National University of Computer & Emerging Sciences, Lahore
Department of Electrical Engineering
(FALL 2024)

EE3005 – Electromagnetic Theory (EMT)
Assignment # 3 (CLO # 03, Marks = 30)

Notes:

1. **The assignment is to be submitted by the student himself / herself at the start of your respective class. No late submissions.**
2. **Any help from internet, solution manuals, or any other source is strictly prohibited. Violation may lead to serious consequences.**
3. **You will get credit for complete working of the solution, copying just the final answers of derivations / integrations will not be acceptable.**

Date of Submission: Monday, December 9, 2024

Drawing diagrams is a must for every Capacitor structure.

Problem Set

Formulate the **i)** *capacitance* and **ii)** *stored energy* for one dimensional change in potential using direct integration method (Laplace's equation) for the following functions:

1. $V(z)$
2. $V(\rho)$
3. $V(\theta)$

Show all the calculations starting from application of Laplacian operator in particular coordinate system along with diagram of the capacitor structure (initial conditions to be applied accordingly). Remember, direct answers are **NOT** acceptable.

Verify your answers using basic definition of capacitance i.e., finding potential from the electric field intensity and known charge distribution:

$$\boxed{C = \frac{Q}{V_o} = \frac{\epsilon \oint_S \mathbf{E} \cdot d\mathbf{S}}{-\int_{init}^{final} \mathbf{E} \cdot d\mathbf{L}}}$$

National University of Computer & Emerging Sciences, Lahore
Department of Electrical Engineering
(FALL 2024)

EE3005 – ELECTROMAGNETIC THEORY (EMT)

Assignment # 3 on CLO # 03

Name:	**Roll Number:**
Date of Submission: December 9, 2024	**Section:**

Instructions

1. Assignment must be handed over to the instructor at the start of the class only.
2. Late submission will not be accepted.
3. Only submit your own work. Do not copy from others. Plagiarism will be dealt seriously and it will be treated according to the University rules.
4. Submit assignment on A4 size paper.
5. Use blue or black ink only. Figures may be in color.
6. Handwriting must be legible. Otherwise, your assignment may be returned un-graded.

Problems ↓	**Score**	**CLO # 03**				
		Exemplary (5)	**Proficient (4)**	**Developing (3)**	**Beginning (2)**	**Novice (1)**
1. [10]						
2. [10]						
3. [10]						
Total						

CLO	**Statement ↓ Score →**	**Exemplary (5)**	**Proficient (4)**	**Developing (3)**	**Beginning (2)**	**Novice (1)**
03	Formulate the capacitance with one dimensional potential variation using direct integration	Find the capacitance by evaluating the total charge on the surface formed by one dimensional capacitor	Find the potential and electric field	Find two unknown constants using boundary conditions	Solve second order Laplace's equation in one dimension	Cannot solve Laplace's equation correctly

Attach this sheet to the front of submitted work.

Laplace Eq. : $\nabla^2 V = 0$ — ①

Capacitance : $C = \dfrac{Q}{V_0}$ — ②

Electric field : $E = -\nabla V$ — ③

Energy Stored: $U = \frac{1}{2} C V_0^2$ — ④

$$C = \frac{Q}{V_0} = \frac{\epsilon \oint_S E \cdot dS}{-\int_{\text{initial}}^{\text{final}} E \cdot dL} \quad \text{— ⑤}$$

①

$V(z)$

Here, V is a function of z only.

