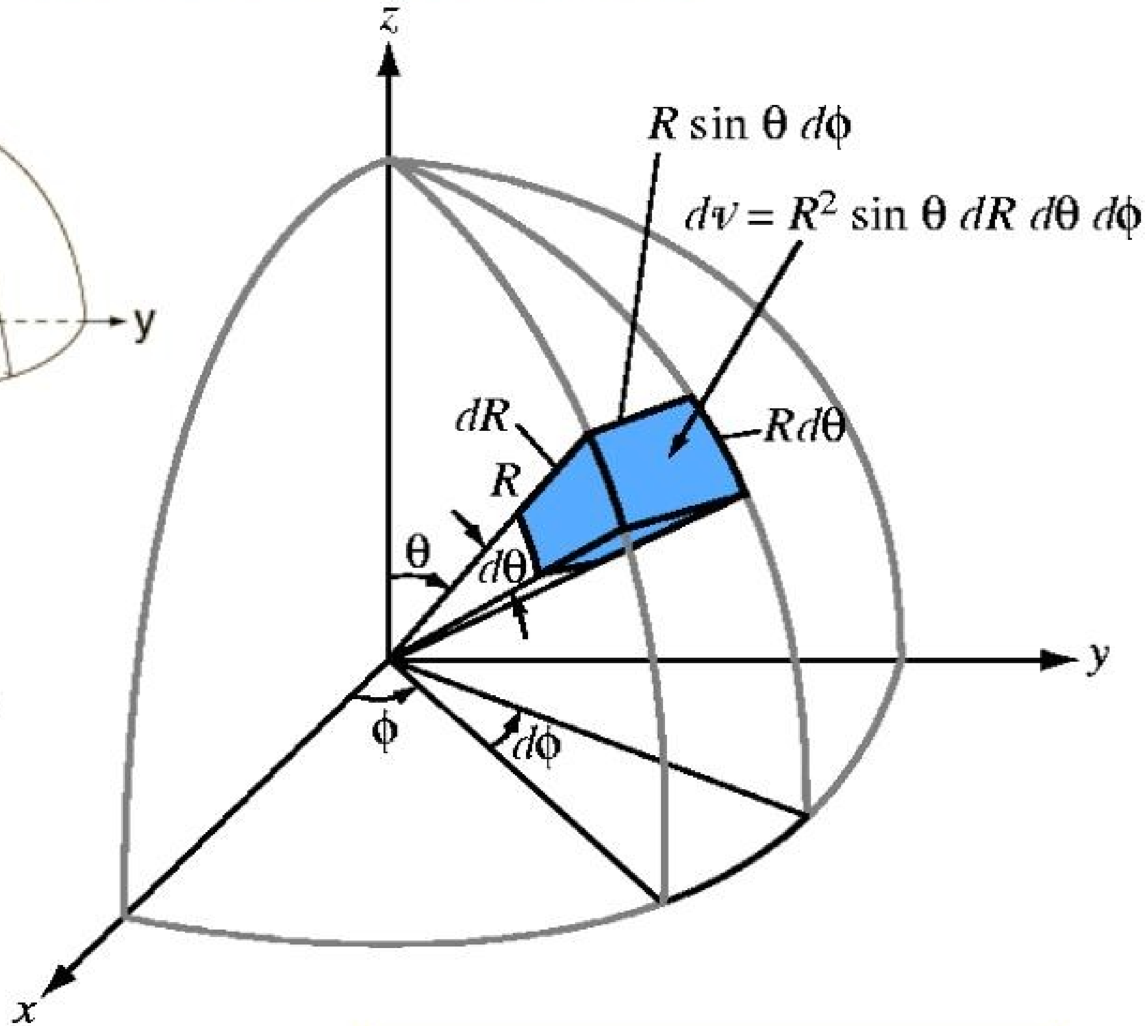
$$ds_\theta = dl_r dl_\varphi = r \sin\theta dr d\varphi$$

$$ds_\varphi = dl_r dl_\theta = r dr d\theta$$

$$d\vec{s} = \hat{r} ds_r + \hat{\theta} ds_\theta + \hat{\varphi} ds_\varphi$$

*Volume Element*

$$dv = r^2 \sin\theta dr d\theta d\varphi$$




*Find the volume and surface area of a sphere of radius R.*

# Points to remember

| System | Coordinates | $dl_1$ | $dl_2$ | $dl_3$ |
|---|---|---|---|---|
| Cartesian | $x, y, z$ | $dx$ | $dy$ | $dz$ |
| Cylinderical | $r, \varphi, z$ | $dr$ | $rd\varphi$ | $dz$ |
| Spherical | $r, \theta, \varphi$ | $dr$ | $rd\theta$ | $r\sin\theta d\varphi$ |

Hence, Volume Element: $dv = dl_1 dl_2 dl_3$

If volume charge density '$\rho$' dependes only on 'r':

$$Q = \int_v \rho dv = \int_l \rho 4\pi r^2 dr$$

Ex: For Circular disc of radius R has a surface charge density $\sigma(\mathrm{r},\varphi)=\sigma_o r^2 \sin^2\varphi\left(C/m^2\right)$ in 2D polar coordinates. Find out the total charge on the disc. (spherical and cylinderical both)

NOTE: $da = rdrd\varphi$ in both the coordinate systems as $\theta=90^0$

***Determine***

***a) Areas S1, S2 and S3.***

***b) Volume covered by these surfaces.***

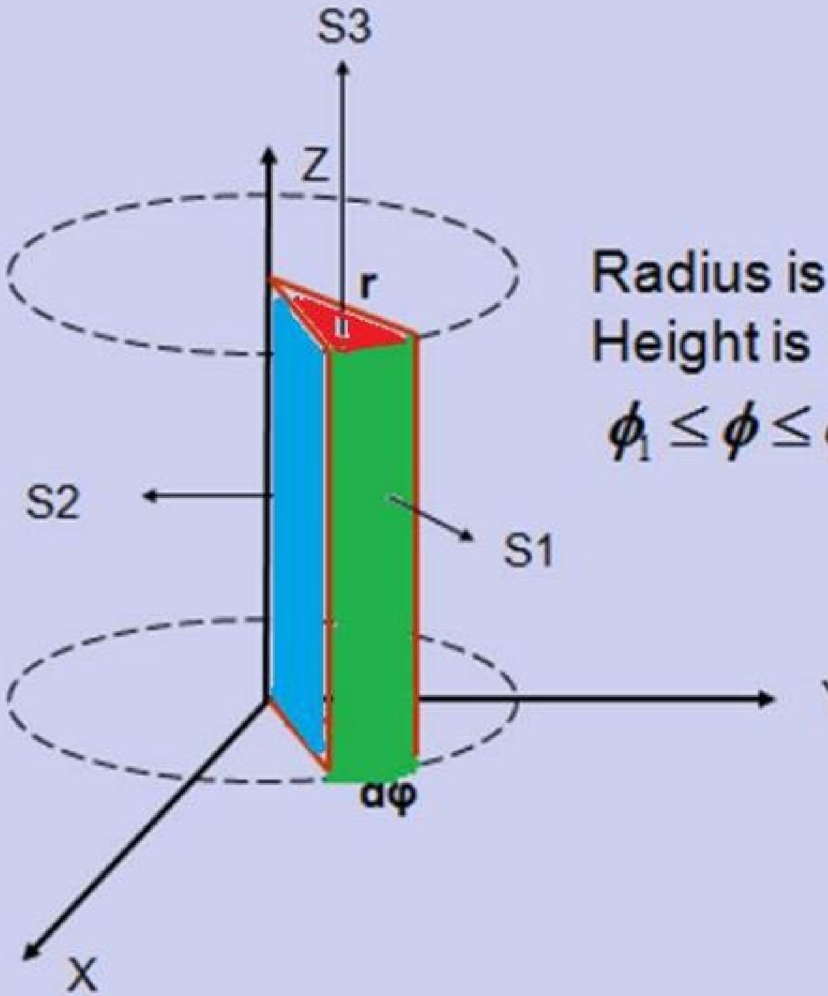
*Solution:*

$$a)\, i)\, S1 = \int_{\varphi_1}^{\varphi_2} r\,d\varphi \int_0^h dz = rh(\varphi_2 - \varphi_1)$$

$$ii)\, S2 = \int_0^r dr \int_0^h dz = rh$$

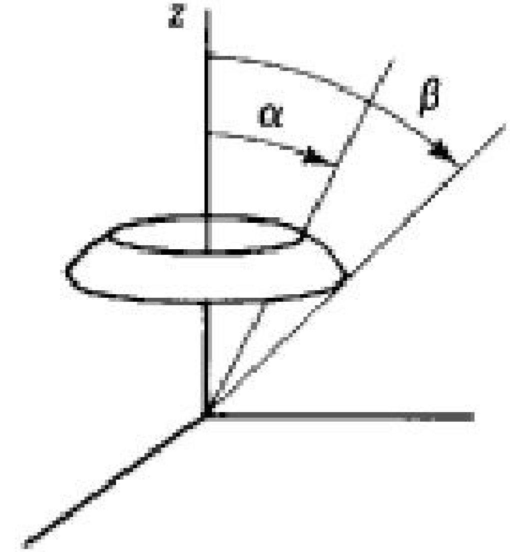
$$iii)\, S3 = \int_{\varphi_1}^{\varphi_2} \int_0^r dr\,r\,d\varphi = \frac{r^2}{2}(\varphi_2 - \varphi_1)$$

$$b)\, V = \int_0^h \int_{\varphi_1}^{\varphi_2} \int_0^r dr.r\,d\varphi.dz = \frac{r^2}{2}(\varphi_2 - \varphi_1)h$$



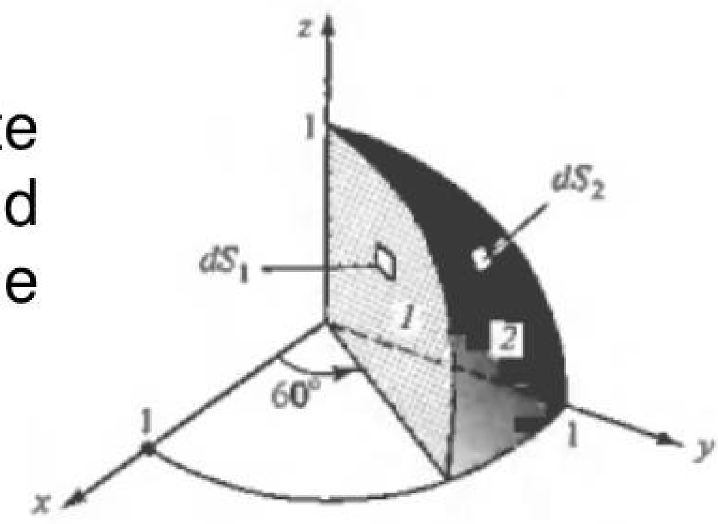

Radius is r,
Height is h,
$\phi_1 \le \phi \le \phi_2$

Ex: Use the spherical coordinate system to find the area of the strip $\alpha \leq \theta \leq \beta$ on the spherical shell of radius a. What results when $\alpha = 0$ and $\beta = \pi$?

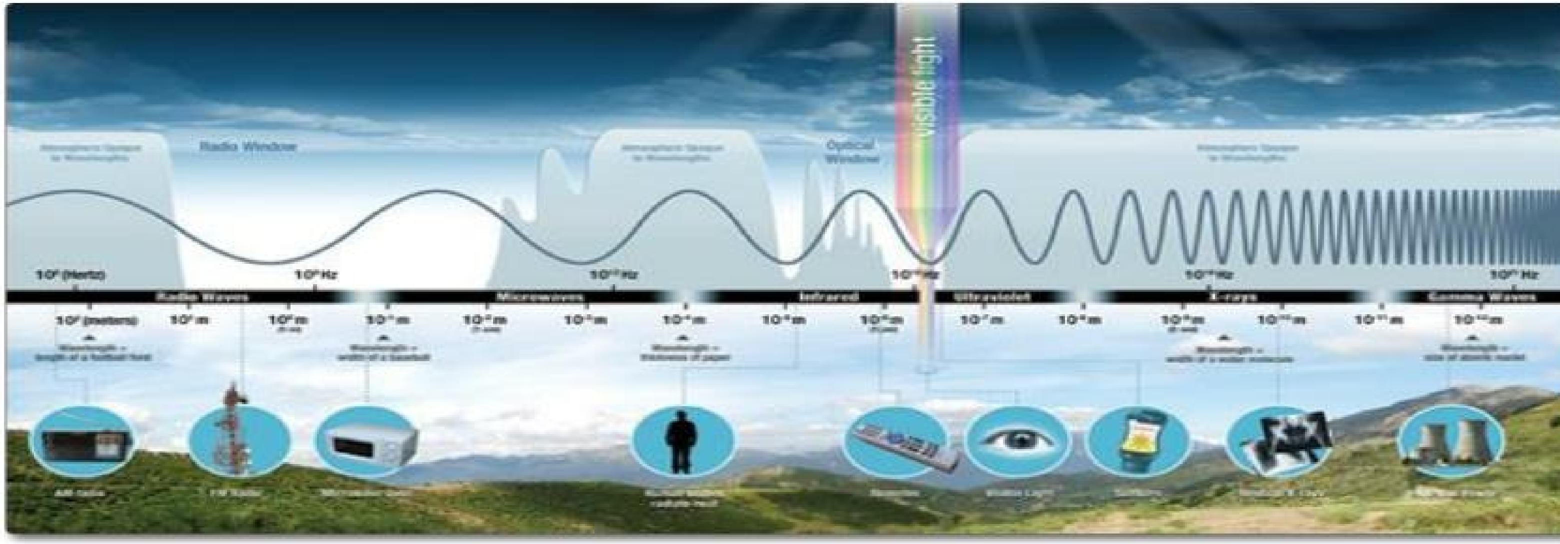



$$A = \int_0^{2\pi} \int_\alpha^\beta a^2 \sin\theta \, d\theta \, d\phi$$
$$= 2\pi a^2(\cos\alpha - \cos\beta)$$

Ex: Use spherical coordinates to write the differential surface areas $dS_1$ and $dS_2$ and then integrate to obtain the areas of the surfaces marked 1 and 2.




Ans : $\pi/4, \pi/6$




## Books:

***Introduction to Electrodynamics***

**by D.J. Griffith**

***Electromagnetics***

**by Edminister (Schuam series)**

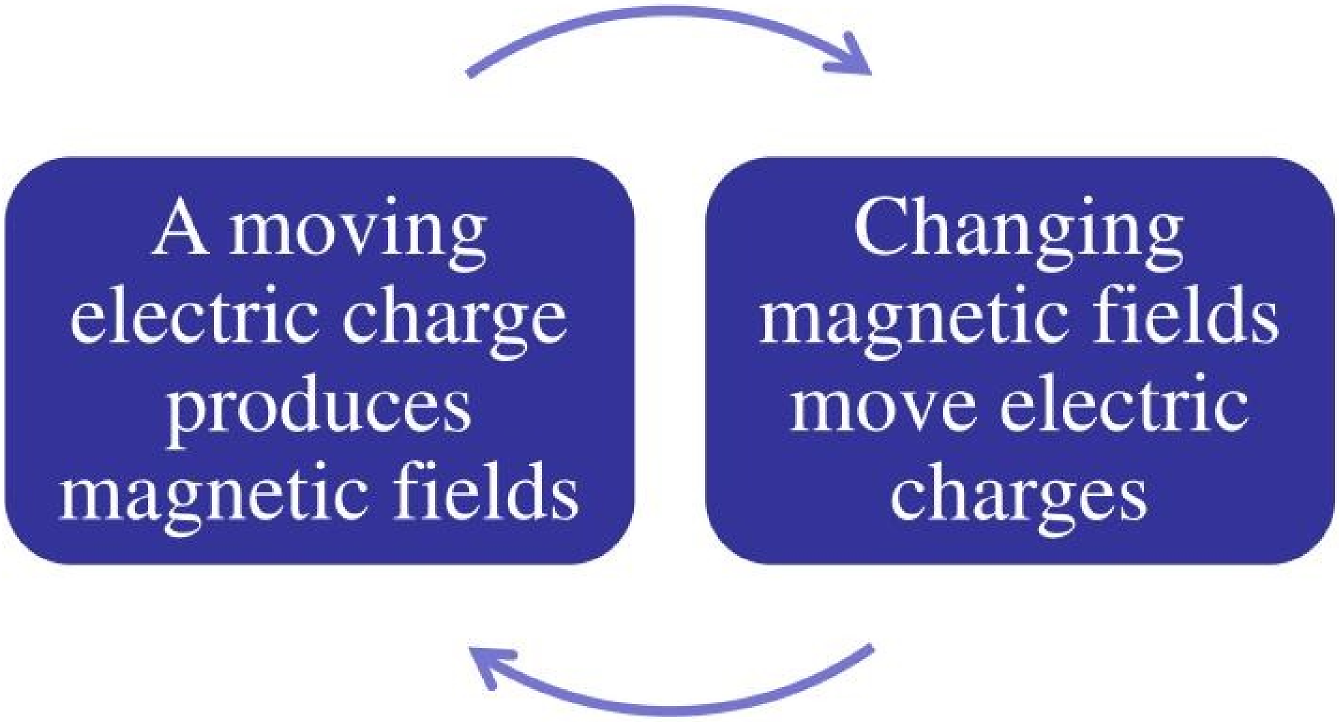
***Principles of Electromagnetics***

***by Matthew N. O. Sadiku***

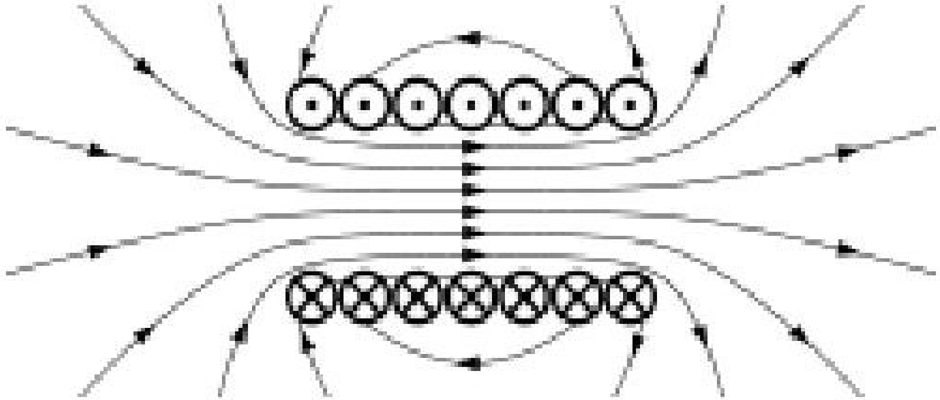
***Engineering Electromagnetic***

***by W H Hayt & J A Buck***

# ELECTRICITY AND MAGNETISM ARE DIFFERENT ASPECTS OF ELECTROMAGNETISM



# Electromagnetism



A <u>fundamental interaction</u> between the magnetic field and the presence and motion of an electric charge

*A "Field" is any physical quantity which takes on different values at different points in space.*

# *Assignment 1:*

## *Basics of fields*

## *Gradient*

## *Divergence and Curl.*

# *Vector Analysis*

What about $\vec{A}.\vec{B}=?$ and $\vec{A}\times\vec{B}=?$

Given $\mathbf{A} = \mathbf{a}_x + \mathbf{a}_y$ ,  $\mathbf{B}=\mathbf{a}_x+2\mathbf{a}_y$,  and  $\mathbf{C}=2\mathbf{a}_y+\mathbf{a}_z$,

find ($\mathbf{A}$ x $\mathbf{B}$) x $\mathbf{C}$ and compare it with $\mathbf{A}$ x ($\mathbf{B}$ x $\mathbf{C}$).

find $\mathbf{A.B}$ x $\mathbf{C}$ and compare it with $\mathbf{A}$ x $\mathbf{B.C}$

## *Scalar and Vector Fields*

*voltage, current, energy, temperature*

*velocity, momentum, acceleration and force*

# Gradient, Divergence and Curl

***The Del Operator*** $\nabla$ $$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

- Gradient of a scalar function is a vector quantity. $\nabla f$

- Divergence of a vector is a scalar quantity. $\nabla . \vec{A}$

- Curl of a vector is a vector quantity. $\nabla \times \vec{A}$

# Operator in Cartesian Coordinate System

**Gradient :** $\nabla T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$ *as* $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

***gradT:*** *points the direction of maximum increase of the function T.*

**Divergence:** $\nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$ where $\overline{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$

**Curl:** $\nabla \times \vec{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}$

# *Operator in Cylindrical Coordinate System*

*Gradient:*

$$\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \phi}\hat{\phi} + \frac{\partial T}{\partial z}\hat{z}$$

$$\overline{V} = V_r\hat{r} + V_\phi\hat{\phi} + V_z\hat{z}$$

*Divergence:*

$$\nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}\left(rV_r\right) + \frac{1}{r}\frac{\partial V_\varphi}{\partial \varphi} + \frac{\partial V_z}{\partial z}$$

*Curl:*

$$\nabla \times \vec{V} = \left(\frac{1}{r}\frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)\hat{\phi} + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rV_\phi\right) - \frac{\partial V_r}{\partial \phi}\right)\hat{z}$$

# Operator in Spherical Coordinate System

**Gradient :** $$\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\,sin\,\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$$

$$\overline{V} = V_r\hat{r} + V_\theta\hat{\theta} + V_\phi\hat{\phi}$$

**Divergence:** $$\nabla \cdot \vec{V} = \frac{1}{r^2}\frac{\partial\left(r^2 V_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta V_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}$$

**Curl:** $$\nabla \times \vec{V} = \frac{1}{r\,sin\,\theta}\left(\frac{\partial}{\partial\theta}\left(sin\,\theta V_\phi\right) - \frac{\partial V_\theta}{\partial\phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{sin\,\theta}\frac{\partial V_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rV_\phi\right)\right)\hat{\theta}$$
$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rV_\theta\right) - \frac{\partial V_r}{\partial\theta}\right)\hat{\phi}$$

# *Fundamental theorem for divergence and curl*

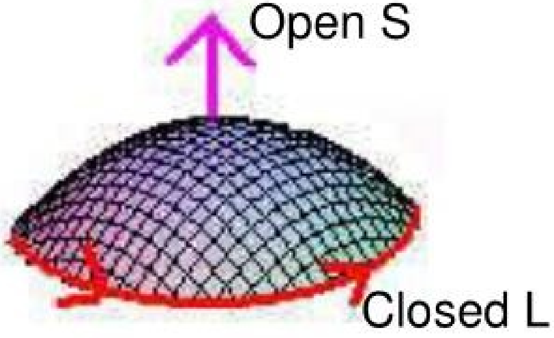
- *Gauss divergence theorem:*

$$\int_v (\nabla . V)\, dv = \oint_s V . ds$$

**Conversion of volume integral to surface integral and vice verse.**

- *Stokes curl theorem*

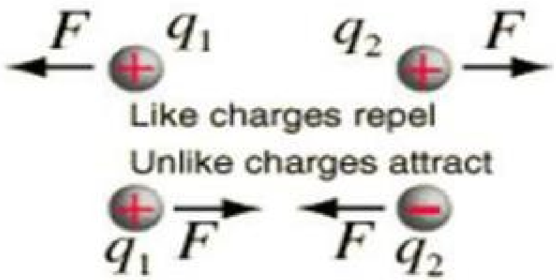
$$\int_s (\nabla \times V) . ds = \oint_l V . dl$$




**Conversion of surface integral to line integral and vice verse.**

# Coulomb's Law

**Like charges repel, unlike charges attract**




$$F = \frac{kq_1q_2}{r^2} = \frac{q_1q_2}{4\pi\varepsilon_o r^2}$$ Coulomb's Law

$$k = \frac{1}{4\pi\varepsilon_o} \approx 9\times 10^9 \ Nm^2/C^2$$ Coulomb's constant

*Exercise:* A charge $Q_1$= 1nC is located at the origin in free space and another charge Q at (2,0,0). If the X-component of the electric field at (3,1,1) is zero, calculate the value of Q. Is the Y component zero at (3,1,1)?

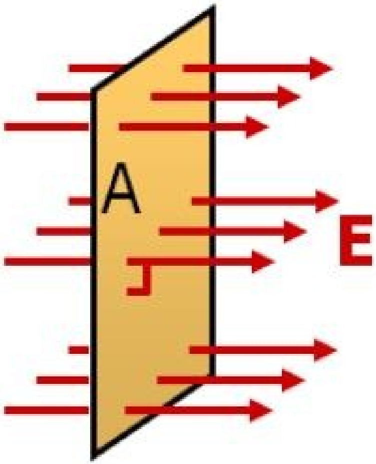
$Ans: -3(3/11)^{1.5}Q_1$

**Calculate E due to**

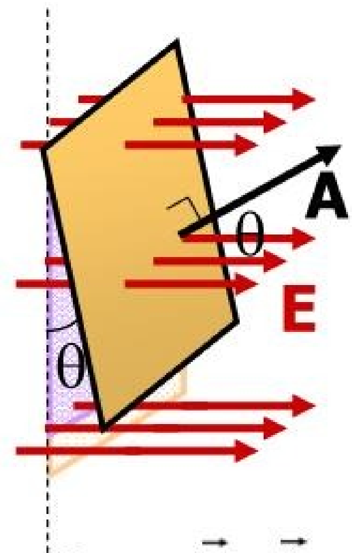
1. Dipole,
2. Rod (line charge), Ring (Line charge),
3. Circular plate (surface charge), Square sheet,
4. Sphere or Cylinder (Volume charge density)

# *Electric Flux*

## *The number of electric field lines through a surface*




$$\Phi_E = EA,$$




$$\Phi_E = \vec{E} \cdot \vec{A} = EACos\theta$$

**Conclusion:**

*The total flux depends on*

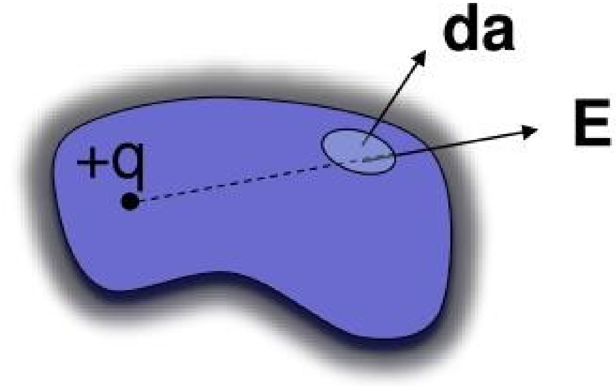
- ✓ *strength of the field,*
- ✓ *the size of the surface area it passes through,*
- ✓ *and on how the area is oriented with respect to the field.*

# Gauss's Law

- **The total of the electric flux out of a closed surface is equal to the charge enclosed divided by the permittivity ($\varepsilon_{o.}$).**

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0} \qquad \textit{Integral Form}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_o} \qquad \textit{Differential Form}$$




$\varepsilon_o$ = the permittivity of free space $8.854 \times 10^{-12}\ C^2/(N\ m^2)$

where

$$\boxed{Q_{enclosed} \approx \sum_i q_i \approx \int_v \rho\, dv \quad or \quad \int_S \sigma\, ds \quad or \quad \int_l \lambda\, dl}$$

# Electric lines of flux and Derivation of Gauss' Law using Coulombs law

- Consider a sphere drawn around a positive point charge. Evaluate the net flux through the closed surface.

Net Flux $\Phi = \oint \vec{E} \cdot d\vec{A} = \oint E \cos\theta dA = \oint E dA$

$E \parallel n$

$\cos 0° = 1$

For a Point charge $E = \dfrac{kq}{r^2}$

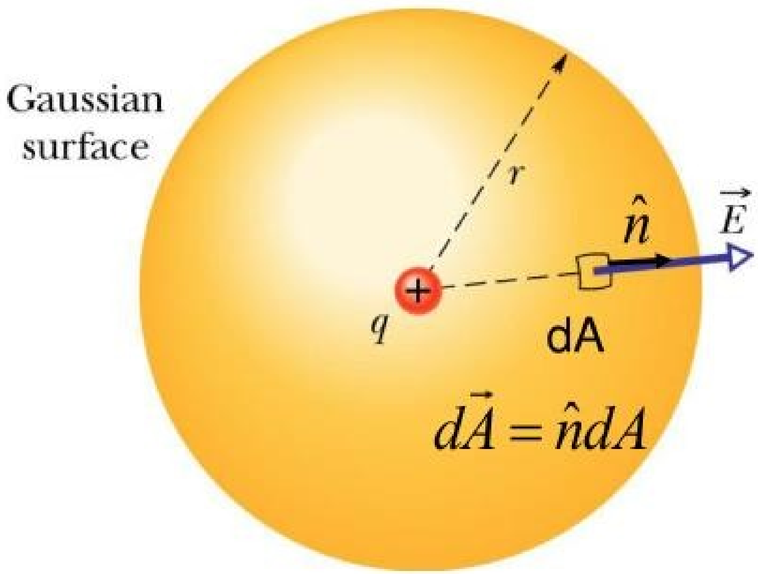
$$\Phi = \oint E dA = \oint \frac{kq}{r^2} dA$$

$$\Phi = \frac{kq}{r^2}(4\pi r^2) = 4\pi kq$$

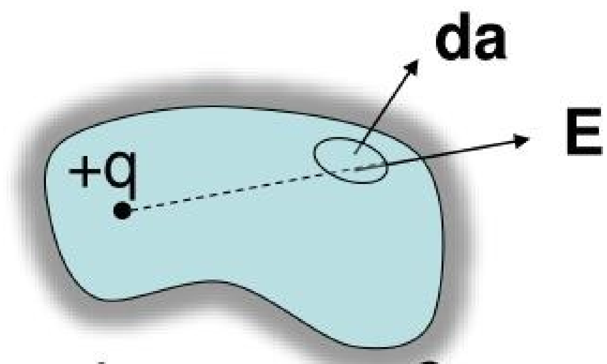
$4\pi k = \dfrac{1}{\varepsilon_0}$ **where** $\varepsilon_0 = 8.85\text{x}10^{-12} \dfrac{C^2}{Nm^2}$

$$\boxed{\Phi_{net} = \frac{q_{enc}}{\varepsilon_0}}$$

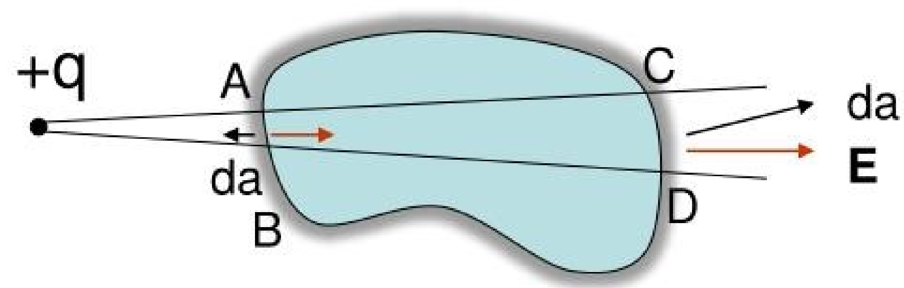
***Gauss' Law***



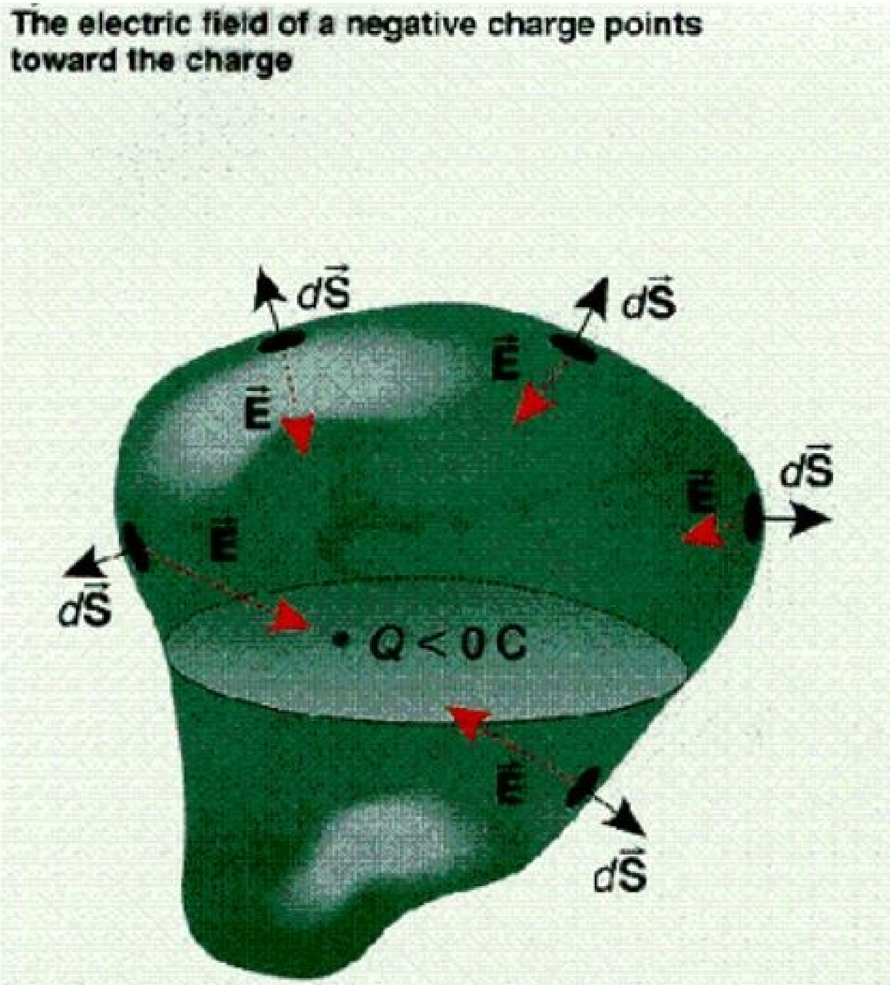
# *Asmnt 2:*
# *Proof of the Gauss's law for the charge inside*



