1 Counting (3 points)

a) How many cards must be selected from a standard deck of cards to guarantee that at least 3 cards of the same suit are selected? Explain your answer.

Solution:

In order to guarantee at least 3 cards of the same suit are selected, we'll start with that in a deck of cards there are 52 cards. 13 cards for each suit. 13x4 = 52.

If we have no more than 2 cards of each suit, that means that we do not have more than $2 \cdot 4 = 8$ cards.

Meaning, if we select the 9th card by the pigeon principle, we will have at least 3 cards of the same suit. (9 cards, 4 suits).

Hence, the least number of cards to select to guarantee 3 cards of the same suit is 9.

b) How many ways are there to select a pair of cards from a standard deck of cards such that one of the cards is red and the other one is black? Your answer can contain factorial or power expressions. Explain your answer

Solution:

To draw a pair of cards where one is black, and one is red:
Since there are 26 cards that are red, and 26 cards that are black, for each of the 26 ways, one black card can be chosen in 26 ways.

Take that number 26 ways for black and multiply it by 26 ways for red, and you end up with

$$26 \cdot 26 = 26^2 = 676$$

c) How many ways are there to divide a standard deck of cards over 4 players? Your answer can contain factorial or power expressions. Explain your answer.

Solution:

There are 52 cards that can be divided to 4 different players. Hence, 4^{52} possible ways to divide the cards among 4 players.

2 Relations (2 points)

a) Let R be a binary relation on the set of integers such that $(a,b) \in R$ if and only if $b = 2a$. What is the composite relation $R \circ R$?

Solution:

R is a binary relation on the set of integers so that $(a,b) \in R$ if $b = 2a$. The composite relation $R \circ R$ is given by:

$$R \circ R = \{(a,c) \mid \exists b \in R : (a,b) \in R \wedge (b,c) \in R\}$$

$$R \circ R = \{(a,c) \mid \exists b \in R : b = 2a \wedge c = 2b$$

$$R \circ R = \{(a,c) \mid \exists b \in R : c = 4a$$

$(a,c) \in R \circ R$ if and only if c = 4a.

Hence, the composite relation $R \circ R = \{(a,c) \mid \exists b \in R : c = 4a$

b) Let *R* be a binary relation on the set of ordered pairs of integers such that *((a,b),(c,d))* ∈ *R* if and only if *ad* = *bc*. Show that *R* is an equivalence relation.

Solution:

$((a,b),(c,d)) \in R$ if and only if *ad* = *bc*.

Symmetric:-

$(a,b)R(c,d)$

$ad = bc$

$bc = ad$

$cb = da$

Since, $\left(\left(c,d\right),\left(a,b\right)\right) \in R$

$\left(\left(a,b\right),\left(c,d\right)\right) \in R$ implies that $\left(\left(c,d\right),\left(a,b\right)\right) \in R$

Hence, it is symmetric.

Transitive:-

R = {((a,b),(c,d))|ad=bc}

Assume ((a,b),(cd)) in R and ((c,d),(e,f)) in R.

Since ad = bc, and cf = de.

Simplify ad = bc and solve for c/d.

ad = bc

ad/bd = bc/bd

a/b = c/d

Simplify cf = de and solve for c/d.

cf = de

cf/fd = de/fd

c/d = e/f

....

a/b = e/f

af = be

Hence, ((a,b),(e,f)) in R

Then, ((a,b), (c,d)) in R and ((c,d), (e,f)) in R implies that ((a,b), (e,f)) in R.

The relation R is transitive

Reflexive:-

$ab = ba$

$ab = ab$

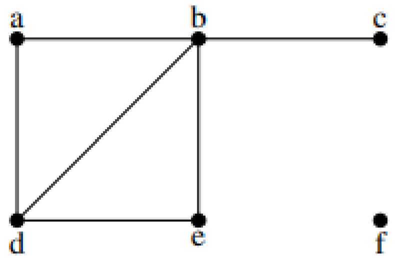
By definition of R.. $((a,b),(a,b)) \in R$

Relation R is reflexive.

$\therefore$ All conditions are fulfilled, and R is therefore an equivalence relation.

3 Graphs (5 points)

a) For the undirected graph shown below, give the number of vertices, the number of edges, and the degree of each vertex, and represent the graph with an adjacency matrix



Solution:

There are 6 vertices, (a,b,c,d,e,f).

The edges are (a,b), (b,c), (a,d), (b,e), (d,e), (d,b) so 6 in total.

The degrees of each vertex are:

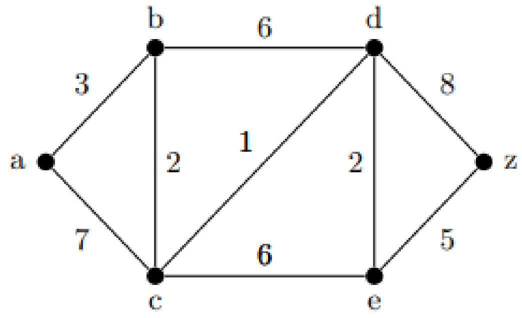
a = 2, b = 4, c = 1,

d = 3, e = 2, f = 0.

The adjacency matrix for the graph is:

0	1	0	1	0	0
1	0	1	1	1	0
0	1	0	0	0	0
1	1	0	0	1	0
0	1	0	1	0	0
0	0	0	0	0	0

b) For the undirected, weighted graph shown below, use Dijkstra's algorithm to find the length of a shortest path from vertex b to all other vertices.



Solution:

1. Set vertices distance to infinity but not the source vertex.
2. Push source vertex from the smallest priority (distance, vertex).
3. The vertex with smallest distance should be removed from the priority queue.
4. Update distance of vertirces to the vertex we removed, push the vertex with a new distance.
5. If the vertex that was removed appears again, ignore it.
6. Continue like this.

Iteration	S	N(S)
0	{b}	{a,c,d}
1	{a,b,c}	{d,e}
2	{a,b,c,d}	{e,z}
3	{a,b,c,d,e}	{e}
n	{a,b,c,d,e,z}	Ø

v	L(v)	L(v)	L(v)	L(v)
a		3	3	3
b	0	0	0	0
c		2	2	2
d		6	3	3
e			8	5
z				10