Week 11- Finite difference

A General Approach to Interpolation

Interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable. The process of computing or finding the value of a function for any value of the independent variable outside the given range is called *extrapolation*. Here, interpolation denotes the method of computing the value of the function $y = f(x)$ for any given value of the independent variable x when a set of values of $y = f(x)$ for certain values of x are known or given.

Forward Differences

The *forward difference* or simply *difference* operator is denoted by Δ and may be defined as

$$\Delta f(x) = f(x + h) - f(x)$$

$$\Delta f(x_i) = f(x_i + h) - f(x_i)$$

$$\Delta y_i = y_{i+1} - y_i \qquad i = 0, 1, 2, \ldots, n - 1$$

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = (y_3 - y_2) - (y_2 - y_1) = y_3 - 2y_2 + y_1$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^{n+1} f(x) = \Delta[\Delta^n f(x)], \text{ i.e., } \Delta^{n+1} y_i = \Delta[\Delta^n y_i], \; n = 0, 1, 2, \ldots$$

$$\Delta^{n+1} f(x) = \Delta^n[f(x + h) - f(x)] = \Delta^n f(x + h) - \Delta^n f(x)$$

Backward Differences

The *backward difference operator* is denoted by ∇ and it is defined as

$$\nabla f(x) = f(x) - f(x - h)$$

$$\nabla y_i = y_i - y_{i-1}, \qquad i = n, n-1, \ldots, 1.$$

$$\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \ldots, \nabla y_n = y_n - y_{n-1}$$

$$\nabla^2 y_2 = \nabla(\nabla y_2) = \nabla(y_2 - y_1) = \nabla y_2 - \nabla y_1 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0.$$

$$\nabla^2 y_3 = y_3 - 2y_2 + y_1, \nabla^2 y_4 = y_4 - 2y_3 + y_2, \text{ and so on.}$$

Generalising, we have

$$\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1},$$

where $\nabla^0 y_i = y_i, \nabla^1 y_i = \nabla y_i.$

Forward Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

Backward Difference Table:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
x_4	y_4		$\nabla^2 y_5$			
		∇y_5				
x_5	y_5					

Example:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	-2			
		1		
1	-1		2	
		3		0
2	2		2	
		5		0
3	7		2	
		7		0
4	14		2	
		9		0
5	23		2	
		11		
6	34			

The Forward Interpolation Formula

We have Newton's forward difference interpolation formula

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$x - x_0 = kh \qquad \text{where} \quad 0 \le k \le n$$

Example (i)

(i) In the following table, use the Newton-Gregory Forward Interpolation formula to find
(a) f(2.4) (b) f(8.7).

x	2	4	6	8	10
f(x)	9.68	10.96	12.32	13.76	15.28

Simple difference table:

x	y=f(x)	Δy	$\Delta^2 y$
2	9.68		
		1.28	
4	10.96		0.08
		1.36	
6	12.32		0.08
		1.44	
8	13.76		0.08
		1.52	
10	15.28		

We have Newton's forward difference interpolation formula

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$x - x_0 = kh \qquad \text{where} \quad 0 \le k \le n$$

(a) $x = 2.4;\ x_0 = 2;\ h = 2;\ k = 0.2$

we get $f(2.4) \cong 9.68 + \frac{2.4-2}{2} \times 1.28 + \frac{(2.4-2)(2.4-4)}{4} \times \frac{0.08}{2}$

so $f(2.4) \cong 9.68 + 0.2 \times 1.28 + 0.1 \times (-1.6) \times 0.04 = 9.9296$

(b) $x = 8.7;\ x_0 = 2;\ h = 2;\ k = 3.35$

we get $f(8.7) \cong 9.68 + 3.35 \times 1.25 + 3.35 \times 2.35 \times 0.04 = 14.2829$

Example (ii)

In the following table of e^x use the Newton-Gregory formula of forward interpolation to calculate

(a) $e^{0.12}$, (b) $e^{2.00}$.

x	0.1	0.6	1.1	1.6	2.1
e^x	1.1052	1.8221	3.0042	4.9530	8.1662

We have Newton's forward difference interpolation formula

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

Simple difference table:

x	$y=e^x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	1.1052				
		0.7169			
0.6	1.8221		0.4652		
		1.1821		0.3015	
1.1	3.0042		0.7667		0.1962
		1.9488		0.4977	
1.6	4.9530		1.2644		
		3.2132			
2.1	8.1662				

(a) With x = 0.12, x_0 = 0.1, h = 0.5, k = 0.04

$$\therefore \qquad e^{0.12} \cong 1.1052 + 0.04 \times 0.7169 + 0.04 \times (-0.96)\frac{0.4652}{2}$$

$$+0.04 \times (-0.96) \times (-1.96)\frac{0.3015}{6} + 0.04 \times (-0.96) \times (-1.96) \times (-2.96)\frac{0.1962}{24}$$