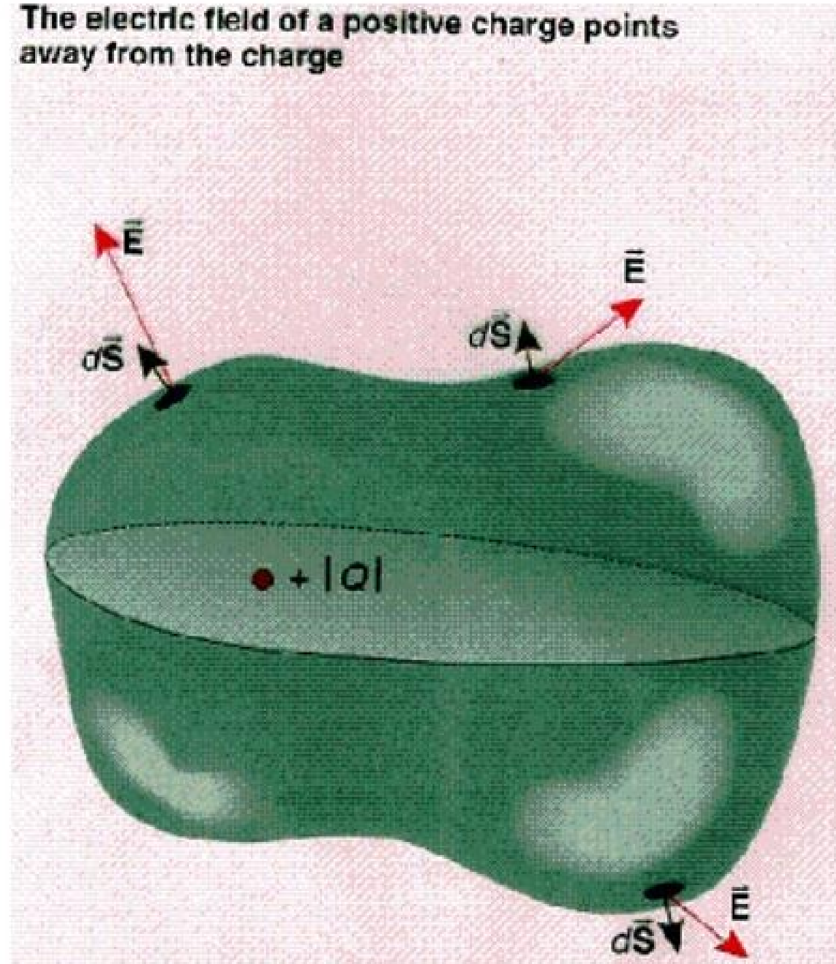

# *and outside the Gaussian surface*




Where dΩ is solid angle

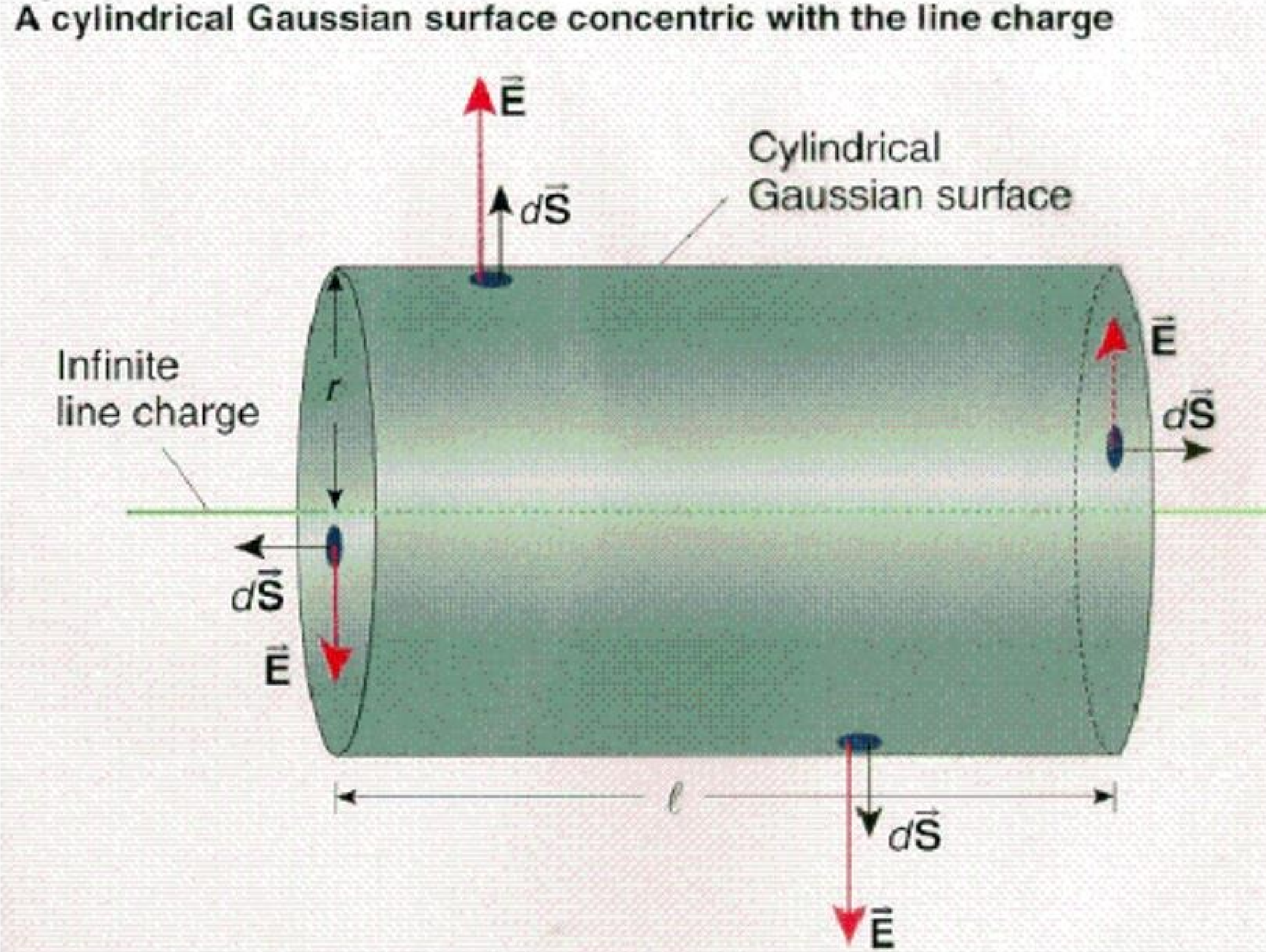



-ve flux




+ve Flux

$$\phi_E = \vec{E}.\vec{A} = E\cos\theta A = E_\perp A$$



# Differential form of Gauss Law: $\boxed{\nabla.E = \frac{\rho}{\varepsilon_0}}$

Proof: Gauss Law $\phi = \oint_{surface} E.da = \frac{Q_{enclosed}}{\varepsilon_0}$

Gauss divergence theorem:

$$\oint_s E.da = \int_V (\nabla.E)dv = \frac{\int_V \rho\, dv}{\varepsilon_0}$$

$$\boxed{Hence\, \nabla.E = \frac{\rho}{\varepsilon_0}} \quad \text{or} \quad \boxed{\nabla.D = \rho}$$

*Ex: If* $\vec{E} = 3x^2\hat{i} + 4y\hat{j} + 5z\hat{k}$
*Calculate volume charge density.*

*Where* $D = \varepsilon_o E$, *called Electric field vector and the* $\rho$ *is volume charge density.*

Note: Gauss law is also known as Maxwell's first equation.

# Quiz:

*1. How electric flux is related to the number of electric field lines?*

*2. Explain Negative and Positive flux.*

*3. Apoint charge of 1$\mu$C is placed at the centre of a hollow cube. Calculate the electric flux diverging through (a) the cube (b) each face*

**(a) 1.12 x $10^5$ V/m, (b) 1.86 x $10^4$ V/m,**

*4. If electric field* $\vec{E} = 3\hat{i} + 4\hat{j} - 5\hat{k}$, *calculates elecrtic flux through the surface* $\vec{S} = 2\times10^{-5}\,\hat{k}\,m^2$

**-$10^{-4}$ V/m**

# Remember

1. Electric Flux ($\varphi = E_{\perp}A = EA\cos\theta = q/\varepsilon_0$)
2. Flux is independent of the distance of a point from position of charge.
3. Electric Flux is the number of electric field lines crossing per unit area.
   I. For $\Phi_{max}$; EA $\Rightarrow \theta = 0^o$.
   II. For $\Phi_{min}$; EA=0 $\Rightarrow \theta = 90^o$
4. Gauss Law is Maxwell's first equation. $\nabla . E = \dfrac{\rho}{\varepsilon_0}$
5. *Conductors in electric Field;*
   I. *$E_{inside} = 0$ as net charge is distributed over the surface of a conductor.*
   II. *At the surface of conductor; perpendicular E only, no parallel component of E.*

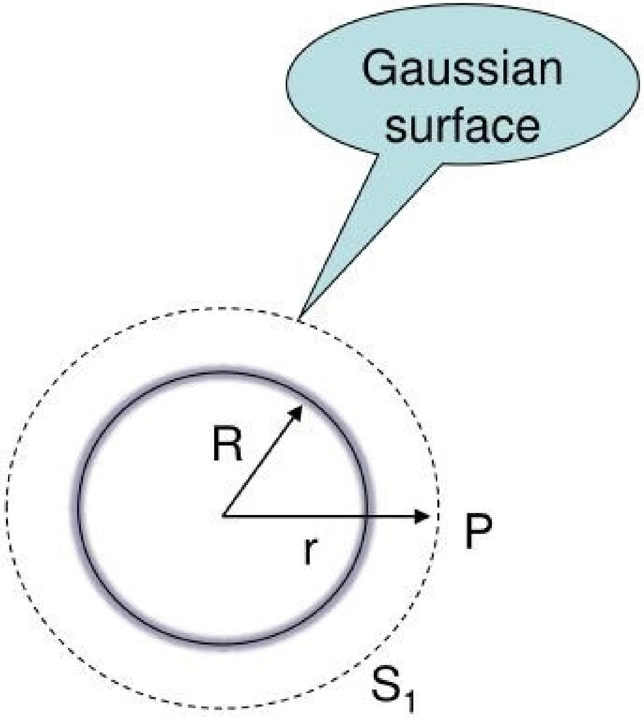
# Applications of Gauss law - Spherical and Cylindrical symmetries

# Applications of Gauss law
## (Spherical distribution systems)

1) Conducting Sphere of charge ‘q’ and radius ‘R’:
   1) E at an external point: $E_o$
   2) E at the surface: $E_s$
   3) E at an internal point: $E_i$
2) Nonconducting Sphere
   1) E at an external point: $E_o$
   2) E at the surface: $E_s$
   3) E at an internal point: $E_i$

# (Spherical systems: Conducting Sphere)

1) Conducting Sphere of charge ‘q’ and radius ‘R’:
   1) E at an external point: $E_o$ $\quad r>R$
   2) E at the surface: $E_s$ $\quad r=R$
   3) E at an internal point: $E_i$ $\quad r<R$




Case-I: E at an external point;
Net electric fux through ‘P’:

$$\oint_S E_o.da = \frac{q_{enclosed}}{\varepsilon_0}$$

$$\oint_{S_1} E_o.da = E_o \oint_{S_1} da = E_o 4\pi r^2 = \frac{q}{\varepsilon_0}$$

*hence* $\boxed{\vec{\mathbf{E}}_o = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}}$

$$\oint_{S_1} da = r^2 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi = 4\pi r^2$$

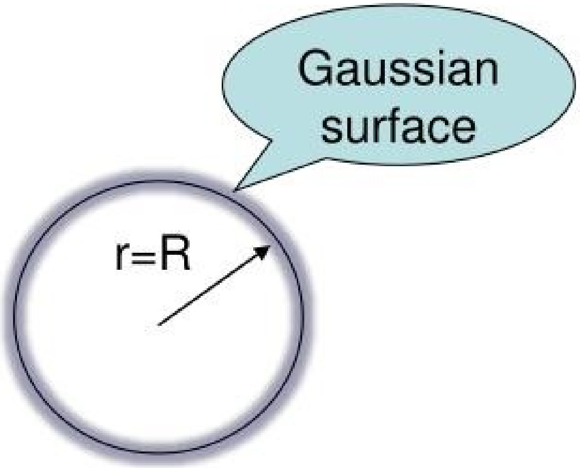
**The Electric field strength at any point outside a spherical charge distribution is the same as through the whole charge were concentrated at the centre.**

# (Spherical systems: Conducting Sphere)

Case-II: E at the Surface;

$$Substitute \quad r = R$$

$$hence \quad \boxed{\vec{E}_s = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\hat{r}}$$

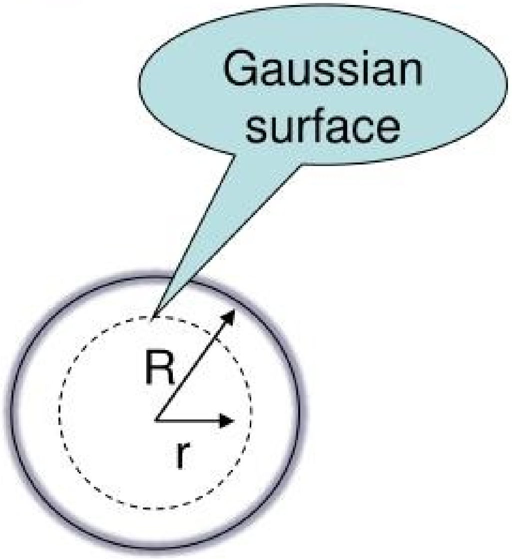



Case-III: E at an internal point;

$$Substitute \quad r < R,$$

$$q = 0 \ inside \ a \ conductor \longleftarrow$$

$$hence \quad \boxed{E_i = 0}$$



# (Spherical systems: Nonconducting Sphere)

1) Nonconducting Sphere of charge 'q' and radius 'R':

1) E at an external point: $E_o$ r>R
2) E at the surface: $E_s$ r=R
3) E at an internal point: $E_i$ r<R

*Volume Charge Density*

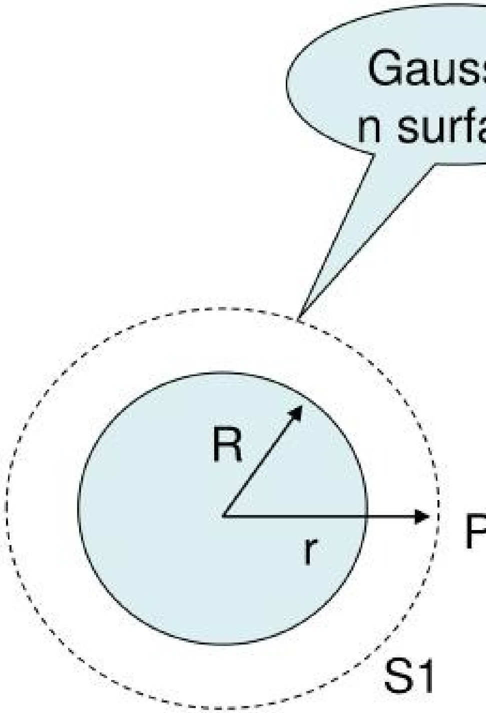
$$\rho = \frac{q}{\frac{4}{3}\pi R^3}$$

Case-I: E at an external point;
Net electric flux through 'P':

$$\oiint_S E_o . da = \frac{q_{enclosed}}{\varepsilon_0}$$

$$\oiint_{S1} E_o . da = E_o \oiint_{S1} da = E_o 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$hence \qquad \vec{E}_o = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

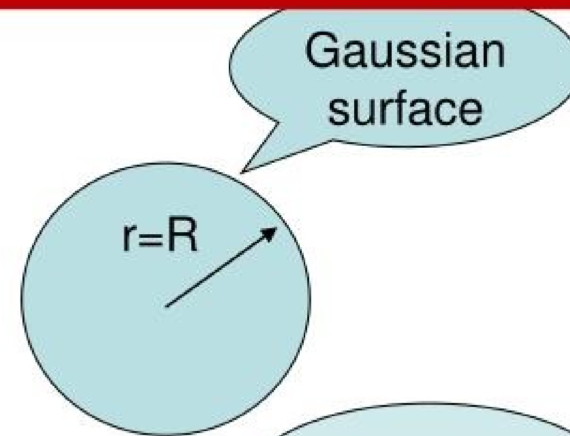


# (Spherical systems: Nonconducting Sphere)

## Case-II: E at the Surface;

$$Substitute \quad r = R$$

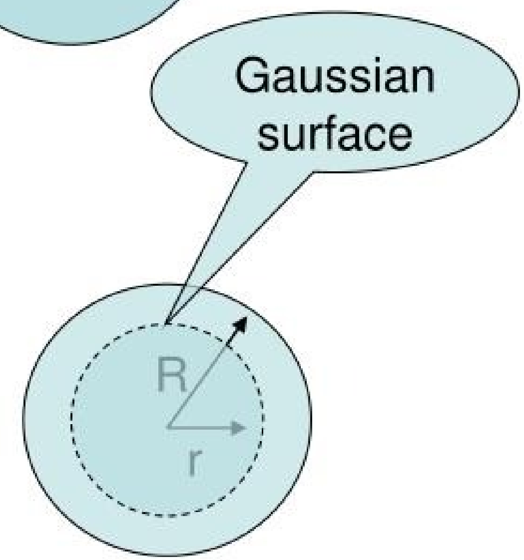
$$hence \quad \boxed{\vec{E}_o = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\hat{r}}$$




## Case-II: E at an internal point;

$$Substitute \quad r < R, \quad q = \rho \times volume$$

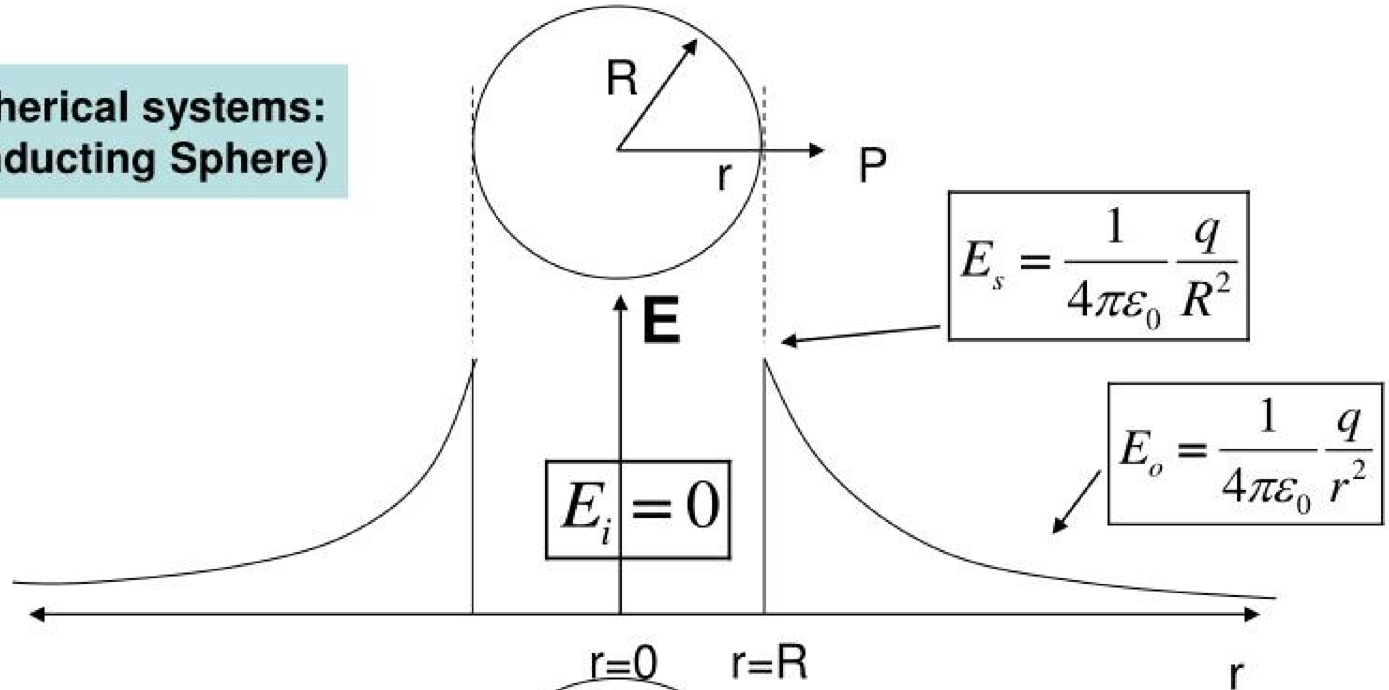
$$q' = \frac{q}{\left(\frac{4}{3}\pi R^3\right)} \times \left(\frac{4}{3}\pi r^3\right) \quad hence$$



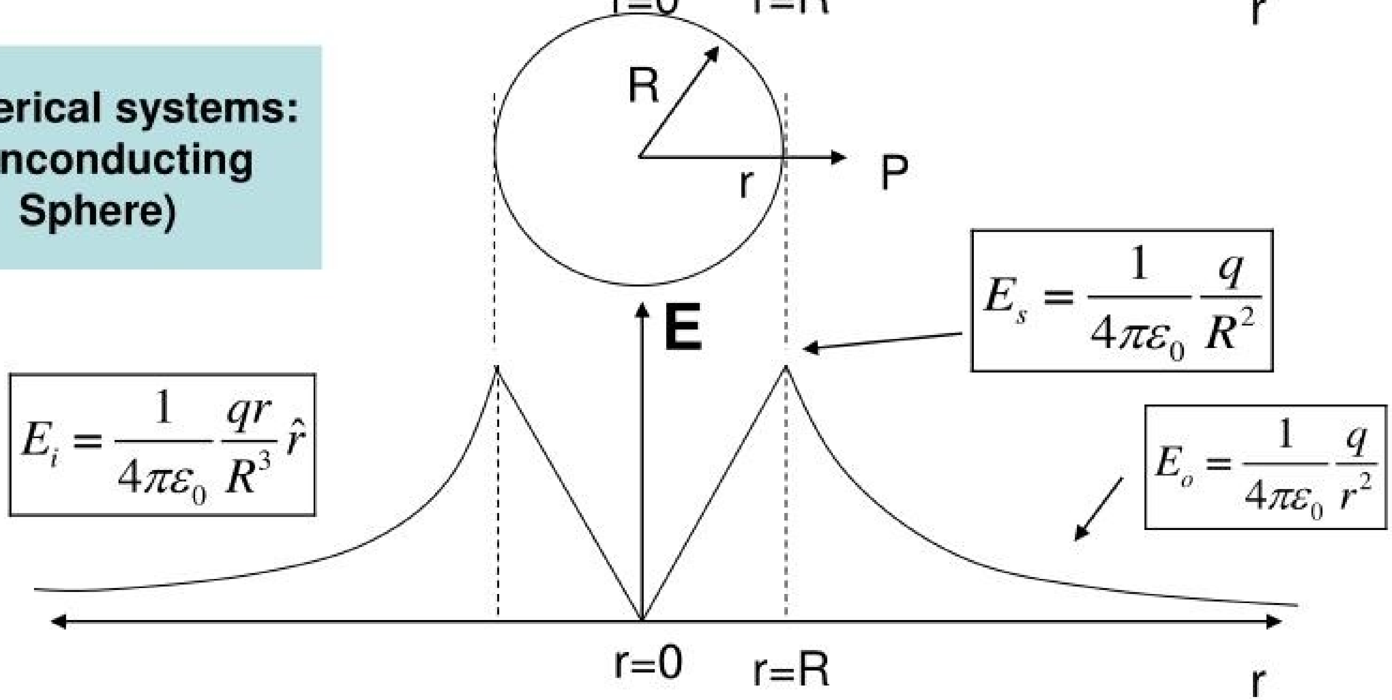

$$E_i \, 4\pi r^2 = \frac{q}{\varepsilon_0}\frac{r^3}{R^3}$$

$$E_i = \frac{q}{4\pi\varepsilon_0}\frac{r}{R^3}\hat{r} \longrightarrow \boxed{E_i \alpha \, r}$$

**(Spherical systems: Conducting Sphere)**



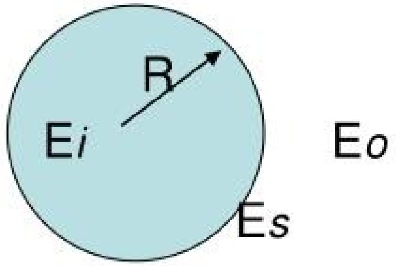

**(Spherical systems: Nonconducting Sphere)**



# Numerical

*For a non conducting solid sphere of radius 'R' and Volume charge density $\rho = kr^2$. Determine Electric Field everywhere by using Gauss Law.*

$$\text{Gauss Law} \quad \oint_S E.da = \frac{q_{enclosed}}{\varepsilon_0} = \frac{\int_v \rho dv}{\varepsilon_0}$$



Solid sphere

*i) For Gaussian surface outside the sphere at $r = r_0$; $E = E_o$*

$$E_0 \oint_S da = \frac{\int_v \rho dv}{\varepsilon_0} = \frac{\int_v kr^2 . r^2 dr . \sin\theta d\theta . d\varphi}{\varepsilon_0}$$

$$E_0 . 4\pi r_0^{\ 2} = \frac{4\pi k}{\varepsilon_0} \int_0^R r^4 dr = \frac{4\pi k R^5}{\varepsilon_0 5}$$

$$\Rightarrow \quad \boxed{\vec{E}_0 = \frac{kR^5}{5\varepsilon_0 r_0^{\ 2}} \hat{r}}$$

*iii) For internal point, $r = r_i$; $E = E_i$*

$$E_i 4\pi r_i^{\ 2} = \frac{4\pi k}{\varepsilon_0} \int_0^{r_i} r^4 dr = \frac{4\pi k r_i^{\ 5}}{\varepsilon_0 5}$$

$$\Rightarrow \quad \boxed{\vec{E}_i = \frac{k r_i^{\ 3}}{\varepsilon_0 5} \hat{r}}$$

*ii) For Gaussian surface at the sphere, $r = r_0 = R$; $E = E_S$*

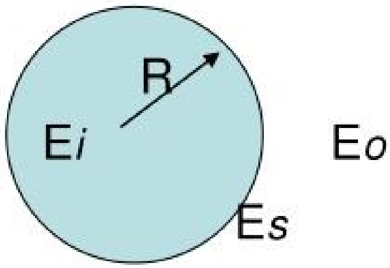
$$\Rightarrow \quad \boxed{\vec{E}_S = \frac{kR^3}{5\varepsilon_0} \hat{r}}$$

# Problems: Spherical Symmetry

**Determine Electric field everywhere by using Gauss Law for the following;**

1. Non conducting solid sphere of radius R and charge density ρ=k/r$^2$, Where k is a constant.

$$\rho = \frac{k}{r^2},$$




Solid sphere

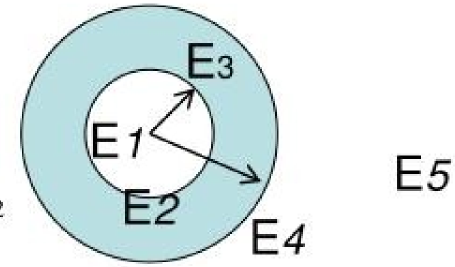
$$\begin{aligned}
&Ans: \\
&At\ r = r_0,\ \vec{E}_0 = \frac{k\,R}{\varepsilon_0 r_0^2}\hat{r}, \\
&At\ r = R,\ \vec{E}_s = \frac{k}{\varepsilon_0 R}\hat{r}, \\
&At\ r = r_i,\ \vec{E}_i = \frac{k\,r_i}{\varepsilon_0 r_i^2}\hat{r} = \frac{k}{\varepsilon_0 r_i}\hat{r},
\end{aligned}$$

2. Non conducting spherical shell of inner radius $r_1$, outer radius $r_2$ and charge density ρ= k/r$^2$ , where k is a constant. Also determine Max E at any value of r .

$$\rho = \frac{k}{r^2},$$

$$where\ \ r_1 \le r \le r_2$$

$$otherwise\ zero.$$




Ans : Spherical shell

$$\begin{aligned}
&At\ \ r_0 > r_2,\ \vec{E}_5 = \frac{k\,[r_2 - r_1]}{\varepsilon_0 r_0^2}\hat{r}, \\
&At\ \ r = r_2,\ \vec{E}_4 = \frac{k\,[r_2 - r_1]}{\varepsilon_0 r_2^2}\hat{r}, \\
&At\ \ r_1 < r < r_2,\ \vec{E}_3 = \frac{k\,[r - r_1]}{\varepsilon_0 r^2}\hat{r}, \\
&At\ \ r \le r_1,\ \vec{E}_2\ or\ \vec{E}_1 = 0, \\
&Max\,E\,is\,at\ \ r = 2r_1\ but\ r_2 > 2r_1, \\
&\text{otherwise at the surface } r_2.
\end{aligned}$$

# Applications of Gauss law
## (Cylindrical distribution systems)

1)<u>*Conducting long Cylinder*</u> *of charge 'q' and radius 'R':*

1) *E at an external point:* $E_o$
2) *E at the surface:* $E_s$
3) *E at an internal point:* $E_i$

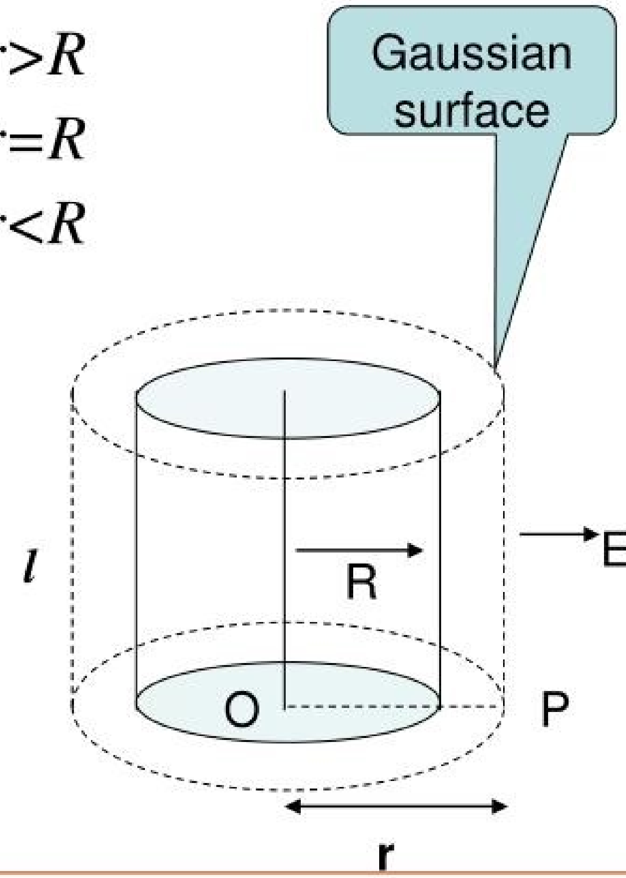
2)<u>*Nonconducting long Cylinder*</u>

1) *E at an external point:* $E_o$
2) *E at the surface:* $E_s$
3) *E at an internal point:* $E_i$

# Cylindrical distribution systems: Conducting Cylinder

*1)Conducting long Cylinder of charge 'q' and radius 'R' :*

1) *E at an external point: $E_o$* $r>R$
2) *E at the surface: $E_s$* $r=R$
3) *E at an internal point: $E_i$* $r<R$




**Case-I: E at an external point; Net electric flux through 'P':**

$$\oint_S E_o.da = \frac{q}{\varepsilon_0}$$

$$\oint_{S1} E_o.da = E_o \oint_{S1} da = E_o 2\pi r l = \frac{q}{\varepsilon_0}$$

$$\text{hence} \qquad \vec{E}_o = \frac{1}{2\pi\varepsilon_0}\frac{q}{r.l}\hat{r}$$

$$q_{enclosed} \approx \sum_i q_i \approx \int_v \rho\, dv \approx \int_S \sigma\, da \approx \int_l \lambda\, dl$$

## Case-II: E at the Surface;

*Substitute* $\quad r = R$

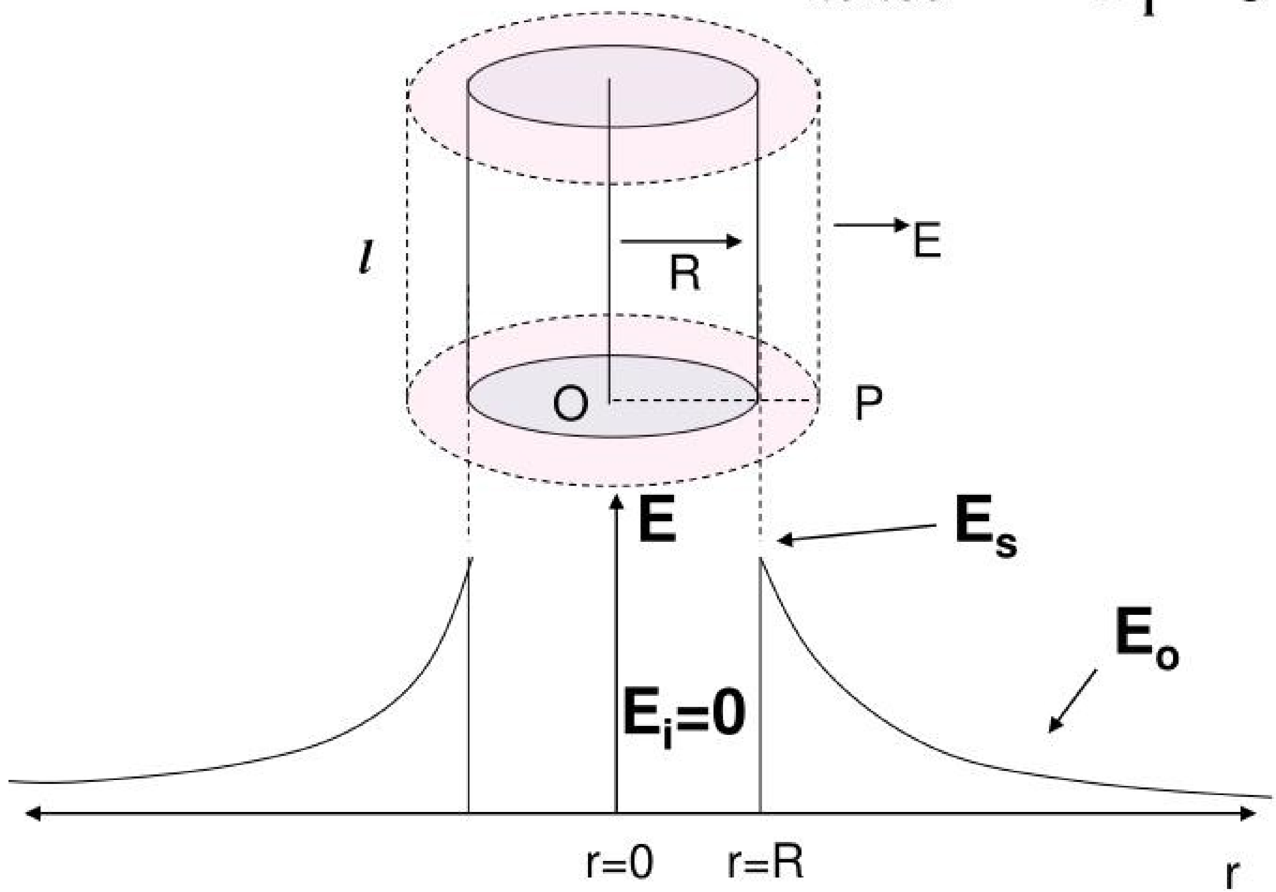
*hence* $\quad \vec{\mathbf{E}}_{\mathbf{S}} = \dfrac{\mathbf{1}}{\mathbf{2\pi}\varepsilon_0} \dfrac{\mathbf{q}}{\mathbf{R}.l} \hat{r}$

