1 Propositional and predicate logic (5 points)

a) Truth table:

p	q	r	p → q	q → r	p → q)∧(q → r)	p → r	[(p → q)∧(q → r)] → (p → r)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

b) Since all submissions in the last column is T (True), the compound proposition [(p → q) ∧ (q → r)] → (p → r) is a tautology.

c) Proving that the compound propositions $(p \wedge \neg q) \rightarrow r$ and $p \rightarrow (q \vee r)$ are logically equivalent:

$(p \wedge \neg q) \rightarrow r \equiv \neg(p \wedge \neg q) \vee r$

($\because a \rightarrow b \equiv \neg a \vee b$) (conditional equivalence)

$\equiv [(\neg p) \vee (\neg(\neg q))] \vee r$ (De-Morgans law)

$\equiv [(\neg p) \vee q] \vee r$ (double negation)

$\equiv (\neg p) \vee (q \vee r)$ (associativity of v)

$\equiv p \rightarrow (q \vee r)$ (conditional equiv.)

Hence, the compound propositions are logically equivalent.

d)

$\forall n \exists m(n+m=0)$

For every integer n there is an integer m so that n+m=0.

Whether or not this a true statement, we can find out by testing with different integers:

n = 2

m = -2

n+m=0

-2+2=0.

Thus, this is a true statement.

$\forall n \exists m(n+m=0)$ is TRUE

Statement 2:

$\exists n \forall m(n<m^2)$

This statement can be read as, there is an integer n that for every integer m would be true in

$n < m^2$

If we try a random integer n

n = 10

And the integer m to be

m = 1

Then,

$n < m^2$

$10 < 1^2$, which isn't true as 1 is not larger than 10.

This, this is a false statement.

$\exists n \forall m(n<m^2)$ is FALSE.

2 Proof methods (2 points)

Prove that if *m* and *n* are integers and *mn* is even, then *m* is even, or *n* is even.

Using contraposition:

If *m* is even or *n* is even is false, that means that *m* is an odd integer as well as *n*.

m = 2x+1, for some integers x,

n = 2y+1, for some integers y,

mn = (2x+1) (2y+1)

= 4xy + 2x + 2y + 1

= 2 (2xy + x + y) + 1

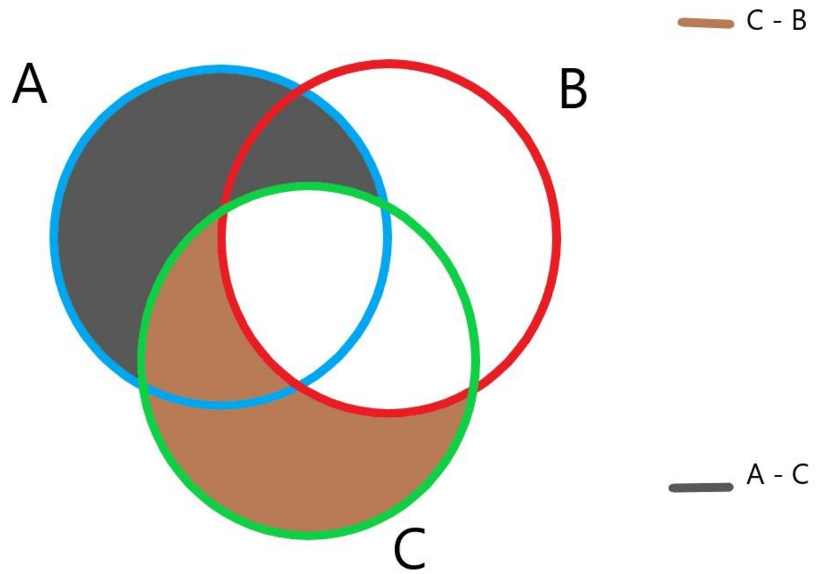
2z + 1 for some integers,

z = 2xy + x + y

From that conclusion, *mn* must be odd. That is a negation if *mn* is even and if *mn* is even, then either *m* is even, or *n* is even.

3 Set theory and functions (3 points)

a) Let A,B,C be sets. Draw a Venn diagram and color the region (A −C)∩(C − B). Prove (using set identities) or disprove (give a counterexample) that (A−C)∩(C −B) = /0.



