Operators Analysis

Shift operator, E

Average Operator, μ :

Differential Operator, D

Forward Operator : Δ

Backward Operator: ∇

Shift operator, E

Let $y = f(x)$ be a function of x, and let x takes the consecutive values x, $x + h$, $x + 2h$, etc. We then define an operator having the property

$$E\,f(x) = f(x+h)$$

Thus, when E operates on $f(x)$, the result is the next value of the function. Here, E is called the shift operator. If we apply the operator E twice on $f(x)$, we get

$$E^2 f(x) = E[E\,f(x)]$$

$$= E[f(x+h)] = f(x+2h)$$

Thus, in general, if we apply the operator 'E' n times on $f(x)$, we get

$$E^n f(x) = f(x+nh)$$

OR

$$E^n y_x = y_{x+nh}$$

$$Ey_0 = y_1, \qquad E^2 y_0 = y_2, \qquad E^4 y_0 = y_4, \qquad \ldots, \qquad E^2 y_2 = y_4$$

$$E^{-1} f(x) = f(x-h)$$

Average Operator, μ;

it is defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left[y_{x+(h/2)} + y_{x-(h/2)}\right]$$

Differential Operator, D

it is defined as

$$\left.\begin{aligned} Df(x) &= \frac{d}{dx} f(x) = f'(x) \\ D^2 f(x) &= \frac{d^2}{dx^2} f(x) = f''(x) \end{aligned}\right\}$$

Important Results Using $\{\Delta, \nabla, \delta, E, \mu\}$

$$\Delta y_x = y_{x+h} - y_x = E y_x - y_x$$
$$= (E-1) y_x$$
$$\Rightarrow \Delta = E - 1$$

Also

$$\nabla y_x = y_x - y_{x-h} = y_x - E^{-1} y_x$$
$$= (1 - E^{-1}) y_x$$

$$\Rightarrow \nabla = 1 - E^{-1} = \frac{E-1}{E}$$

Proofs for the Relations among the Operators:

1. $$\Delta = E - 1$$

Since $$\Delta f(x) = f(x+h) - f(x)$$

or $$\Delta f(x) = E[f(x)] - f(x) = (E-1)f(x)$$

Since $f(x)$ is arbitrary, so ignoring it, we have

$\Delta = E - 1$ or $E = 1 + \Delta$

2. $$\nabla = 1 - E^{-1}$$

We have
$$\begin{aligned}\nabla f(x) &= f(x) - f(x-h)\\ &= f(x) - E^{-1}[f(x)]\\ &= (1 - E^{-1})f(x)\end{aligned}$$

Hence $$\nabla = 1 - E^{-1}$$

3. $$\delta = E^{1/2} - E^{-1/2}$$

We have
$$\begin{aligned}\delta[f(x)] &= f(x+h/2) - f(x-h/2)\\ &= E^{1/2}\cdot[f(x)] - E^{-1/2}\cdot[f(x)]\\ &= (E^{1/2} - E^{-1/2})f(x)\end{aligned}$$

Hence $$\delta = E^{1/2} - E^{-1/2}$$

4. $\Delta = E\nabla = \nabla E = \delta E^{1/2}$

We have $E\nabla[f(x)] = E[f(x) - f(x-h)]$

$= E[f(x)] - E[f(x-h)]$

$= f(x+h) - f(x) = \Delta f(x)$

Hence $E\nabla = \Delta$

Again, $\nabla E[f(x)] = \nabla f(x+h) = f(x+h) - f(x) = \Delta f(x)$

Hence $\nabla E = \Delta$

Also, $\delta E^{1/2} \cdot [f(x)] = \delta[f(x+h/2)]$

$= f(x+h) - f(x) = \Delta f(x)$

Hence $\delta E^{1/2} = \Delta$

5. $$E = e^{hD}$$

where $$D = \frac{d}{dx}$$

We know $E[f(x)] = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$, by Taylor's series

$$= f(x) + hDf(x) + \frac{h^2}{2!}D^2 f(x) + \cdots = \left(1 + hD + \frac{h^2 D^2}{2!} + \cdots\right) f(x) = e^{hD} \cdot f(x)$$

Hence $$E = e^{hD}.$$

Prove that

1) $$1+\delta^2\mu^2=\left(1+\frac{\delta^2}{2}\right)^2$$

2) $$E^{1/2}=\mu+\frac{\delta}{2}$$

3) $$\Delta=\frac{\delta^2}{2}+\delta\sqrt{1+(\delta^2/4)}$$

4) $$\mu\delta=\frac{\Delta E^{-1}}{2}+\frac{\Delta}{2}$$

5) $$\mu\delta=\frac{\Delta+\nabla}{2}$$

(1) $$\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1})$$

LHS= $$\therefore \quad 1 + \mu^2\delta^2 = 1 + \frac{1}{4}(E^2 - 2 + E^{-2}) = \frac{1}{2}(E + E^{-1})^2$$

RHS= $$1 + \frac{\delta^2}{2} = 1 + \frac{1}{2}(E^{1/2} - E^{-1/2})^2 = \frac{1}{2}(E + E^{-1})^2$$

(2)

$$\mu + (\delta / 2)$$
$$= \frac{1}{2}(E^{1/2} + E^{-1/2} + E^{1/2} - E^{-1/2}) = E^{1/2}$$

(3)

RHS= $$\frac{\delta^2}{2}+\delta\sqrt{1+(\delta^2/4)}=\frac{\left(E^{1/2}-E^{-1/2}\right)^2}{2}+\frac{\left(E^{1/2}-E^{-1/2}\right)\sqrt{1+\frac{1}{4}\left(E^{1/2}-E^{-1/2}\right)^2}}{1}$$

$$=\frac{E-2+E^{-1}}{2}+\frac{1}{2}(E^{1/2}-E^{-1/2})(E^{1/2}+E^{-1/2})$$

$$=\frac{E-2+E^{-1}}{2}+\frac{E-E^{-1}}{2}$$

$$=E-1=\Delta$$ =LHS

(4)

$$\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1})$$
$$= \frac{1}{2}(1 + \Delta - E^{-1}) = \frac{\Delta}{2} + \frac{1}{2}(1 - E^{-1})$$
$$= \frac{\Delta}{2} + \frac{1}{2}\left(\frac{E-1}{E}\right) = \frac{\Delta}{2} + \frac{\Delta}{2E}$$

(5)

LHS = $\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})$

$$= \frac{1}{2}(E - E^{-1})$$
$$= \frac{1}{2}(1 + \Delta - 1 + \nabla) = \frac{1}{2}(\Delta + \nabla)$$ = RHS

Relationship among the operators

	E	Δ	∇	δ
E	E	$\Delta + 1$	$(1-\nabla)^{-1}$	$1+\frac{1}{2}\delta^2+\delta\sqrt{\left(1+\frac{1}{4}\delta^2\right)}$
Δ	$E-1$	Δ	$(1-\nabla)^{-1}-1$	$\frac{1}{2}\delta^2+\delta\sqrt{\left(1+\frac{1}{4}\delta^2\right)}$
∇	$1-E^{-1}$	$1-(1+\Delta)^{-1}$	∇	$-\frac{1}{2}\delta^2+\delta\sqrt{\left(1+\frac{1}{4}\delta^2\right)}$
δ	$E^{1/2}-E^{-1/2}$	$\Delta(1+\Delta)^{-1/2}$	$\nabla(1-\nabla)^{-1/2}$	δ