## Case-III: E at an internal point;

*Substitute* $\quad r < R,$

$q = 0$ *for a conductor*

*hence* $\quad \mathbf{E_i} = \mathbf{0}$



# *Cylindrical distribution systems: Nonconducting Cylinder*

1) Nonconducting Cylinder of radius 'R', height '$l$' *and charge density 'ρ'*:

$$\rho = \frac{q}{\pi r^2 l}$$

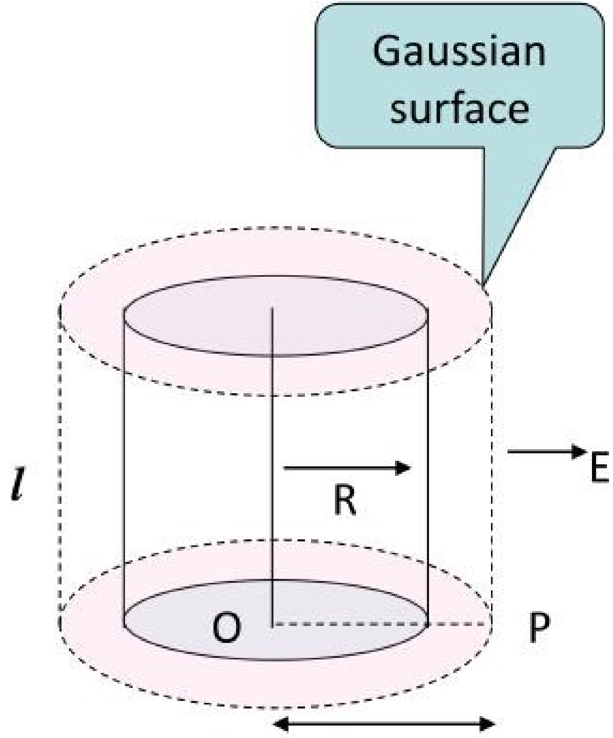
1) E at an external point: $E_o$ r>R
2) E at the surface: $E_s$ r=R
3) E at an internal point: $E_i$ r<R

Case-I: E at an external point;
Net electric flux through 'P':

$$\oint_S \vec{E}_o . d\vec{A} = \frac{q}{\varepsilon_0},$$

$$\oint_{S1} \vec{E}_o . d\vec{A} = E_o \oint_{S1} dA = E_o \, 2\pi r l = \frac{q}{\varepsilon_0}$$

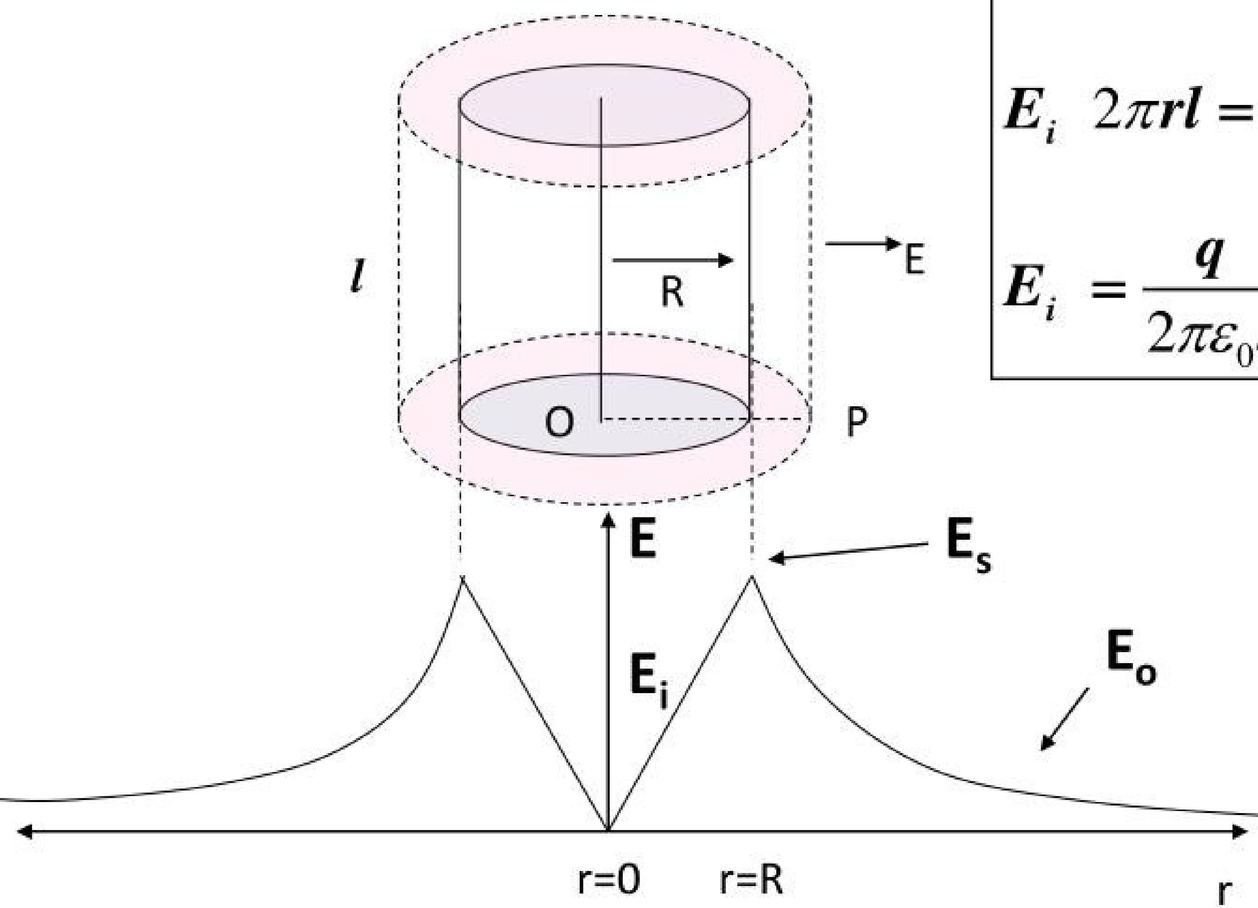
$$\textit{hence} \qquad \vec{\mathbf{E}}_o = \frac{1}{2\pi\varepsilon_0} \frac{\mathbf{q}}{\mathbf{r}.l} \hat{r}$$



Case-II: E at the Surface;

*Substitute* $\quad r = R$

*hence* $\quad \vec{\mathbf{E}}_S = \dfrac{1}{2\pi\varepsilon_0}\dfrac{q}{R.l}\hat{r}$




Case-III: E at an internal point;

*Substitute* $\quad r < R, \quad q' = \rho \times volume$

$$q' = \frac{q}{\pi R^2 l} \times \pi r^2 l \quad hence$$

$$E_i \; 2\pi r l = \frac{q}{\varepsilon_0}\frac{r^2}{R^2}$$

$$E_i = \frac{q}{2\pi\varepsilon_0 l}\frac{r}{R^2}\hat{r}$$

## For infinite long line charge density 'λ'

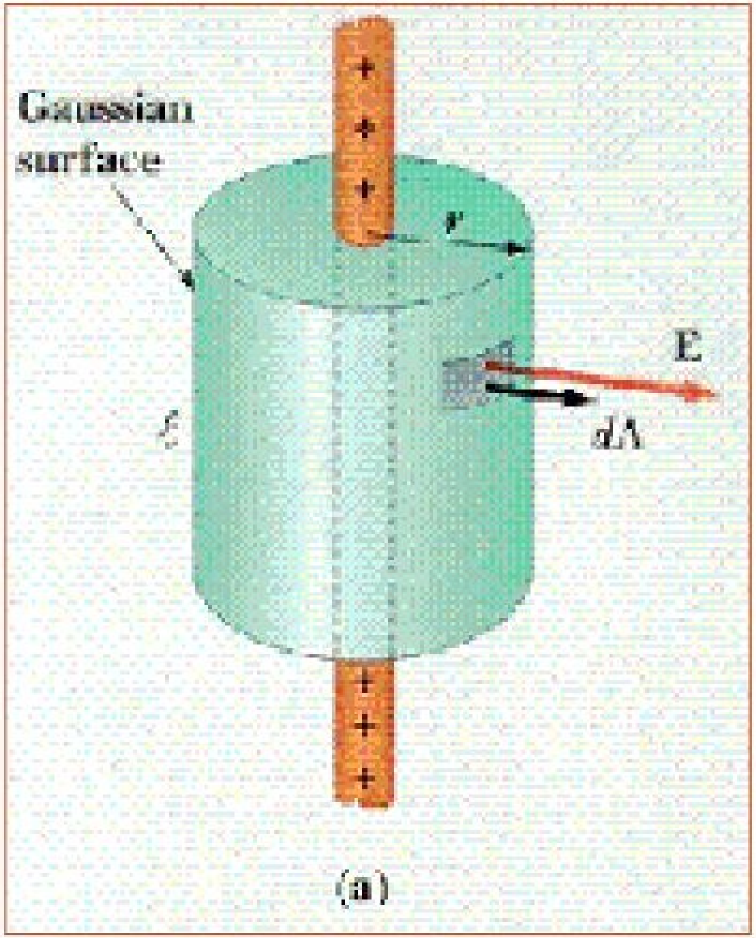
$$\oint E.da = \frac{q}{\varepsilon_0} = \frac{\int_0^h \lambda dl}{\varepsilon_0}$$

$$E.\int_A r\, d\phi\, dz = \frac{\lambda.h}{\varepsilon_0}$$

$$E.r\int_0^{2\pi} d\phi \int_0^h dz = \frac{\lambda.h}{\varepsilon_0}$$

$$E.r.2\pi.h = \frac{\lambda.h}{\varepsilon_0}$$

$$\boxed{E = \frac{\lambda}{2\pi r \varepsilon_0}}$$



# Numerical:
# Non conducting Cylindrical shell ($r_1$, $r_2$ and height h) having volume charge density ρ=k/r. Determine E everywhere.

*Case-I: E at an external point $r_0$; $E_0$*

$$\oint_S E_o.da = \frac{\int_v \frac{k}{r} \cdot r\,dr\,d\varphi\,dz}{\varepsilon_0},$$

$$E_o \oint_{S1} r_0 d\varphi dz = \frac{2\pi hk \int_{r_1}^{r_2} dr}{\varepsilon_0}$$

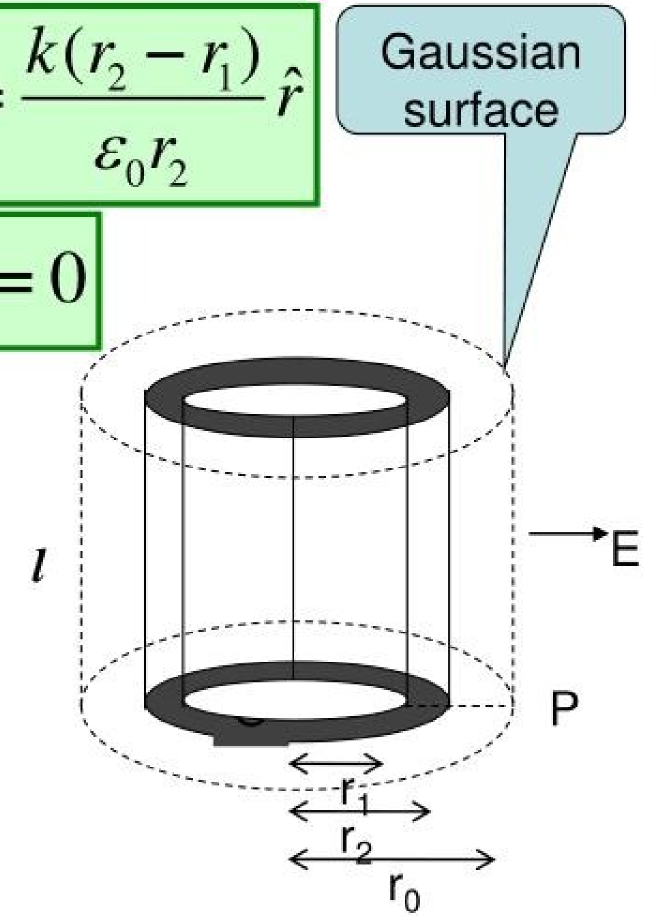
$$E_o .2\pi r_0 h = \frac{2\pi hk(r_2 - r_1)}{\varepsilon_0}$$

$$hence \qquad \vec{\mathbf{E}}_{\mathbf{o}} = \frac{\mathbf{k}(r_2 - r_1)}{\varepsilon_0 \mathbf{r_0}} \hat{r}$$

$$Case\ II: \quad \vec{E}_{r_1<r<r_2} = \frac{k(r_i - r_1)}{\varepsilon_0 r_i} \hat{r}$$

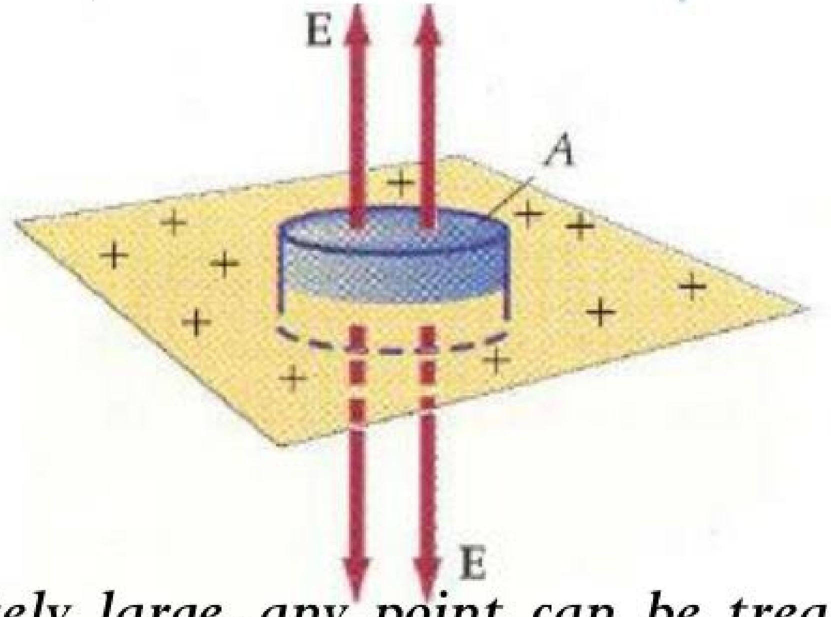
$$Case\ III: \quad \vec{E}_{r=r_2} = \frac{k(r_2 - r_1)}{\varepsilon_0 r_2} \hat{r}$$

$$Case\ IV: \quad \vec{E}_{r \le r_1} = 0$$



# *Applications of Gauss law (Infinitely long sheet of Charge)*

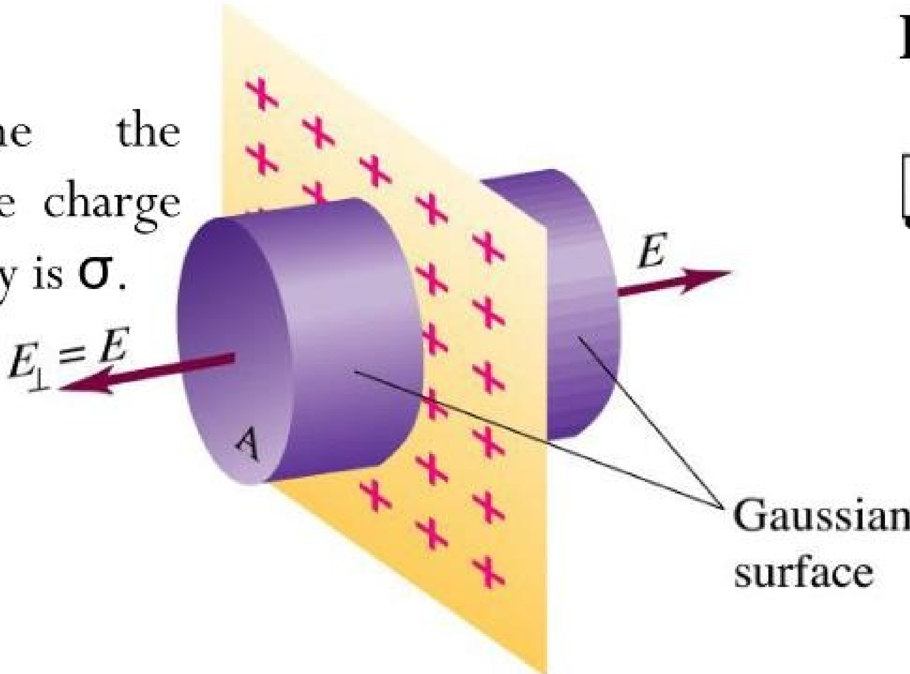
Planar symmetry




***The plane is infinitely large, any point can be treated as the center point of the plane, so E at that point must be normal to the surface and must have the same magnitude at all points equidistant from the plane.***

*A cylindrical Gaussian surface is used to find the electric field of an infinite plane sheet of charge.*

Assume the surface charge density is σ.



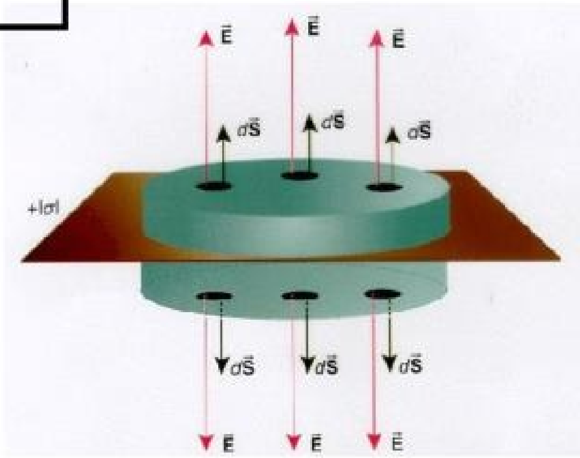

**Flux**

$$\oiint \vec{E}\cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_o}$$

$$EA + EA = \frac{\sigma A}{\varepsilon_o}$$

$$\boxed{E = \frac{\sigma}{2\varepsilon_o}}$$

***'E' does not depend on the distance 'r'. Therefore, the field is uniform everywhere.***

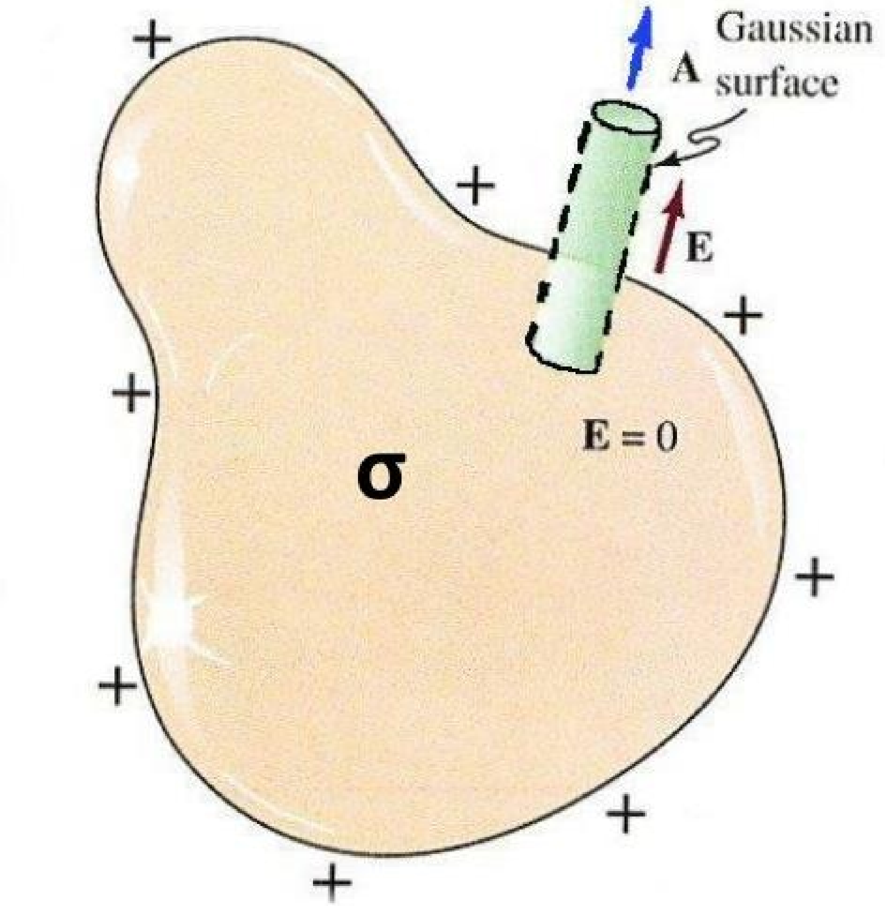


# *Electric field just outside the surface of a charged conductor*

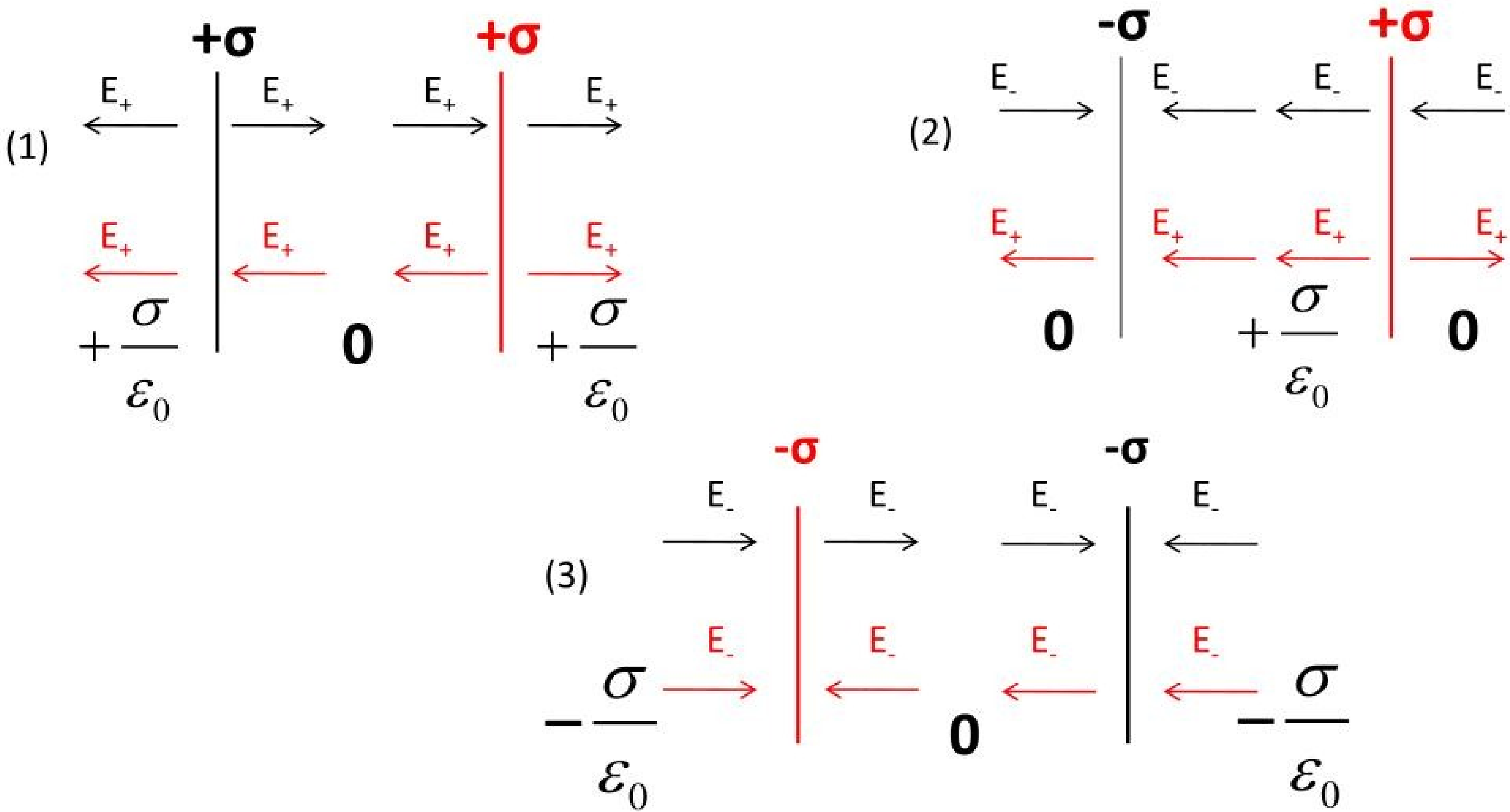
*Assume the surface charge density is $\sigma$.*

$$\Phi_E = EA = \frac{Q_{in}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$\text{or} \quad E = \frac{\sigma}{\varepsilon_0}$$



Find the field in each of three regions
(i) to the left of both
(ii) between them
(iii) to the right of both



# Quiz

- *For a conducting sphere: with surface charge density 'σ' and radius R, determine $E_o$, $E_s$ and $E_i$.*

$$Ans: \quad \vec{E}_{0(r>R)} = \frac{\sigma.R^2}{\varepsilon_0 r^2}\hat{r},\ \vec{E}_{s(r=R)} = \frac{\sigma}{\varepsilon_0}\hat{r},\ E_{i(r<R)} = 0$$

- *For a spherical shell volume charge density is $\rho = k/r$ for $a \leq r \leq b$ otherwise zero. determine E for each region.*

$$Ans: \quad \vec{E}_{0(r>R)} = \frac{k.R^3}{3\varepsilon_0 r}\hat{r},\ \vec{E}_{s(r=R)} = \frac{k.R^2}{3\varepsilon_0}\hat{r},\ E_{i(r<R)} = \frac{k.r^2}{3\varepsilon_0}\hat{r}$$

- *For a cylinder of radius 'R' and height 'h' volume charge density is $\rho = kr$. Determine $E_o$, $E_s$ and $E_i$*

- *Electric field in inside and just outside (very close) of the surface of a charged conductor*
  - *Inside $E = 0$, Outside $E = \sigma/\varepsilon_0$*

# Line integral of Electric field: Electric Potential

*Electric field at a field point* **r**, *due to a point charge at origin:* $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

$$\int_a^b \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\int_a^b \frac{q}{r^2}dr = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_a}-\frac{1}{r_b}\right) = V(a)-V(b)$$

$$\Rightarrow -\int_a^b \vec{E}\cdot d\vec{l} = V(b)-V(a)$$

*As fundamental theorem for gradient is*

$$\Rightarrow V(b)-V(a) = \int_a^b \left(\vec{\nabla}V\right)\cdot d\vec{l}$$

*And if a=∞, b=r*

$$V(\vec{r}) = -\int_\infty^r \vec{E}\cdot d\vec{l}$$

$$\boxed{\vec{E} = -\vec{\nabla}V}$$

*The* ***electric potential*** *at a distance* **r** *from the point charge is the work done per unit charge in bringing a test charge from infinity to that point.*

- *Surface over which Potential is constant is called an equi-potential surface.*
- *Reference point : $V(\infty)=0$ convention at infinity.*
- *Superposition principle: $V=V_1+V_2+\ldots..$*
- *Unit: Nm/C or Joule/C or VOLT*

---

## Curl of E ?

$$\Rightarrow \oint \vec{E}\cdot d\vec{l} = 0$$

***The integral around a closed path is zero***

Using Stokes' theorem

$$\int_s \left(\vec{\nabla}\times\vec{E}\right)\cdot d\vec{a} = \oint_l \vec{E}.d\vec{l} = 0$$

$$\vec{\nabla}\times\vec{E} = 0$$

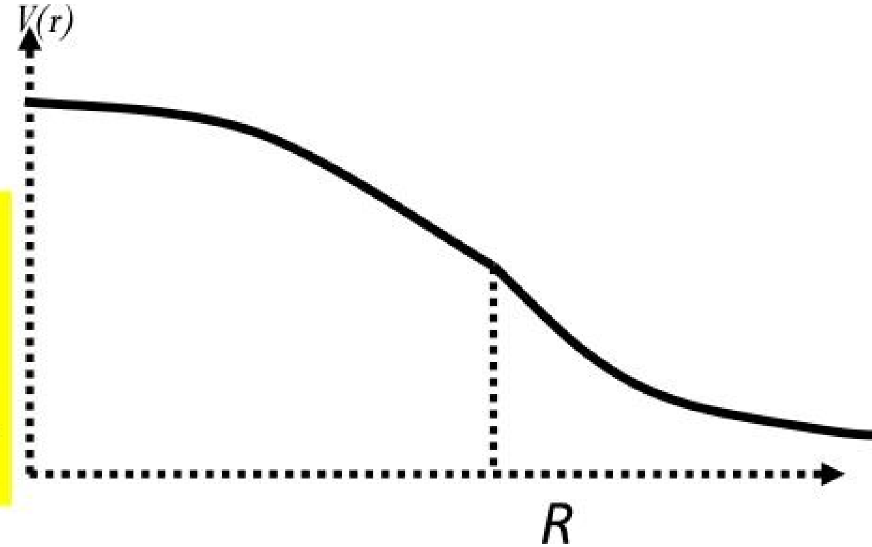
***In electrostatics only.***
*→ **no moving charge or current***

Which of these is an impossible electrostatic field and why?

a. $\bar{E} = xy\,\hat{\imath} + 2yz\,\hat{\jmath} + 3xz\,\hat{k}$ b. $\bar{E} = y^2\,\hat{\imath} + (2xy + z^2)\,\hat{\jmath} + 2yz\,\hat{k}$

# Numerical

*Find the potential inside and outside a uniformly charged solid sphere of radius R and total charge q. Use infinity as your reference point. Sketch V(r) .*




**Electric field at $r > R$**

$$E = \frac{kq}{r^2}$$

$r < R$

$$E = \frac{kqr}{R^3}$$

**Therefore, Electric potential**

at $r > R$

$$V = -\int_{\infty}^{r} \frac{kq}{r^2} dr = \frac{kq}{r}$$

$r < R$

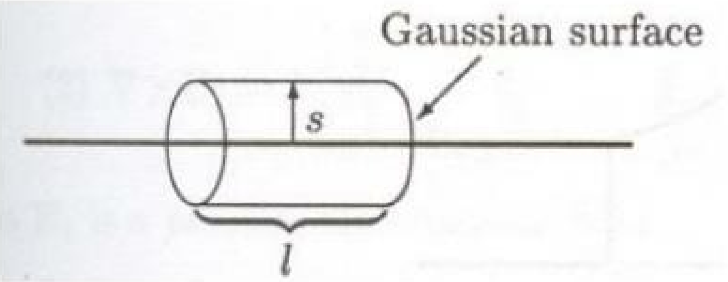
$$V = -\int_{\infty}^{R} \frac{kq}{r^2} dr - \int_{R}^{r} \frac{kq}{R^3} r dr$$

$$\Rightarrow V = \frac{kq}{R} - \frac{kq}{2R^3}\left(r^2 - R^2\right) = \frac{kq}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

**Find the potential a distance *s* from an infinitely long straight wire that carries uniform line charge λ.**

**Solution:**

$$E = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{s}\hat{s}$$




$$V(s) = -\int_a^s \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{s}ds$$

*In this case we cannot set the reference point at ∞, since the charge itself extends to ∞. Let's set it at s = a. Then*

$$V(s) = -\frac{1}{4\pi\varepsilon_0}2\lambda\ln\left(\frac{s}{a}\right)$$

# Poisson's and Laplace's Equation

- ***The electric field is related to the charge density***

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

- ***The electric field is related to the electric potential***

$$E = -\nabla V$$

- ***Therefore the potential is related to the charge density by***

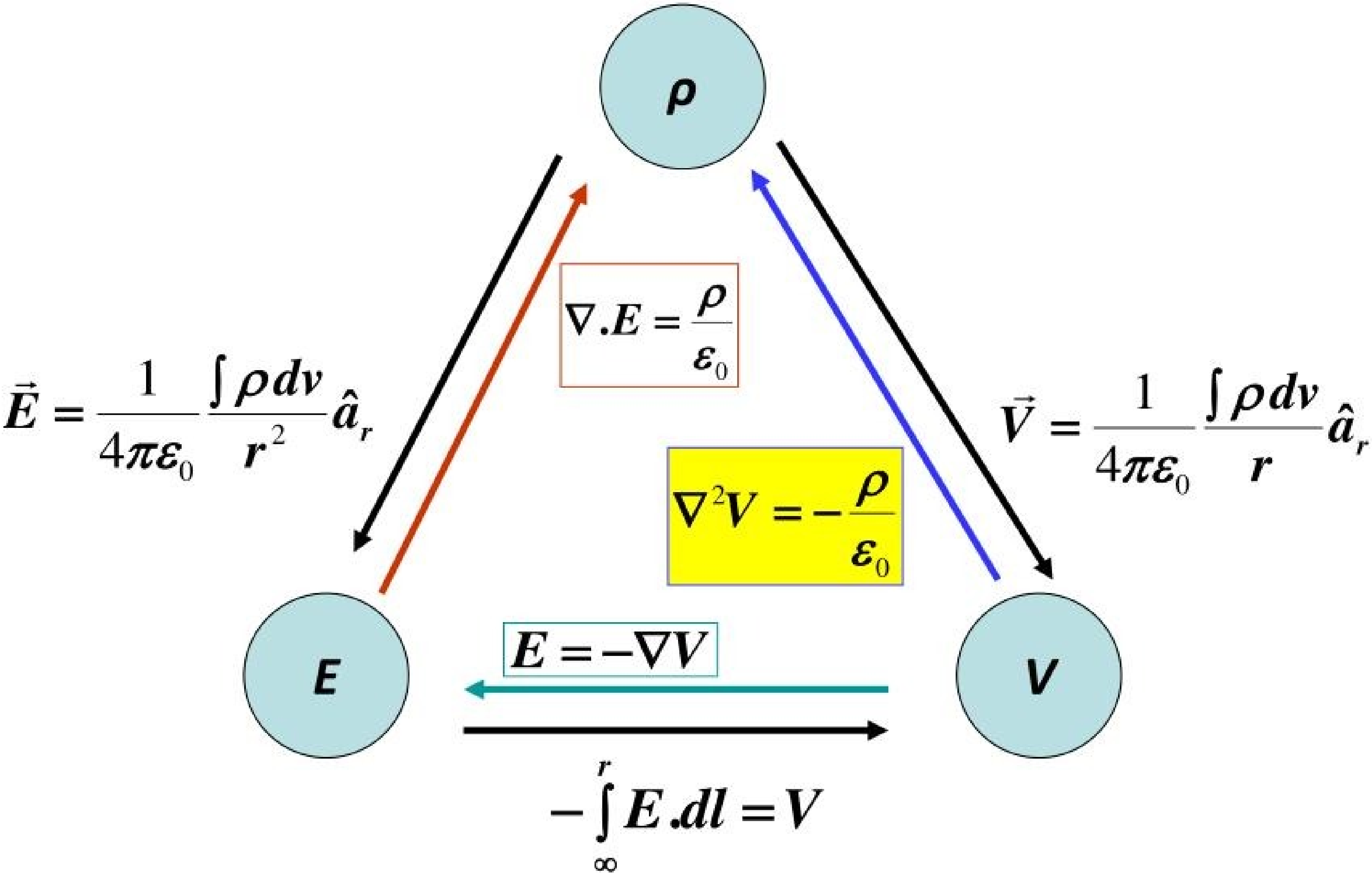
$$\boxed{\nabla \cdot \nabla V = \nabla^2 V = \frac{-\rho}{\varepsilon_0}} \text{ Poisson's equation}$$