Hence, eq ① becomes

$$\frac{d^2V}{dz^2} = 0$$

Integrating b.s, we get

$$\frac{dV}{dz} = A$$

and again $V = Az + B$ — ⓐ

Applying boundary conditions on ⓐ

At $z = 0$; $V(0) = V_0$

$\Rightarrow V_0 = A(0) + B$

$B = V_0$

At $z = d$; $V(d) = 0$

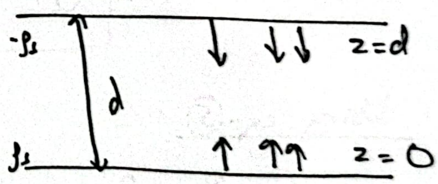
$\Rightarrow 0 = Ad + V_0$

$A = -\dfrac{V_0}{d}$

Therefore, eq ⓐ becomes

$$V(z) = -\frac{V_0}{d} z + V_0$$

Parallel-plate capacitor



From eq ③ $E_z = -\dfrac{dV}{dz}$

$$= -\frac{d}{dz}\left(-\frac{V_o}{d}z + V_o\right)$$

$$= \frac{V_o}{d}$$

As $Q = \sigma A$ and $\sigma = \epsilon E_z = \epsilon \dfrac{V_o}{d}$

$$\Rightarrow \quad Q = \epsilon \frac{V_o S}{d}$$

From eq ② $C = \dfrac{Q}{V_o}$

$$= \frac{\frac{\epsilon \cancel{V_o} S}{d}}{\cancel{V_o}}$$

$$\boxed{C = \frac{\epsilon S}{d}}$$

From eq ④ $U = \dfrac{1}{2} C V_o^2$

$$\boxed{U = \frac{1}{2}\frac{\epsilon S}{d} V_o^2}$$

<u>Using eq. ⑤</u>

$$E = \frac{V_o}{d}\hat{z}$$

$$\Rightarrow Q = \epsilon \int_S E \cdot dS = \epsilon \int_S \frac{V_o}{d} dS = \epsilon \frac{V_o}{d} S$$

$$\Rightarrow V_o = -\int_{\text{initial}}^{\text{final}} E \cdot dL = -\int_0^d \frac{V_o}{d} dz = \frac{V_o}{\cancel{d}}(\cancel{d} - 0) = V_o$$

$$\Rightarrow C = \frac{Q}{V_o} = \frac{\epsilon \frac{\cancel{V_o}}{d} S}{\cancel{V_o}} = \frac{\epsilon S}{d} \quad \Rightarrow U = \frac{1}{2} C V_o^2 = \frac{1}{2}\left(\frac{\epsilon S}{d}\right) V_o^2$$

<u>Verified</u>.

(2)

$V(\rho)$

Here, V is a function of ρ only.

Hence, eq (1) becomes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)=0$$

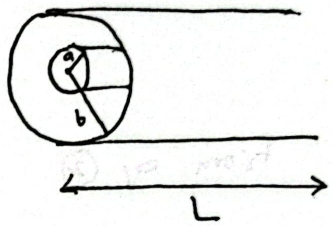
$$\frac{\partial^2 V}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial\rho}=0$$

Integrating b.s, we get

$$\rho\frac{dV}{d\rho}=A$$

$$\Rightarrow \frac{\partial V}{\partial\rho}=\frac{A}{\rho}$$

Coaxial Capacitor



and again

$$V=A\ln(\rho)+B \quad \text{—(b)}$$

Applying boundary condition on (b)

Put $\rho=a \Rightarrow V(a)=V_0$

$$\Rightarrow V_0=A\ln(a)+B \quad \text{—(c)}$$

Put $\rho=b \Rightarrow V(b)=0$

$$\Rightarrow 0=A\ln(b)+B$$

$$B=-A\ln(b) \quad \text{—(d)}$$

Put in (c)

$$V_0=A\ln(a)-A\ln(b)$$

$$V_0=A\ln\left(\frac{a}{b}\right)$$

$$A=\frac{V_0}{\ln\left(\frac{a}{b}\right)}$$

From a to b,

$$A=\frac{V_0}{\ln\left(\frac{b}{a}\right)}$$

Put in (d)

$$B=-\left(\frac{V_0}{\ln\left(\frac{b}{a}\right)}\right)\ln(b)$$

$$=-\frac{V_0\ln(b)}{\ln\left(\frac{b}{a}\right)}$$

Hence, eq (b) becomes

$$V(\rho)=\frac{V_0}{\ln\left(\frac{b}{a}\right)}\ln\left(\frac{b}{\rho}\right)$$