$\therefore$ $e^{0.12}$=1.1269 (correct value to 5 d.p. is 1.12750)

$$p(x) = y_0 + k\Delta y_0 + k(k-1)\frac{\Delta^2 y_0}{2!} + \ldots + k(k-1)(k-2)\ldots(k-n+1)\frac{\Delta^n y_0}{n!}$$

(b) x = 2, x_0=0.1, h = 0.5, k = 3.8

$$e^2 \cong 1.1052 + 2.72422 + 2.47486 + 0.96239 + 0.12525$$

$\therefore e^2 \cong 7.3919$ (to 4 dp). (correct value 7.3891 to 4 dp)

Example

Consider the following table of values

x	.2	.3	.4	.5	.6
F(x)	.2304	.2788	.3222	.3617	.3979

Find f (.36) using Newton's Forward Difference Formula.

Simple difference table:

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.2	0.2304	0.0484	-0.005	0.0011	-0.0005
0.3	0.2788	0.0434	-0.0039	0.0006	
0.4	0.3222	0.0395	-0.0033		
0.5	0.3617	0.0362			
0.6	0.3979				

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$
$$+\frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots + \frac{p(p-1)(p-2)........(p-n+1)}{n!}\Delta^n y_0$$

Where

$x_0 = 0.2,\quad y_0 = 0.2304,\quad \Delta y_0 = 0.0484,$

$\Delta^2 y_0 = -0.005,\quad \Delta^3 y_0 = 0.0011\,,\ \Delta^4 y_0 = -.0005$ $\quad p = \dfrac{x - x_0}{h} = \dfrac{0.36 - 0.2}{0.1} = 1.6$

$$y_x = 0.2304 + 1.6(0.0484) + \frac{1.6(1.6-1)}{2!}(-0.005) + \frac{1.6(1.6-1)(1.6-2)}{3!}(0.0011) + \frac{1.6(1.6-1)(1.6-2)(1.6-3)}{4!}(-.0005)$$

$$= 0.2304 + .077441 - .0024 + \frac{1.6(.6)(-.4)}{6}(.0011) + \frac{1.6(.6)(-.4)(-1.4)}{24}(-.0005)$$

$$= 0.3078 - .0024 - .00007 - .00001$$

$$= .3053$$

Example:

Evaluate $f(15)$, given the following table of values:

x	10	20	30	40	50
$y = f(x)$	46	66	81	93	101

Simple difference table:

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	46				
20	66	20			
30	81	15	-5		
40	93	12	-3	2	
50	101	8	-4	-1	-3

We have Newton's forward difference interpolation formula

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$x_0 = 10, \quad y_0 = 46, \quad \Delta y_0 = 20,$$

$$\Delta^2 y_0 = -5, \quad \Delta^3 y_0 = 2, \quad \Delta^4 y_0 = -3$$

$$p = \frac{x - x_0}{h} = \frac{15-10}{10} = 0.5$$

$$f(15) = y_{15} = 46 + (0.5)(20) + \frac{(0.5)(0.5-1)}{2}(-5)$$

$$+\frac{(0.5)(0.5-1)(0.5-2)}{6}(2) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}(-3)$$

$$= 46 + 10 + 0.625 + 0.125 + 0.1172$$

$f(15) = 56.8672$ correct to four decimal places.

The Newton-Gregory Backward Interpolation Formula

$$y = f(x) \cong y_0 + k\nabla y_0 + k(k+1)\frac{\nabla^2 y_0}{2!} + k(k+1)(k+2)\frac{\nabla^3 y_0}{3!} + \ldots + k(k+1)\ldots(k+n-1)\frac{\nabla^n y_0}{n!}$$

Example

Apply the backward formula to find $e^{2.00}$ in Example (ii) of the previous section.

Example (ii)

In the following table of e^x use the Newton-Gregory formula of forward interpolation to calculate

(a) $e^{0.12}$, (b) $e^{2.00}$.

x	0.1	0.6	1.1	1.6	2.1
e^x	1.1052	1.8221	3.0042	4.9530	8.1662

Simple difference table:

x	$y=e^{2.00}$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	1.1052				
		0.7169			
0.6	1.8221		0.4652		
		1.1821		0.3015	
1.1	3.0042		0.7667		0.1962
		1.9488		0.4977	
1.6	4.9530		1.2644		
		3.2132			
2.1	8.1662				