***$\nabla^2$ : Laplacian operator.***

In a charge-free region of space, $\nabla^2 V = 0$

$\boxed{\text{Laplace's equation}}$

# Conversion from one to another



# Note: Laplace's Operator

$\nabla^2$ operator ( $\nabla \cdot \nabla$ ) in different coordinate systems:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

*in Cartesian.*

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

*in Cylinderical*

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

*in Spherical*

**Cartesian.** $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

**Cylindrical.** $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

**Spherical.** $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $\quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

*Gradient* :
$$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

*Divergence* :
$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

*Curl* :
$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

*Laplacian* :
$$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

***Show that potential function V(x,y,z) or V(r) satisfies the Laplace's equation.***

$$V(x,y,z) = k\frac{q}{\sqrt{x^2+y^2+z^2}} \quad or \quad k\frac{q}{r}$$

Cartesian: $$\nabla^2 V = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)V(x,y,z)\ldots\ldots \quad \ldots\ldots (1)$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = k\left[\frac{3qx^2}{(x^2+y^2+z^2)^{5/2}} - \frac{q}{(x^2+y^2+z^2)^{3/2}}\right]\ldots(2)$$

similarly, $\dfrac{\partial^2 V}{\partial y^2}$ and $\dfrac{\partial^2 V}{\partial z^2}$

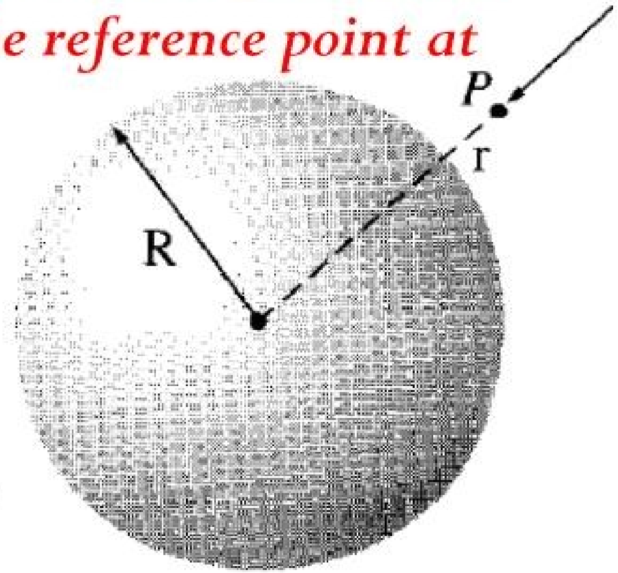
Substituting values, we get $$\nabla^2 V = \left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}\right) = 0$$

Spherical: $$\nabla^2 V(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}V(r)\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{kq}{r}\right)\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(-\frac{kq}{r^2}\right)\right)$$

$$= 0$$

*Find the potential inside and outside a spherical shell of radius R, which carries a uniform surface charge. Set the reference point at infinity.*




$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

*Where q is the total charge on the sphere.*

*The field inside is zero. For points outside the sphere*

$$V(r) = -\int_{\infty}^{r} \mathbf{E}.d\mathbf{l} = \frac{-1}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}dr' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

***r < R***

$$V(r) = \frac{-1}{4\pi\varepsilon_0}\int_{\infty}^{R}\frac{q}{r'^2}dr' - \int_{R}^{r}(0)dr' = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$$

*Notice that the potential is not zero inside the shell , even though the field is. V is constant in this region, so that $\nabla V = 0$*

*Calculate the numerical value for V and $\rho_v$ at point P in free space if* ***(D 7.1 (page 175, 7th Ed. Hayt)***

(a) $V = \dfrac{4yz}{x^2+1}$ at $P(1,2,3)$

Hint : $\nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$

(b) $V = 5r^2 \cos 2\phi$ at $P\ (r=3, \phi=\pi/3, z=2)$

$$\boxed{\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}}$$

(c) $V = \dfrac{2\cos\phi}{r^2}$ at $P\ (r=0.5, \theta=\pi/4, \phi=\pi/3)$

$$\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}}$$

**Ans: (a) 12V, - 106.2 $pC/m^3$**

**(b) -22.5V, 0**

**(c) 4V, -141.7$pC/m^3$**

# *Numerical*

- *Does potential function $2(x^2-y^2+z)$ satisfies Laplace's equation?*

Ans: Yes

$$\boxed{\nabla^2 V = 0}$$

- *Determine potential outside a charged conducting sphere of radius R,* ***using Laplace's equation.***

*Given* $V=V_0$ *at* $r=R$

$= 0$ *at* $r=$*infinite.*

$$\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}}$$

**Solution:** $As\, V\ depends\ on\ r\ only;$

$$\nabla^2 V(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0 \quad ........ (1)$$

$$or \qquad \left(r^2\frac{\partial V}{\partial r}\right) = Constant\,(say\,'C_1')$$

$$or \qquad \left(\frac{\partial V}{\partial r}\right) = \frac{C_1}{r^2},$$

$$by\ integrating\ ,we\ get\ \ V(r) = -\frac{C_1}{r} + C_2 \quad ......... (2)$$

$Boundary\ conditions;$

$$V = 0\ \ at\ \ r = \infty \ \Rightarrow\ C_2 = 0$$

$$V = V_0\ \ at\ \ r = R\ \Rightarrow V_0 = -\frac{C_1}{R}\ \Rightarrow C_1 = -V_0 R;$$

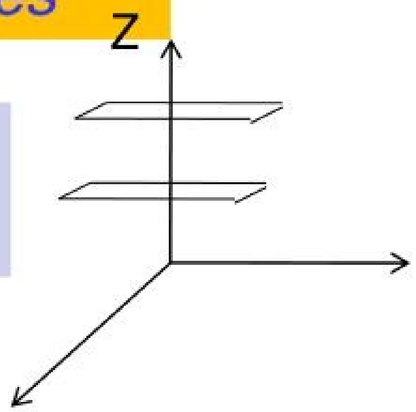
$Substituting\ \ C_1\, and\, C_2\ \ in\ \ equation\,(2), we\ get$

$$V(r) = V_0\frac{R}{r};$$

$$We\ know\ that\ \ V_0 = \frac{1}{4\pi\varepsilon_0}\frac{q}{R} \quad on\ the\ sphere, \quad \Rightarrow V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

# Applications: Laplace's and Poisson's equation

## In Cartesian or Rectangular coordinates




### One dimensional solution of Laplace's Equation in rectangular coordinate system

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad (cartesian) \qquad ....(1)$$

Let V be a function of z only. Then Laplace's Equation reduces to

$$\nabla^2 V = \frac{\partial^2 V}{\partial z^2} = 0 \qquad ....(2)$$

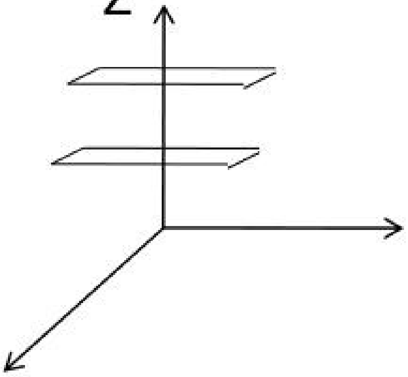
$$\text{Solution of this equation,} \quad V = Az + B \quad ....(3)$$

$$\text{where } A, B \text{ are constants}$$

***Equation (3) represents <u>a family of equi – potential surfaces</u> with z taking up constant values.***

# *Applications: Laplace's and Poisson's equation*

***Consider two such equi – potential surfaces one at $z = z_1$ and the other at $z = z_2$.***




*Let $\underline{V = V_1 \text{ at } z = z_1}$ and $V = \underline{V_2 \text{ at } z = z_2}$*

*This is the case with a parallel plate capacitor with a plate separation of $z_1 \sim z_2 = d$ and a potential difference $V_1 \sim V_2$.*

Applying the above two conditions, called boundary conditions, we get,

$$V = V_1 = Az_1 + B \quad ....(4) \quad V = V_2 = Az_2 + B \quad ...(5)$$

Let , $V_1 = 0$ at $z_1 = 0$ and $V_2 = V_a$ at $z_2 = z$

$$\boxed{V = \frac{V_a}{d} z}$$

***We find that V is a linear function of z***


Fig 1 Parallel plate capacitor

*Similarly, V as a function of x or y the solution of the Laplace's equation can be solved.*

# *Applications: Laplace's and Poisson's equation*

## *In Cylindrical coordinates: r dependent only*

Now, in cylindrical coordinates, the Laplace's equation becomes

$$\nabla^2 V = \left( \frac{1}{r}\frac{\partial}{\partial r}\left( r\frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \right) V \qquad \text{in Cylindrical}$$

***Consider V is a function of r only . Thus the Laplace's equation reduces to***

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left( r\frac{\partial V}{\partial r} \right) = 0 \qquad ------(1)$$

$$or \qquad \left( r\frac{\partial V}{\partial r} \right) = A$$

$$or \qquad V = A\ln r + B$$

From this equation, we observe that equipotential surfaces are given by <u>r = constant</u> and are cylinders. Example of the problem is that of a coaxial capacitor or coaxial cable.

# *Applications: Laplace's and Poisson's equation*

## ***In Cylindrical coordinates: r dependent only***

Let us create the boundary conditions by choosing

$V = V_a$ at $r = a$ and $V = V_b$ at $r = b$, $b > a$.

*Thus the equation becomes,*

$$V = V_a = A\, l n\, a + B \quad ; \quad V = V_b = A\, l n\, b + B$$

*Solving these two equations for A and B we get*

$$A = \frac{V_a - V_b}{\ln\left(\frac{a}{b}\right)}$$

$$B = \frac{V_b \ln a - V_a \ln b}{\ln\left(\frac{a}{b}\right)}$$

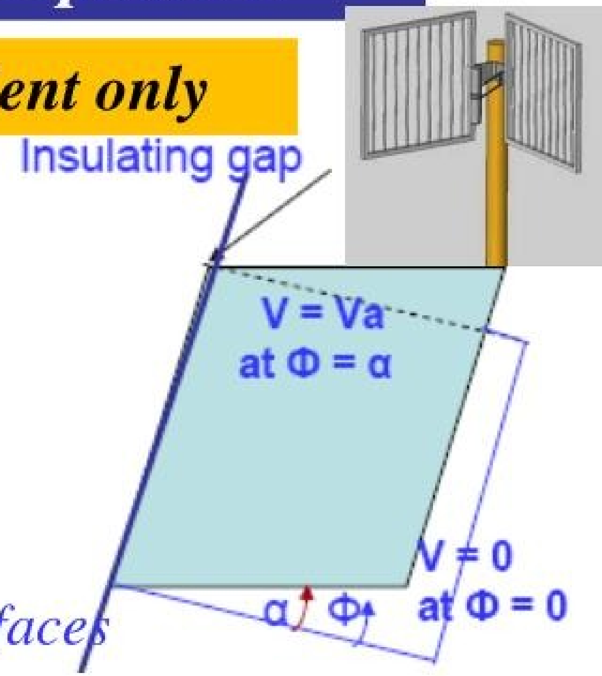
$$V = \frac{V_a - V_b}{\ln\left(\frac{a}{b}\right)} \ln r + \frac{V_b \ln a - V_a \ln b}{\ln\left(\frac{a}{b}\right)}$$

*Let the boundary at r=b be grounded then,*

$$V = V_a \frac{\ln\left(\frac{b}{r}\right)}{\ln\left(\frac{b}{a}\right)}$$

# *Applications: Laplace's and Poisson's equation*

## ***In Cylindrical coordinates: $\varphi$ dependent only***




Consider V as a function of $\Phi$ only.

$$\nabla^2 V = \left( \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) V = 0 \quad \textit{in Cylindrical}$$

$$V = A\,\phi + B$$

*From this equation, we observe that equipotential surfaces are given by $\Phi$ = constant planes,*

*Choose two such equipotential surfaces, $V = V_a$ at $\Phi = \alpha$ and $V = 0$ at $\Phi = 0$.*

*Example: corner reflector antenna ,Used in communication systems.*

$$V = V_a = A\,\alpha + B$$

$$V = 0 = A\,0 + B$$

$$\therefore B = 0 \;\; ; \quad A = \frac{V_a}{\phi}$$

Thus the general expression for V becomes

$$\boxed{V = \frac{V_a}{\alpha}\phi}$$

# *Applications: Laplace's and Poisson's equation*

## *In Spherical coordinates: r dependent only*

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

*Consider that V is a function of r only. Laplace's equation reduces to*

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0 \quad or, \quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

$$\therefore \quad r^2 \frac{\partial V}{\partial r} = A$$

$$or \quad \frac{\partial V}{\partial r} = \frac{A}{r^2}$$

*On further integration,* $V = -\dfrac{A}{r} + B$


*where A and B are arbitrary constants to be evaluated. This equation represents a family of equi – potential surfaces for r = constant.*

Let us choose two such equi-potential surfaces at r = a and r = b, b > a , such that at r = a, $V = V_a$ and at r = b, $V = V_b$

# Applications: Laplace's and Poisson's equation

## In Spherical coordinates: r dependent only

$$V_a = -\frac{A}{a} + B \quad .....(e) \qquad V_b = -\frac{A}{b} + B \quad .....(f)$$

**This is an example of concentric spheres or Spherical capacitor**

Solving these two equations , we get

$$\boxed{A = \frac{V_a - V_b}{\left(\frac{1}{b} - \frac{1}{a}\right)} \quad \text{and} \quad B = \frac{V_a \frac{1}{b} - V_b \frac{1}{a}}{\left(\frac{1}{b} - \frac{1}{a}\right)}}$$

Substituting the values of A and B in equation (d), we get,

$$\boxed{V = \frac{V_a - V_b}{\left(\frac{1}{a} - \frac{1}{b}\right)}\left(\frac{1}{r}\right) + \frac{V_b\left(\frac{1}{a}\right) - V_a\left(\frac{1}{b}\right)}{\left(\frac{1}{a} - \frac{1}{b}\right)}} \quad ....(g)$$

Let $V_b = 0$ Then equation becomes

$$\boxed{V = V_a \frac{\left(\frac{1}{r} - \frac{1}{b}\right)}{\left(\frac{1}{a} - \frac{1}{b}\right)}}$$

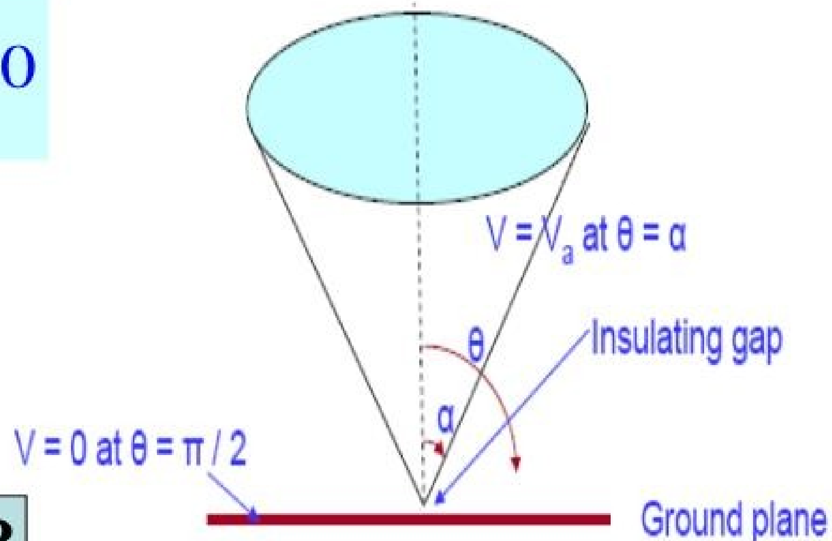
# *Applications: Laplace's and Poisson's equation*

## *In Spherical coordinates: $\theta$ dependent only*

Finally let us consider V as a function of θ only . In this case the Laplace's equation reduces to

$$\nabla^2 \mathbf{V} = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left( \sin\theta \frac{\partial V}{\partial \theta} \right) = 0$$

$$\frac{\partial V}{\partial \theta} = \frac{A}{\sin\theta}$$




Integrating once again, we get,

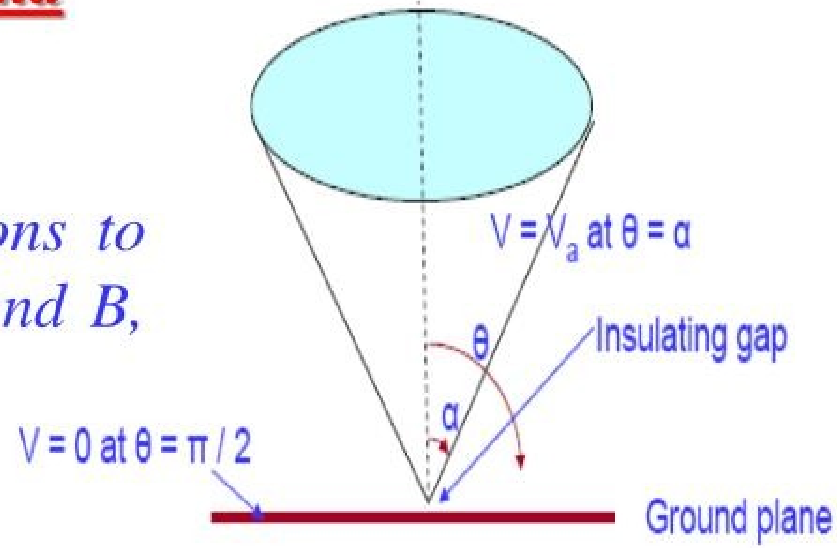
$$\boxed{V = A \ln(\tan\theta / 2) + B}$$

This equation represents a family of equi-potential surfaces for constant θ. Let us consider two such equi-potential surfaces at $\theta = \pi/2$, $V = 0$ and at $\theta = \alpha$, $V = V_a$.

# *Applications: Laplace's and Poisson's equation*

## ***In Spherical coordinates: $\theta$ dependent only***

*The equi-potential surfaces are cones as shown in figure below. Such a system is called a **conical antenna***

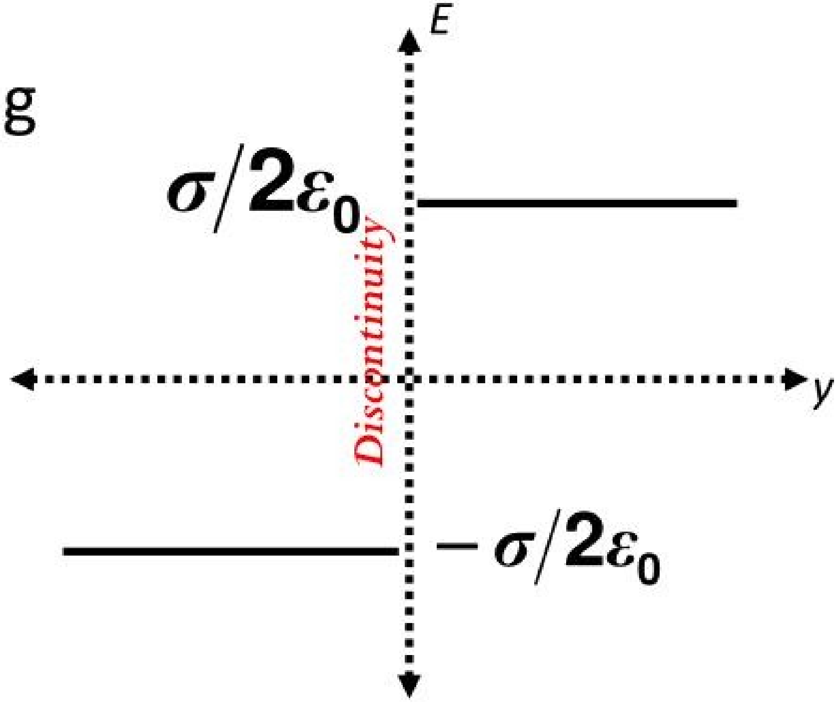


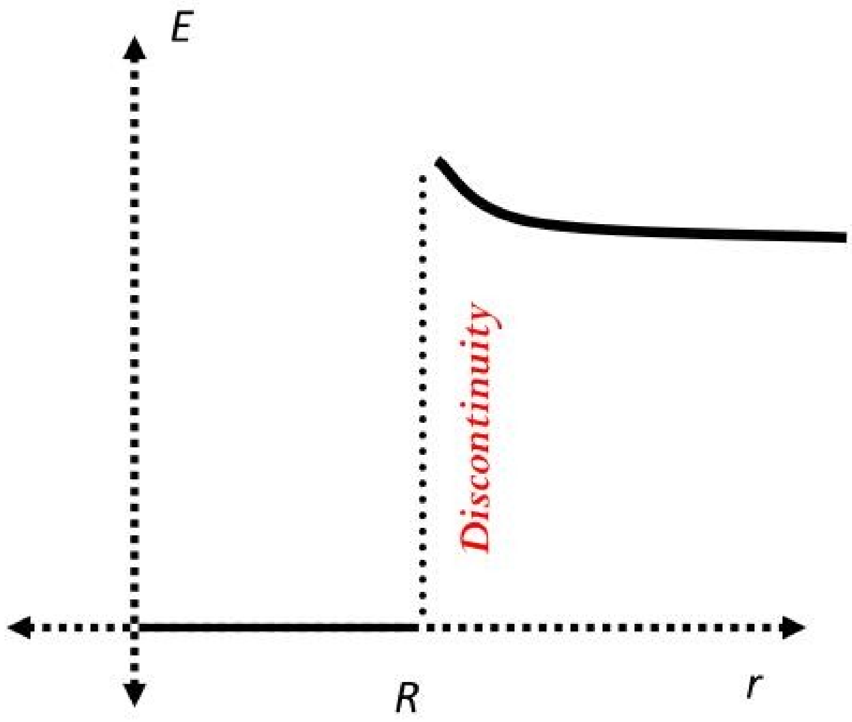

*Applying these two boundary conditions to the equation (i) and solving it for A and B, substituting these values in (i),we get,*

$$V = V_a \frac{\ln(\tan\theta/2)}{\ln(\tan\alpha/2)}$$

# Electrostatic Boundary Conditions

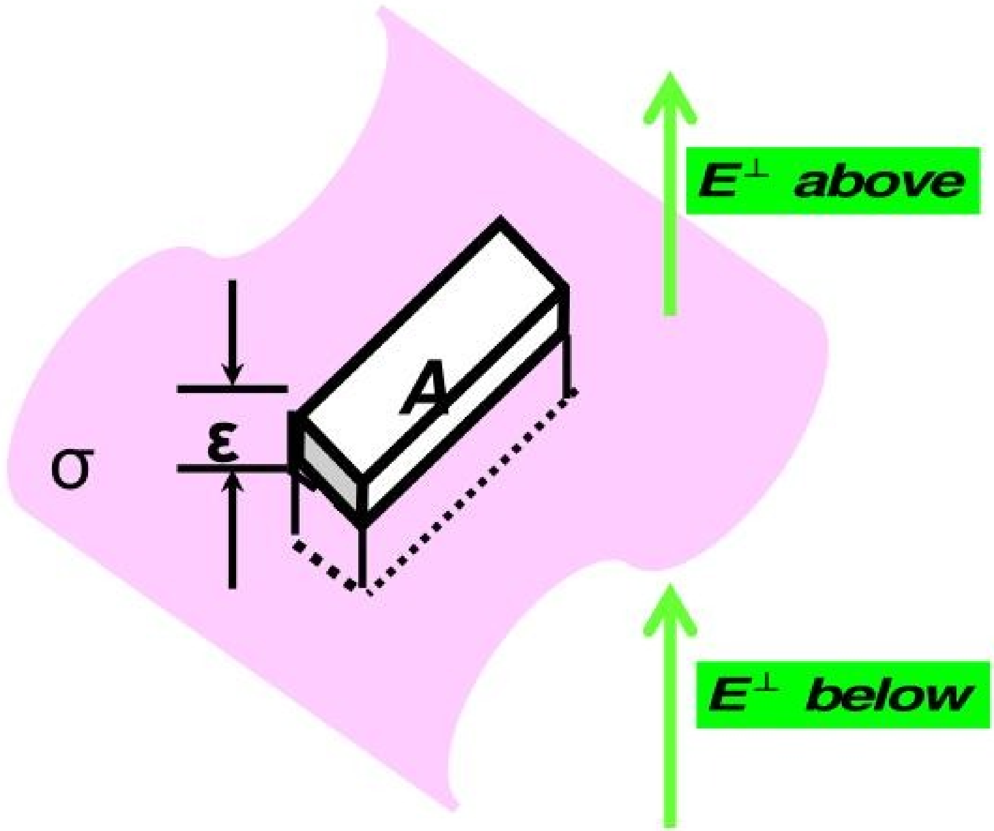
For an infinite plane carrying uniform surface charge σ




For a spherical shell of radius R, carrying uniform surface charge σ

# *Electrostatic Boundary Conditions*




From Gauss's Law

$$\oint_S \mathbf{E}\cdot d\mathbf{s} = \frac{1}{\varepsilon_0} Q_{enc} = \frac{1}{\varepsilon_0}\sigma A$$

$$E^{\perp}_{above} A - E^{\perp}_{below} A = \frac{1}{\varepsilon_0}\sigma A$$

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{1}{\varepsilon_0}\sigma$$

**Conclusion** *: The normal component of E is discontinuous by an amount* $\boldsymbol{\sigma}/\boldsymbol{\varepsilon}_0$ *at any boundary.*

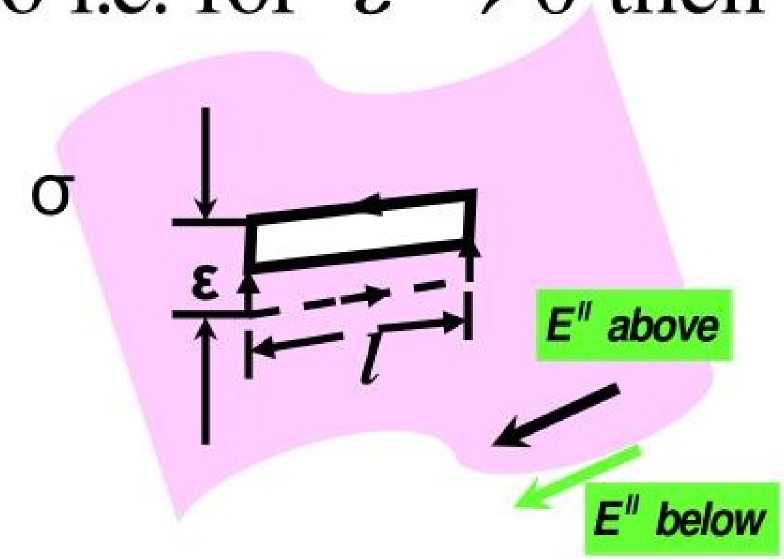
# Electrostatic Boundary Conditions

The line integral of the static electric field E around a closed path is zero i.e. for $\varepsilon \to 0$ then

$$\oint E.dl = 0$$

$$E^{\parallel}_{above} l - E^{\parallel}_{below} l = 0$$

$$E^{\parallel}_{above} = E^{\parallel}_{below}$$




The tangential component of E by contrast is always continuous.

# *Force on the surface of conductor*

***Electric Field outside a conductor***

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

*Note1: Force on a charge q (or surface charge density σ) placed in an external field E :*

$$F = qE \;\; or \;\; F = \sigma A E, \;\Rightarrow p = \frac{F}{A} = \sigma E$$

***Note 2: " But E is discontinuous across a surface charge distribution" Therefore***

***On the surface, force per unit area :***

$$P = \frac{1}{2}\sigma\left(\vec{E}_{above} + \vec{E}_{below}\right)$$

***In our case: $E_{above}=\sigma/\varepsilon_0$ and $E_{below}$ is zero. Hence Force (per unit area) on the conductor surface:***

$$P = \frac{1}{2\varepsilon_0}\sigma^2\hat{n} \;\; \text{or} \;\; P = \frac{1}{2}\varepsilon_0 E^2$$

Where P is outward electrostatic Pressure on the conductor surface.

# Prob. 2.37:

Two large metal plates (each of area *A*) are held a distance *d* apart. Suppose we put a charge *Q* on each plate, what is the electrostatic pressure on the plates ?