From eq ③ $$E = -\nabla V = -\frac{dV}{d\rho}$$

$$= -\frac{V_0}{\ln\left(\frac{b}{a}\right)} \cdot \left(-\frac{1}{\rho}\right)$$

$$= \frac{V_0}{\ln\left(\frac{b}{a}\right)\rho}$$

Charge per unit length : $\lambda = \epsilon E_\rho \cdot \rho \Big|_{\rho=a}$

$$= \epsilon \frac{V_0}{\ln\left(\frac{b}{a}\right)\cancel{\rho}} \cdot \cancel{\rho} = \epsilon \frac{V_0}{\ln\left(\frac{b}{a}\right)}$$

As $\lambda = \frac{Q}{L}$

$$\Rightarrow Q = \lambda L = \frac{\epsilon V_0 L}{\ln\left(\frac{b}{a}\right)}$$

~~As~~ from eq ② $$C = \frac{Q}{V_0} = \frac{\frac{\epsilon \cancel{V_0} L}{\ln\left(\frac{b}{a}\right)}}{\cancel{V_0}}$$

$$\boxed{C = \frac{\epsilon L}{\ln\left(\frac{b}{a}\right)}}$$

From eq ④ $U = \frac{1}{2} C V_0^2 \Rightarrow \boxed{U = \frac{1}{2} \frac{\epsilon L}{\ln\left(\frac{b}{a}\right)} V_0^2}$

<u>Using eq ⑤</u>

$E = \frac{\lambda}{2\pi\epsilon\rho}\hat{\rho}$, S = area of inner cylinder $\rho = a$ with Length L

$$\Rightarrow Q = \epsilon \int_S E \cdot dS = \epsilon \int_0^{2\pi}\int_0^{L} \frac{\lambda}{2\pi\epsilon a}(a\,d\phi)\,dz = \frac{\cancel{\epsilon}\lambda \cancel{a}}{2\pi\cancel{\epsilon}\cancel{a}}\int_0^{2\pi} d\phi \int_0^{L} dz$$

$$= \frac{\lambda}{\cancel{2\pi}}(\cancel{2\pi})(L) = \lambda L$$

$$\Rightarrow V_0 = -\int_{ini}^{final} E \cdot dL = -\int_a^b \frac{\lambda}{2\pi\epsilon\rho}\,d\rho = \frac{\lambda}{2\pi\epsilon}\ln\left(\frac{b}{a}\right)$$

$$\Rightarrow C = \frac{Q}{V_0} = \frac{\cancel{\lambda} L}{\frac{\cancel{\lambda}}{2\pi\epsilon}\ln\left(\frac{b}{a}\right)} = \frac{2\pi\epsilon L}{\ln\left(\frac{b}{a}\right)} \Rightarrow U = \frac{1}{2}CV_0^2 = \frac{1}{2}\left(\frac{2\pi\epsilon L}{\ln\left(\frac{b}{a}\right)}\right)\left(\frac{\lambda}{2\pi\epsilon}\ln\frac{b}{a}\right)^2$$

(3)

$V(\theta)$

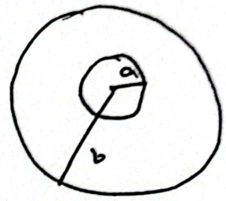
Here, V is a function of θ only.

Hence, eq (i) becomes

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

Spherical Capacitor



Integrating
$$\sin\theta\frac{\partial V}{\partial\theta}=A$$

$$\Rightarrow \quad \frac{\partial V}{\partial\theta}=\frac{A}{\sin\theta}$$

and again
$$V = A\ln(\sin\theta)+B \quad \text{—— (e)}$$

Applying boundary conditions

At $\theta=0$, $V(0)=V_0$

$$\Rightarrow V_0 = A\ln(\sin 0)+B$$

As $\ln(\sin 0) \Rightarrow \infty$ (undefined)

$\Rightarrow C_1 = 0$

At $\theta=\pi$, $\quad V(\pi) = C_2 = 0$

$\Rightarrow C_2 = 0$

This implies $\quad V(\theta)=0$

Therefore the potential does not vary with 'θ'.