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

$$+ \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n y_n + \text{Error}$$

$$p = \frac{x - x_n}{h}$$

P

We have: x = 2.0, x_o = 2.10, h = 0.5 and k = -0.2

n

$$e^{2.00} \simeq 8.1662 - 0.2 \times 3.2132 - 0.2 \times 0.8 \times \frac{1.2644}{2} - 0.2 \times 0.8 \times 1.8 \times \frac{0.4977}{6} + \ldots$$

$$= 7.3920.$$

Example

The sales for the last five years is given in the table below. Estimate the sales for the year 1979

Year	1974	1976	1978	1980	1982
Sales (in lakhs)	40	43	48	52	57

Newton's backward difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1974	40				
1976	43	3			
1978	48	5	2		
1980	52	4	-1	3	
1982	57	5	1	2	5

The Newton-Gregory Backward Interpolation Formula

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

$$+\frac{p(p+1)(p+2)\cdots(p+n-1)}{n!}\nabla^n y_n + \text{Error}$$

$$p = \frac{x - x_n}{h}$$

$$p = \frac{1979 - 1982}{2} = -1.5$$

$$\nabla y_n = 5, \quad \nabla^2 y_n = 1,$$
$$\nabla^3 y_n = 2, \quad \nabla^4 y_n = 5$$

$$y_{1979} = 57 + (-1.5)5 + \frac{(-1.5)(-0.5)}{2}(1) + \frac{(-1.5)(-0.5)(0.5)}{6}(2)$$
$$+ \frac{(-1.5)(-0.5)(0.5)(1.5)}{24}(5)$$
$$= 57 - 7.5 + 0.375 + 0.125 + 0.1172$$

$$y_{1979} = 50.1172$$

Home Activity:

The values of sin x are given below for different values of x. Find the value of sin 42°.

x	40	45	50	55	60
y = f(x) sin x	0.6428	0.7071	0.7660	0.8192	0.8660

Calculate the value of f (84) for the data given in the table below:

x	40	50	60	70	80	90
f(x)	204	224	246	270	296	324

Find the missing entry in the following tables:

(*a*)

x	0	1	2	3	4
y = f(x)	1	3	13	—	81

(*b*)

x	0	1	2	3	4
y = f(x)	1	0	—	28	69

Try to solve questions (Q3 ,Q4,Q5) from exercise 3.3 [burden faires]

Stirling's Formula

Centered Differences

Consider the mean of the Gauss's forward and backward interpolation formula given l we get

$$y_p = y_0 + u\left[\frac{\Delta y_{-1} + \Delta y_0}{2}\right] + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$+ \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1)(u^2-4)}{5!}\left[\Delta^5 y_{-2} + \Delta^5 y_{-3}\right]$$

$$u = \frac{x - x_0}{h}$$

Use Stirling's interpolation formula to find y_{28}, given that $y_{20} = 48234$, $y_{25} = 47354$, $y_{30} = 46267$, $y_{35} = 44978$ and $y_{40} = 43389$.

Here $x = 30$ as origin and $h = 5$. Therefore $u = \dfrac{28-30}{5} = -0.4$. The difference table is shown below:

x	$u = \dfrac{x-30}{5}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
20	–2	48234				
			–880			
25	–1	47354		–207		
			–1087		5	
30	0	46267		–202		–103
			–1289		–98	
35	1	44978		–300		
			–1589			
40	2	43389				

$$P_n(x) = P_{2m+1}(x) = f[x_0] + \frac{sh}{2}(f[x_{-1}, x_0] + f[x_0, x_1]) + s^2h^2 f[x_{-1}, x_0, x_1]$$
$$+ \frac{s(s^2-1)h^3}{2} f[x_{-2}, x_{-1}, x_0, x_1] + f[x_{-1}, x_0, x_1, x_2])$$
$$+ \cdots + s^2(s^2-1)(s^2-4)\cdots(s^2-(m-1)^2)h^{2m} f[x_{-m}, \ldots, x_m]$$

Old formula DDT

The Stirling's interpolation formula is

$$y_u = y_0 + u\left[\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right] + \frac{u^2\Delta^2 y_{-1}}{2} + \frac{u(u^2-1)}{6}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{24}\Delta^4 y_{-2} + \cdots$$

$$= 46267 + (-0.4)\left[\frac{-1087 - 1289}{2}\right] + \frac{(-0.4)^2}{2}(202) + \frac{(-0.4)(-0.4^2-1)}{6}\left[\frac{5-98}{2}\right]$$
$$+ \frac{(-0.4)^2(-0.4^2-1)}{24}(-103) = 46724.0128$$

Example 2 Consider the table of data given in the previous examples. Use Stirling's formula to approximate $f(1.5)$ with $x_0 = 1.6$.

x	$f(x)$
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

f (1.5) = 0.5118200 use of DDT

Now use SDT and find f(1.5)


ANY
QUESTIONS