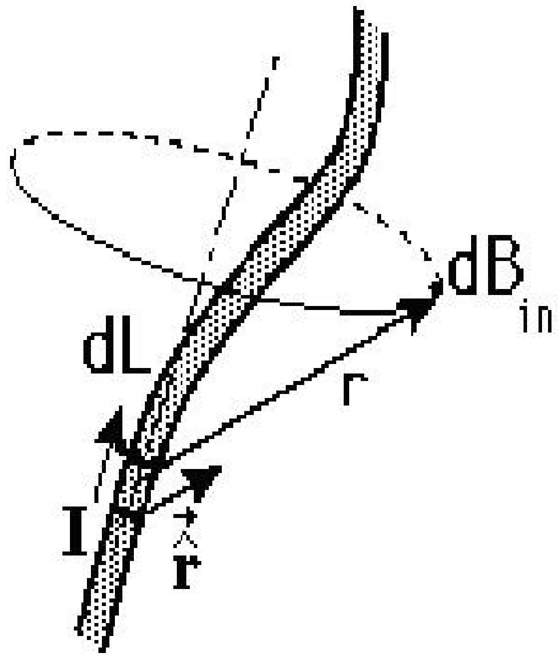
$$\boxed{P = \frac{Q^2}{2\varepsilon_0 A^2}}$$

# Assignment:

Define Biot-Savart law and Ampere's circuital law

# Biot-Savart Law

- Currents, i.e. moving electric charges, produce magnetic fields. **There are no magnetic charges**



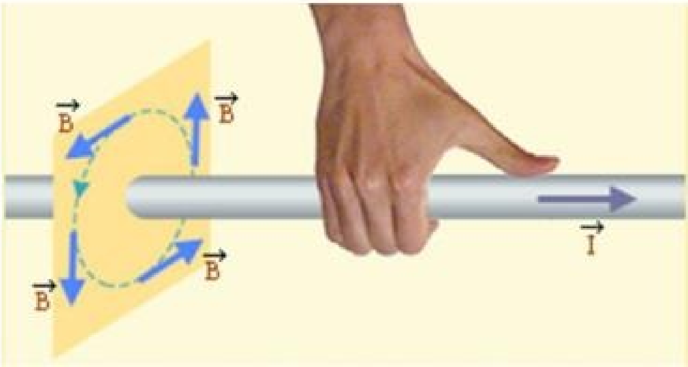

Magnetic field of a current element $\quad d\vec{B} = \dfrac{\mu_0 I d\vec{L} \times \vec{r}}{4\pi\, r^3}$

where

$d\vec{L}$ = infinitesmal length of conductor carrying electric current $\mathbf{I}$

$\vec{\mathbf{r}}$ = vector to specify direction of the vector distance $\mathbf{r}$ from the current to the field point.

$$\mu_0 = 4\pi\times10^{-7}\,\frac{Tm}{A} = 4\pi\times10^{-7}\,\frac{Weber}{Am} = 4\pi\times10^{-7}\,\frac{Henry}{m}$$



# MAGNETIC FLUX

It is defined as the magnetic lines of force produced in the medium surrounding electric currents or magnets and is expressed as surface integral of the magnetic flux density.

$$\varphi = \int_s \vec{B} \cdot d\vec{S}$$

$$\varphi = \oint \vec{B} \cdot d\vec{S} = 0 \quad \Rightarrow \nabla . B = 0$$

The unit of magnetic flux is T.m$^2$ ( *weber).*

**B** is defined as magnetic flux per unit area (Magnetic flux density) through a loop of small area

Gauss' law for magnetic fields** says there can not be a net magnetic flux through the surface since there can be no net magnetic charge enclosed by the surface. $\Rightarrow$ *magnetic monopoles do not exist.*

**** Maxwell's II equation**

*If the magnetic flux density in a medium is given by* $B = \frac{1}{r}\cos\varphi\hat{a}_r$, *what is the flux crossing the surface defined by* $-\frac{\pi}{4} < \varphi \le \frac{\pi}{4}, 0 \le z \le 2m.$

**Solution:**

$$\mathbf{B} = \frac{1}{r} cos\phi \ \hat{a}_r,$$

$$\phi = \int_s \mathbf{B} \cdot d\mathbf{S} \qquad Where \qquad d\mathbf{S} = r \ d\phi \ dz \ \hat{a}_r$$

$$\phi = \int_s \frac{1}{r} cos\phi \ \hat{a}_r \cdot rd\phi \ dz \ \hat{a}_r$$

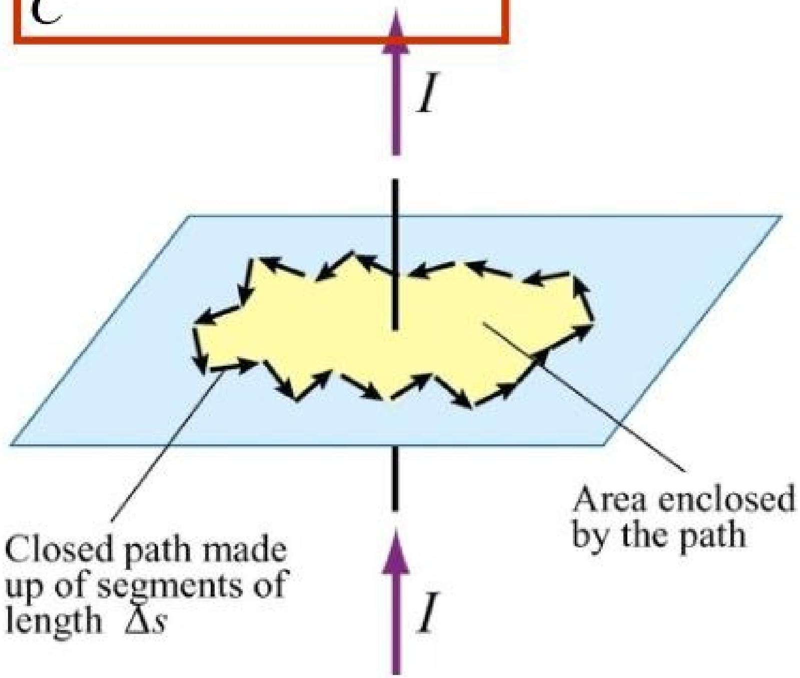
$$\varphi = \int_0^2 \int_{-\pi/4}^{\pi/4} \cos\varphi d\varphi dz$$

$\boxed{\phi = 2.83 \text{ wb}}$

# *Ampere's Circuital Law** in Integral Form*

***Ampere's Circuital Law*** *-"the circulation of the magnetic flux density in free space is proportional to the total current through the surface bounding the path over which the circulation is computed."*

$$\boxed{\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{encl}}$$




*$I_{encl}$ is current through S:*

$$I_{encl} = \oiint_S J \cdot dS$$

*Where J is defined as current density*

$$\boxed{\nabla \times \mathrm{B} = \mu_0 \mathrm{J}}$$

$$\nabla \times H = J \neq 0$$

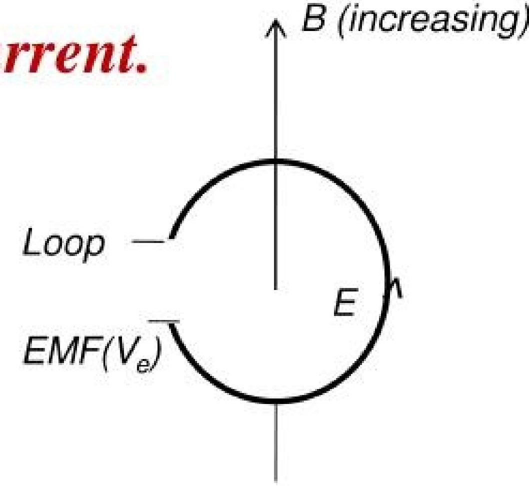
*This shows that magnetostatic field is nonconservative in nature.*

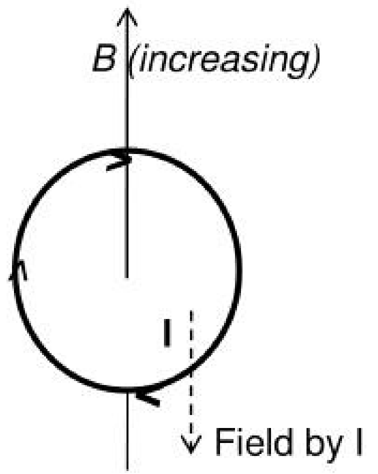
**Note: ** Ampere's circuital Law is Maxwell's IV equation.**

# 3. Faraday's Law **

***Changing magnetic field gives rise to electric current.***

*Induced emf if the loop is open-circuited* $\boxed{\varepsilon = -\frac{\partial \varphi_B}{\partial t}}$




*Induced emf in the loop is* $\boxed{\varepsilon = \oint_P E \cdot dl}$

$$\oint_P \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial \phi_B}{\partial t}$$

Integral form

$$or \;\; \nabla \times E = -\frac{\partial B}{\partial t}$$

Differential form

***Note: ** Faraday's Law is Maxwell's III equation.***

*Four fundamental Laws:*

1. $$\boxed{\oint_S E \cdot dS = \frac{Q_{enclosed}}{\varepsilon_o}}$$

2. $$\boxed{\oint_S B \cdot dS = 0}$$

3. $$\boxed{\oint_P E \cdot dl = \boxed{0} - \frac{\partial \varphi_B}{\partial t}}$$

4. $$\boxed{\oint_P B \cdot dl = \mu_0 i_{encl} + \boxed{\mu_0 \varepsilon_0 \frac{\partial \varphi_E}{\partial t}}}$$

***Need modification in ampere's law***

# Modification to Ampere's Law:

-Ampere's law must be wrong!

$$\oint \vec{B} \cdot dl = \mu_0 I_{enc}$$

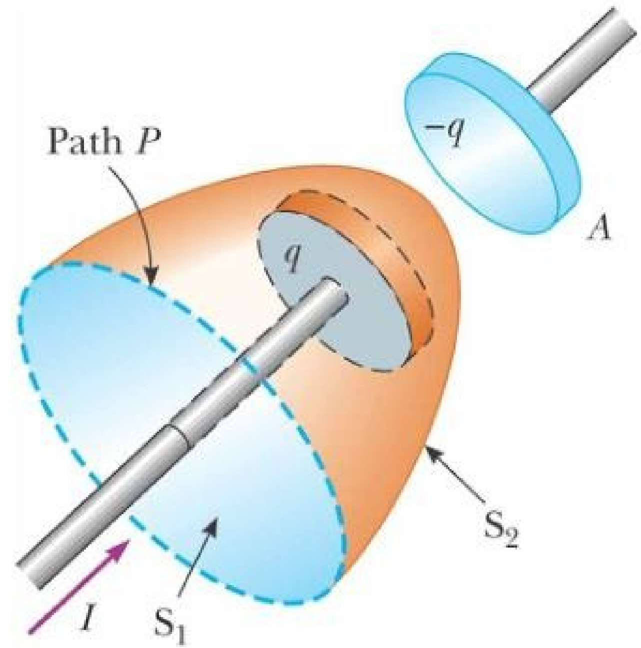
- it depends on what "enclosed" means

Surface S1 encloses a current

Surface S2 does not!

What if we moved S1 into the gap?

How can we modify the rule
to handle all situations?



# *Ampere's Law (constant currents):*

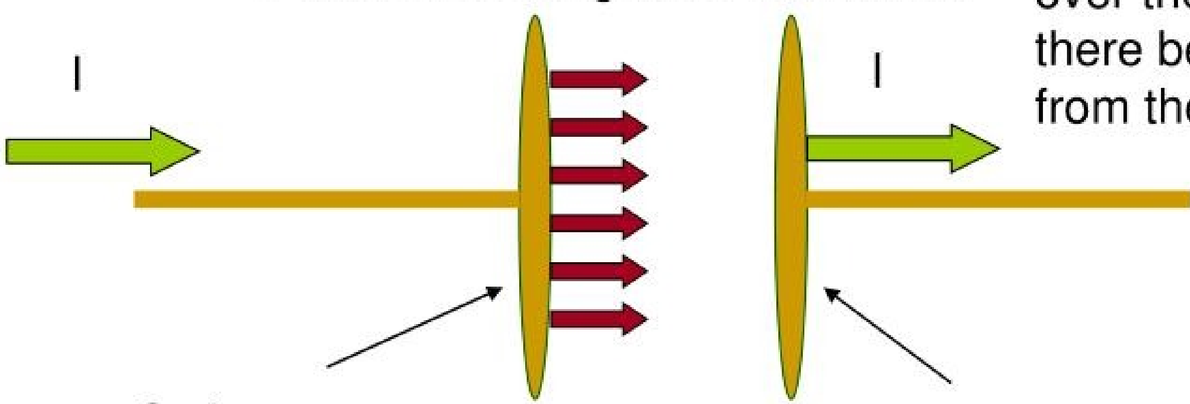
$$\oint \vec{B}.\vec{dl} = \mu_0 I_{enc}$$

Ampere's Law for constant currents.

## *What about currents which are not continuous?*
## *Displacement current in a capacitor*

The capacitor holds a charge Q over the two plates. How can there be a current emerging from the capacitor plates?

E-field increasing as Q increases!




+Q charge deposits on the plate

-Q charge induced by E-field

# Modified Ampere's Law

$$\oint \vec{B}.d\vec{l} = \mu_0\left(I_{enc} + I_d\right)$$

To see how the displacement current comes about, one has to consider the electric flux through the capacitor's plate (Gauss's Law).

$$\oint \vec{E}.d\vec{A} = \Phi_E = \frac{Q_{enc}}{\varepsilon_0} \quad \Rightarrow Q_{enc} = \varepsilon_0 \oint \vec{E}.d\vec{A}$$

Q increases on the capacitor, the electric flux also increases at the same rate.

$$\boxed{\frac{dQ_{enc}}{dt} = \varepsilon_0 \frac{d}{dt}\Phi_E = \varepsilon_0 \frac{d}{dt}\oint \vec{E}.d\vec{A} \equiv I_d}$$

# *CONTINUITY EQUATION*

From the *principle of charge conservation*, the time rate of decrease of charge with in a given volume must be equal to the net outward current flow through the surface of the volume. Thus current $I_{out}$ coming out of the closed surface is

$$I_{out} = \oint \mathbf{J} \cdot d\mathbf{S} = \frac{-dQ_{in}}{dt}$$

*$Q_{in}$ is the total charge enclosed by the closed surface.*

*Using the divergence theorem*

$$\oint_S J \cdot dS = \int_v \nabla \cdot J dv \longleftarrow$$

$$\text{and} \quad -\frac{dQ_{in}}{dt} = -\frac{d}{dt}\int_v \rho_v dv = -\int_v \frac{\partial \rho_v}{\partial t} dv \longleftarrow$$

$$\int_v \nabla \cdot \mathbf{J} dv = -\int_v \frac{\partial \rho_v}{\partial t} dv \longrightarrow \boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}}$$

***This equation is called continuity equation. It is derived from <u>principle of conservation of charge</u> and shows that there can be no accumulation of charge at any point.***

$$For\ steady\ current\ \frac{\partial \rho_v}{\partial t}=0 \qquad \Rightarrow \nabla \cdot \mathbf{J}=0$$

***Hence the total charge leaving a volume is the same as the total charge entering it.***

## Maxwell's modification of Ampere's Law

*From Ampere's Law* $\oint_P B \cdot dl = \mu_0 I_{encl}$ *where* $I_{encl} = \oint_S J \cdot dS$

*Applying Stroke's theorem to left-hand side of above equation*

$$\mu_0 I_{encl} = \oint_L \mathbf{B} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S}$$

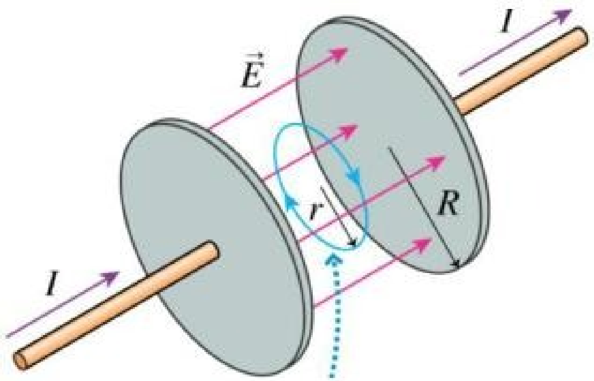
$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \mu_0 \oint_S \mathbf{J} \cdot d\mathbf{S}$$

***Differential form of Ampere's Law*** $\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$

*Taking divergence of above equation*

$$\nabla\cdot(\nabla\times\mathbf{B}) = 0 = \mu_0(\nabla\cdot\mathbf{J})$$

*Since R.H.S. of above equation is not zero**, so let us apply Continuity equation and Gauss's law (Electrostatics) we get*




$$\nabla\cdot\mathrm{J} = -\frac{\partial\rho_v}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0\nabla\cdot\mathrm{E}) = -\nabla\cdot\left(\varepsilon_0\frac{\partial\mathrm{E}}{\partial\mathrm{t}}\right)$$

$$\nabla\cdot\mathrm{J} + \nabla\cdot\left(\varepsilon_0\frac{\partial\mathrm{E}}{\partial\mathrm{t}}\right) = 0 \quad or\ \nabla.\left(\mathrm{J} + \varepsilon_0\frac{\partial\mathrm{E}}{\partial\mathrm{t}}\right) = 0$$

*Hence* $$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

*The term* $\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$ *is called as displacement current density.* $\mathbf{J}_d = \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$

*Hence modified Ampere's Law (differential form)*

$$\boxed{\nabla\times\mathbf{B} = \mu_0(\mathbf{J} + \mathbf{J}_d)}$$

---

***Note: Fundamental theorem of vector analysis;* $\nabla\cdot(\nabla\times\mathrm{B}) = 0$

***Exercise:***

***Write Maxwell's equations in integral form and obtain their differential form by using vector analysis. Write statement and physical significance of every equation.***

# *Maxwell's Equations in Integral Form*

1. $\oint_S E \cdot dS = \dfrac{Q_{enclosed}}{\varepsilon_o}$

"Gauss Law in Electrostatics" : Electric flux coming out from the surface of the body is equivalent to the charge enclosed by the body

2. $\oint_S B \cdot dS = 0$

"Gauss Law in Magnetostatics" : Magneticflux coming out from the closed surface of the body is zero as no magnetic monopole exist.

3. $\oint_L E \cdot dl = -\dfrac{\partial \varphi_B}{\partial t}$

"Faraday'sLaw":Changing Magnetic flux produce electric current (or field) in a closed loop.where Magnetic flux $\varphi_B = \oint_S \vec{B}.d\vec{s}$

4. $\oint_L B \cdot dl = \mu_0 \left( i + \varepsilon_0 \dfrac{\partial \varphi_E}{\partial t} \right)$

"Modified Ampere's Law"or "Maxwell-Ampere's Law": Changing electric flux can produce magnetic field in a discontinuous circuit to hold Ampere's circuital Law.where Electric flux $\varphi_E = \oint_S \vec{E}.d\vec{s}$

# ***Proof: Differential form***

Use equation 1 and apply Gauss divergence theorem

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \oint_V (\nabla\cdot\mathbf{E})\,dv \longrightarrow \oint_V (\nabla\cdot\mathbf{E})\,dv = \frac{Q_{enclosed}}{\varepsilon_o} = \oint_V \frac{\rho\,dv}{\varepsilon_0} \rightarrow \boxed{\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}}$$

Use equation 2 and apply Gauss divergence theorem

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = \oint_V (\nabla\cdot\mathbf{B})\,dv = 0 \qquad \boxed{\nabla\cdot\mathbf{B} = 0}$$

Use equation 3 and apply Stoke's theorem

$$\oint_P \mathbf{E}\cdot d\mathbf{l} = \oint_S (\nabla\times\mathbf{E})\cdot d\mathbf{S}$$

$$\longrightarrow \oint_S (\nabla\times\mathrm{E})\cdot d\mathrm{S} = -\frac{\partial\phi}{\partial t} = -\oint_S \frac{\partial \mathrm{B}}{\partial t}\cdot d\mathrm{S} \longrightarrow \boxed{\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}}$$

Use equation 4 and apply Stoke's theorem

$$\oint_P \mathbf{B} \cdot d\mathbf{l} = \oint_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S}$$

$$\oint_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \mu_0 \left( JA + \varepsilon_0 \frac{\partial \phi_E}{\partial t} \right)$$

$$\oint_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \mu_0 \left( JA + \varepsilon_0 \frac{\partial (EA)}{\partial t} \right)$$

$$\oint_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \oint_S \mu_0 \left( \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{S} \qquad Where\ A = \int dS$$

$$\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \qquad Since\ \mathbf{D} = \varepsilon_0 \mathbf{E} \qquad \boxed{\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right)}$$

Curl of **B** is due to current flow and a changing electric field.

# Maxwell's Equations with Physical Interpretation

1. Relates net electric flux to net enclosed electric charge.

$$\boxed{\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{enclosed}}{\varepsilon_o}}$$

$$\boxed{\nabla \cdot E = \frac{\rho_v}{\varepsilon_0}}$$

2. Relates net magnetic flux to net enclosed magnetic charge.

$$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{S} = 0}$$

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

3. Relates induced electric field to changing magnetic flux.

$$\boxed{\oint_P \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial \phi_B}{\partial t}}$$

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

4. Relates induced magnetic field to changing electric flux to and to current.

$$\boxed{\oint_P \mathbf{B} \cdot d\mathbf{l} = \mu_0 \left( i + \varepsilon_0 \frac{\partial \phi_E}{\partial t} \right)}$$

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right)}$$

# Case 1: Maxwell equations in free space* (no free charges and no currents)

$$q = 0 \Rightarrow \rho = 0, \quad i = 0 \Rightarrow J = 0,$$

$$\nabla \cdot E = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

* Helpful to understand Electromagnetic waves in free space.

# Electromagnetic waves in free space or vacuum

1. $\nabla\cdot\mathbf{E}=0$

2. $\nabla\cdot\mathbf{B}=0$

3. $\nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}$

4. $\nabla\times\mathbf{B}=\mu_0\varepsilon_0\dfrac{\partial\mathbf{E}}{\partial t}$

Take curl of equation 3

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\times(\nabla\times\mathbf{E})=\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}$$

$$-\nabla^2\mathbf{E}=\nabla\times\left(-\frac{\partial\mathbf{B}}{\partial t}\right)$$

$$-\nabla^2\mathbf{E}=-\frac{\partial}{\partial t}(\nabla\times\mathbf{B})$$

$$=-\mu_o\varepsilon_o\frac{\partial^2\mathbf{E}}{\partial t^2}$$

$$\nabla^2\mathbf{E}=\mu_o\varepsilon_o\frac{\partial^2\mathbf{E}}{\partial t^2}$$

$$\nabla\times\mathbf{B}=\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\nabla\times(\nabla\times\mathbf{B})=\nabla(\nabla\cdot\mathbf{B})-\nabla^2\mathbf{B}$$

$$-\nabla^2\mathbf{B}=\nabla\times\left(\mu_o\varepsilon_o\frac{\partial\mathbf{E}}{\partial t}\right)$$

$$-\nabla^2\mathbf{B}=\mu_o\varepsilon_o\frac{\partial}{\partial t}(\nabla\times\mathbf{E})$$

$$=-\mu_o\varepsilon_o\frac{\partial^2\mathbf{B}}{\partial t^2}$$

$$\nabla^2\mathbf{B}=\mu_o\varepsilon_o\frac{\partial^2\mathbf{B}}{\partial t^2}$$

$$\nabla^2 \mathbf{E} = \mu_o \varepsilon_o \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \mathbf{B} = \mu_o \varepsilon_o \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

*Standard Wave equation:*

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

$$\frac{1}{v^2} = \mu_o \varepsilon_o \qquad v = \frac{1}{\sqrt{\mu_o \varepsilon_o}}$$

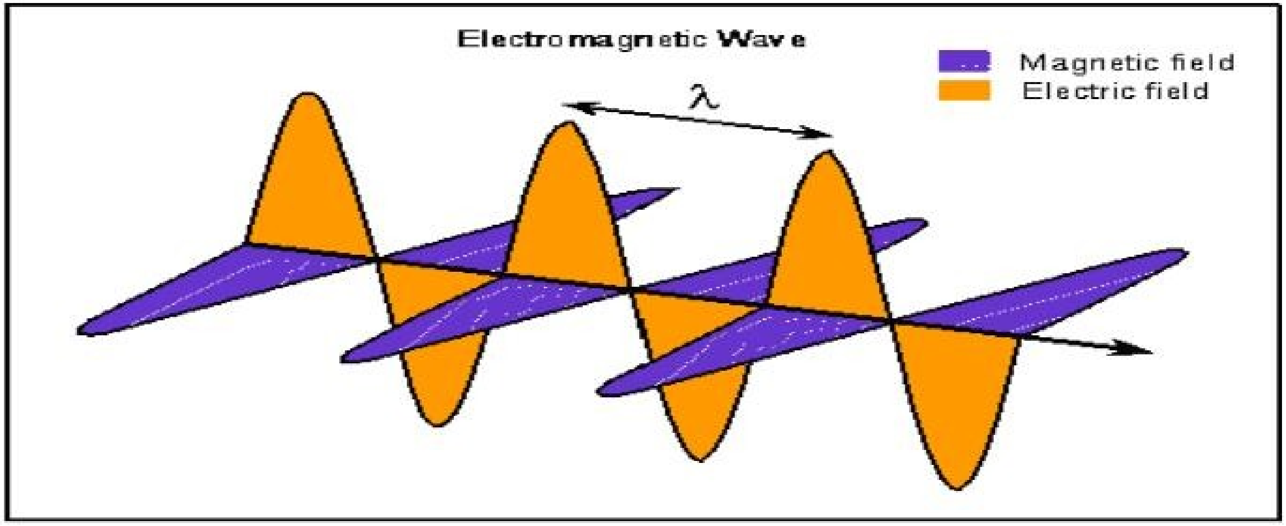
$$\mu_o = 4\pi \times 10^{-7} \text{ weber / amp} - \text{m}$$

$$\varepsilon_o = 8.542 \times 10^{-12} \text{ Farad / m}$$

$$\text{v} = 3 \times 10^8 \text{ m / sec} = \text{c}$$

***Thus we conclude that light is electromagnetic in nature with electric vector E and magnetic vector B oscillating as a wave and propagating with a velocity of light in free space .***

***Maxwell's equations also imply that empty space supports the propagation of electromagnetic waves, traveling with speed of light.***




- *Derivation of Electromagnetic wave equation*
- *Properties of E.M. Waves.*
  - ***E is prependicular to k (direction of propagation); k.E=0***
  - ***B is prependicular to k (direction of propagation); k.B=0***
  - ***E is prependicular to B too;*** $k \times E = ?$
  - ***Wave Impedance*** $E_0/B_0 = c$

- *Poynting Theorem.*

# Solution of Electromagnetic waves in free space

$$\nabla^2 \mathbf{E} = \mu_o \varepsilon_o \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \mathbf{B} = \mu_o \varepsilon_o \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

The harmonic solutions to the wave equations

$$\begin{cases} E = E_o e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \\ B = B_o e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \end{cases}$$

Where, $k=2\pi/\lambda$ is propagation vector,
$\omega=2\pi\nu$ is angular frequency

where $E_o$ and $B_o$ (amplitudes of wave) space and time independent vectors but may in general be complex.

$$\vec{E}(r,t) = \mathbf{E}_o e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

$$\vec{E}_o = E_{ox}\hat{a}_x + E_{oy}\hat{a}_y + E_{oz}\hat{a}_z$$

$$\vec{k} = k_x\hat{a}_x + k_y\hat{a}_y + k_z\hat{a}_z$$

$$\vec{r} = x\hat{a}_x + y\hat{a}_y + y\hat{a}_z$$

*Prove-Transverse Nature*

*k is perpendicular to E and B.*

$k \cdot E = 0$ *or* $k \cdot B = 0$

*and E, B and k are orthogonal*

1. $\nabla \cdot \mathbf{E} = 0$
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
4. $\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

$$\nabla.E = (\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial x} + \hat{a}_z \frac{\partial}{\partial x})E_o e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$\vec{k}.\vec{r} = (k_x\hat{a}_x + k_y\hat{a}_y + k_z\hat{a}_z).(x\hat{a}_x + y\hat{a}_y + y\hat{a}_z)$$

$$= k_x x + k_y y + k_z z$$

$$\nabla.E = i(k_x E_{ox} + k_y E_{oy} + k_z E_{oz})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$= i(k_x\hat{a}_x + k_y\hat{a}_y + k_z\hat{a}_z).(E_{ox}\hat{a}_x + E_{oy}\hat{a}_y + E_{oz}\hat{a}_z)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$= i\vec{k}.\vec{E}$$

**Similarly,**

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{B} = \mathrm{i}\mathbf{k} \cdot \mathbf{B}_\mathrm{o} \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega \mathrm{t})}$$

$$= \mathrm{i}\mathbf{k} \cdot \mathbf{B} = \mathbf{0}$$

*Now Prove*

*E, B and k are orthogonal*

Proof: $$\nabla \times \mathbf{E} = i(\mathbf{k} \times \mathbf{E}) \; and, \; \frac{\partial \mathbf{B}}{\partial t} = -i\omega \mathbf{B}$$

$$\boxed{i\mathrm{k} \times \mathrm{E} = i\omega \mathrm{B}}$$

$$\nabla \times E = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

where, $\mathrm{E}_x = \mathrm{E}_{ox} e^{i(\bar{k}.\bar{r} - \omega t)}$

$\mathrm{E}_y = \mathrm{E}_{oy} e^{i(\bar{k}.\bar{r} - \omega t)}$

$\mathrm{E}_z = \mathrm{E}_{oz} e^{i(\bar{k}.\bar{r} - \omega t)}$

$$\nabla \times E = \hat{a}_x \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) - \hat{a}_y \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) + \hat{a}_z \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

$$\nabla \times E = \hat{a}_x \left( ik_y E_z - ik_z E_y \right) - \hat{a}_y \left( ik_x E_z - ik_z E_x \right) + \hat{a}_z \left( ik_x E_y - ik_y E_x \right)$$

$$= ik \times E \text{ (L.H.S)}$$

Solving for R.H.S.

$$\frac{\partial \mathrm{B}}{\partial t} = -i\omega \mathrm{B}$$

Similarly from equation 4, we can show that

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\boxed{i\mathbf{k} \times \mathbf{B} = -i\mu_0 \varepsilon_0 \omega \mathbf{E}}$$

Since

$\bar{B} = \mu_0 \bar{H}$

$\bar{D} = \varepsilon_0 \bar{E}$

or

$$\boxed{\mathrm{k} \times \mathrm{H} = -\omega \mathrm{D}}$$

i.e. E is perpendicular to the Plane formed by k and B !!

Thus, k, E and B vectors are mutually perpendicular to each other.

*Refractive index of the medium is defined as*

$$\boxed{n = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = \sqrt{\mu_r\varepsilon_r}}$$

*In a dielectric medium,* $\boxed{\mu = \mu_r\mu_o}$ $\boxed{\varepsilon = \varepsilon_r\varepsilon_o}$

*Since dielectric is non magnetic,* $\boxed{\mu_r = 1}$

$$\boxed{n = \sqrt{\varepsilon_r}}$$

$$\boxed{v = \frac{1}{\sqrt{\mu_o\varepsilon}} = \frac{1}{n\sqrt{\mu_o\varepsilon_o}} = \frac{c}{n}}$$

*That is the speed of electromagnetic waves in an isotropic dielectric is less than the speed of electromagnetic waves in free space.*

*Exercise: The relative permittivity of distilled water is 81. Calculate refractive index and velocity of light in it .*

## Assignment : Obtain Algebraic form of Maxwell's Equations in free space

$\vec{k}.\vec{E}=0$ $\quad$ *(i.e. $\vec{E}$ is perpendicular to $\vec{k}$)*

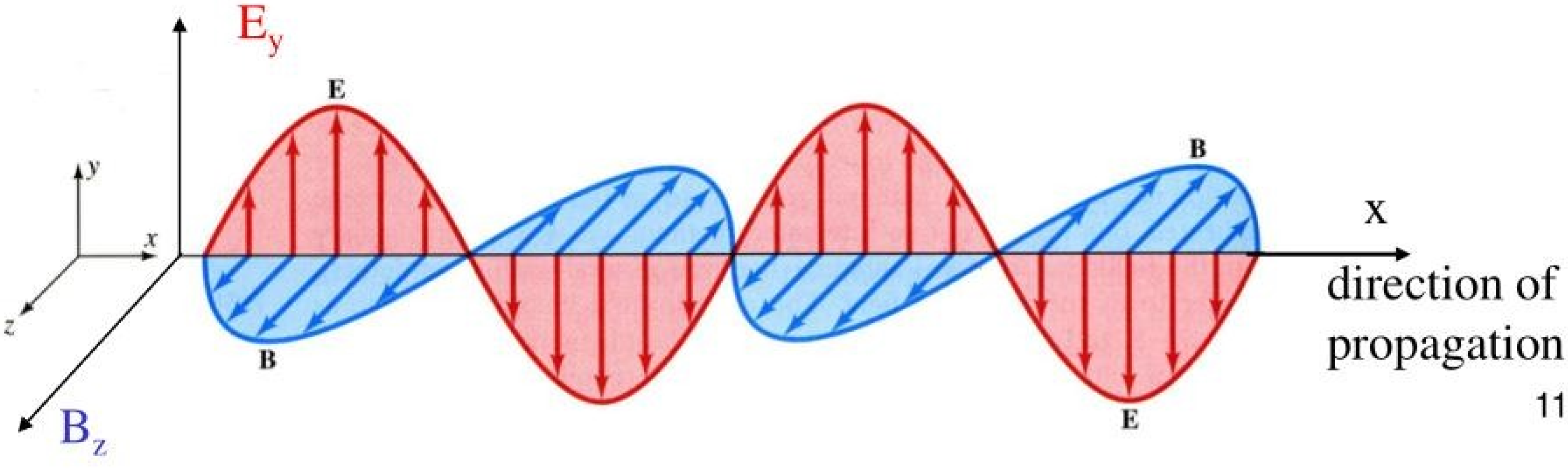
$\vec{k}.\vec{B}=0$ $\quad$ *(i.e. $\vec{B}$ is perpendicular to $\vec{k}$)*

$$\left.\begin{array}{l}\vec{k}\times\vec{E}=\omega\vec{B}\\ \\ \vec{k}\times\vec{B}=-\varepsilon_o\mu_o\omega\vec{E}\end{array}\right\}(i.e.\ \vec{E},\ \vec{B}\ and\ \vec{k}\ are\ orthoganal)$$

# Plane Waves

we conclude that electromagnetic field vectors **E** and **B** are both perpendicular to the direction of propagation vector **k**. This shows that Electromagnetic waves are transverse waves, but are not mechanical waves (they do not need medium to vibrate in).

*Therefore, electromagnetic waves can propagate in free space.*

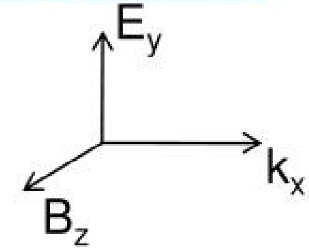


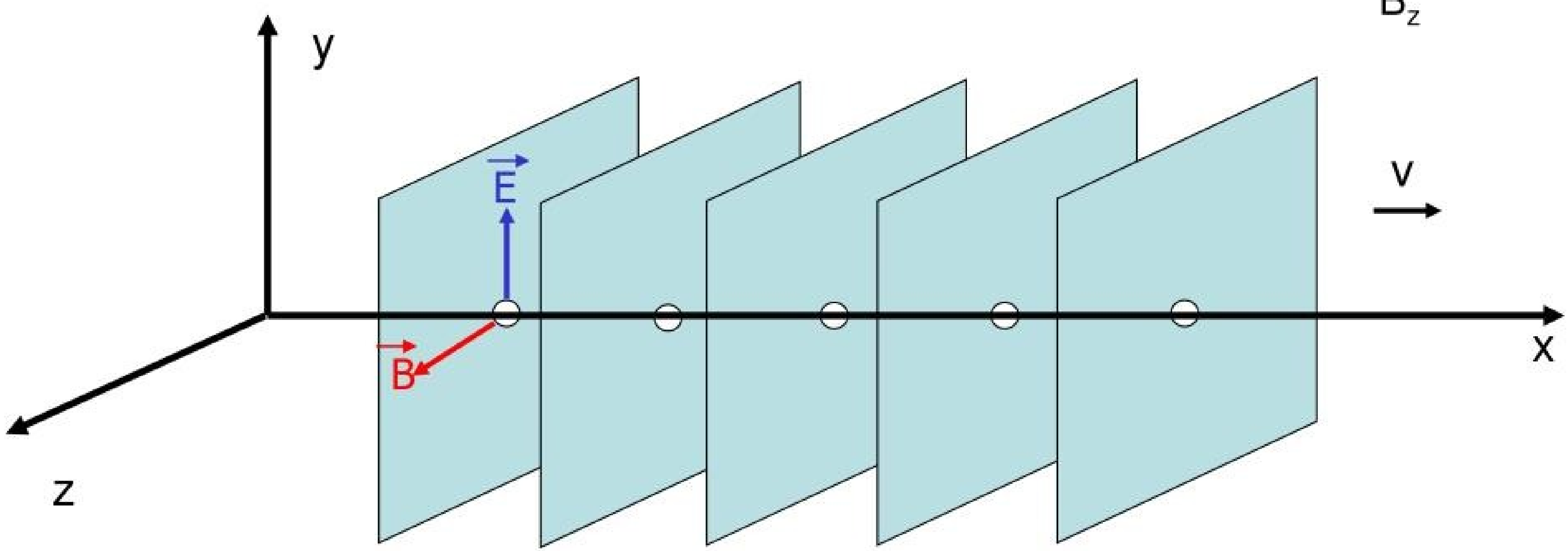
# Monochromatic Plane Waves

Plane wave because the fields are uniform over every plane perpendicular to the direction of propagation (i.e. $x$ = constant plane) as shown in the figure below.

$$E_y = E_o e^{i(kx-\omega t)} \qquad E_y = E_o \cos(kx-\omega t) \qquad E_y = E_o \sin(kx-\omega t)$$

$$B_z = B_o e^{i(kx-\omega t)} \quad \text{or} \quad B_z = B_o \cos(kx-\omega t) \quad \text{or} \quad B_z = B_o \sin(kx-\omega t)$$



## Show that E and B of plane wave are in same phase at any time in space

$$E_y = E_o e^{i(kx-\omega t)} \qquad B_z = B_o e^{i(kx-\omega t)}$$

From equation 3 $\quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \Rightarrow \quad \mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$

$$\omega = 2\pi\upsilon, \quad k = \frac{2\pi}{\lambda}$$

Since $\left.\begin{matrix} \mathbf{E} \to \mathbf{E_y} \\ \mathbf{B} \to \mathbf{B_z} \\ \mathbf{k} \to \mathrm{k}_\mathbf{x} \end{matrix}\right\} \Rightarrow \left.\begin{matrix} k_x E_y = \omega B_z \\ E_y = \frac{\omega}{k_x} B_z \end{matrix}\right\} \Rightarrow \boxed{E_y = cB_z} \Leftarrow \frac{\omega}{k} = \frac{2\pi\upsilon}{\frac{2\pi}{\lambda}} = \upsilon\lambda = c$

But $\left.\begin{matrix} E_y = E_o e^{i(kx-\omega t)} \\ B_z = B_o e^{i(kx-\omega t)} \end{matrix}\right\} \Rightarrow \boxed{\mathrm{B_o} = \frac{\mathrm{E_o}}{\mathrm{c}}}$

$$\boxed{\frac{E_o}{B_o} = \frac{\omega}{k} = c = \frac{1}{\sqrt{\mu_o \varepsilon_o}}}$$

Since $\omega/\boldsymbol{k}$ is a real number, the electric and magnetic vectors are in phase; thus if at any instant, E is zero then B is also zero, similarly when E attains its maximum value, B also attains its maximum value, etc.

$$\frac{E_o}{H_o} = \frac{E_o}{B_o}\mu_o = \mu_o c = \frac{\mu_o}{\sqrt{\mu_o \varepsilon_o}} = \sqrt{\frac{\mu_o}{\varepsilon_o}} = \eta_o = 376.72\Omega$$

Here $\eta_o$ is universal const. and called as characterstic or intrinsic wave impedance of the free space.