As seen in this picture, C – B has been drawn as the brown color, while A – C has been drawn as the black/grey color.

(A −C)∩(C – B) denotes the region that both C – B and A – C have in common.

That region does not exist.

(Written in MathType vers. 7.4.4.516)

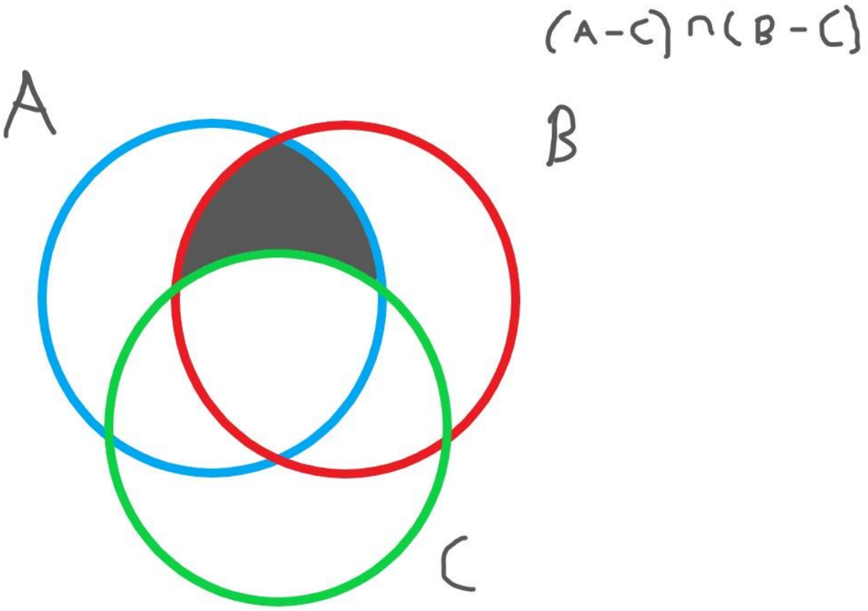
$$\therefore (A-C)\cap(C-B)=\phi$$

This can be proven using set identities.

$$
\begin{aligned}
&(A-C)\cap(C-B)\\
&= A\cap C^C)\cap(C\cap B^C)(\because X-Y=X\cap Y^C)\\
&= A\cap C^C\cap C\cap B^C(\because (X\cap Y)\cap Z=X\cap Y\cap Z)\\
&= A\cap\phi\cap B^C(\because X\cap X^C=\phi)
\end{aligned}
$$

$$=\phi(\because x\cap\phi=\phi=\phi\cap X)$$

b)



$(A-C)\cap(B-C)\neq\phi$

Proven:

$A=\{1,2,3,4\}$,
$B=\{5,3,4,8\}$
$C=\{9,3,5,4\}$
$A-C=\{1,2,5\}$
$B-C=\{8,1\}$
$(A-C)\cap(B-C)=\{1\}\neq\phi$

c) Let $f(x) = x^2$ be a function from the set of real numbers to the set of real numbers. Is *f* one-to-one (injective)? Onto (surjective)? A one-to-one correspondence (bijective)? Explain why/why not.

Statement 1: Injection.

The function f is injective if

$\forall a, b \in X \ (f(a) = f(b) \Rightarrow a = b)$

Let

$$a = 2$$
$$b = -2$$

$$f(a) = f(2) = 2^2 = 4,$$
$$f(b) = f(-2) = (-2)^2 = 4$$

$$4 = 4$$

$$f(a) = f(b)$$

Hence,

$$2 \neq -2,$$
$$a \neq b$$

It is therefore <u>not</u> injective.

Statement 2: Surjection.

The function f is surjective if

$y \in Y, \exists x \in X$ so that $f(x) = y$

So,

$$y = -3 \in \mathbb{R},$$
$$\exists x \in \mathbb{R}, f(x) = y,$$
$$x^2 = -3$$

However, $x^2 \geq 0 \forall x \in \mathbb{R}$

$$x^2 \neq -3$$

We have now argued two different things, and that is a contradiction. **<u>Which means that this function is not surjective. (By definition).</u>**

Lastly, whether the function is bijective. **<u>Which it cannot be</u>** as the function is not injective nor surjective.