**Both $E_y$ and $B_z$ are in same phase.**

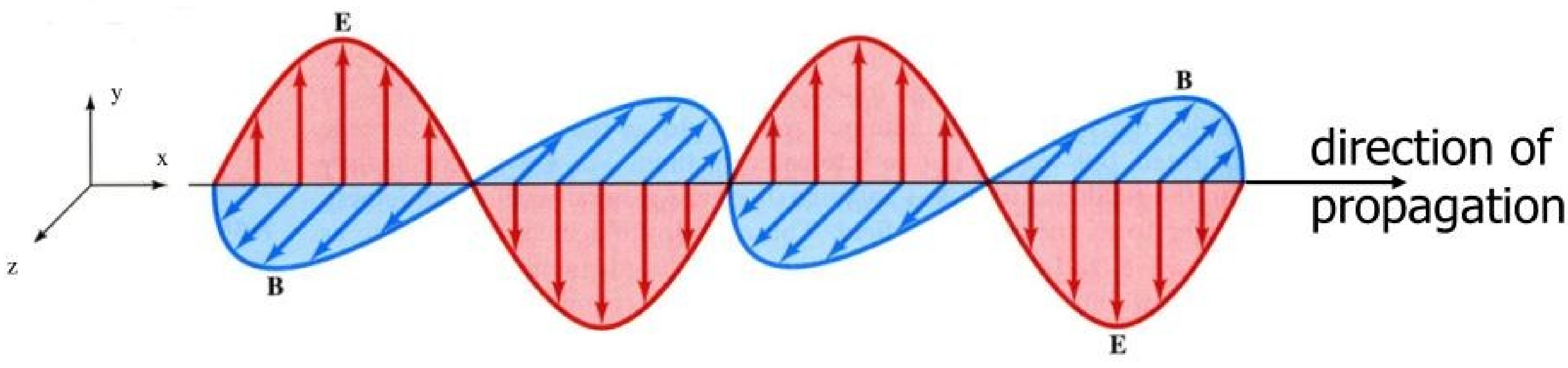
# **Summary** of Important Properties of Electromagnetic Waves

→ The solutions (plane wave) of Maxwell's equations are wave-like with both E and B satisfying a wave equation.

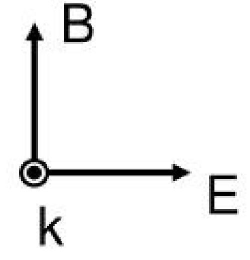
$$\nabla^2 f = \mu_o \varepsilon_o \frac{\partial^2 f}{\partial t^2} \qquad \rightarrow \qquad \begin{aligned} E_y &= E_o \cos(kx - \omega t) \\ B_z &= B_o \cos(kx - \omega t) \end{aligned}$$

→ Electromagnetic waves travel through empty space with the speed of light $c = 1/(\mu_0 \varepsilon_0)^{1/2}$.

→ The plane wave as represented by above is said to be linearly polarized because the electric vector is always along y-axis and, similarly, the magnetic vector is always along z-axis.



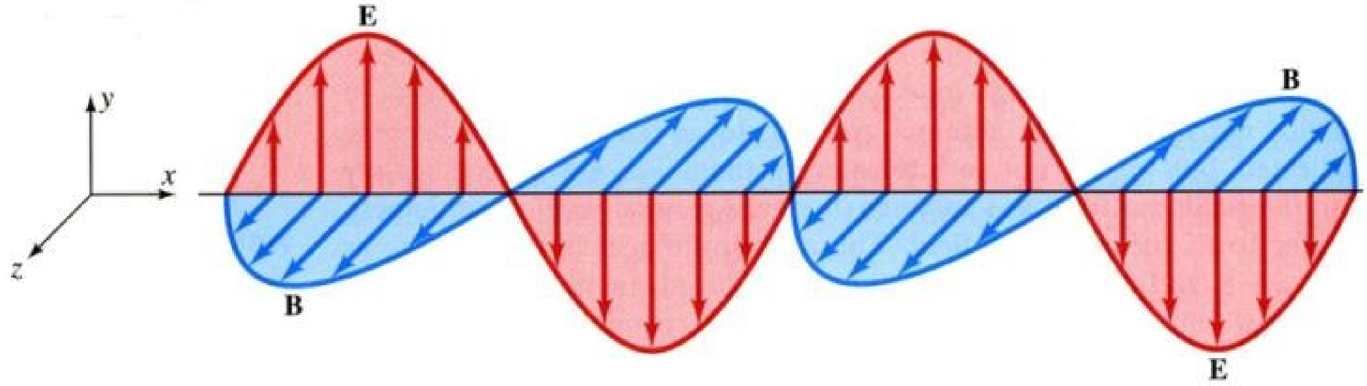
The components of the electric and magnetic fields of plane EM waves are perpendicular to each other and perpendicular to the direction of wave propagation. The latter property says that EM waves are transverse waves.




The magnitudes of **E** and **B** in empty space are related by E/B = c.

$$\frac{E_o}{B_o} = \frac{E}{B} = \frac{\omega}{k} = c$$

Mutual generation of electric and magnetic fields result in the propagation of the EM waves.




direction of propagation

# Numerical

In free space the Electric field is given as

$$\vec{E} = 10\sin(2x - 100t)\,\hat{j}.$$

Determine D, B and H by using Maxwell's equations.

Sol: Wave is propagating along x direction.

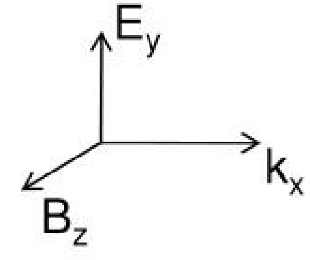
(1) $\vec{D} = \varepsilon_0 \vec{E} = 10\varepsilon_0 \sin(2x - 100t)\,\hat{j}$

(2) $$Using\quad \nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t},\ or\quad \bar{k} \times \bar{E} = \omega \bar{B}$$

$$20\,Cos(2x - 100t)\,\hat{k} = -\frac{\partial B}{\partial t},$$

$$\boxed{\vec{B} = \frac{1}{5} Sin(2x - 100t)\,\hat{k}}$$

(3) $$\vec{H} = \frac{\vec{B}}{\mu_0} = \frac{1}{5\mu_0} Sin(2x - 100t)\,\hat{k}$$



# Energy in EM Waves: Poynting Theorem and Poynting Vector

# *Energy Carried by Electromagnetic Waves*

*Electromagnetic waves carry energy, and as they propagate through space they can transfer energy to objects in their path. The rate of flow of energy in an electromagnetic wave is described by a vector* ***S*****,** *called the* ***Poynting vector*****.***

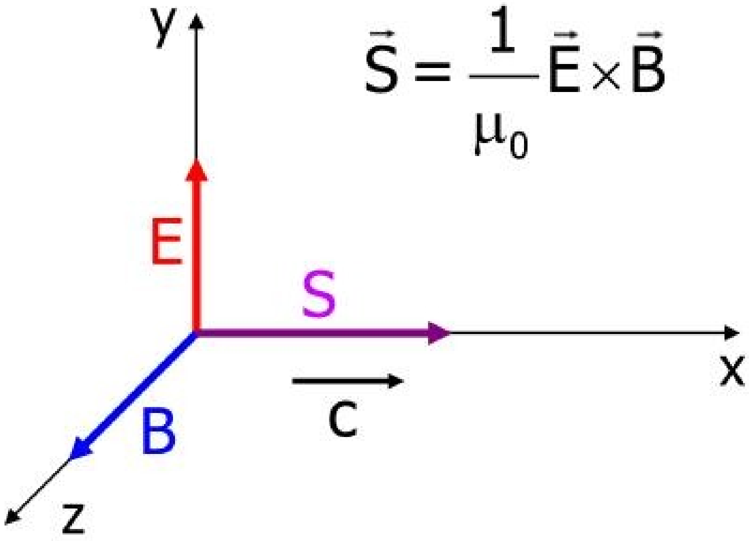
$$\boxed{\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}}$$

The magnitude S represents the rate at which energy flows through a unit surface area perpendicular to the direction of wave propagation.

Thus, **S** represents **power per unit area**. The direction of S is along the direction of wave propagation. The units of S are $\mathbf{J/(s \cdot m^2) = W/m^2}$.

*J. H. Poynting, 1884.

# *Energy Carried by Electromagnetic Waves*




$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \quad \text{or} \quad S = \frac{EB}{\mu_0} \quad \text{or} \quad \boxed{S = \frac{E^2}{\mu_0 c} = \frac{cB^2}{\mu_0}.}$$

Because B = E/c *for EM waves*

*EM waves are sinusoidal*

$$E_y = E_o cos(kx - \omega t)$$

$$B_z = B_o cos(kx - \omega t)$$

The average of S over one or more cycles is called the **wave intensity I**.

The time average of $\cos^2(kx - \omega t)$ is 1/2, so

$$I = S_{average} = \langle S \rangle = \frac{E_o B_o}{2\mu_o} = \frac{E_o^2}{2\mu_o c} = \frac{cB_o^2}{2\mu_o} = \frac{c\mu_o H_o^2}{2} = \frac{c\varepsilon_o E_o^2}{2}$$

# *Energy Density*

The **energy densities** (energy per unit volume) associated with electric field and magnetic fields are:

$$u_E = \frac{1}{2}\varepsilon_0 E^2 \qquad\qquad u_B = \frac{1}{2}\frac{B^2}{\mu_0}$$

Using $B = E/c$ and $c = 1/(\mu_0\varepsilon_0)^{½}$ we can write

$$\boxed{u_B = \frac{1}{2}\frac{B^2}{\mu_0} = \frac{1}{2}\frac{\left(E/c\right)^2}{\mu_0} = \frac{1}{2}\frac{\mu_0\varepsilon_0 E^2}{\mu_0} = \frac{1}{2}\varepsilon_0 E^2 = u_E}$$

Conclusion: For an electromagnetic wave, the instantaneous energy density associated with the magnetic field equals the instantaneous energy density associated with the electric field.

Hence, in a given volume the energy is equally shared by the two fields.

The **total energy density** is equal to the sum of the energy densities associated with the electric and magnetic fields:

$$u = u_B + u_E = \varepsilon_0 E^2 = \frac{B^2}{\mu_0}$$

# *Relation between S and U*

When we average this instantaneous energy density over one or more cycles of an electromagnetic wave, we again get a factor of ½ from the time average of $\cos^2(kx - \omega t)$.

$$\langle u_E \rangle = \frac{1}{4}\varepsilon_0 E_{max}^2 \ , \ \langle u_B \rangle = \frac{1}{4}\frac{B_{max}^2}{\mu_0}, \text{ and } \boxed{\langle u \rangle = \frac{1}{2}\varepsilon_0 E_{max}^2 = \frac{1}{2}\frac{B_{max}^2}{\mu_0}}$$

Recall $S_{average} = \langle S \rangle = \dfrac{1}{2}\dfrac{E_{max}^2}{\mu_0 c} = \dfrac{1}{2}\dfrac{B_{max}^2}{\mu_0}c \quad \Rightarrow \quad \boxed{\langle S \rangle = c\langle U \rangle.}$

*Hence* $\boxed{\textit{Energy flux} = \textit{energy density} \times c}$

*The intensity of an electromagnetic wave equals the average energy density multiplied by the speed of light.*

# *Momentum Transport- Radiation Pressure*

*Relativistic energy and Momentum*

$$E^2 = (pc)^2 + \left(m_o c^2\right)^2$$

*Energy* *momentum* *rest mass energy*

*For light* $m_o = 0$

$$\boxed{E = U = pc}$$

*When em wave strikes a surface , its momentum changes. Rate of change of momentum is equal to Force. This force acting per unit area of the surface exert a pressure, called*

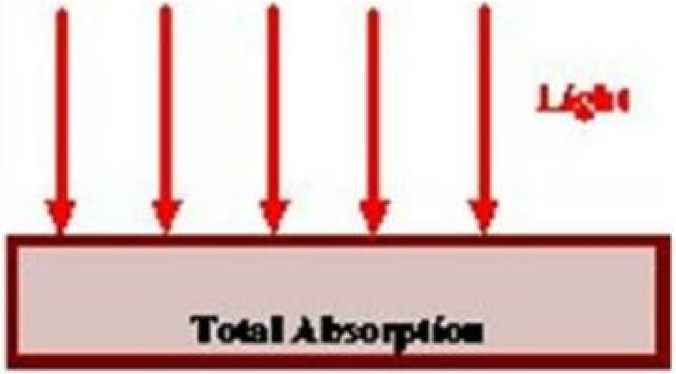
**RADIATION PRESSURE**

$$\frac{1}{A}\frac{dp}{dt} = \frac{1}{A}\frac{d}{dt}\left(\frac{U}{c}\right) = \frac{1}{c}\left(\frac{dU/dt}{A}\right) = \frac{1}{c}S$$

## Total Absorption

$$F = \frac{dp}{dt} = \frac{1}{c}\frac{dU}{dt} = \frac{1}{c} IA$$

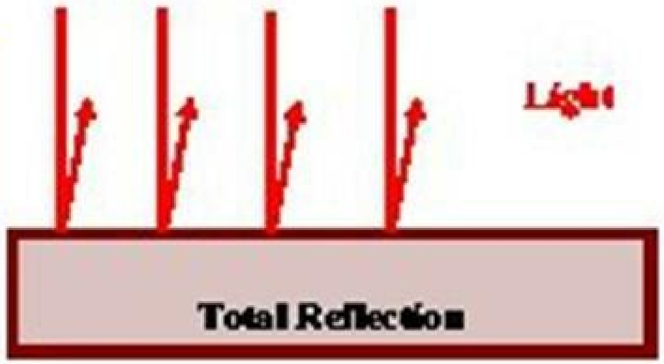
$$\boxed{P = \frac{F}{A} = \frac{1}{c} I} \qquad (radiation\ pressure)$$




## Total Reflection

$$F = \frac{dp}{dt} = \frac{2}{c}\frac{dU}{dt} = \frac{2}{c} IA$$

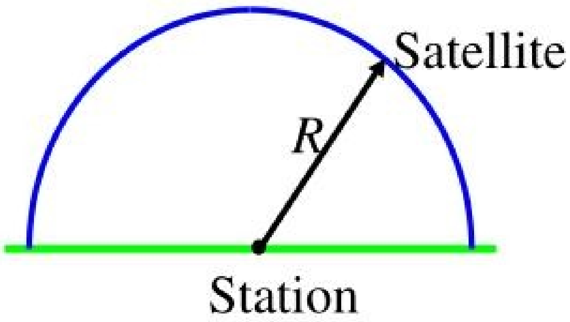
$$\boxed{P = \frac{F}{A} = \frac{2}{c} I} \qquad (radiation\ pressure)$$




**1 pascal (Pa) = 1 N/m$^2$ = 1 kg/(m·s$^2$)**

Ex: a radio station on the surface of the earth radiates a sinusoidal wave with an average total power of 50 kW. Assuming the wave is radiated equally in all directions above the ground, find the amplitude of the electric and magnetic fields detected by a satellite 100 km from the antenna.

All the radiated power passes through the **hemispherical surface*** so the average power per unit area (the intensity) is




$$I = \left(\frac{\text{power}}{\text{area}}\right)_{av} = \frac{P}{2\pi R^2} = 7.96\times 10^{-7}\ \text{W/m}^2$$

$$I = \langle S\rangle = \frac{1}{2}\frac{E_{max}^2}{\mu_0 c} \Rightarrow E_{max} = \sqrt{2\mu_0 c I} = 2.45\times 10^{-2}\, V/m$$

$$B_{max} = \frac{E_{max}}{c} = 8.17\times 10^{-11}\,\text{T}$$

Exercise: calculate the average energy densities associated with the electric and magnetic field.

$$\langle u_B\rangle = \langle u_E\rangle = 1.33\times 10^{-15}\,\text{J/m}^3$$

*In problems like this you need to ask whether the power is radiated into all space or into just part of space.

# Poynting's Theorem
# "Conservation of Energy"

***"The work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed out through the surface"***

$$\boxed{\int_V \left(\underline{E}\cdot\underline{J}_i\right)dv = -\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2\right)dv-\oint_S\left(\underline{E}\times\underline{H}\right)\cdot d\,\underline{s}}$$

Poynting Theorem
Chapter 8, Page 346, Griffith

# Physical Interpretation of the Terms in Poynting's Theorem

- Hence, the terms

$$\int_V \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right) dv$$

represent the ***total electromagnetic energy stored*** in the volume *V*.

- The term

$$\oint_S \left(\overline{E} \times \overline{H}\right) \cdot d\overline{s}$$

represents ***the flow of instantaneous power*** out of the volume *V* through the surface *S*.

- The term

$$\int_V \left(\underline{E} \cdot \underline{J}\right) dv$$

represents the *total electromagnetic energy generated (Rate of work done) by the sources* in the volume *V*.

# Differential form of Poynting's Theorem

$$\int_V \underline{E} \cdot \underline{J}\, dv = -\frac{\partial}{\partial t}\int_V \left( \frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2 \right) dv - \oint_S \left( \underline{E} \times \underline{H} \right) \cdot d\,\underline{s}$$

$$\int_V \frac{\partial U_{mec}}{\partial t} dv = -\frac{\partial}{\partial t}\int_V U_{EM}\, dv - \oint_S S \cdot d\,\underline{s}$$

$$\int_V \frac{\partial}{\partial t}(U_{mec} + U_{EM})\, dv = -\int_V \nabla . S\, dv$$

$$\boxed{\nabla . S = -\frac{\partial}{\partial t}(U_{mec} + U_{EM})}$$

Shows Conservation of Energy

# Poynting Vector in the Time Domain

- We define a new vector called the (instantaneous) ***Poynting vector*** as

$$\overline{S} = \overline{E} \times \overline{H}$$

- **The Poynting vector has units of $W/m^2$.**

- The Poynting vector has the same direction as the direction of propagation.

- The Poynting vector at a point is equivalent to the power density of the wave at that point.

If the earth receives 2cal $min^{-1}cm^{-2}$ solar energy, what are the amplitudes of electric and magnetic fields of radiation?

$E_0$=1026.8V/m

$H_0$=2.726A-turn/m

The electric field in an em wave is given by

$E=E_0\sin\omega(t-x/c)$, where $E_0$=1000Newton/coul. Find the energy contained in a cylinder of cross section $10^{-3}m^2$ and length 100cm along x-axis.

U=2.2x10-11 Joule

# Boundary conditions, Page 333, Ch. 7

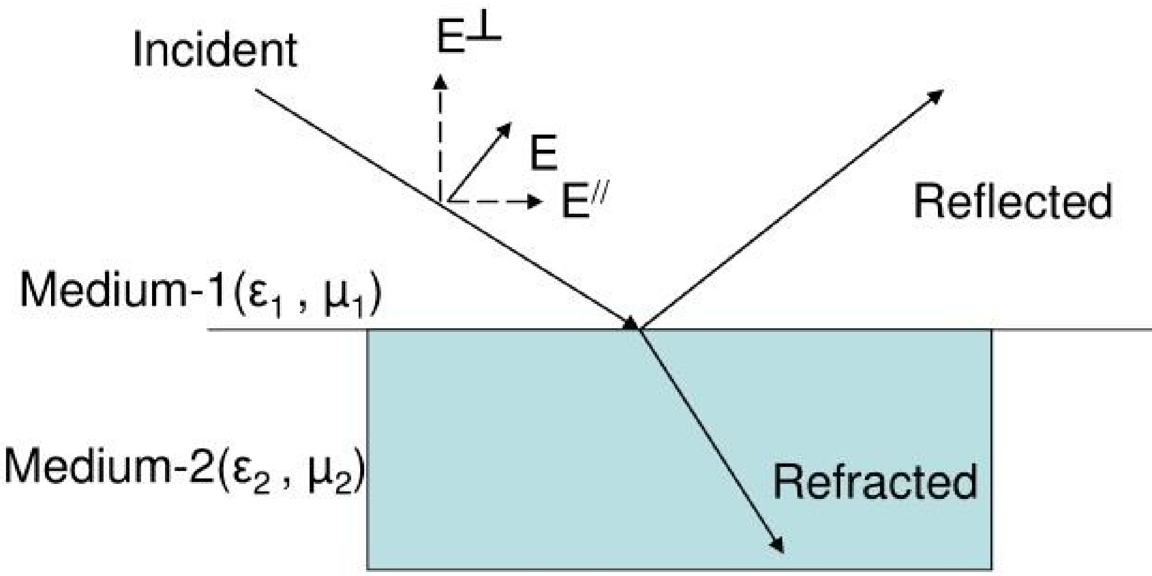
- If there is no free charge or free current at the interface of two medium, then

$$1.\ \varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp}$$

$$2.\ B_1^{\perp} = B_2^{\perp}$$

$$3.\ E_1^{//} = E_2^{//}$$

$$4.\ \frac{B_1^{//}}{\mu_1} = \frac{B_2^{//}}{\mu_2}$$



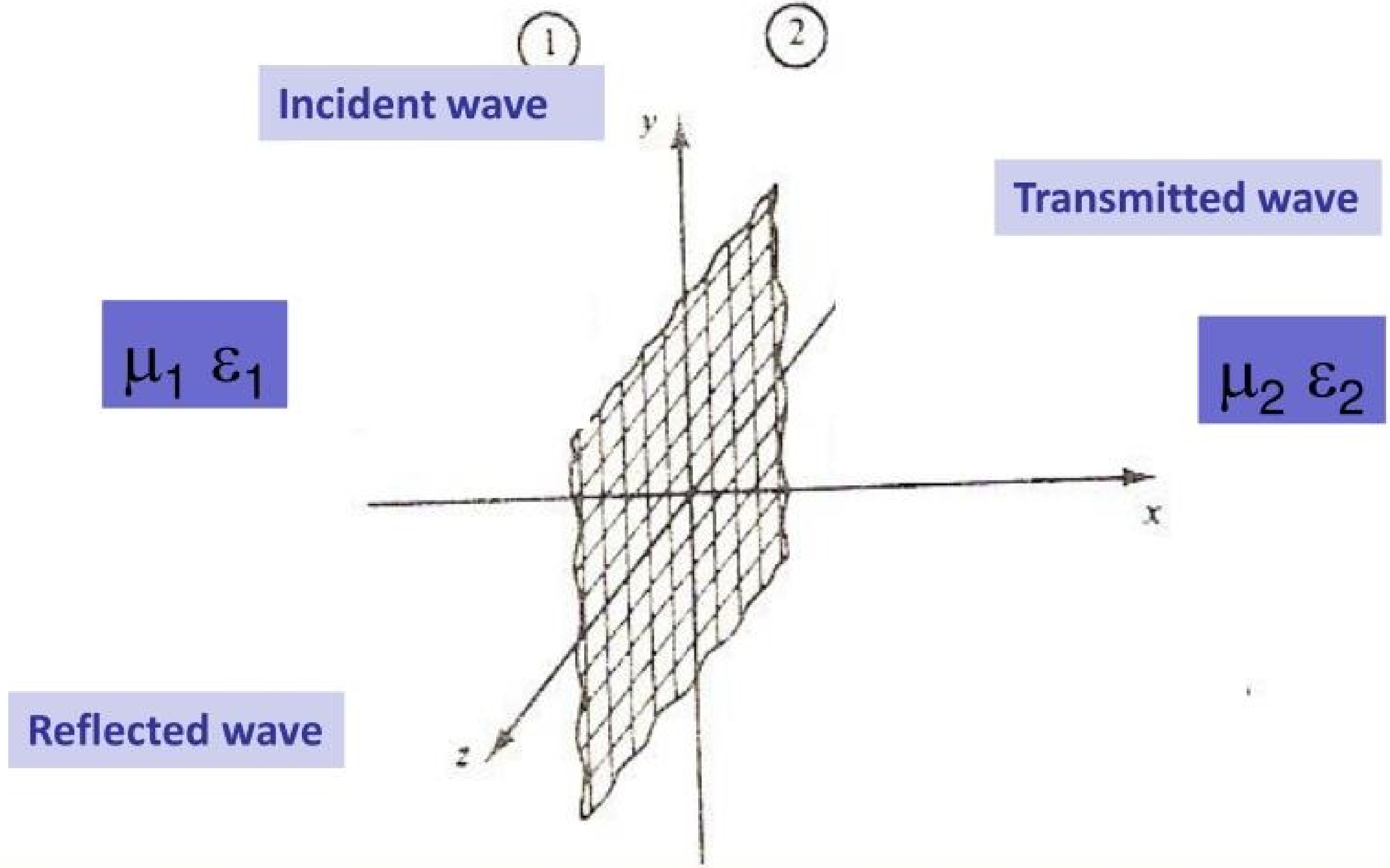
# Reflection and Transmission at **Normal incidence**

Reference

Introduction to Electrodynamics

By D. J. Griffith

# Reflection and Transmission at
# **Normal incidence**

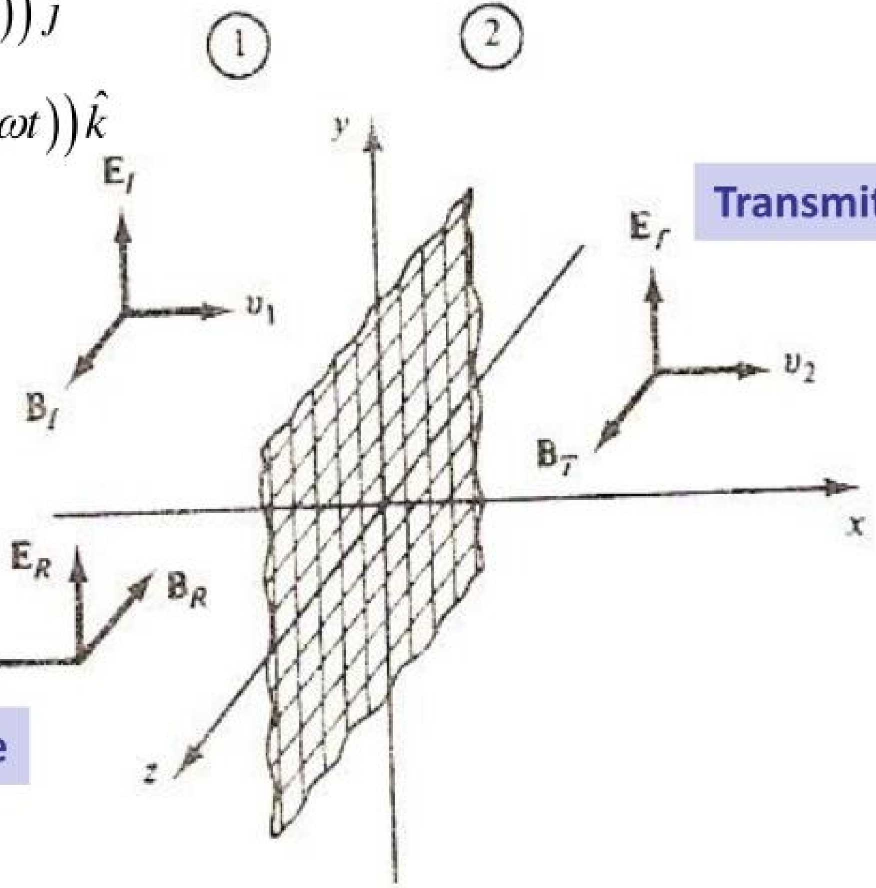



Suppose yz plane forms the boundary between two linear media. A plane wave of frequency ω traveling in The x direction (from left) and polarized along y direction, approaches the interface from left (see figure)

# Reflection and Transmission at **Normal incidence**

$$\vec{E}_I(x,t) = E_{0I} \exp\left(i\left(k_1 x - \omega t\right)\right)\hat{j}$$

$$\vec{B}_I(x,t) = \frac{1}{v_1} E_{0I} \exp\left(i\left(k_1 x - \omega t\right)\right)\hat{k}$$




Transmitted wave

$\mu_1\ \varepsilon_1$

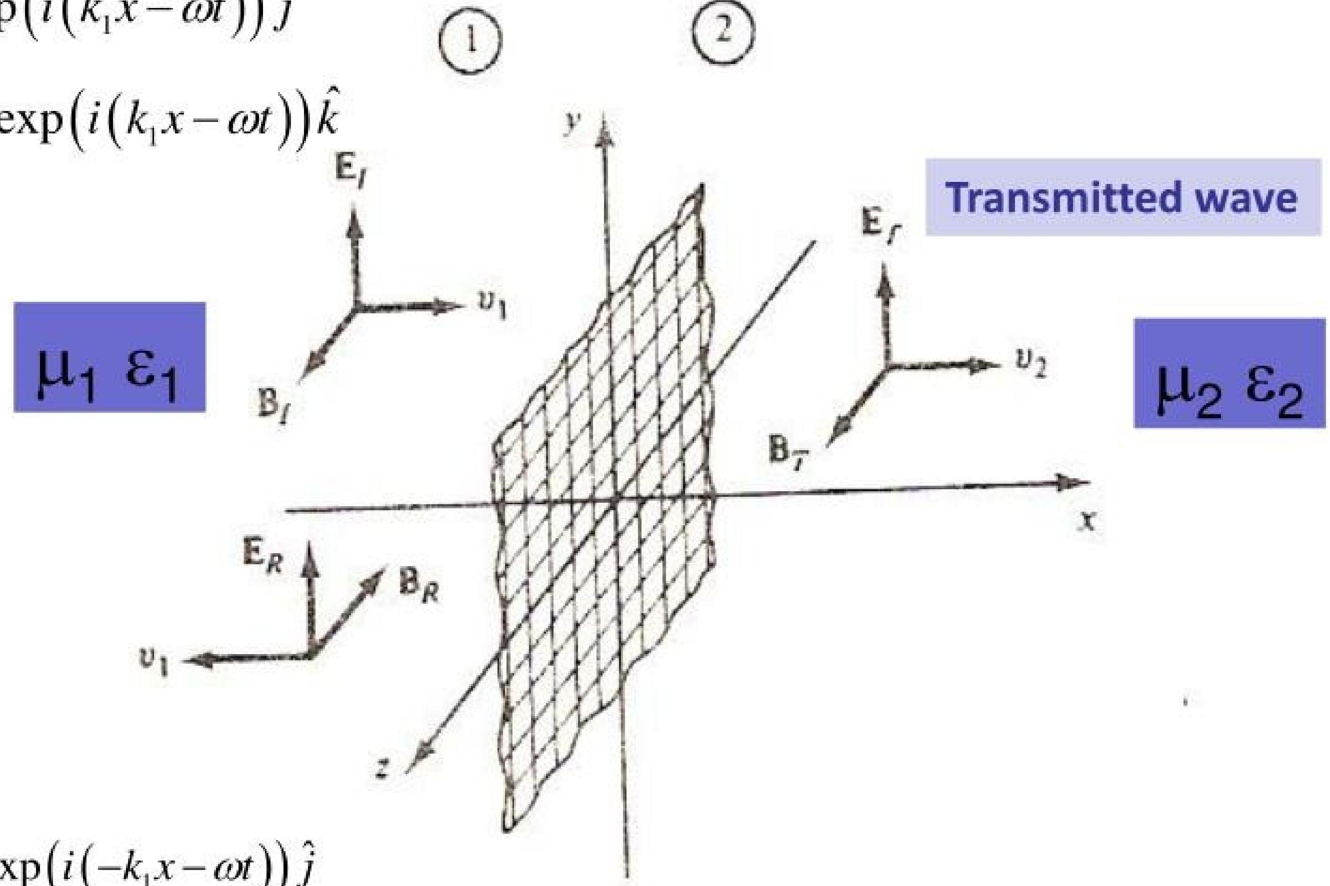
$\mu_2\ \varepsilon_2$

Reflected wave

# Reflection and Transmission at **Normal incidence**

$$\vec{E}_I(x,t) = E_{0I}\exp\left(i\left(k_1 x - \omega t\right)\right)\hat{j}$$

$$\vec{B}_I(x,t) = \frac{1}{v_1}E_{0I}\exp\left(i\left(k_1 x - \omega t\right)\right)\hat{k}$$




$$\vec{E}_R(x,t) = E_{0R}\exp\left(i\left(-k_1 x - \omega t\right)\right)\hat{j}$$

$$\vec{B}_R(x,t) = -\frac{1}{v_1}E_{0R}\exp\left(i\left(-k_1 x - \omega t\right)\right)\hat{k}$$

# Reflection and Transmission at **Normal incidence**

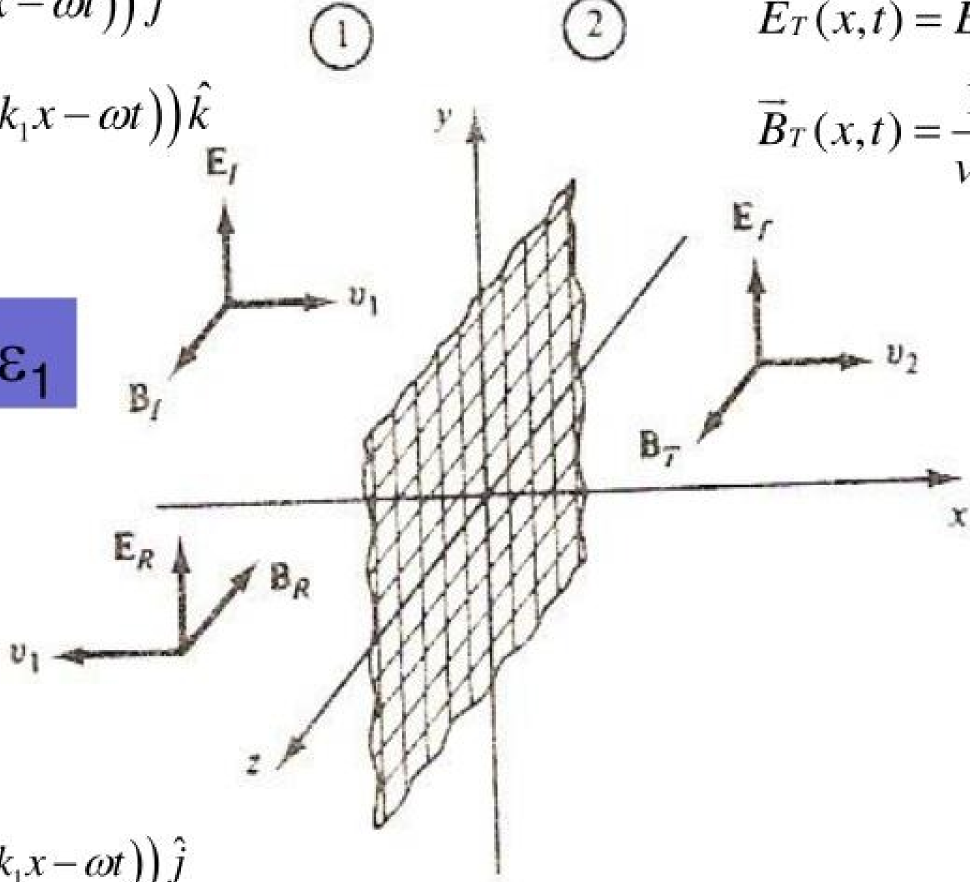
$$\vec{E}_I(x,t) = E_{0I}\exp\left(i\left(k_1 x - \omega t\right)\right)\hat{j}$$

$$\vec{B}_I(x,t) = \frac{1}{v_1}E_{0I}\exp\left(i\left(k_1 x - \omega t\right)\right)\hat{k}$$

$$\vec{E}_T(x,t) = E_{0T}\exp\left(i\left(k_2 x - \omega t\right)\right)\hat{j}$$

$$\vec{B}_T(x,t) = \frac{1}{v_2}E_{0T}\exp\left(i\left(k_2 x - \omega t\right)\right)\hat{k}$$

$\mu_1\ \varepsilon_1$




$\mu_2\ \varepsilon_2$

$$\vec{E}_R(x,t) = E_{0R}\exp\left(i\left(-k_1 x - \omega t\right)\right)\hat{j}$$

$$\vec{B}_R(x,t) = -\frac{1}{v_1}E_{0R}\exp\left(i\left(-k_1 x - \omega t\right)\right)\hat{k}$$

# Reflection and Transmission at **Normal incidence**

**Incident wave**

$$\vec{E}_I(x,t) = E_{0I}\exp\left(i\left(k_1x-\omega t\right)\right)\hat{j}$$

$$\vec{B}_I(x,t) = \frac{1}{v_1}E_{0I}\exp\left(i\left(k_1x-\omega t\right)\right)\hat{k}$$

$$\vec{E}_R(x,t) = E_{0R}\exp\left(i\left(-k_1x-\omega t\right)\right)\hat{j}$$

$$\vec{B}_R(x,t) = -\frac{1}{v_1}E_{0R}\exp\left(i\left(-k_1x-\omega t\right)\right)\hat{k}$$

**Reflected wave**

$$\vec{E}_T(x,t) = E_{0T}\exp\left(i\left(k_2x-\omega t\right)\right)\hat{j}$$

**Transmitted Wave**

$$\vec{B}_T(x,t) = \frac{1}{v_2}E_{0T}\exp\left(i\left(k_2x-\omega t\right)\right)\hat{k}$$

# Reflection and Transmission at **Normal incidence**

At **x=o** the **combined fields** to the left

$\mathbf{E}_I+\mathbf{E}_R$ and $\mathbf{B}_I+\mathbf{B}_R$, must join the fields to the right $\mathbf{E}_T$ and $\mathbf{B}_T$ in accordance to the **boundary condition**.

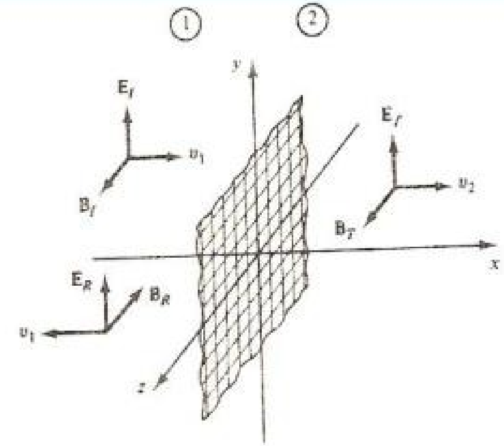
Since there are **no components perpendicular** to the surface so boundary conditions (i) and (ii) are trivial. **However last two [(iii) & (iv)] yields:**

$$\overline{D}_1^{\perp}(x=0)=\overline{D}_2^{\perp}(x=0) \qquad \text{(i)}$$

$$\overline{B}_1^{\perp}(x=0)=\overline{B}_2^{\perp}(x=0) \qquad \text{(ii)}$$

$$\overline{E}^{//}(x=0)=\overline{E}^{//}(x=0) \qquad \text{(iii)}$$

$$\frac{\overline{B}_1^{//}}{\mu_1}(x=0)=\frac{\overline{B}_2^{//}}{\mu_2}(x=0) \qquad \text{(iv)}$$



$$E_{0R}+E_{0I}=E_{0T}$$

$$\frac{1}{\mu_1}\left(\frac{1}{v_1}E_{0I}-\frac{1}{v_1}E_{0R}\right)=\frac{1}{\mu_2}\left(\frac{1}{v_2}E_{0T}\right)$$

$$\frac{1}{\mu_1}\left(\frac{1}{v_1}E_{0I}-\frac{1}{v_1}E_{0R}\right)=\frac{1}{\mu_2}\left(\frac{1}{v_2}E_{0T}\right)$$

$$\rightarrow E_{0I}-E_{0R}=\frac{\mu_1 v_1}{\mu_2 v_2}E_{0T}$$

$$E_{0R}+E_{0I}=E_{0T} \quad \text{-------------------------} \quad (1)$$

$$or, E_{0I}-E_{0R}=\beta E_{0T} \quad \text{-------------------------} \quad (2)$$

$$\text{where } \beta=\frac{\mu_1 v_1}{\mu_2 v_2}=\sqrt{\frac{\mu_1 \varepsilon_2}{\mu_2 \varepsilon_1}}=\frac{\mu_1 n_2}{\mu_2 n_1}$$

Using (1) and (2)

$$\therefore \boxed{E_{0R}=\left(\frac{1-\beta}{1+\beta}\right)E_{0I}} \text{ and } \boxed{E_{0T}=\left(\frac{2}{1+\beta}\right)E_{0I}}$$

$$\text{If } \mu_r \rightarrow 1 \text{(nonmagnetic media) then } \beta=\frac{v_1}{v_2}$$

$$\text{thus we have, } E_{0R}=\left(\frac{v_2-v_1}{v_2+v_1}\right)E_{0I} \text{ and } E_{0T}=\left(\frac{2v_2}{v_2+v_1}\right)E_{0I}$$

**The reflected wave is in phase if $v_2>v_1$ or $n_1>n_2$**

**and**

**out of phase if $v_2<v_1$. or $n_1<n_2$**

The real amplitudes are related by

$$E_{0R} = \left|\frac{v_2 - v_1}{v_2 + v_1}\right| E_{0I} \text{ and } E_{0T} = \left(\frac{2v_2}{v_2 + v_1}\right) E_{0I}$$

in terms of refractive index $n = \frac{c}{v}$

$$\boxed{E_{0R} = \left|\frac{n_1 - n_2}{n_2 + n_1}\right| E_{0I} \text{ and } E_{0T} = \left(\frac{2n_1}{n_2 + n_1}\right) E_{0I}}$$

Reflected wave is 180° out of phase when reflected from a denser medium. This fact was encountered by you during Last semester optics course. Now you have a proof!!!

# Reflectance (R) and Transmittance (T)

- Wave Intensity (average power per unit area is given by):

$$I = \frac{1}{2}\frac{B_0 E_0}{\mu} = \frac{1}{2}\varepsilon v E_0^2$$

- If $\mu_1 = \mu_2 = \mu_{0,}$ i.e $\mu_r = 1$ , then the ratio of the reflected intensity to the incident intensity is

$$\boxed{R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2}$$

Where as the ratio of transmitted intensity to incident intensity is

Use **ε α (n)²**

$$\boxed{T = \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \frac{n_2}{n_1}\left(\frac{2n_1}{n_1 + n_2}\right)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2}}$$

*Traveling **E** and **H** waves in free space ( region 1) are normally incident on the interface with a perfect dielectric (region 2) for which $\varepsilon_r$=3.0. Compare the magnitudes of the incident, reflected and transmitted **E** and **H** waves at the interface.*

Find out Reflectance and Transmittance .

Check the result.

Ans:

$$\frac{E_{0R}}{E_{0I}} = -0.268; \frac{E_{0T}}{E_{0I}} = 0.732$$

$$\frac{H_{0R}}{H_{0I}} = 0.268; \frac{H_{0T}}{H_{0I}} = 1.268$$

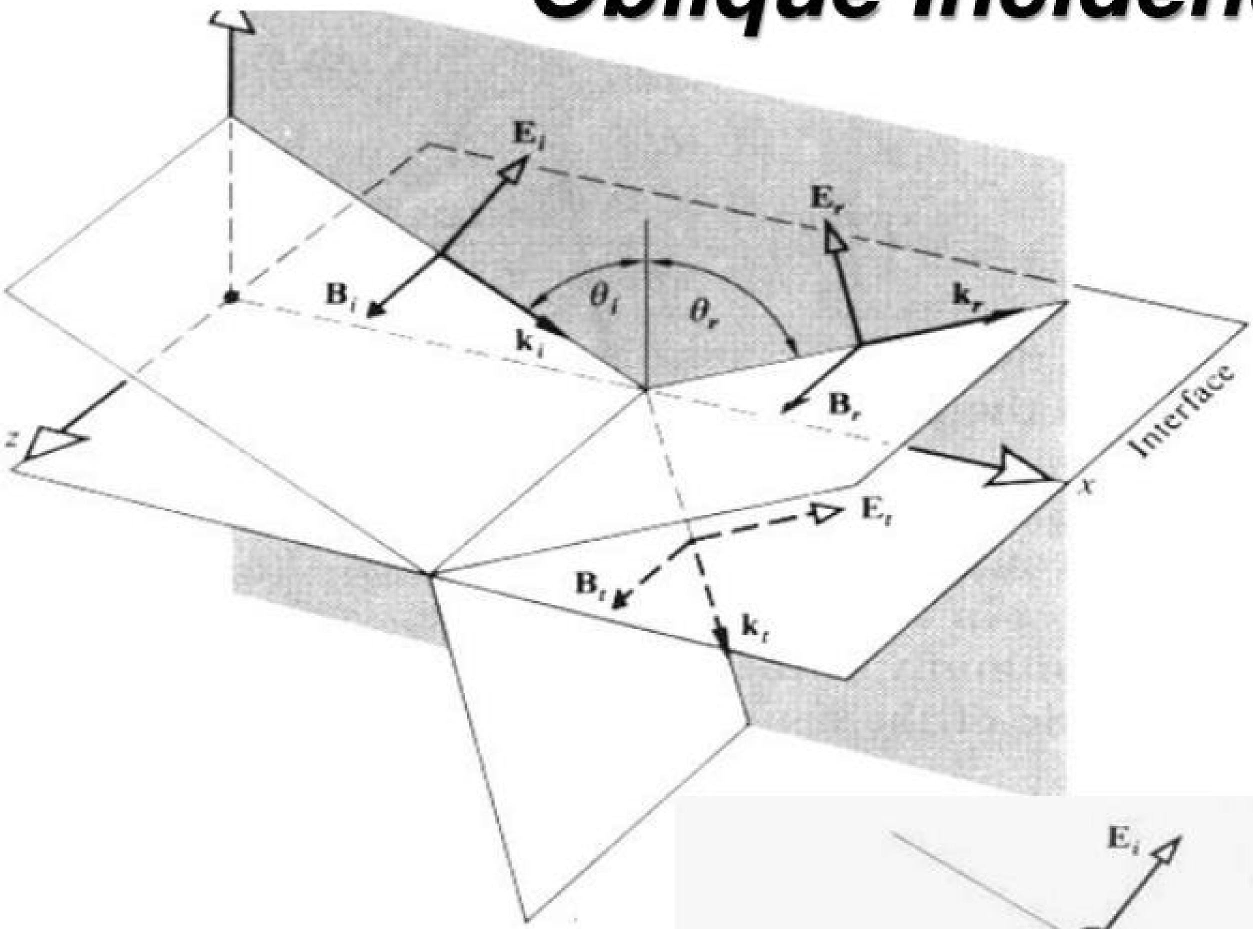
$$R = 0.072; T = 0.928$$

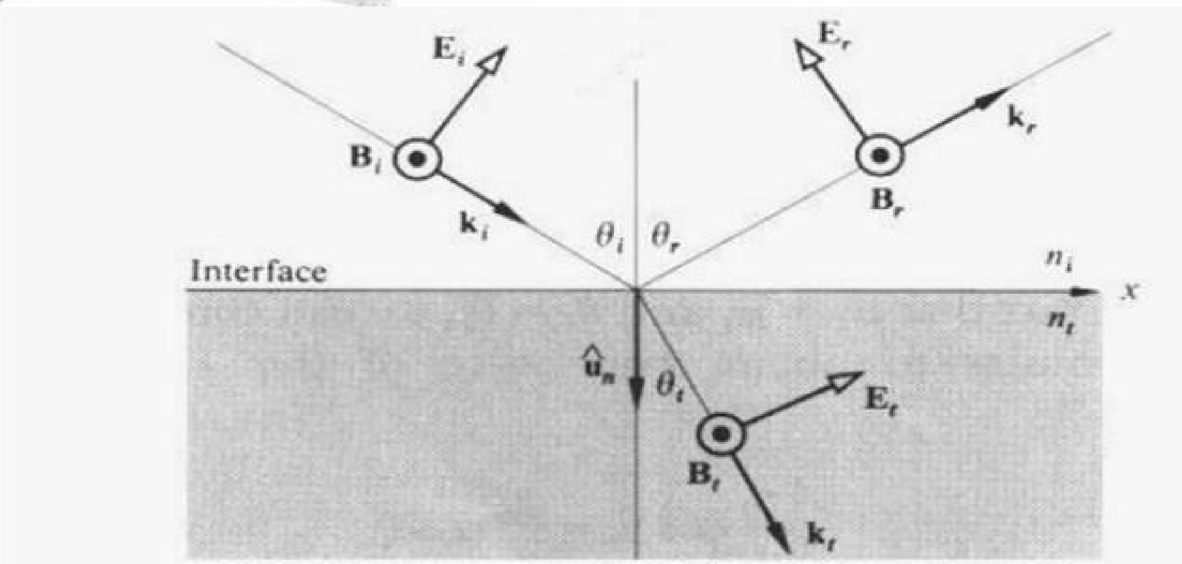
NOTE: R+T=1 => conservation of energy

# REFLECTION AND REFRACTION OF ELECTROMAGNETIC WAVES

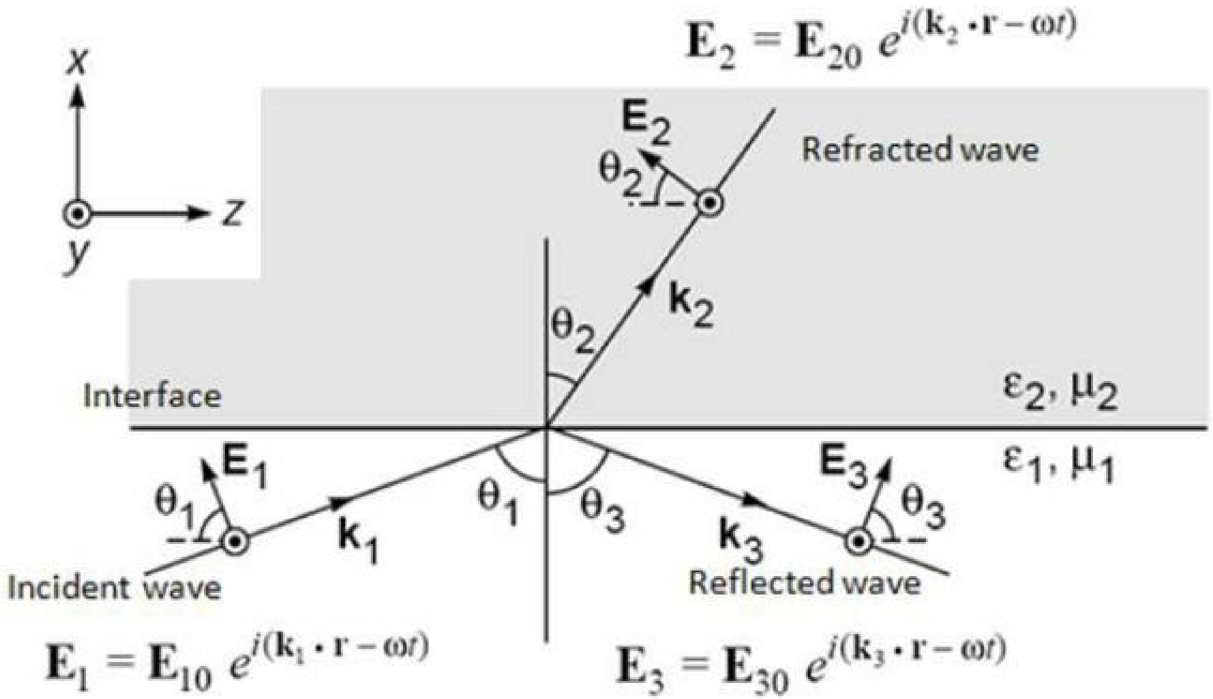
**Chapter 24, Optics by GHATAK 4th Ed.**

# *Oblique Incidence*



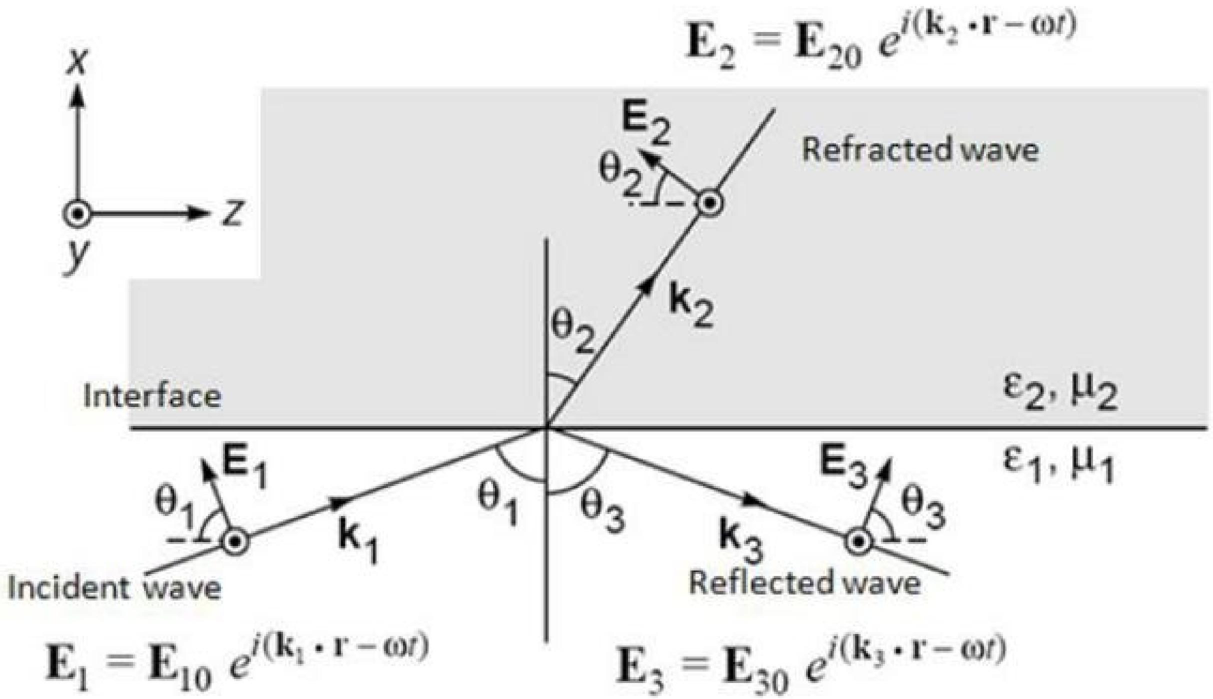
# Case 1: E parallel to the plane of incidence




***The reflection of a plane wave with its electric vector parallel to the plane of incidence.***

The parallel polarization (**or the *p polarization***) *is also called the transverse magnetic (or the TM) polarization as the magnetic field* is perpendicular to the plane of incidence.

# Case 1: E parallel to the plane of incidence




$$\overline{D_1}^{\perp}(x=0) = \overline{D_2}^{\perp}(x=0)$$
$$\overline{B_1}^{\perp}(x=0) = \overline{B_2}^{\perp}(x=0)$$
$$\overline{E_1}^{//}(x=0) = \overline{E_2}^{//}(x=0)$$
$$\overline{H_1}^{//}(x=0) = \overline{H_2}^{//}(x=0)$$

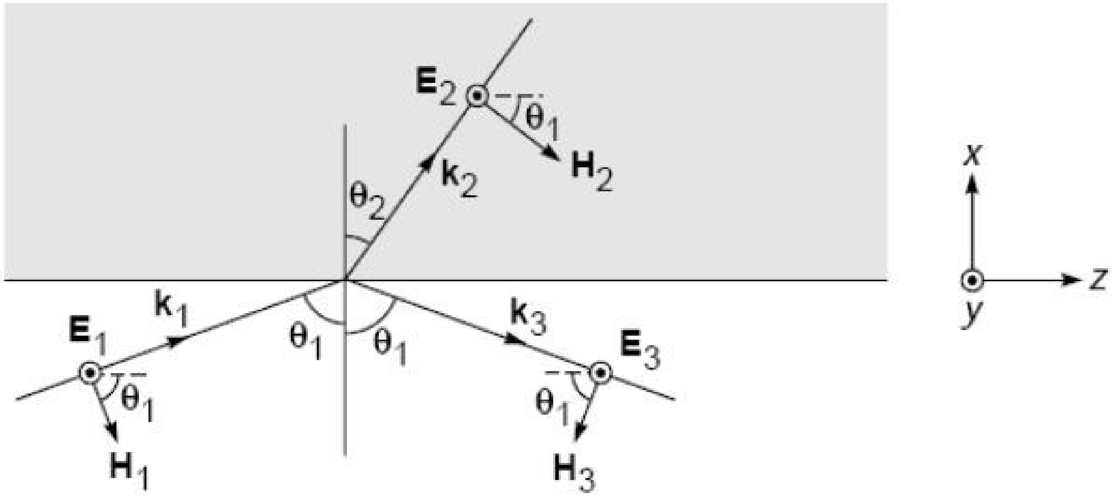
*The z component of the electric field represents a tangential component which should be continuous across the surface. Thus at x=0*

$$E_{1z}+E_{3z}=E_{2z}$$

*Further, the normal component of D must also be continuous, and since D = εE,*

$$\varepsilon_1 E_{1x}+ \varepsilon_1 E_{3x}= \varepsilon_2 E_{2x}$$

# Case 2. E perpendicular to the plane of incidence




$$\overline{D_1}^{\perp}(x=0)=\overline{D_2}^{\perp}(x=0)$$
$$\overline{B_1}^{\perp}(x=0)=\overline{B_2}^{\perp}(x=0)$$
$$\overline{E_1}^{//}(x=0)=\overline{E_2}^{//}(x=0)$$
$$\overline{H_1}^{//}(x=0)=\overline{H_2}^{//}(x=0)$$

*The reflection and refraction of a plane wave with the electric vector lying perpendicular to the plane of incidence.* ***(s-polarized)***

---

Since, the *y axis is tangential to the interface, the y component of* ***E must*** be continuous across the interface; consequently

$$E_{10}+E_{30}=E_{20}$$

*the z -component of the magnetic field to be continuous,*

$$H_{10}\cos\theta_1 - H_{30}\cos\theta_1 = H_{20}\cos\theta_2 \quad or \quad \frac{k_1}{\omega\mu_1}(E_{10}-E_{30})\cos\theta_1 = \frac{k_2}{\omega\mu_2}E_{20}\cos\theta_2$$

***We summarize the amplitude reflection and transmission coefficients for the two cases; the results are valid for nonmagnetic media:***

$$r_{\parallel} = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2}$$

$$= \frac{(n_2/n_1)^2 \cos\theta_1 - \sqrt{(n_2/n_1)^2 - \sin^2\theta_1}}{(n_2/n_1)^2 \cos\theta_1 + \sqrt{(n_2/n_1)^2 - \sin^2\theta_1}}$$

$$= \frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)}$$

$$r_{\perp} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$$

$$= \frac{\cos\theta_1 - \sqrt{(n_2/n_1)^2 - \sin^2\theta_1}}{\cos\theta_1 + \sqrt{(n_2/n_1)^2 - \sin^2\theta_1}}$$

$$= -\frac{\sin(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)}$$

$$t_{\parallel} = \frac{2 n_1 \cos\theta_1}{n_2 \cos\theta_1 + n_1 \cos\theta_2}$$

$$= \frac{2 \cos\theta_1 \sin\theta_2}{\sin\theta_1 \cos\theta_1 + \sin\theta_2 \cos\theta_2}$$

$$t_{\perp} = \frac{2 n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2} = \frac{2 \cos\theta_1 \sin\theta_2}{\sin(\theta_1 + \theta_2)}$$

***Equations are known as the Fresnel equations.***

*Ex 1: Light is incident at 60° on a boundary separated by media of refractive index $n_1 = 1$ and $n_2 = sqrt(3)$ . Find the amplitude of reflection coefficient r when electric field is perpendicular to the plane of incidence.*

$$\boxed{\begin{array}{l} r_{\perp} = \dfrac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} \\ n_1 \sin\theta_i = n_2 \sin\theta_t \end{array}}$$

*Ex 2: Consider a linearly polarized electromagnetic wave (with its electric field vector along y-direction) of magnitude 5 V/m propagating in vacuum. It is incident on a dielectric interface at x = 0 at an angle of incidence of 30°. The frequency associated with the wave is 6×10$^{14}$ Hz and refractive index of the dielectric is 1.5.*
*Show that R+ T=1.*

$$r_{\perp} = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} = 0.2404$$

$$t_{\perp} = \frac{2\cos\theta_i \sin\theta_t}{\sin(\theta_i + \theta_t)} = 0.7596 \; ; \; T_{\perp} = \frac{n_2 \cos\theta_2}{n_1 \cos\theta_1} \left| t_{\perp} \right|^2$$

$$\Rightarrow R_{\perp} = rr' = 0.057796$$

$$\Rightarrow T_{\perp} = \frac{4 n_1 n_2 \cos\theta_1 \cos\theta_2}{\left( n_1 \cos\theta_1 + n_2 \cos\theta_2 \right)^2} = 0.942204$$

*Hence Proved,* $\boxed{R_{\perp} + T_{\perp} = 1}$

# FIBER OPTICS

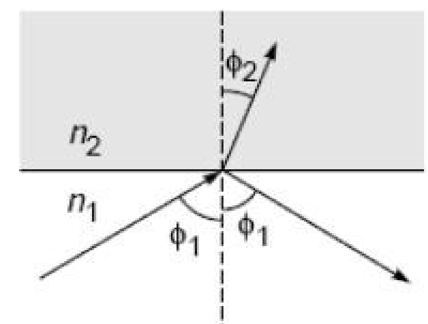
I have heard a ray of light laugh and sing. We may talk by light to any visible distance without any conducting wire.

—Alexander Graham Bell (1880),
after succeeding in transmitting a
voice signal over 200 m using light as the signal carrier

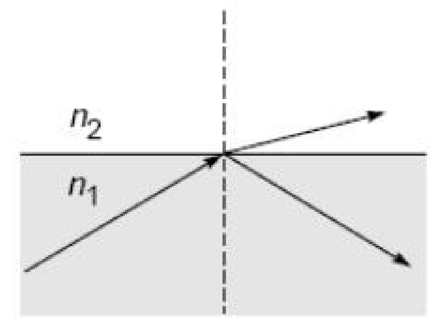
***REF: Optics by Ajoy Ghatak***
***Chapter -27***

# *TOTAL INTERNAL REFLECTION*

(a) For a ray incident on a denser medium ($n_2 > n_1$), the angle of refraction is less than the angle of incidence.



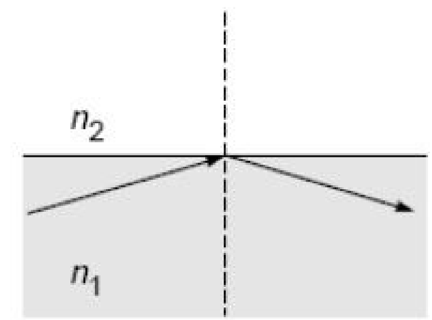

(b) For a ray incident on a rarer medium ($n_2 < n_1$), the angle of refraction is greater than the angle of incidence.




(c) The angle of incidence, for which the angle of refraction is 90°, is known as the critical angle and is denoted by $\phi_c$. *Thus, when*

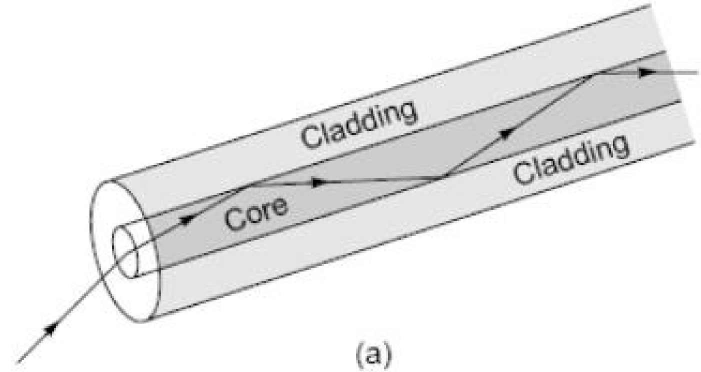
$$\phi_1 = \phi_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

the angle of refraction $\phi_2$ = 90°. When the angle of incidence exceeds the critical angle (i.e., when $\phi_1 > \phi_c$), *there is no refracted* ray and we have what is known as total internal reflection.



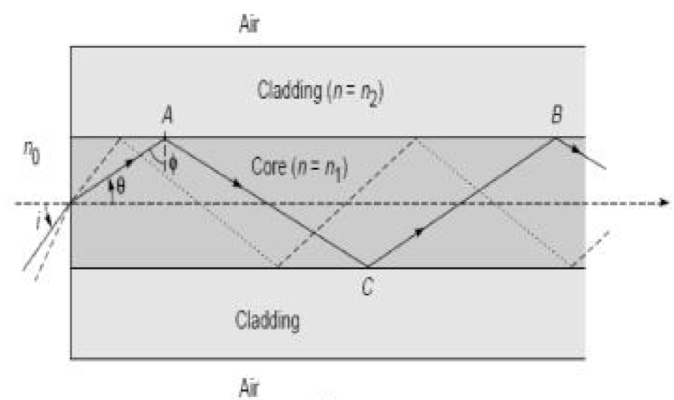
# THE OPTICAL FIBER

A dielectric (glass) fiber consists of a cylindrical central core cladded by a material of slightly lower refractive index.



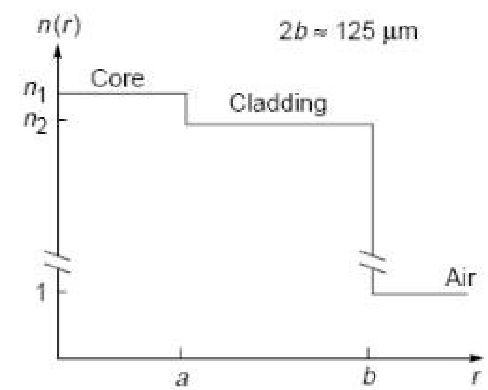
(a)

Light rays incident on the core-cladding interface at an angle greater than the critical angle are trapped inside the core of the fiber.




Refractive index distribution for a step-index fiber.

$$n = \begin{cases} n_1 & 0<r<a \\ n_2 & r>a \end{cases}$$



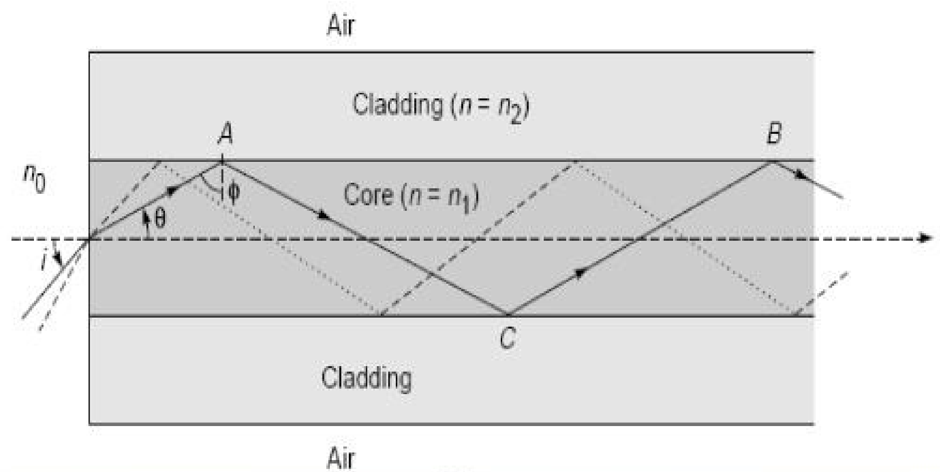

where $n_1$ and $n_2$ ($< n_1$) represent, respectively, the refractive indices of core and cladding and a represents the radius of the core.

Now, for a ray entering the fiber, if the angle of incidence (at the core-cladding interface) is greater than the critical angle $\phi_c$, *then the ray will undergo TIR at that interface. Thus,* for TIR to occur at the core-cladding interface

$$\phi > \phi_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

Or $\theta$ should be less than $\theta_c$:

$$\theta < \theta_c = \cos^{-1}\left(\frac{n_2}{n_1}\right)$$

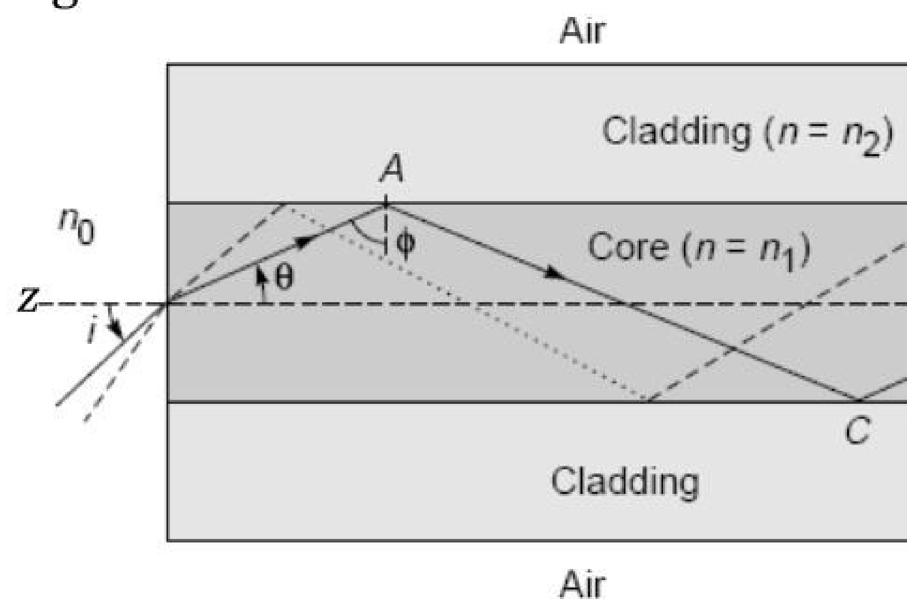



Further, because of the cylindrical symmetry in the fiber structure, the ray will suffer **TIR** at the lower interface also and therefore get guided through the core by repeated total internal reflections.

# THE NUMERICAL APERTURE

Consider a ray which is incident on the entrance aperture of the fiber, making an angle *i with the* axis. Assuming the outside medium to have a refractive index $n_o$

$$\frac{\sin i}{\sin \theta} = \frac{n_1}{n_0}$$



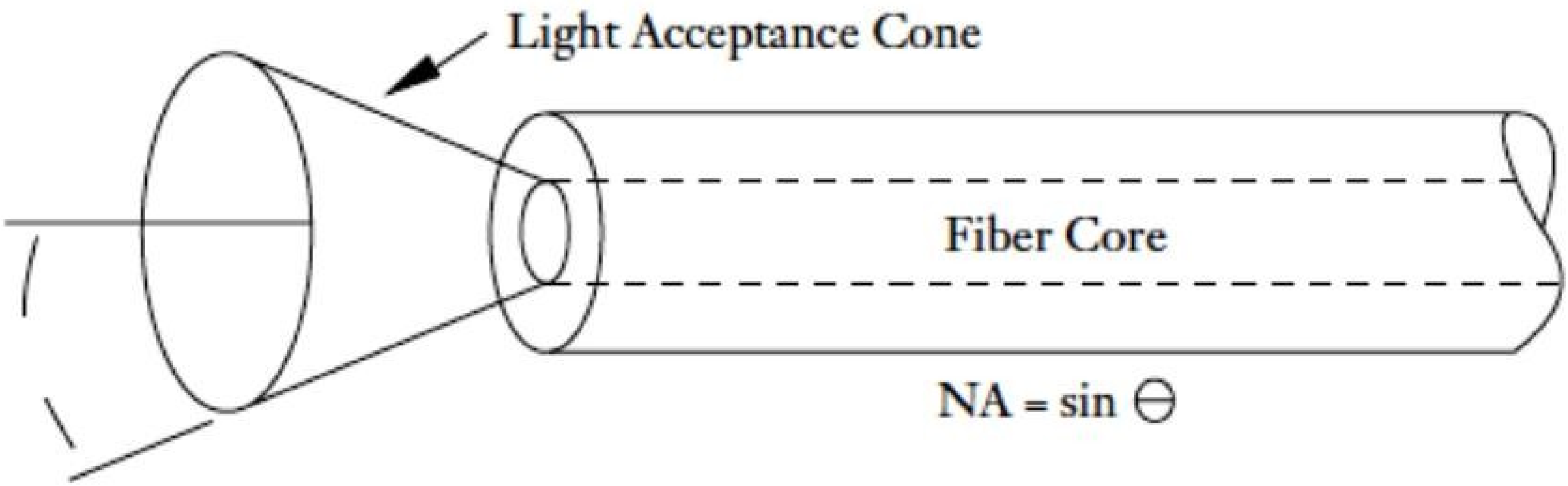

ray has to suffer total internal reflection. If the outside medium is air, i.e., $n_o = 1$; *then* the maximum value of sin $i$ *for a ray to be guided is given by*

$$\sin \phi (= \cos \theta) > \frac{n_2}{n_1}$$

$$\sin\,\theta < \frac{n_1}{n_0}\sqrt{1 - \left(\frac{n_2}{n_1}\right)^2} \qquad or, \quad \sin\,i < \sqrt{\frac{n_1^2 - n_2^2}{n_o^2}}$$

If the outside medium is air, i.e., $n_o = 1$; *and therefore* the maximum value of sin $i$ *for a ray to be guided is given by*

$$N.A. = \sin i_m = \sqrt{n_1^2 - n_2^2}$$




Thus, if a cone of light is incident on one end of the fiber, it will be guided through it provided the semi angle of the cone is less than $i_m$ ***(Acceptance Angle)***

***The quantity sin $i_m$ is known as the numerical aperture (NA) of the fiber and is a measure of the light-gathering power of the fiber.***

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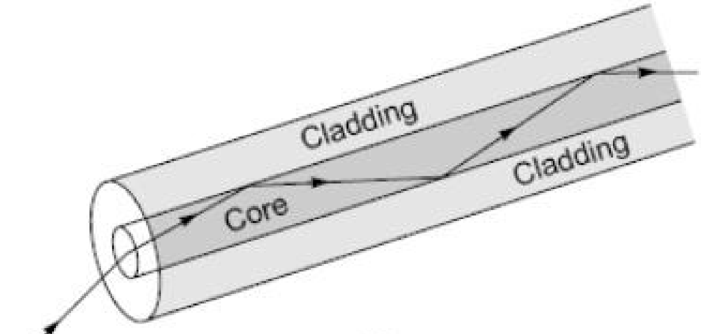


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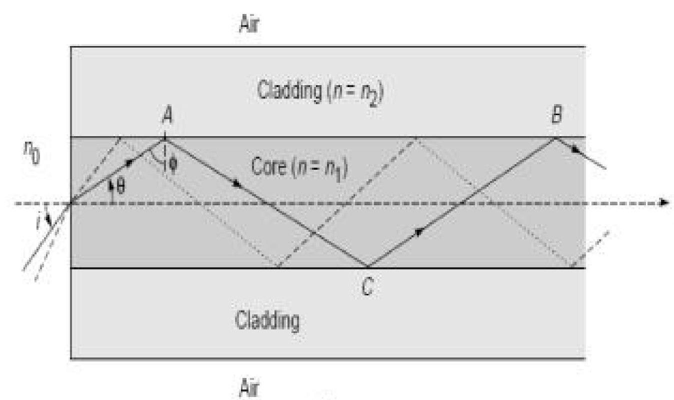


Refractive index distribution for a step-index fiber.

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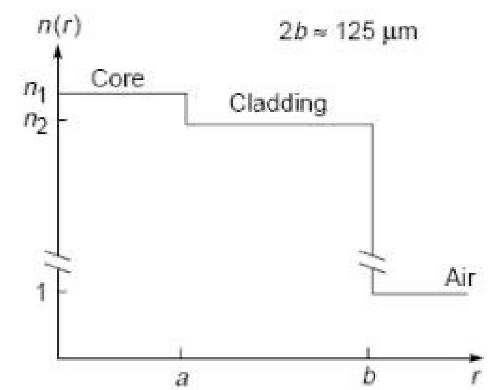
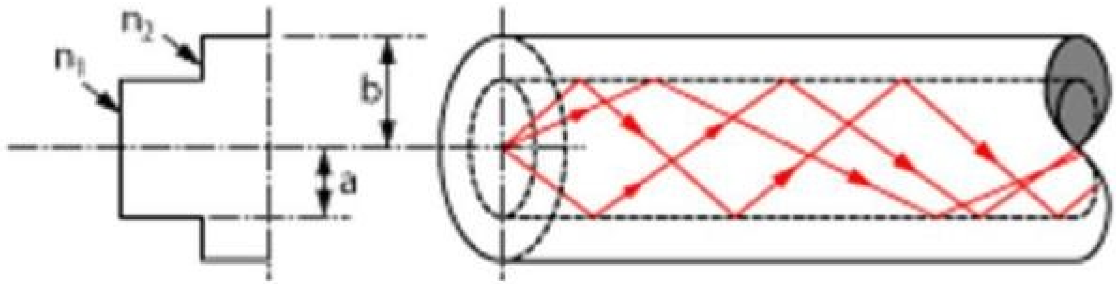



where $n_1$ and $n_2$ ($< n_1$) represent, respectively, the refractive indices of core and cladding and a represents the radius of the core.

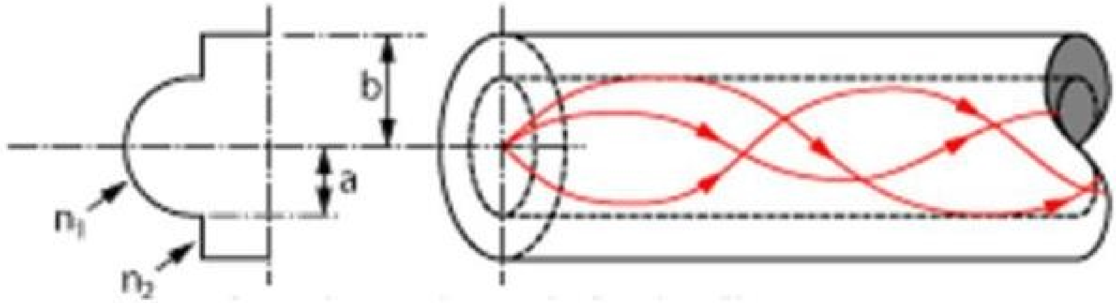
# Types of Fibers

*On the basis of refractive index profile of the core and the way in which light signal propagates down the core, the fibers are broadly divided into three categories.*

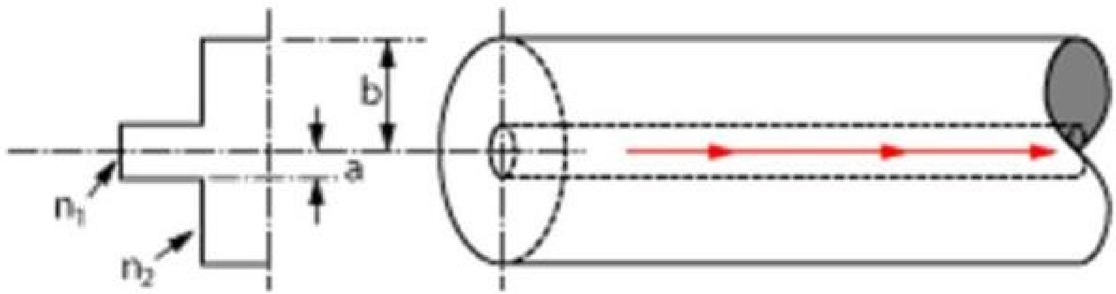
**Step index multimode fibers (MMF)**




**Graded Index multimode fibers (GRIN)**

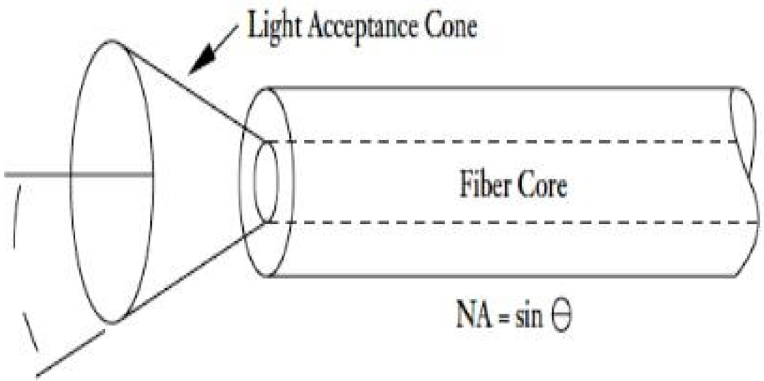



**Step index single mode or monomode fibers (SMF)**



# *Numerical Aperture for Step Index Fiber*

$$N.A. = \sin i_m = \sqrt{n_1^2 - n_2^2}$$




$$N.A. = n_1 \sqrt{\frac{n_1^2 - n_2^2}{n_1^2}} = n_1 \sqrt{2\Delta}$$

*where* $\Delta$ *is relative refractive index ,*

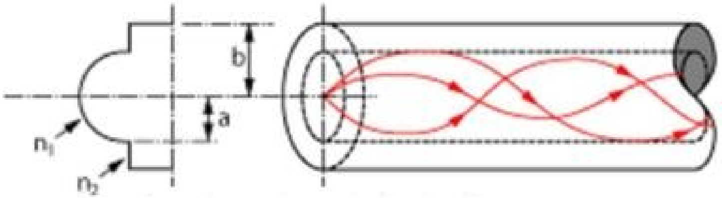
$$\Delta = \frac{n_1 - n_2}{n_1}$$

# Numerical Aperture for graded Index Fiber

*In the graded index fiber, the numerical aperture is a function of position across the core.*

$$NA = \sqrt{n^2(r) - n_2^2} = NA(r=0)\sqrt{1-\left(\frac{r}{a}\right)^x} \text{ for } r \le a$$

$$=0 \text{ for } r>a$$




*x : the refractive index profile variation*

*NA(r=0): numerical aperture at the centre of the fibre core.*

$$NA(0)=\sqrt{n^2(0) - n_2^2} = \sqrt{n_1^2 - n_2^2}$$

*For a graded index fiber numerical aperture decrease from axial numerical aperture NA(r=0) to zero as r increases from zero to core radius' a'.*

# *Number of Modes and Cut-off Parameters of Fibers*

The number of modes supported by an optical fiber is obtained by an important parameter associated with the cut-off condition is called, cut-off parameter such as normalized frequency of cut-off, which is referred to V parameter or V-number.

Mathematically V- number is expressed as, $V = \frac{\pi d}{\lambda_o}\sqrt{n_1^2 - n_2^2} \quad or, \; V \cong \frac{\pi d}{\lambda_o}(NA)$

The approximate total number of modes which the fiber will support is expressed as,

$$Number\ of\ Modes\ (N) \cong \frac{V^2}{2}$$

provided the *V-number > 1* . Out of all the modes only those modes will be propagated for which cut-off frequencies are less than the *V-number*.

If the external medium around the fiber has a refractive index $n_o$ then

$$\boxed{V = \frac{\pi d}{\lambda_o} n_o (NA) \qquad as \quad NA = \frac{\sqrt{n_1^2 - n_2^2}}{n_o}}$$

*Numerical : Compute the maximum value of Δ ( relative refractive index) and $n_2$ (cladding) of a single mode fibre of core diameter 10 μm and core refractive index 1.5. The fibre is coupled to a light source with a of 1.3μm. V cut-off for single mode propagation is 2.405. Also calculate the acceptance angle.*

$$\textit{relative difference of index} \quad \Delta = \frac{n_1 - n_2}{n_1}$$

Δ=0.0022
$n_2$ *(cladding)=1.497*
Acceptance angle = 5.71deg

# *Graded Index Fiber*

The variation of refractive index of the fibre core of the graded index multimode fiber with radius *a measured from the centre of the core is expressed as.*

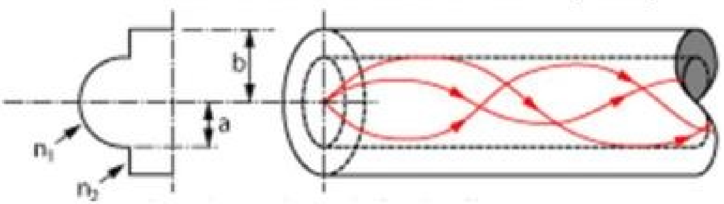
$$n(r) = n_1\left(1 - 2\Delta\left(\frac{r}{a}\right)^x\right)^{1/2}$$

*x :the refractive index profile variation*

In the graded index multimode fibre. the number of modes is expressed (provided the number modes is more than 50) as.

$$N = \left(\frac{x}{x+2}\right)\frac{2\pi^2 a^2 (NA)^2}{\lambda^2}$$

Graded Index multimode fibers (GRIN)




***N is doubled to account for the two possible polarizations.***

***Q3/Tut 4.*** *A graded index fibre has a core diameter 40 μm, NA = 0.21 and index profile = 1.85. Compute the number of modes at operating wavelength of 1.3 μm.*

No. of Modes: 197

# *Attenuation and Signal Losses in Optical Fibers*

*The reduction in amplitude (or power) and intensity of a signal as it is guided through an optical fibre is called attenuation.*

# Attenuation coefficient

*Attenuation losses in optical fibers are generally measured in terms of the decibel* $(dB)$.

*Due to attenuation, the power output*$(P_{out})$ *at the end of 1 km of optical fibre drops to some fraction* $(say\ k)$ *of the input power* $(P_{in})$ *, that is,*

$$P_{out} = k\,P_{in}$$

*After 2 km* $\qquad P_{out} = k^2 P_{in}$

*Similarly after L km* $\qquad P_{out} = k^L P_{in} \qquad or, \qquad \dfrac{P_{out}}{P_{in}} = k^L$

*Taking log of both sides and then multiply by 10 gives power loss in dB as*

$$\text{Power Loss } (dB) = 10 \log \frac{P_{out}}{P_{in}} = 10 \log k^L = L\,\underline{\underline{10 \log k}} = \underline{\underline{\alpha}}L$$

*where* $\alpha$ *is the attenuation coefficient of the fiber in dB/km.*

$$\alpha = \frac{10}{L} \log\left(\frac{P_{out}}{P_{in}}\right) dB / km$$

*To indicate loss we introduce negative sign in the expression* $\boxed{\alpha = -\dfrac{10}{L} \log\left(\dfrac{P_{out}}{P_{in}}\right) dB / km}$

# Attenuation coefficient

*The optical power, after propagating through a fibre that is 500 m long is reduced to 25% of its original value. Calculate the fibre loss in dB/km.*

**12.042 dB/km**

## *Attenuation and Signal Losses in Optical Fibers*

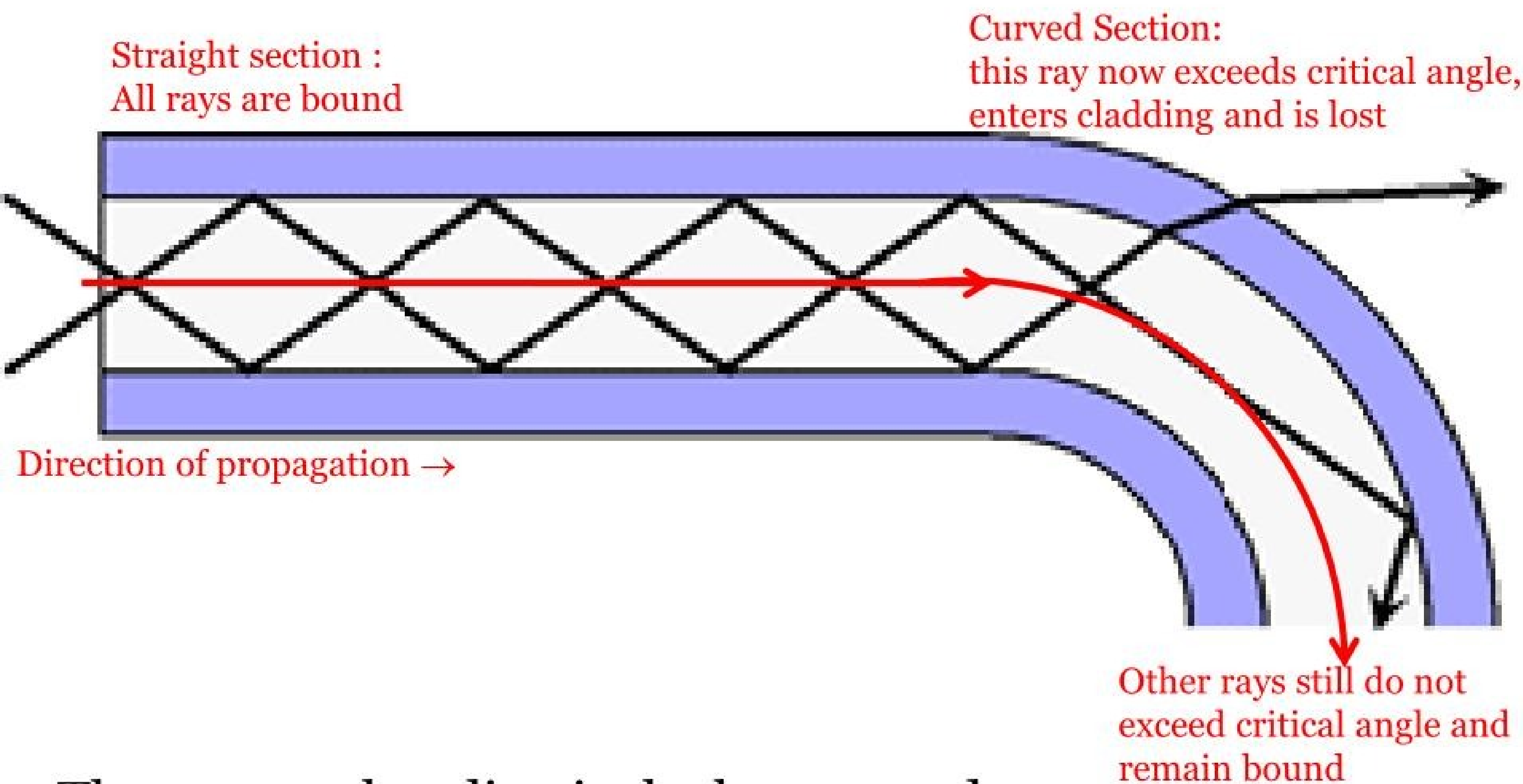
*The reduction in amplitude (or power) and intensity of a signal as it is guided through an optical fibre is called attenuation.*

*The loss of optical power and decrease in signal strength along a fibre are due to*

- *Absorption losses*
- *Rayleigh scattering losses*
- *Waveguide scattering losses*
- *bending losses,*
- *connector loss, splice loss, loss at terminals etc.*

# Attenuation and Signal Losses in Optical Fibers

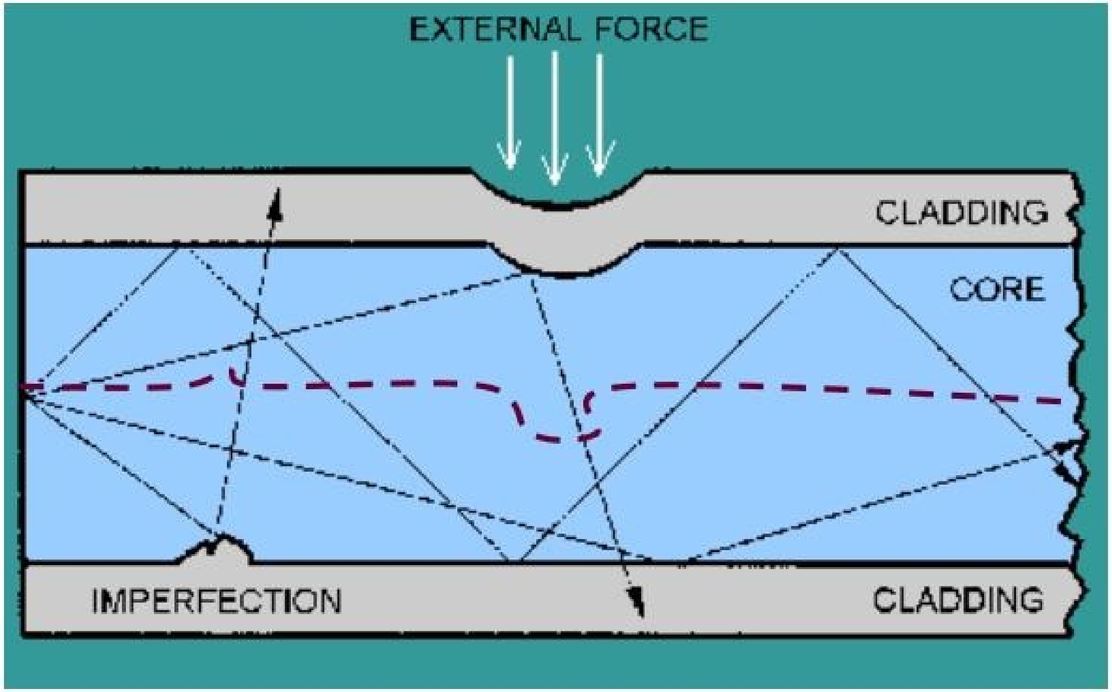
## Bending: Macrobending




Thus macro-bending is the loss caused by the curvature of the entire fiber axis

# Attenuation and Signal Losses in Optical Fibers

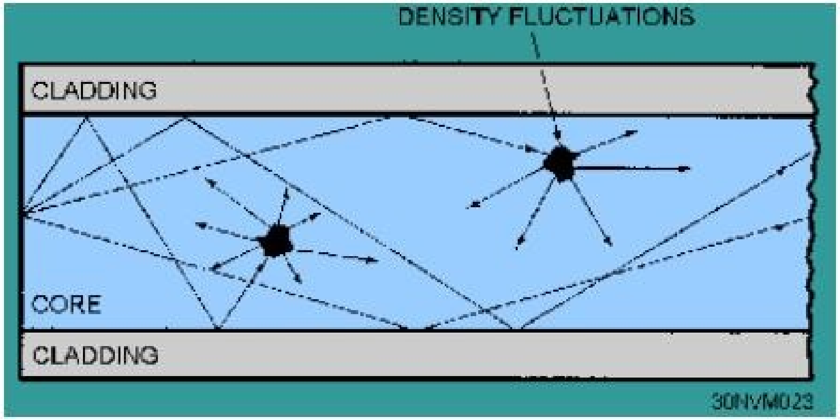
Bending: Macrobending and Microbending losses

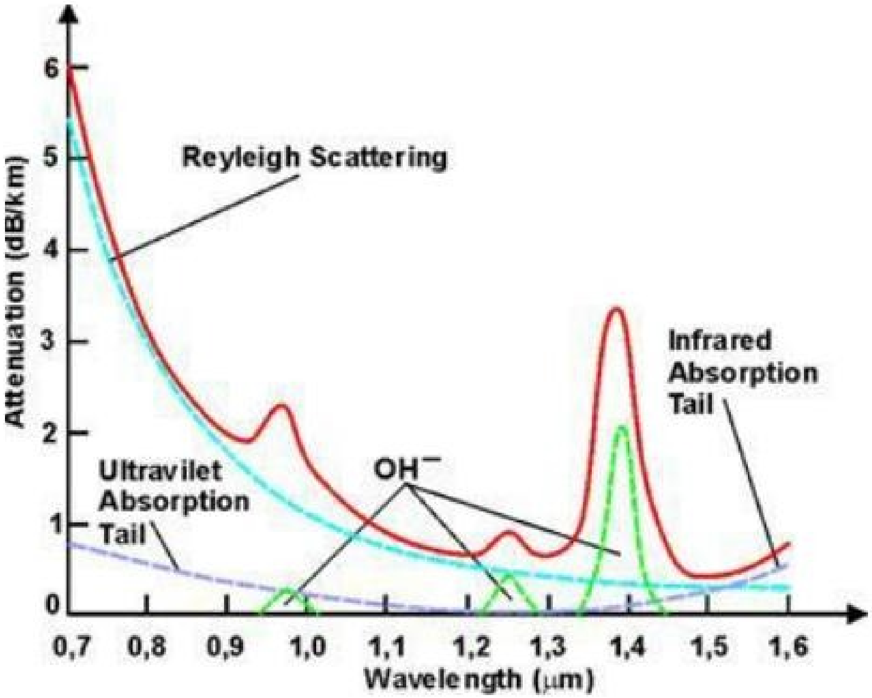



Thus micro-bending is the loss caused by micro deformations of the fiber axis.

# Attenuation and Signal Losses in Optical Fibers

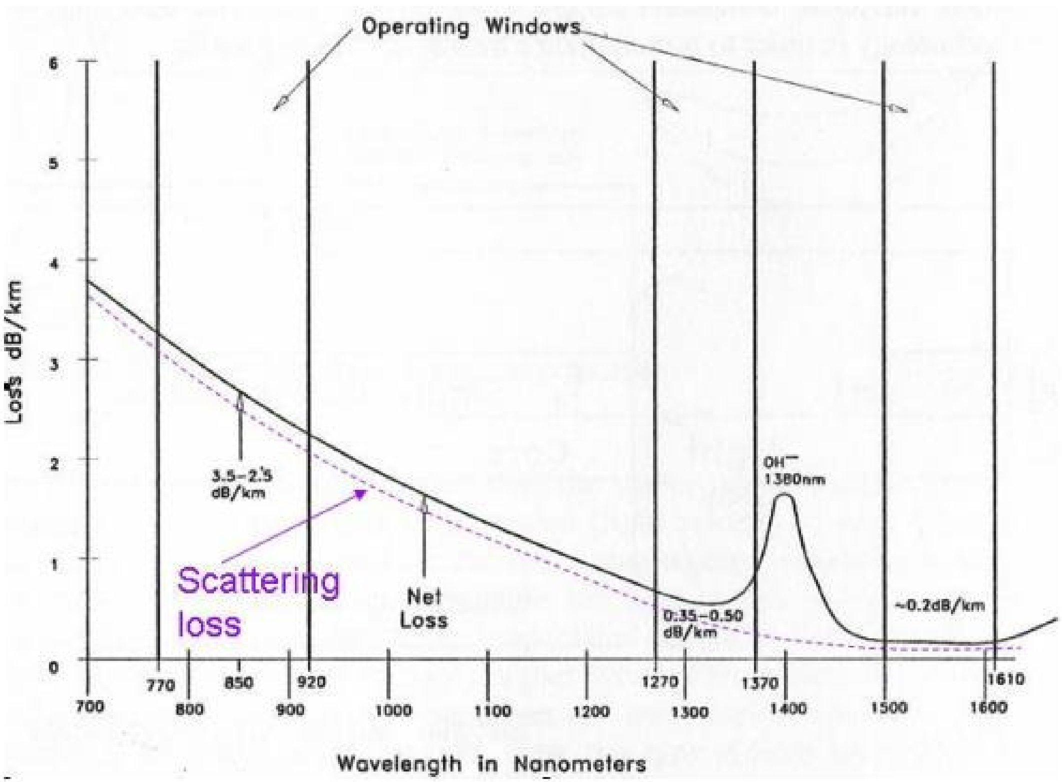
## Scattering losses




Even a very small change in core's refractive index will be seen by a traveling beam as an optical obstacle which will change the direction of the original beam.

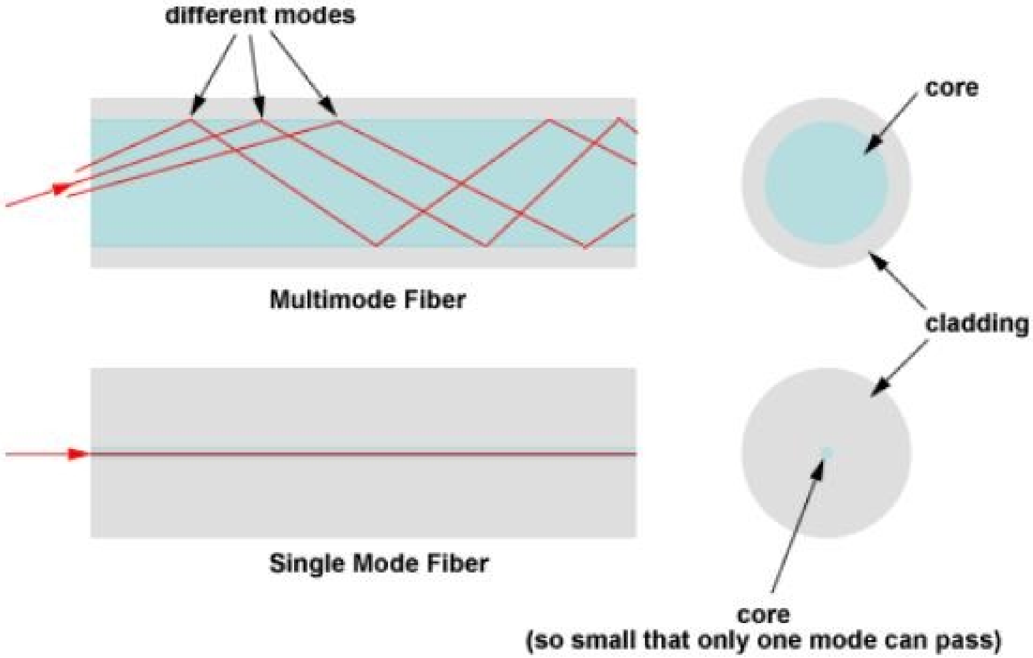
## Absorption losses



# SINGLE MODE FIBER

Model dispersion problem can be solved by restricting the number of modes i.e. by using single mode fibers. For a single mode fiber the condition is given as

$$V \leq 2.405$$




However, single mode fiber is the costliest one and also it is very difficult to maintain an accurate core size.

The core size of single mode fiber varies from 4 to 11 micrometer.